Representation and Stability concerning General Nonic Functional Equation

Ick-Soon Chang, Yang-Hi Lee and Jaiok Roh

Department of Mathematics, Chungnam National University,
Daejeon 34134, Republic of Korea
E-mail: ischang@cnu.ac.kr
Department of Mathematics Education, Gongju National University of Education,
Gongju 32553, Republic of Korea
E-mail: yanghi22@naver.com
Ilsong Liberal Art Schools (Mathematics), Hallym University,
Chuncheon, Kangwon-Do 24252, Republic of Korea
E-mail: joroh@hallym.ac.kr

Abstract. In this paper, we introduce a way of representing a given mapping as the sum of odd and even mappings. Then, by using this representation, we investigate the stability of various forms for the following general nonic functional equation

$$\sum_{i=0}^{10} {}_{10}C_i(-1)^{10-i} f(x+iy) = 0.$$

AMS Subject Classification: 39B82; 39B52.

Key Words: stability, hyperstability, general nonic functional equation.

1 Introduction

The various types of stability for functional equations are very interesting material in the field of mathematical analysis. The stability problems of functional equations have been developed by some authors ; see [8, 10, 21, 24]. In paricular, Gilányi [9] has investigated the stability of the monomial functional equation in real normed spaces. Subsequent studies have improved the results of Gilányi (for example, [6, 11, 13]). Moreover, the hyperstability of the monomial functional equation can be found in [1, 13, 14]. The hyperstability of functional equation means that arbitrary mappings satisfying equation approximately (in some sense) must be really solution to it (refer to [19]).

In this paper, we discuss the general functional equation :

$$\sum_{i=0}^{n} {}_{n}C_i(-1)^{n-i} f(x+iy) = 0. \tag{1.1}$$

This functional equation is called as the general nonic functional equation, specially, for $n = 10$. The function $f(x) := \sum_{i=0}^{n-1} a_i x^i$ is a particular solution of (1.1), while the function $f(x) := ax^n$ is a particular solution of the n-monomial functional equation

$$\sum_{i=0}^{n} {}_nC_i(-1)^{n-i} f(x+iy) - n!f(x) = 0. \tag{1.2}$$

One can find more details in [3]. The functional equation (1.2) is said to be a nonic functional equation, specially, for $n = 10$. That is the reason that we call the equation (1.1) as the general nonic functional equation, specially, for $n = 10$.

On the other hand, the rest of this paper is organized as follows. In section 2, we study a way of representing a given mapping as the sum of odd and even mappings. In section 3, we investigate the hyperstability of the general nonic functional equation, that is, (1.1) for $n = 10$. And then in section 4, we discuss the stability problem of the general nonic functional equation.

Not much study has been conducted on the general nonic functional equations. The big advantage of this paper is the uniqueness of the solution in the stability of the general nonic functional equation. The uniqueness of the solution in the stability of the monomial functional equation have been discussed in many researches. But, the uniqueness of the solution in the stability of the general nonic functional equations is more complicated problem. Considering the special representation of a given mapping, we then solved this problem.

In the papers [2, 4, 5, 12, 23], one can see the hyperstability result of the functional equations. The recent results of the stability of the general functional equation (1.1) can be found in [7, 15, 16, 17, 18, 20, 22].

Lastly, before going into the content of the paper, readers should recall that X is a normed space and Y is a Banach space throughout this paper.

2 Representation of a given mapping

In this section, we will introduce a way of representing a given mapping as the sum of odd and even mappings.

For a given mapping $f : X \to Y$, we denote

$$f_o(x) := \frac{f(x) - f(-x)}{2}, \qquad f_e(x) := \frac{f(x) + f(-x)}{2}.$$

Let us consider the following system of nonhomogeneous linear equations

$$\begin{cases} f_1(x) & +f_3(x) & +f_5(x) & +f_7(x) & +f_9(x) & = f_o(x), \\ 2^1 f_1(x) & +8^1 f_3(x) & +32^1 f_5(x) & +128^1 f_7(x) & +512 f_9(x) & = f_o(2x), \\ 2^2 f_1(x) & +8^2 f_3(x) & +32^2 f_5(x) & +128^2 f_7(x) & +512^2 f_9(x) & = f_o(4x), \\ 2^3 f_1(x) & +8^3 f_3(x) & +32^3 f_5(x) & +128^3 f_7(x) & +512^3 f_9(x) & = f_o(8x)), \\ 2^4 f_1(x) & +8^4 f_3(x) & +32^4 f_5(x) & +128^4 f_7(x) & +512^4 f_9(x) & = f_o(16x) \end{cases}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 29 May 2023 doi:10.20944/preprints202305.1988.v1

and

$$\begin{cases} f_2(x) & +f_4(x) & +f_6(x) & +f_8(x) & = f_e(x), \\ 4^1 f_2(x) & +16^1 f_4(x) & +64^1 f_6(x) & +256^1 f_8(x) & = f_e(2x), \\ 4^2 f_2(x) & +16^2 f_4(x) & +64^2 f_6(x) & +256^2 f_8(x) & = f_e(4x), \\ 4^3 f_2(x) & +16^3 f_4(x) & +64^3 f_6(x) & +256^3 f_8(x) & = f_e(8x) \end{cases}$$

for all $x \in X$. Then, we obtain the following lemmas by the uniqueness of solution stated in Cramer's rule.

Lemma 2.1 *Let $f : X \to Y$ be a given mapping and*

$$M := \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 8 & 32 & 128 & 512 \\ 2^2 & 8^2 & 32^2 & 128^2 & 512^2 \\ 2^3 & 8^3 & 32^3 & 128^3 & 512^3 \\ 2^4 & 8^4 & 32^4 & 128^4 & 512^4 \end{vmatrix}.$$

Then, we have the mappings $f_1, f_3, f_5, f_7, f_9 : X \to Y$ defined by formulas

$$f_1(x) = \frac{\begin{vmatrix} f_o(x) & 1 & 1 & 1 & 1 \\ f_o(2x) & 8 & 32 & 128 & 512 \\ f_o(4x) & 8^2 & 32^2 & 128^2 & 512^2 \\ f_o(8x) & 8^3 & 32^3 & 128^3 & 512^3 \\ f_o(16x) & 8^4 & 32^4 & 128^4 & 512^4 \end{vmatrix}}{M}, \quad f_3(x) = \frac{\begin{vmatrix} 1 & f_o(x) & 1 & 1 & 1 \\ 2 & f_o(2x) & 32 & 128 & 512 \\ 2^2 & f_o(4x) & 32^2 & 128^2 & 512^2 \\ 2^3 & f_o(8x) & 32^3 & 128^3 & 512^3 \\ 2^4 & f_o(16x) & 32^4 & 128^4 & 512^4 \end{vmatrix}}{M},$$

$$f_5(x) = \frac{\begin{vmatrix} 1 & 1 & f_o(x) & 1 & 1 \\ 2 & 8 & f_o(2x) & 128 & 512 \\ 2^2 & 8^2 & f_o(4x) & 128^2 & 512^2 \\ 2^3 & 8^3 & f_o(8x) & 128^3 & 512^3 \\ 2^4 & 8^4 & f_o(16x) & 128^4 & 512^4 \end{vmatrix}}{M}, \quad f_7(x) = \frac{\begin{vmatrix} 1 & 1 & 1 & f_o(x) & 1 \\ 2 & 8 & 32 & f_o(2x) & 512 \\ 2^2 & 8^2 & 32^2 & f_o(4x) & 512^2 \\ 2^3 & 8^3 & 32^3 & f_o(8x) & 512^3 \\ 2^4 & 8^4 & 32^4 & f_o(16x) & 512^4 \end{vmatrix}}{M},$$

$$f_9(x) = \frac{\begin{vmatrix} 1 & 1 & 1 & 1 & f_o(x) \\ 2 & 8 & 32 & 128 & f_o(2x) \\ 2^2 & 8^2 & 32^2 & 128^2 & f_o(4x) \\ 2^3 & 8^3 & 32^3 & 128^3 & f_o(8x) \\ 2^4 & 8^4 & 32^4 & 128^4 & f_o(16x) \end{vmatrix}}{M}$$

for all $x \in X$. Furthermore, $f_o(x) = f_1(x) + f_3(x) + f_5(x) + f_7(x) + f_9(x)$ for all $x \in X$.

Lemma 2.2 *Let $f : X \to Y$ be a given mapping and*

$$M' := \begin{vmatrix} 1 & 1 & 1 & 1 \\ 4 & 16 & 64 & 256 \\ 4^2 & 16^2 & 64^2 & 256^2 \\ 4^3 & 16^3 & 64^3 & 256^3 \end{vmatrix}.$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 29 May 2023 doi:10.20944/preprints202305.1988.v1

Then, we have the mappings $f_2, f_4, f_6, f_8 : X \to Y$ *defined by formulas*

$$f_2(x) = \frac{\begin{vmatrix} f_e(x) & 1 & 1 & 1 \\ f_e(2x) & 16 & 64 & 256 \\ f_e(4x) & 16^2 & 64^2 & 256^2 \\ f_e(8x) & 16^3 & 64^3 & 256^3 \end{vmatrix}}{M'}, \quad f_4(x) = \frac{\begin{vmatrix} 1 & f_e(x) & 1 & 1 \\ 4 & f_e(2x) & 64 & 256 \\ 4^2 & f_e(4x) & 64^2 & 256^2 \\ 4^3 & f_e(8x) & 64^3 & 256^3 \end{vmatrix}}{M'},$$

$$f_6(x) = \frac{\begin{vmatrix} 1 & 1 & f_e(x) & 1 \\ 4 & 16 & f_e(2x) & 256 \\ 4^2 & 16^2 & f_e(4x) & 256^2 \\ 4^3 & 16^3 & f_e(8x) & 256^3 \end{vmatrix}}{M'}, \quad f_8(x) = \frac{\begin{vmatrix} 1 & 1 & 1 & f_e(x) \\ 4 & 16 & 64 & f_e(2x) \\ 4^2 & 16^2 & 64^2 & f_e(4x) \\ 4^3 & 16^3 & 64^3 & f_e(8x) \end{vmatrix}}{M'}$$

for all $x \in X$ *and* $f_e(x) = f_2(x) + f_4(x) + f_6(x) + f_8(x)$ *for all* $x \in X$.

Remark 2.3 *By Lemma* 2.1 *and Lemma* 2.2*, we have the following results: For all* $x \in X$,

$$\begin{aligned}
f_1(x) &:= \frac{f_o(16x) - 680f_o(8x) + 91392f_o(4x) - 2785280f_o(2x) + 16777216f_o(x)}{722925 \cdot 16}, \\
f_2(x) &:= \frac{f_e(8x) - 336f_e(4x) - 21504f_e(2x) - 262144f_e(x)}{2835 \cdot 64}, \\
f_3(x) &:= -\frac{5440(f_o(16x) - 674f_o(8x) + 87360f_o(4x) - 2269184f_o(2x) + 4194304f_o(x))}{722925 \cdot 65536}, \\
f_4(x) &:= -\frac{f_e(8x) - 324f_e(4x) - 17664f_e(2x) - 65536f_e(x))}{135 \cdot 1024}, \\
f_5(x) &:= \frac{1428(f_o(16x) - 650f_o(8x) + 71952f_o(4x) - 665600f_o(2x) + 1048576f_o(x)}{722925 \cdot 65536}, \\
f_6(x) &:= \frac{f_e(8x) - 276f_e(4x) - 5184f_e(2x) - 16384f_e(x))}{135 \cdot 4096}, \\
f_7(x) &:= -\frac{85(f_o(16x) - 554f_o(8x) + 21840f_o(4x) - 172544f_o(2x) + 262144f_o(x))}{722925 \cdot 65536}, \\
f_8(x) &:= -\frac{f_e(8x) - 84f_e(4x) - 1344f_e(2x) - 4096f_e(x))}{2835 \cdot 4096}, \\
f_9(x) &:= \frac{f_o(16x) - 170f_o(8x) + 5712f_o(4x) - 43520f_o(2x) + 65536f_o(x)}{722925 \cdot 65536}.
\end{aligned}$$

Moreover,

$$f(x) = \sum_{i=1}^{9} f_i(x) \quad \textit{for all } x \in X.$$

From now on, we define the mappings needed to prove main theorems.

Definition 2.4 *For a given mapping $f : X \to Y$, we define as*

$$
\begin{aligned}
\tilde{f}(x) &:= f(x) - f(0),\\
Df(x,y) &:= \sum_{i=0}^{10} {}_{10}C_i(-1)^{10-i} f(x+iy),\\
\Gamma f(x) :=& Df_o(12x,4x) + 10Df_o(8x,4x) + 55Df_o(4x,4x) + 220Df_o(10x,2x)\\
&+ 2200Df_o(8x,2x) + 9988Df_o(6x,2x) + 27280Df_o(4x,2x)\\
&+ 45352Df_o(2x,2x) + 17920Df_o(5x,x) + 179200Df_o(4x,x)\\
&+ 760320Df_o(3x,x) + 1689600Df_o(2x,x) + 1790976Df_o(x,x),\\
\Delta f(x) :=& Df_e(6x,2x) + 10Df_e(4x,2x) + 55Df_e(2x,2x) + 110Df_e(0,2x)\\
&+ 320Df_e(3x,x) + 3200Df_e(2x,x) + 12992Df_e(x,x) + 12160Df_e(0,x)
\end{aligned}
$$

for all $x, y \in X$.

As results of tedious calculation, we obtain the following lemmas:

Lemma 2.5 *Let $f : X \to Y$ be an arbitrarily given mapping. Then, the equalities*

$$
\begin{aligned}
&\tilde{f}(x) = \sum_{i=1}^{9} \tilde{f}_i(x),\\
&D\tilde{f}(x,y) = Df(x,y),\\
&\Gamma\tilde{f}(x) = f_o(32x) - 682f_o(16x) + 92752f_o(8x) - 2968064f_o(4x),\\
&\qquad + 22347776f_o(2x) - 33554432f_o(x),\\
&\Delta\tilde{f}(x) = f_e(16x) - 340f_e(8x) + 22848f_e(4x) - 348160f_e(2x) + 1048576f_e(x)
\end{aligned}
$$

hold for all $x, y \in X$.

Lemma 2.6 *Let $f : X \to Y$ be an arbitrarily given mapping. Then, we have that*

$$
\begin{aligned}
&\tilde{f}_1(x) - \frac{\tilde{f}_1(2x)}{2} = \frac{\Gamma\tilde{f}_o(x)}{722925 \cdot 32}, &&\tilde{f}_2(x) - \frac{\tilde{f}_2(2x)}{4} = \frac{\Delta\tilde{f}_e(x)}{2835 \cdot 256},\\
&\tilde{f}_3(x) - \frac{\tilde{f}_3(2x)}{8} = -\frac{5440\Gamma\tilde{f}_o(x)}{722925 \cdot 65536 \cdot 8}, &&\tilde{f}_4(x) - \frac{\tilde{f}_4(2x)}{16} = -\frac{21\Delta\tilde{f}_e(x)}{2835 \cdot 256 \cdot 64},\\
&\tilde{f}_5(x) - \frac{\tilde{f}_5(2x)}{32} = \frac{1428\Gamma\tilde{f}_o(x)}{722925 \cdot 32 \cdot 65536}, &&\tilde{f}_6(x) - \frac{\tilde{f}_6(2x)}{64} = \frac{21\Delta\tilde{f}_e(x)}{2835 \cdot 256 \cdot 1024},\\
&\tilde{f}_7(x) - \frac{\tilde{f}_7(2x)}{128} = -\frac{85\Gamma\tilde{f}_o(x)}{722925 \cdot 65536 \cdot 128}, &&\tilde{f}_8(x) - \frac{\tilde{f}_8(2x)}{256} = -\frac{\Delta\tilde{f}_e(x)}{2835 \cdot 256 \cdot 4096},\\
&\tilde{f}_9(x) - \frac{\tilde{f}_9(2x)}{512} = \frac{\Gamma\tilde{f}_o(x)}{722925 \cdot 65536 \cdot 512}
\end{aligned}
$$

are fulfilled for all $x, y \in X$.

Lemma 2.7 *If $f : X \to Y$ is a mapping such that $Df(x,y) = 0$ for all $x, y \in X$ with $f(0) = 0$, then for each $i \in \{1, 2, \ldots, 9\}$, the mappings $f_i(x)$ in Remark 2.3 satisfy the equalities $Df_i(x,y) = 0$ for all $x \in X$ and $f_i(2x) = 2^i f_i(x)$ for all $x, y \in X$.*

Proof. Since $Df(x,y) = 0$ for all $x, y \in X$, by the definition of $f_i, \Gamma f$ and Δf, we have

$$Df_i(x,y) = 0, \ \Delta f(x) = 0, \ \Gamma f(x) = 0.$$

Applying Lemma 2.6, we arrive at $f_i(2x) = 2^i f_i(x)$. □

3 Hyperstability of the general nonic functional equation

In this section, we will prove the hyperstability of the general nonic functional equation. To prove main theorem, we will use the functions introduced in previous section.

Theorem 3.1 *Let $p < 0$ be a real number. Suppose that $f : X \to Y$ is a mapping such that*

$$\frac{\|Df(x,y)\|}{\|x\|^p + \|y\|^p} \leq \theta \ \text{ for all } x, y \in X \backslash \{0\}. \tag{3.1}$$

Then, $Df(x,y) = 0$ is fulfilled for all $x, y \in X$.

Proof. • STEP $1°$: Using the definition of $\tilde{f}$ together with

$$\sum_{i=0}^{10} {}_{10}C_i (-1)^{10-i} = 0,$$

we have that

$$D\tilde{f}(x,y) = Df(x,y), \quad \tilde{f}(0) = 0.$$

From the expression (3.1) and the definitions of Γf and Δf, we get

$$\|\Gamma \tilde{f}(x)\| \leq 47377612800K'\theta\|x\|^p, \quad \|\Delta \tilde{f}(x)\| \leq 11612160K\theta\|x\|^p$$

for all $x \in X \backslash \{0\}$, where

$$\begin{aligned} K' :=& \frac{220 \cdot 20^p + 55 \cdot 16^p + 10 \cdot 12^p + 17920 \cdot 10^p + 45353 \cdot 8^p + 27280 \cdot 6^p}{47377612800} \\ &+ \frac{1801030 \cdot 4^p + 1689600 \cdot 3^p + 847560 \cdot 2^p + 4617216}{47377612800}, \\ K :=& \frac{110 \cdot 10^p + 55 \cdot 8^p + 10 \cdot 6^p + 12160 \cdot 5^p + 12993 \cdot 4^p + 3200 \cdot 3^p + 496 \cdot 2^p + 28672}{11612160}. \end{aligned}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 29 May 2023 doi:10.20944/preprints202305.1988.v1

Then, Lemma 2.5 and the definitions of f_k $(k \in \{1, 2, \ldots, 9\})$ ensure the inequalities

$$\begin{aligned}
\left\| \frac{\tilde{f}_1(2^i x)}{2^i} - \frac{\tilde{f}_1(2^{i+1}x)}{2^{i+1}} \right\| &= \left\| -\frac{4096\Gamma \tilde{f}_o(x)}{47377612800 \cdot 2^{i+1}} \right\| \le \frac{4096K'\theta\|x\|^p}{2^{i+1}}, \\
\left\| \frac{\tilde{f}_2(2^i x)}{4^i} - \frac{\tilde{f}_2(2^{i+1}x)}{4^{i+1}} \right\| &= \left\| -\frac{64\Delta \tilde{f}_e(x)}{11612160 \cdot 4^{i+1}} \right\| \le \frac{64K\theta\|x\|^p}{4^{i+1}}, \\
\left\| \frac{\tilde{f}_3(2^i x)}{8^i} - \frac{\tilde{f}_3(2^{i+1}x)}{8^{i+1}} \right\| &= \left\| \frac{5440\Gamma \tilde{f}_o(x)}{47377612800 \cdot 8^{i+1}} \right\| \le \frac{5440K'\theta\|x\|^p}{8^{i+1}}, \\
\left\| \frac{\tilde{f}_4(2^i x)}{16^i} - \frac{\tilde{f}_4(2^{i+1}x)}{16^{i+1}} \right\| &= \left\| \frac{84\Delta \tilde{f}_e(x)}{11612160 \cdot 16^{i+1}} \right\| \le \frac{84K\theta\|x\|^p}{16^{i+1}} \\
\left\| \frac{\tilde{f}_5(2^i x)}{32^i} - \frac{\tilde{f}_5(2^{i+1}x)}{32^{i+1}} \right\| &= \left\| -\frac{1428\Gamma \tilde{f}_o(x)}{47377612800 \cdot 32^{i+1}} \right\| \le \frac{1428K'\theta\|x\|^p}{32^{i+1}}, \\
\left\| \frac{\tilde{f}_6(2^i x)}{64^i} - \frac{\tilde{f}_6(2^{i+1}x)}{64^{i+1}} \right\| &= \left\| -\frac{21\Delta \tilde{f}_e(x)}{11612160 \cdot 64^{i+1}} \right\| \le \frac{21K\theta\|x\|^p}{64^{i+1}}, \\
\left\| \frac{\tilde{f}_7(2^i x)}{128^i} - \frac{\tilde{f}_7(2^{i+1}x)}{128^{i+1}} \right\| &= \left\| \frac{85\Gamma \tilde{f}_o(x)}{47377612800 \cdot 128^{i+1}} \right\| \le \frac{85K'\theta\|x\|^p}{128^{i+1}}, \\
\left\| \frac{\tilde{f}_8(2^i x)}{256^i} - \frac{\tilde{f}_8(2^{i+1}x)}{256^{i+1}} \right\| &= \left\| \frac{\Delta \tilde{f}_e(x)}{11612160 \cdot 256^{i+1}} \right\| \le \frac{K\theta\|x\|^p}{256^{i+1}}, \\
\left\| \frac{\tilde{f}_9(2^i x)}{512^i} - \frac{\tilde{f}_9(2^{i+1}x)}{512^{i+1}} \right\| &= \left\| \frac{\Gamma \tilde{f}_o(x)}{47377612800 \cdot 512^{i+1}} \right\| \le \frac{K'\theta\|x\|^p}{512^{i+1}}
\end{aligned}$$

for all $x \in X \backslash \{0\}$. Note that

$$\sum_{k=1}^{9} \frac{\tilde{f}_k(2^n x)}{2^{kn}} - \sum_{k=1}^{9} \frac{\tilde{f}_k(2^{n+m}x)}{2^{k(n+m)}} = \sum_{i=n}^{n+m-1} \left(\sum_{k=1}^{9} \frac{\tilde{f}_k(2^i x)}{2^{ki}} - \sum_{k=1}^{9} \frac{\tilde{f}_k(2^{i+1}x)}{2^{k(i+1)}} \right).$$

This implies that

$$\begin{aligned}
\left\| \sum_{k=1}^{9} \frac{\tilde{f}_k(2^n x)}{2^{kn}} - \sum_{k=1}^{9} \frac{\tilde{f}_k(2^{n+m}x)}{2^{k(n+m)}} \right\| &\le \sum_{i=n}^{n+m-1} \left(\frac{4096K'}{2^{i+1}} + \frac{64K}{4^{i+1}} + \frac{5440K'}{8^{i+1}} \right. \\
&\quad \left. + \frac{84K}{16^{i+1}} + \frac{1428K'}{32^{i+1}} + \frac{21K}{64^{i+1}} + \frac{85K'}{128^{i+1}} + \frac{K}{256^{i+1}} + \frac{K'}{512^{i+1}} \right) \cdot \theta\|x\|^p \qquad (3.2)
\end{aligned}$$

for all $x \in X \backslash \{0\}$ and $n, m \in \mathbb{N} \cup \{0\}$. Due to $p < 0$, the sequences $\{\frac{\tilde{f}_1(2^n x)}{2^n}\}, \cdots, \{\frac{\tilde{f}_9(2^n x)}{2^{9n}}\}$ and $\{\sum_{k=1}^{9} \frac{\tilde{f}_k(2^n x)}{2^{kn}}\}$ are Cauchy for all $x \in X \backslash \{0\}$. Since Y is complete and $\tilde{f}(0) = 0$, the sequences $\{\frac{\tilde{f}_1(2^n x)}{2^n}\}, \cdots, \{\frac{\tilde{f}_9(2^n x)}{2^{9n}}\}$ and $\{\sum_{k=1}^{9} \frac{\tilde{f}_k(2^n x)}{2^{kn}}\}$ converge for all $x \in X$. Hence, for

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 29 May 2023 doi:10.20944/preprints202305.1988.v1

each $k \in \{1, 2, \ldots, 9\}$, we can define mappings $F_k, F : X \to Y$ by

$$F_k(x) := \lim_{n\to\infty} \frac{\tilde{f}_k(2^n x)}{2^{kn}}, \ F(x) := \lim_{n\to\infty} \sum_{k=1}^{9} \frac{\tilde{f}_k(2^n x)}{2^{kn}} \qquad (x \in X).$$

• STEP 2°: By (3.1) and the definitions of Df, F_1 and f_1, since $Df_1(x,y) = D\tilde{f}_1(x,y)$, we then have

$$\begin{aligned}
\|DF_1(x,y)\| &= \lim_{n\to\infty} \left\| \frac{D\tilde{f}_1(2^n x, 2^n y)}{2^n} \right\| = \lim_{n\to\infty} \left\| \frac{Df_1(2^n x, 2^n y)}{2^n} \right\| \\
&= \lim_{n\to\infty} \left\| \frac{16777216 Df_o(2^n x, 2^n y)}{11566800 \cdot 2^n} - \frac{2785280 Df_o(2^{n+1}x, 2^{n+1}y)}{11566800 \cdot 2^n} \right. \\
&\quad \left. + \frac{91392 f_o(2^{n+2}x, 2^{n+2}y)}{11566800 \cdot 2^n} - \frac{680 Df_o(2^{n+3}x, 2^{n+3}y)}{11566800 \cdot 2^n} + \frac{Df_o(2^{n+4}x, 2^{n+4}y)}{11566800 \cdot 2^n} \right\| \\
&\le \lim_{n\to\infty} \left(\frac{16777216 + 2785280 \cdot 2^p + 91392 \cdot 2^{2p} + 680 \cdot 2^{3p} + 2^{4p}}{11566800} \right) \cdot \frac{2^{np}}{2^n} \theta\big(\|x\|^p + \|y\|^p\big) \\
&=0
\end{aligned} \tag{3.3}$$

for all $x, y \in X \backslash \{0\}$. On the other hand, from the definition of DF_1, we have

$$DF_1(x,0) = \sum_{i=0}^{10} {}_{10}C_i (-1)^{10-i} F_1(x + i0) = F_1(x) \sum_{i=0}^{10} {}_{10}C_i (-1)^{10-i} = 0 \tag{3.4}$$

for all $x \in X$. And, in view of (3.3), we obtain

$$DF_1(0,y) = DF_1(-10y, y) = 0 \qquad \text{for all } y \in X \backslash \{0\}. \tag{3.5}$$

Therefore, the relations (3.3), (3.4) and (3.5) yield that $DF_1(x,y) = 0$ for all $x, y \in X$. Similarly, we can show that $DF_k(x,y) = 0$ for each $k \in \{2, 3, \cdots, 9\}$ and all $x, y \in X$. Since $DF(x,y) = \sum_{k=1}^{9} DF_k(x,y)$ for all $x, y \in X$, we get $DF(x,y) = 0$ for all $x, y \in X$.

• STEP 3°: Observe that for all $x \in X$,

$$\begin{aligned}
&D\tilde{f}(x,y) - DF(x,y) = \sum_{i=0}^{10} {}_{10}C_i (-1)^{10-i} \big(\tilde{f}(x + iy) - F(x + iy)\big), \\
&DF((1-n)x, nx) = 0.
\end{aligned}$$

Then, we see that

$$\begin{aligned}
D\tilde{f}((1-n)x, nx) &= \tilde{f}((1-n)x) - F((1-n)x) - 10\tilde{f}(x) + 10F(x) \\
&\quad + \sum_{i=2}^{10} {}_{10}C_i (-1)^{10-i} \big(\tilde{f}((1-n)x + inx) - F((1-n)x + inx)\big)
\end{aligned} \tag{3.6}$$

 doi:10.20944/preprints202305.1988.v1

for any $n \in N$ and $x \in X\backslash\{0\}$. Moreover, by letting $n = 0$ and taking the limit $m \to \infty$ in (3.2), we get the inequalities

$$\|\tilde{f}(x) - F(x)\| \leq \Big(\frac{4096K'}{2-2^p} + \frac{64K}{4-2^p} + \frac{5440K'}{8-2^p} + \frac{84K}{16-2^p} + \frac{1428K'}{32-2^p} + \frac{21K}{64-2^p} + \frac{85K'}{128-2^p} + \frac{K}{256-2^p} + \frac{K'}{512-2^p} \Big) \cdot \theta\|x\|^p \quad (3.7)$$

for all $x \in X$.

Since $p < 0$, by (3.1), (3.6) and (3.7), we are forced to conclude that

$$\begin{aligned} 10 \cdot \|\tilde{f}(x) - F(x)\| \leq & \lim_{n\to\infty} \|D\tilde{f}((1-n)x, nx)\| + \lim_{n\to\infty} \|\tilde{f}((1-n)x) - F((1-n)x\| \\ & + \sum_{i=2}^{10} \lim_{n\to\infty} \big\|{}_{10}C_i(\tilde{f}(((i-1)n+1)x) - F(((i-1)n+1)x)\big\| \\ & \leq \lim_{n\to\infty} \big((n-1)^p + n^p\big) + M(n-1)^p + \sum_{i=2}^{10} {}_{10}C_i\big((i-1)n+1)^p\big) \cdot \theta\|x\|^p \\ & = 0 \end{aligned}$$

for all $x \in X\backslash\{0\}$, where

$$M := \frac{4096K'}{2-2^p} + \frac{64K}{4-2^p} + \frac{5440K'}{8-2^p} + \frac{84K}{16-2^p} + \frac{1428K'}{32-2^p} + \frac{21K}{64-2^p} + \frac{85K'}{128-2^p} + \frac{K}{256-2^p} + \frac{K'}{512-2^p}.$$

So, we have $\|\tilde{f}(x) - F(x)\| = 0$ for all $x \in X\backslash\{0\}$. Since $\tilde{f}(0) = 0 = F(0)$, we have $\tilde{f}(x) = F(x)$ for all $x \in X$. Therefore, $D\tilde{f}(x,y) = DF(x,y) = 0$ for all $x, y \in X$. From the fact that $D\tilde{f}(x,y) = Df(x,y)$, we finally have

$$Df(x,y) = D\tilde{f}(x,y) = DF(x,y) = 0,$$

which completes the proof. □

4 Stability of the general nonic functional equation

In this section, we will consider the stability of the general nonic functional equation

$$\sum_{i=0}^{10} {}_{10}C_i(-1)^{10-i} f(x+iy) = 0.$$

Theorem 4.1 *Let $p \neq 1,2,3,4,5,6,7,8,9$ be a nonnegative real number. Suppose that $f : X \to Y$ is a mapping such that for all $x, y \in X$,*

$$\|Df(x,y)\| \leq \theta\big(\|x\|^p + \|y\|^p\big). \quad (4.1)$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 29 May 2023 doi:10.20944/preprints202305.1988.v1

Then, there exists a unique mapping F satisfying $DF(x,y)=0$ and

$$\|\tilde{f}(x)-F(x)\| \le \left(\frac{4096}{|2-2^p|}+\frac{5440}{|8-2^p|}+\frac{1428}{|32-2^p|}+\frac{85}{|128-2^p|}+\frac{1}{|512-2^p|}\right)\cdot\frac{K'\theta\|x\|^p}{4096} + \left(\frac{64}{|4-2^p|}+\frac{84}{|16-2^p|}+\frac{21}{|64-2^p|}+\frac{1}{|256-2^p|}\right)\cdot\frac{K\theta\|x\|^p}{64}, \tag{4.2}$$

for all $x\in X$, where $K := 6^p+10\cdot 4^p+320\cdot 3^p+3431\cdot 2^p+41664$ and

$$K' := 12^p+220\cdot 10^p+2210\cdot 8^p+9988\cdot 6^p+17920\cdot 5^p+206601\cdot 4^p+760320\cdot 3^p+1819992\cdot 2^p+6228992.$$

Proof. From the definition of $\tilde{f}$, we get $D\tilde{f}(x,y)=Df(x,y)$ and $\tilde{f}(0)=0$. By (4.1) and the definition of Γf and Δf, we have that

$$\begin{aligned}\|\Gamma\tilde{f}(x)\| =& \big\|Df_o(12x,4x)+10Df_o(8x,4x)+55Df_o(4x,4x)+220Df_o(10x,2x)\\ &+2200Df_o(8x,2x)+9988Df_o(6x,2x)+27280Df_o(4x,2x)\\ &+45352Df_o(2x,2x)+17920Df_o(5x,x)+179200Df_o(4x,x)\\ &+760320Df_o(3x,x)+1689600Df_o(2x,x)+1790976Df_o(x,x)\big\|\\ \le&\big(12^p+4^p+10\cdot 8^p+10\cdot 4^p+110\cdot 4^p+220\cdot 10^p+220\cdot 2^p\\ &+2200\cdot 8^p+2200\cdot 2^p+9988\cdot 6^p+9988\cdot 2^p+27280\cdot 4^p+27280\cdot 2^p\\ &+90704\cdot 2^p+17920\cdot 5^p+17920+179200\cdot 4^p+179200\\ &+760320\cdot 3^p+760320+1689600\cdot 2^p+1689600+3581952\big)\cdot\theta\|x\|^p\\ \le&722925\cdot 16K'\theta\|x\|^p, \end{aligned}\tag{4.3}$$

$$\begin{aligned}\|\Delta\tilde{f}(x)\| =& \big\|Df_e(6x,2x)+10Df_e(4x,2x)+55Df_e(2x,2x)+110Df_e(0,2x)\\ &+320Df_e(3x,x)+3200Df_e(2x,x)+12992Df_e(x,x)+12160Df_e(0,x)\big\|\\ \le&\big(6^p+2^p+10\cdot 4^p+10\cdot 2^p+110\cdot 2^p+110\cdot 2^p\\ &+320\cdot 3^p+320+3200\cdot 2^p+3200+25984+12160\big)\cdot\theta\|x\|^p\\ \le&2835\cdot 64K\theta\|x\|^p \end{aligned}\tag{4.4}$$

for all $x\in X$.

Next, for $i\in\{1,2,\dots,9\}$, we will find F_k to make $F(x)=\sum_{k=1}^{9}F_k(x)$. For each given $p\neq 1,2,3,4,5,6,7,8,9$, we will use different approach to find the functions F_k.

- **Setting** F_1 : Let $0\le p<1$. It follows from Lemma 2.6 and (4.3) that

$$\left\|\frac{\tilde{f}_1(2^ix)}{2^i}-\frac{\tilde{f}_1(2^{i+1}x)}{2^{i+1}}\right\| = \left\|\frac{\Gamma\tilde{f}(2^ix)}{722925\cdot 32\cdot 2^i}\right\| \le \frac{K'\theta\|x\|^p}{2\cdot 2^i},$$

for all $x\in X$. Notice that for all $x\in X$,

$$\frac{\tilde{f}_1(2^nx)}{2^n}-\frac{\tilde{f}_1(2^{n+m}x)}{2^{n+m}} = \sum_{i=n}^{n+m-1}\left(\frac{\tilde{f}_1(2^ix)}{2^i}-\frac{\tilde{f}_1(2^{i+1}x)}{2^{i+1}}\right),$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 29 May 2023 doi:10.20944/preprints202305.1988.v1

which implies that

$$\left\| \frac{\tilde{f}_1(2^n x)}{2^n} - \frac{\tilde{f}_1(2^{n+m} x)}{2^{n+m}} \right\| \le \sum_{i=n}^{n+m-1} \frac{K' 2^{ip} \theta \|x\|^p}{2 \cdot 2^i} \tag{4.5}$$

for all $x \in X$ and $n, m \in \mathbb{N} \cup \{0\}$. By (4.5), the sequence $\{\frac{\tilde{f}_1(2^n x)}{2^n}\}$ is Cauchy for all $x \in X$, because of the fact that $0 \le p < 1$. Since Y is complete, the sequence $\{\frac{\tilde{f}_1(2^n x)}{2^n}\}$ converges. Hence, we may define a mapping $F_1 : X \to Y$ by

$$F_1(x) := \lim_{n\to\infty} \frac{\tilde{f}_1(2^n x)}{2^n} \quad \text{for all } x \in X.$$

Moreover, letting $n = 0$ and taking the limit $m \to \infty$ in (4.5), we get the inequality

$$\|\tilde{f}_1(x) - F_1(x)\| \le \frac{K'\theta\|x\|^p}{2 - 2^p} \quad \text{for all } x \in X.$$

By the definition of F_1, we easily get $F_1(2x) = 2F_1(x)$ for all $x \in X$ and

$$\begin{aligned}
\|DF_1(x,y)\| &= \lim_{n\to\infty} \left\| \frac{Df_1(2^n x, 2^n y)}{2^n} \right\| \\
&= \lim_{n\to\infty} \left\| \frac{16777216 Df_o(2^n x, 2^n y)}{11566800 \cdot 2^n} - \frac{2785280 Df_o(2^{n+1} x, 2^{n+1} y)}{11566800 \cdot 2^n} \right. \\
&\quad \left. + \frac{91392 Df_o(2^{n+2} x, 2^{n+2} y)}{11566800 \cdot 2^n} - \frac{680 Df_o(2^{n+3} x, 2^{n+3} y)}{11566800 \cdot 2^n} - \frac{Df_o(2^{n+4} x, 2^{n+4} y)}{11566800 \cdot 2^n} \right\| \\
&\le \lim_{n\to\infty} \frac{16777216 \cdot 2^{np}\theta(\|x\|^p + \|y\|^p)}{11566800 \cdot 2^n} + \lim_{n\to\infty} \frac{2785280 \cdot 2^{(n+1)p}\theta(\|x\|^p + \|y\|^p)}{11566800 \cdot 2^n} \\
&\quad + \lim_{n\to\infty} \frac{91392 \cdot 2^{(n+2)p}\theta(\|x\|^p + \|y\|^p))}{11566800 \cdot 2^n} + \lim_{n\to\infty} \frac{680 \cdot 2^{(n+3)p}\theta(\|x\|^p + \|y\|^p))}{11566800 \cdot 2^n} \\
&\quad + \lim_{n\to\infty} \frac{2^{(n+4)p}\theta(\|x\|^p + \|y\|^p))}{11566800 \cdot 2^n} = 0
\end{aligned}$$

for all $x, y \in X$.

Let $p > 1$. It follows from Lemma 2.6 and (4.3) that

$$\left\| 2^i \tilde{f}_1(2^{-i} x) - 2^{i+1} \tilde{f}_1(2^{-i-1} x) \right\| \le \left\| \frac{2^i \Gamma \tilde{f}(2^{-i-1} x)}{722925 \cdot 16} \right\| \le \frac{K' 2^i \theta \|x\|^p}{2^{(i+1)p}}$$

for all $x \in X$. Because of the fact that

$$2^n \tilde{f}_1(2^{-n} x) - 2^{n+m} \tilde{f}_1(2^{-n-m} x) = \sum_{i=n}^{n+m-1} \Big(2^i \tilde{f}_1(2^{-i} x) - \big(2^{i+1} \tilde{f}_1(2^{-i-1} x) \Big)$$

for all $x \in X$, we have

$$\left\| 2^n \tilde{f}_1(2^{-n} x) - 2^{n+m} \tilde{f}_1(2^{-n-m} x) \right\| \le \sum_{i=n}^{n+m-1} \frac{K' 2^i \theta \|x\|^p}{2^{(i+1)p}} \tag{4.6}$$

 doi:10.20944/preprints202305.1988.v1

for all $x \in X$ and $n, m \in \mathbb{N} \cup \{0\}$. Since $p > 1$, by (4.6), the sequence $\{2^n \tilde{f}_1(2^{-n}x)\}$ is Cauchy for all $x \in X$. By the completeness of Y, we know that the sequence $\{2^n \tilde{f}_1(2^{-n}x)\}$ converges. Hence, we can define a mapping $F_1 : X \to Y$ by

$$F_1(x) := \lim_{n\to\infty} 2^n \tilde{f}_1(2^{-n}x) \text{ for all } x \in X.$$

However, letting $n = 0$ and passing the limit $m \to \infty$ in (4.6) we get the inequality

$$\|\tilde{f}_1(x) - F_1(x)\| \le \frac{K'\theta\|x\|^p}{2^p - 2} \quad \text{for all } x \in X.$$

From the definition of F_1, we easily get $F_1(2x) = 2F_1(x)$ for all $x \in X$ and $DF_1(x, y) = 0$ for all $x, y \in X$.

- **Setting** F_2 : Let $p < 2$. It follows from Lemma 2.6 and (4.4) that

$$\left\|\frac{\tilde{f}_2(2^n x)}{4^n} - \frac{\tilde{f}_2(2^{n+m}x)}{4^{n+m}}\right\| = \left\|\frac{\Delta\tilde{f}_e(2^i x)}{2835\cdot 256\cdot 4^i}\right\| \le \sum_{i=n}^{n+m-1} \frac{K2^{ip}\theta\|x\|^p}{4\cdot 4^i} \tag{4.7}$$

for all $x \in X$ and $n, m \in \mathbb{N} \cup \{0\}$. Since $p < 2$, we have from (4.7) that the sequence $\{\frac{\tilde{f}_2(2^n x)}{4^n}\}$ is Cauchy for all $x \in X$. By the completeness of Y, the sequence $\{\frac{\tilde{f}_2(2^n x)}{4^n}\}$ converges. Hence, we can define a mapping $F_2 : X \to Y$ by

$$F_2(x) := \lim_{n\to\infty} \frac{\tilde{f}_2(2^n x)}{4^n} \quad \text{for all } x \in X.$$

Now, letting $n = 0$ and passing the limit $m \to \infty$ in (4.7), we obtain that

$$\|\tilde{f}_2(x) - F_2(x)\| \le \frac{K\theta\|x\|^p}{4 - 2^p} \quad \text{for all } x \in X. \tag{4.8}$$

From the definition of F_2, we then have $F_2(2x) = 4F_2(x)$ for all $x \in X$ and $DF_2(x, y) = 0$ for all $x, y \in X$.

Let $p > 2$. It follows from Lemma 2.6 and (4.4) that

$$\left\|4^n \tilde{f}_2(2^{-n}x) - 4^{n+m}\tilde{f}_2(2^{-n-m}x)\right\| \le \sum_{i=n}^{n+m-1} \frac{K4^i\theta\|x\|^p}{2^{(i+1)p}} \tag{4.9}$$

for all $x \in X$ and $n, m \in \mathbb{N} \cup \{0\}$. Since $p > 2$, we have by (4.9) that the sequence $4^n \tilde{f}_2(2^{-n}x)$ is Cauchy for all $x \in X$. Since Y is complete, the sequence $\{4^n \tilde{f}_2(2^{-n}x)\}$ converges. Then, we can define a mapping $F_2 : X \to Y$ by

$$F_2(x) := \lim_{n\to\infty} 4^n \tilde{f}_2(2^{-n}x) \quad \text{for all } x \in X.$$

In (4.9), puting $n = 0$ and passing the limit $m \to \infty$, one obtains that

$$\|\tilde{f}_2(x) - F_2(x)\| \le \frac{K\theta\|x\|^p}{2^p - 4} \quad \text{for all } x \in X.$$

 doi:10.20944/preprints202305.1988.v1

According the definition of F_2, we arrive at $F_2(2x) = 4F_2(x)$ for all $x \in X$ and $DF_2(x,y) = 0$ for all $x, y \in X$.

- **Setting** F_3 : Let $p < 3$. It follows by Lemma 2.6 and (4.3) that

$$\left\| \frac{\tilde{f}_3(2^n x)}{8^n} - \frac{\tilde{f}_3(2^{n+m} x)}{8^{n+m}} \right\| = \left\| \frac{5440\Gamma \tilde{f}(2^i x)}{722925 \cdot 524288 \cdot 8^i} \right\| \le \sum_{i=n}^{n+m-1} \frac{5440K' 2^{ip} \theta \|x\|^p}{32768 \cdot 8^i} \tag{4.10}$$

for all $x \in X$ and $n, m \in \mathbb{N} \cup \{0\}$. Since $p < 3$, we see by (4.10) that the sequence $\{\frac{\tilde{f}_3(2^n x)}{8^n}\}$ is Cauchy for all $x \in X$. From the completeness of Y, the sequence $\{\frac{\tilde{f}_3(2^n x)}{8^n}\}$ converges. So, we may define a mapping $F_3 : X \to Y$ by

$$F_3(x) := \lim_{n\to\infty} \frac{\tilde{f}_3(2^n x)}{8^n} \quad \text{for all } x \in X.$$

Taking $n = 0$ and sending the limit $m \to \infty$ in (4.10), we find that

$$\|\tilde{f}_3(x) - F_3(x)\| \le \frac{5440K'\theta\|x\|^p}{4096(8 - 2^p)}$$

for all $x \in X$. Based on the definition of F_3, we yield that $F_3(2x) = 8F_3(x)$ for all $x \in X$ and $DF_3(x,y) = 0$ for all $x, y \in X$.

Let $p > 3$. The lemma 2.6 and (4.3) guarantee that

$$\left\| 8^n \tilde{f}_3(2^{-n}x) - 8^{n+m} \tilde{f}_3(2^{-n-m}x) \right\| \le \sum_{i=n}^{n+m-1} \frac{5440K' 8^i \theta \|x\|^p}{4096 \cdot 2^{(i+1)p}} \tag{4.11}$$

for all $x \in X$ and $n, m \in \mathbb{N} \cup \{0\}$. Since $p > 3$, we have from (4.11) that $\{8^n \tilde{f}_3(2^{-n}x)\}$ is a Cauchy sequence for all $x \in X$. So, because Y is complete, the sequence $\{8^n \tilde{f}_3(2^{-n}x)\}$ converges. Then, we may define a mapping $F_3 : X \to Y$ by

$$F_3(x) := \lim_{n\to\infty} 8^n \tilde{f}_3(2^{-n}x) \quad \text{for all} \quad x \in X.$$

Let $n = 0$ and take the limit $m \to \infty$ in (4.11) and then

$$\|\tilde{f}_3(x) - F_3(x)\| \le \frac{5440K'\theta\|x\|^p}{4096(2^p - 8)} \quad \text{for all } x \in X.$$

The definition of F_3 gives that $F_3(2x) = 8F_3(x)$ for all $x \in X$ and $DF_3(x,y) = 0$ for all $x, y \in X$.

- **Setting** F_4 : Let $p < 4$. It follows from Lemma 2.6 and (4.4) that

$$\left\| \frac{\tilde{f}_4(2^n x)}{16^n} - \frac{\tilde{f}_4(2^{n+m} x)}{16^{n+m}} \right\| = \left\| \frac{21\Delta \tilde{f}_e(2^i x)}{2835 \cdot 16384 \cdot 16^i} \right\| \le \sum_{i=n}^{n+m-1} \frac{21K 2^{ip} \theta \|x\|^p}{256 \cdot 16^i} \tag{4.12}$$

for all $x \in X$ and $n, m \in \mathbb{N} \cup \{0\}$. Considering $p < 4$ and (4.12), the sequence $\{\frac{\tilde{f}_4(2^n x)}{16^n}\}$ is Cauchy for all $x \in X$. Since Y is complete, the sequence $\{\frac{\tilde{f}_4(2^n x)}{16^n}\}$ converges. Thereby, we can define a mapping $F_4 : X \to Y$ by

$$F_4(x) := \lim_{n\to\infty} \frac{\tilde{f}_4(2^n x)}{16^n} \quad \text{for all } x \in X.$$

Now, by letting $n = 0$ and passing the limit $m \to \infty$ in (4.12), we obtain the inequality

$$\|\tilde{f}_4(x) - F_4(x)\| \le \frac{21K\theta\|x\|^p}{16(16 - 2^p)} \quad \text{for all } x \in X.$$

We have from the definition of F_4 that $F_4(2x) = 16F_4(x)$ for all $x \in X$ and $DF_4(x, y) = 0$ for all $x, y \in X$.

Let $p > 4$. Lemma 2.6 and (4.4) imply that

$$\left\|16^n \tilde{f}_4(2^{-n}x) - 16^{n+m}\tilde{f}_4(2^{-n-m}x)\right\| \le \sum_{i=n}^{n+m-1} \frac{21K16^i\theta\|x\|^p}{16 \cdot 2^{(i+1)p}} \tag{4.13}$$

for all $x \in X$ and $n, m \in \mathbb{N} \cup \{0\}$. Since $p > 4$, we have by (4.13) that $16^n\tilde{f}_4(2^{-n}x)$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{16^n\tilde{f}_4(2^{-n}x)\}$ converges. Thus, we can define a mapping $F_4 : X \to Y$ by

$$F_4(x) := \lim_{n\to\infty} 16^n\tilde{f}_4(2^{-n}x) \quad \text{for all } x \in X.$$

Meanwhile, in (4.13), letting $n = 0$ and passing the limit $m \to \infty$ we then have the inequality

$$\|\tilde{f}_4(x) - F_4(x)\| \le \frac{21K\theta\|x\|^p}{16(2^p - 16)} \quad \text{for all } x \in X.$$

From the definition of F_4, we see that $F_4(2x) = 16F_4(x)$ for all $x \in X$ and $DF_4(x, y) = 0$ for all $x, y \in X$.

- **Setting** F_5 : Let $p < 5$. It follows from Lemma 2.6 and (4.3) that

$$\left\|\frac{\tilde{f}_5(2^n x)}{32^n} - \frac{\tilde{f}_5(2^{n+m}x)}{32^{n+m}}\right\| = \left\|\frac{1428\Gamma\tilde{f}(2^i x)}{722925 \cdot 2097152 \cdot 32^i}\right\| \le \sum_{i=n}^{n+m-1} \frac{1428K'2^{ip}\theta\|x\|^p}{131072 \cdot 32^i} \tag{4.14}$$

for all $x \in X$ and $n, m \in \mathbb{N} \cup \{0\}$. We have assume that $p < 5$. So, we have by (4.14) that $\{\frac{\tilde{f}_5(2^n x)}{32^n}\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{\tilde{f}_5(2^n x)}{32^n}\}$ converges. Hence, we can define a mapping $F_5 : X \to Y$ by

$$F_5(x) := \lim_{n\to\infty} \frac{\tilde{f}_5(2^n x)}{32^n} \quad \text{for all } x \in X.$$

 doi:10.20944/preprints202305.1988.v1

Moreover, letting $n = 0$ and passing the limit $m \to \infty$ in (4.14), we get the inequality

$$\|\tilde{f}_5(x) - F_5(x)\| \leq \frac{1428K'\theta\|x\|^p}{4096(32-2^p)} \quad \text{for all } x \in X.$$

With the help of the definition of F_5, we obtain that $F_5(2x) = 32F_5(x)$ for all $x \in X$ and $DF_5(x, y) = 0$ for all $x, y \in X$.

Let $p > 5$. In view of Lemma 2.6 and (4.3), we have that

$$\left\|32^n \tilde{f}_5(2^{-n}x) - 32^{n+m}\tilde{f}_5(2^{-n-m}x)\right\| \leq \sum_{i=n}^{n+m-1} \frac{1428 \cdot 32^i K'\theta\|x\|^p}{4096 \cdot 2^{(i+1)p}} \tag{4.15}$$

for all $x \in X$ and $n, m \in \mathbb{N} \cup \{0\}$. It follows from $p > 5$ and (4.15) that the sequence $\{32^n \tilde{f}_5(2^{-n}x)\}$ is Cauchy for all $x \in X$. Since Y is complete, we see that the sequence $\{32^n \tilde{f}_5(2^{-n}x)\}$ converges. So, one can define a mapping $F_5 : X \to Y$ by

$$F_5(x) := \lim_{n\to\infty} 32^n \tilde{f}_5(2^{-n}x) \quad \text{for all} \quad x \in X.$$

Furthermore, setting $n = 0$ and sending the limit $m \to \infty$ in (4.15), we lead to

$$\|\tilde{f}_5(x) - F_5(x)\| \leq \frac{1428K'\theta\|x\|^p}{4096(2^p-32)} \quad \text{for all} \quad x \in X.$$

By virtue of the definition of F_5, we find that $F_5(2x) = 32F_5(x)$ for all $x \in X$ and $DF_5(x, y) = 0$ for all $x, y \in X$.

- **Setting** F_6 : Let $p < 6$. Combine Lemma 2.6 and (4.4) to find that

$$\left\|\frac{\tilde{f}_6(2^n x)}{64^n} - \frac{\tilde{f}_6(2^{n+m}x)}{64^{n+m}}\right\| = \left\|\frac{21\Delta \tilde{f}_e(2^i x)}{2835 \cdot 262144 \cdot 64^i}\right\| \leq \sum_{i=n}^{n+m-1} \frac{21K2^{ip}\theta\|x\|^p}{4096 \cdot 64^i} \tag{4.16}$$

for all $x \in X$ and $n, m \in \mathbb{N}\cup\{0\}$. Then, since $p < 6$, it follows by (4.16) that $\{\frac{\tilde{f}_6(2^n x)}{64^n}\}$ is a Cauchy sequence for all $x \in X$. The completeness of Y ensures that the sequence $\{\frac{\tilde{f}_6(2^n x)}{64^n}\}$ converges, so that, we define a mapping $F_6 : X \to Y$ by

$$F_6(x) := \lim_{n\to\infty} \frac{\tilde{f}_6(2^n x)}{64^n} \quad \text{for all} \quad x \in X.$$

But then, let $n = 0$ and take the limit as $m \to \infty$ in (4.16) to get

$$\|\tilde{f}_6(x) - F_6(x)\| \leq \frac{21K\theta\|x\|^p}{64(64-2^p)} \quad \text{for all} \quad x \in X.$$

According to the definition of F_6, we have shown that $F_6(2x) = 64F_6(x)$ for all $x \in X$ and $DF_6(x, y) = 0$ for all $x, y \in X$.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 29 May 2023 doi:10.20944/preprints202305.1988.v1

Let $p>6$. It follows from Lemma 2.6 and (4.4) that

$$\left\|64^n\tilde{f}_6(2^{-n}x)-64^{n+m}\tilde{f}_6(2^{-n-m}x)\right\|\leq\sum_{i=n}^{n+m-1}\frac{21\cdot 64^iK\theta\|x\|^p}{64\cdot 2^{(i+1)p}} \tag{4.17}$$

for all $x\in X$ and $n,m\in\mathbb{N}\cup\{0\}$. Since $p>6$, we see from (4.17) that the sequence $\{64^n\tilde{f}_6(2^{-n}x)\}$ is Cauchy for all $x\in X$. Thus, by the completeness of Y, we find that the sequence $\{64^n\tilde{f}_6(2^{-n}x)\}$ converges. Hence, we can define a mapping $F_6:X\to Y$ by

$$F_6(x):=\lim_{n\to\infty}64^n\tilde{f}_6(2^{-n}x)\quad\text{for all}\;\;x\in X.$$

In addition, put $n=0$ and then let $m\to\infty$ in (4.17) to have

$$\|\tilde{f}_6(x)-F_6(x)\|\leq\frac{21K\theta\|x\|^p}{64(2^p-64)}\;\;\text{for all}\;\;x\in X.$$

From the definition of F_6, we get $F_6(2x)=64F_6(x)$ for all $x\in X$ and $DF_6(x,y)=0$ for all $x,y\in X$.

- **Setting** F_7 : Let $p<7$. We know by Lemma 2.6 and (4.3) that

$$\left\|\frac{\tilde{f}_7(2^nx)}{128^n}-\frac{\tilde{f}_7(2^{n+m}x)}{128^{n+m}}\right\|=\left\|\frac{85\Gamma\tilde{f}(2^ix)}{722925\cdot 8388608\cdot 128^i}\right\|\leq\sum_{i=n}^{n+m-1}\frac{85K'2^{ip}\theta\|x\|^p}{524504\cdot 128^i} \tag{4.18}$$

for all $x\in X$ and $n,m\in\mathbb{N}\cup\{0\}$. Based on the fact that $p<7$ and (4.18), the sequence $\{\frac{\tilde{f}_7(2^nx)}{128^n}\}$ is Cauchy for all $x\in X$. Since Y is complete, the sequence $\{\frac{\tilde{f}_7(2^nx)}{128^n}\}$ converges. So, one can define a mapping $F_7:X\to Y$ by

$$F_7(x):=\lim_{n\to\infty}\frac{\tilde{f}_7(2^nx)}{128^n}\quad\text{for all}\;\;x\in X.$$

Moreover, letting $n=0$ and passing the limit $m\to\infty$ in (4.18), we yield that

$$\|\tilde{f}_7(x)-F_7(x)\|\leq\frac{85K'\theta\|x\|^p}{4096(128-2^p)}\;\;\text{for all}\;\;x\in X.$$

By the definition of F_7, we have that $F_7(2x)=128F_7(x)$ for all $x\in X$ and $DF_7(x,y)=0$ for all $x,y\in X$.

Let $p>7$. Note that, by Lemma 2.6 and (4.3), we have

$$\left\|128^n\tilde{f}_7(2^{-n}x)-128^{n+m}\tilde{f}_7(2^{-n-m}x)\right\|\leq\frac{85\cdot 128^iK'\theta\|x\|^p}{4096\cdot 2^{(i+1)p}} \tag{4.19}$$

for all $x\in X$ and $n,m\in\mathbb{N}\cup\{0\}$. On the basis of the assumption $p>7$ and (4.19), we see that $\{128^n\tilde{f}_7(2^{-n}x)\}$ is a Cauchy sequence for all $x\in X$. Since Y is complete, the sequence $\{128^n\tilde{f}_7(2^{-n}x)\}$ converges for all $x\in X$. Therefore, we can define a mapping $F_7:X\to Y$ by

$$F_7(x):=\lim_{n\to\infty}128^n\tilde{f}_7(2^{-n}x)\;\;\text{for all}\;\;x\in X.$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 29 May 2023 doi:10.20944/preprints202305.1988.v1

In (4.19), set $n = 0$ and then let $m \to \infty$ to find

$$\|\tilde{f}_7(x) - F_7(x)\| \leq \frac{85K'\theta\|x\|^p}{4096(2^p - 128)} \quad \text{for all} \quad x \in X.$$

By the definition of F_7, it is shown that $F_7(2x) = 128F_7(x)$ for all $x \in X$ and $DF_7(x, y) = 0$ for all $x, y \in X$.

- **Setting** F_8 : Let $p < 8$. It follows from Lemma 2.6 and (4.4) that

$$\left\|\frac{\tilde{f}_8(2^n x)}{256^n} - \frac{\tilde{f}_8(2^{n+m}x)}{256^{n+m}}\right\| = \left\|\frac{\Delta\tilde{f}_e(2^i x)}{2835 \cdot 256 \cdot 4096 \cdot 256^i}\right\| \leq \sum_{i=n}^{n+m-1} \frac{K2^{ip}\theta\|x\|^p}{16384 \cdot 256^i} \tag{4.20}$$

for all $x \in X$ and $n, m \in \mathbb{N} \cup \{0\}$. By the assumption $p < 8$ and (4.20), the sequence $\{\frac{\tilde{f}_8(2^n x)}{256^n}\}$ is Cauchy for all $x \in X$. Since Y is complete, the sequence $\{\frac{\tilde{f}_8(2^n x)}{256^n}\}$ converges. Hence, we can define a mapping $F_8 : X \to Y$ by

$$F_8(x) := \lim_{n\to\infty} \frac{\tilde{f}_8(2^n x)}{256^n} \quad \text{for all} \quad x \in X.$$

Letting $n = 0$ and passing the limit $m \to \infty$ in (4.20), we then have the following inequality

$$\|\tilde{f}_8(x) - F_8(x)\| \leq \frac{K\theta\|x\|^p}{64(256 - 2^p)} \quad \text{for all} \quad x \in X.$$

From the definition of F_8, we are forced to conclude that $F_8(2x) = 256F_8(x)$ for all $x \in X$ and $DF_8(x, y) = 0$ for all $x, y \in X$.

Let $p > 8$. Observe that, by Lemma 2.6 and (4.4), we obtain that

$$\left\|256^n\tilde{f}_8(2^{-n}x) - 256^{n+m}\tilde{f}_8(2^{-n-m}x)\right\| \leq \sum_{i=n}^{n+m-1} \frac{256^i K\theta\|x\|^p}{64 \cdot 2^{(i+1)p}} \tag{4.21}$$

for all $x \in X$ and $n, m \in \mathbb{N} \cup \{0\}$. It follows from the assumption $p > 8$ and (4.21) that $\{256^n\tilde{f}_8(2^{-n}x)\}$ is a Cauchy sequence for all $x \in X$. The completeness of Y implies that the sequence $\{256^n\tilde{f}_8(2^{-n}x)\}$ converges, so that, we can define a mapping $F_8 : X \to Y$ by

$$F_8(x) := \lim_{n\to\infty} 256^n\tilde{f}_8(2^{-n}x) \quad \text{for all} \quad x \in X.$$

Put $n = 0$ and then take $m \to \infty$ in (4.21) to get

$$\|\tilde{f}_8(x) - F_8(x)\| \leq \frac{K\theta\|x\|^p}{64(2^p - 256)} \quad \text{for all} \quad x \in X.$$

According to the definition of F_8, we find that $F_8(2x) = 256F_8(x)$ for all $x \in X$ and $DF_8(x, y) = 0$ for all $x, y \in X$.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 29 May 2023 doi:10.20944/preprints202305.1988.v1

- **Setting** F_9 : Let $p<9$. It follows from Lemma 2.6 and (4.3) that

$$\left\|\frac{\tilde{f}_9(2^n x)}{512^n}-\frac{\tilde{f}_9(2^{n+m}x)}{512^{n+m}}\right\|=\left\|\frac{\Gamma\tilde{f}(2^i x)}{722925\cdot 33554432\cdot 512^i}\right\|\leq\sum_{i=n}^{n+m-1}\frac{K'2^{ip}\theta\|x\|^p}{2097152\cdot 512^i}\tag{4.22}$$

for all $x\in X$ and $n,m\in\mathbb{N}\cup\{0\}$. We then have by the assumption $p<9$ and (4.22) that $\{\frac{\tilde{f}_9(2^n x)}{512^n}\}$ is a Cauchy sequence for all $x\in X$. Since Y is complete, the sequence $\{\frac{\tilde{f}_9(2^n x)}{512^n}\}$ converges. Then, one can define a mapping $F_9 : X\to Y$ by

$$F_9(x):=\lim_{n\to\infty}\frac{\tilde{f}_9(2^n x)}{512^n}\quad\text{for all}\quad x\in X.$$

On the other hand, letting $n=0$ and passing the limit $m\to\infty$ in (4.22), we deduce that

$$\|\tilde{f}_9(x)-F_9(x)\|\leq\frac{K'\theta\|x\|^p}{4096(512-2^p)}\quad\text{for all}\quad x\in X.$$

We have from the definition of F_9 that $F_9(2x)=512F_9(x)$ for all $x\in X$ and $DF_9(x,y)=0$ for all $x,y\in X$.

Let $p>9$. Using Lemma 2.6 and (4.3), we have

$$\left\|512^n\tilde{f}_9(2^{-n}x)-512^{n+m}\tilde{f}_9(2^{-n-m}x)\right\|\leq\frac{512^i K'\theta\|x\|^p}{4096\cdot 2^{(i+1)p}}\tag{4.23}$$

for all $x\in X$ and $n,m\in\mathbb{N}\cup\{0\}$. From $p>9$ and (4.23), it follows that the sequence $\{512^n\tilde{f}_9(2^{-n}x)\}$ is Cauchy for all $x\in X$. By the completeness of Y, the sequence $\{512^n\tilde{f}_9(2^{-n}x)\}$ converges. Thereby, we can define a mapping $F_9 : X\to Y$ by

$$F_9(x):=\lim_{n\to\infty}512^n\tilde{f}_9(2^{-n}x)\quad\text{for all}\quad x\in X.$$

In particular, put $n=0$ and then let $m\to\infty$ in (4.23) to have

$$\|\tilde{f}_9(x)-F_9(x)\|\leq\frac{K'\theta\|x\|^p}{4096(2^p-512)}\quad\text{for all}\quad x\in X.$$

With the aid of the definition of F_9, one obtains that $F_9(2x)=512F_9(x)$ for all $x\in X$ and $DF_9(x,y)=0$ for all $x,y\in X$.

Finally, we set a mapping F as

$$F(x):=\sum_{k=1}^{9}F_k(x)\qquad\text{for all } x\in X.$$

Since $DF_k(x,y)=0$ for all $k\in\{1,2,\ldots,9\}$, we have

$$DF(x,y)=\sum_{k=1}^{9}DF_k(x,y)=0\qquad\text{for all } x,y\in X.$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 29 May 2023 doi:10.20944/preprints202305.1988.v1

Next, we are in the position to prove that the mapping F satisfies the inequality (4.2). Since $\tilde{f}(x) = \sum_{k=1}^{9} \tilde{f}_k(x)$, we have

$$\|\tilde{f}(x) - F(x)\| \leq \sum_{k=1}^{9} \|\tilde{f}_k(x) - F_k(x)\| \quad \text{for all} \quad x \in X.$$

and so we obtain the desired result (4.2).

It remains to prove that F is unique: Suppose that $F' : V \to Y$ be another mapping with $F'(0) = 0$ satisfying the relation $DF'(x, y) = 0$ and the inequality (4.2). We have by Lemma 2.7 that for each $k \in \{1, 2, \cdots, 9\}$, the mappings $F'_k : X \to Y$ satisfy

$$F'(x) = \sum_{k=1}^{9} F'_k(x), \quad F'_k(2x) = 2^k F'_k(x) \quad (x \in X).$$

To verify the uniqueness of F, we want to prove it only if $2 < p < 3$. This is because other cases of p can be showed in a similar fashion. Therefore, let us assume that $2 < p < 3$. Then, we see that for $2 < p < 3$,

$$\begin{aligned}
&\left\|4^n \tilde{f}_2\left(\frac{x}{2^n}\right) - F'_2(x)\right\| = \left\|4^n \tilde{f}_2\left(\frac{x}{2^n}\right) - 4^n F'_2\left(\frac{x}{2^n}\right)\right\| \\
&= \frac{4^n}{181440}\left\|262144\tilde{f}_e\left(\frac{x}{2^n}\right) - 21504\tilde{f}_e\left(\frac{2x}{2^n}\right) + 336\tilde{f}_e\left(\frac{4x}{2^n}\right) - \tilde{f}_e\left(\frac{8x}{2^n}\right)\right. \\
&\quad \left. - 262144F'_e\left(\frac{x}{2^n}\right) + 21504F'_e\left(\frac{2x}{2^n}\right) - 336F'_e\left(\frac{4x}{2^n}\right) + F'_e\left(\frac{8x}{2^n}\right)\right\| \\
&\leq \frac{262144 \cdot 4^n}{181440}\left\|\tilde{f}_e\left(\frac{x}{2^n}\right) - F'_e\left(\frac{x}{2^n}\right)\right\| + \frac{21504 \cdot 4^n}{181440}\left\|\tilde{f}_e\left(\frac{2x}{2^n}\right) - F'_e\left(\frac{2x}{2^n}\right)\right\| \\
&\quad + \frac{336 \cdot 4^n}{181440}\left\|\tilde{f}_e\left(\frac{4x}{2^n}\right) - F'_e\left(\frac{4x}{2^n}\right)\right\| + \frac{4^n}{181440}\left\|\tilde{f}_e\left(\frac{8x}{2^n}\right) - F'_e\left(\frac{8x}{2^n}\right)\right\| \\
&\leq \frac{262144 + 21504 \cdot 2^p + 336 \cdot 4^p + 8^p}{181440} \cdot \frac{4^n}{2^{np}} M\theta\|x\|^p,
\end{aligned}$$

and

$$\begin{aligned}
\left\|\frac{\tilde{f}_3(2^n x)}{8^n} - F'_3(x)\right\| &= \left\|\frac{\tilde{f}_3(2^n x)}{8^n} - \frac{F'_3(2^n x)}{8^n}\right\| \\
&= \frac{5440}{722925 \cdot 65536}\left\|\frac{4194304((F'_o - \tilde{f}_o)(2^n x)) - 2269184((F'_o - \tilde{f}_o)(2^{n+1}x))}{8^n}\right. \\
&\quad \left. + \frac{87360((F'_o - \tilde{f}_o)(2^{n+2}x)) - 674((F'_o - \tilde{f}_o)(2^{n+3}x)) + ((F'_o - \tilde{f}_o)(2^{n+4}x))}{8^n}\right\| \\
&\leq \left(\frac{4194304 + 2269184 \cdot 2^p + 87360 \cdot 2^{2p} + 674 \cdot 2^{3p} + 2^{4p}}{722925 \cdot 65536}\right) \cdot \frac{5440 \cdot 2^{np} M\theta\|x\|^p}{8^n}
\end{aligned}$$

for all $x \in X$ and all positive integer n, where

$$M := \left(\frac{4096}{|2-2^p|} + \frac{5440}{|8-2^p|} + \frac{1428}{|32-2^p|} + \frac{85}{|128-2^p|} + \frac{1}{|512-2^p|} \right) \cdot \frac{K'}{4096} + \left(\frac{64}{|4-2^p|} + \frac{84}{|16-2^p|} + \frac{21}{|64-2^p|} + \frac{1}{|256-2^p|} \right) \cdot \frac{K}{64}.$$

Taking the limit in the above relations as $n \to \infty$, we obtain the equality

$$F_2'(x) = \lim_{n\to\infty} \tilde{4}^n \tilde{f}_2\left(\frac{x}{2^n}\right), \quad F_3'(x) = \lim_{n\to\infty} \frac{\tilde{f}_3(2^n x)}{8^n} \quad (x \in X),$$

which means that $F_2(x) = F_2'(x)$ and $F_3(x) = F_3'(x)$ for all $x \in X$. Employing the similar way, it is easily shown that for each $k \in \{1, 4, 5, 6, 7, 8, 9\}$, the equalities $F_k = F_k'$ hold. Note that

$$F(x) = \sum_{k=1}^{9} F_k(x) = \sum_{k=1}^{9} F_k'(x) = F'(x).$$

This completes the proof of the uniqueness of F for $2 < p < 3$.

For other p cases, the uniqueness proof of F can be proved very same to the proof for $2 < p < 3$. □

Acknowledgment

This work was supported by the Hallym University Research Fund (HRF-202205-007)

Conflict of interest

The authors declare that they have no competing interests.

Author Contributions

Ick-Soon Chang, Yang-Hi Lee and Jaiok Roh contributed in the writing, review and editing of this paper.

Data Availability Statement

No data was gathered for this article.

References

[1] M. Almahalebi, A. Charifi and S. Kabbaj, *Hyperstability of a monomial functional equation*, JSRR **3 (20)** (2014), 2685-2693

[2] A. Bahyrycz and J. Olko, *Hyperstability of some functional equations on restricted domain*, J. Funct. Spaces **2017** (2017), 6 pp.

[3] J. Baker, *A general functional equation and its stability*, Proc. Amer. Math. Soc. **133 (6)** (2005), 1657-1664.

[4] A. Bodaghi and A. Fošner, *Characterization, stability and hyperstability of multi-quadratic-cubic mappings*, J. Inequal. Appl. **49** (2021), 12 pp.

[5] J. Brzdek, *A hyperstability result for the Cauchy equation*, Bull. Aust. Math. Soc. **89** (2014), 33-40.

[6] L. Cădariu and V. Radu, *The fixed points method for the stability of some functional equations*, Carpathian J. Math. **23** (2007), 63-72

[7] I.-S. Chang, Y.-H. Lee and J. Roh, *Nearly general septic functional equation*, J. Funct. Spaces 2021 (2021), 7pp.

[8] P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), 431-436.

[9] A. Gilányi, *On the stability of monomial functional equations*, Publ. Math. Debrecen **56** (2000), 201-212.

[10] D.H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. USA **27** (1941), 222-224.

[11] Z. Kaiser, *On stability of the monomial functional equation in normed spaces over fields with valuation*, J. Math. Anal. Appl. **322** (2006), 1188-1198

[12] G.-H. Kim and Y.-H Lee, *Stability of an additive-quadratic-quartic functional equation*, Demonstr. Math. **53 (1)** (2020), 1-7.

[13] Y.-H. Lee, *On the stability of the monomial functional equation*, Bull. Korean Math. Soc. **45** (2008), 397-403.

[14] Y.-H. Lee, *Stability of a monomial functional equation on a restricted domain*, Mathematics **5** (2017), 11pp.

[15] Y.-H. Lee, *On the Hyers-Ulam-Rassias stability of a general quintic functional equation and a general sextic functional equation*, Mathematics **7** (2019), 14pp.

[16] Y.-H. Lee and K.W. Jun, *A generalization of the Hyers-Ulam-Rassias stability of Jensen's equation*, J. Math. Anal. Appl. **238 (1)** (1999), 305-315.

[17] Y.-H. Lee and S.-M. Jung, *General uniqueness theorem concerning the stability of additive, quadratic, and cubic functional equations*, Adv. Differ. Equ. (2016) **75**, 12pp.

[18] Y.-H. Lee, S.-M. Jung and M.T. Rassias, *Uniqueness theorems on functional inequalities concerning cubic-quadratic-additive equation*, J. Math. Inequal. **12** (2018), 43-61.

[19] G. Maksa and Z. Páles, *Hyperstability of a class of linear functional equations*, Acta Math. Acad. Paedagog. Nyházi. (N.S.) **17 (2)** (2001), 107-112.

[20] M.B. Moghimi and Abbas Najati, *Some hyperstability and stability results for the Cauchy and Jensen equations*, J. Inequal. Appl. **2022 (97)** (2022), 12pp.

[21] Th.M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297-300.

[22] J. Roh, Y.-H. Lee and S.-M. Jung, *The stability of a general sextic functional equation by fixed point theory*, J. Funct. Spaces **2020** (2020), 8pp.

[23] A. Ponmana Selvan and A. Najati, *Hyers-Ulam stability and hyperstability of a Jensen-type functional equation on 2-Banach spaces*, J. Inequal. Appl. **2022 (32)** (2022), 11pp.

[24] S.M. Ulam, *A Collection of Mathematical Problems*, Interscience, New York, (1960).