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*Article*

# Randomly Stopped Sums with Generalized Subexponential Distribution

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**Abstract:** Let $\{\xi_1, \xi_2 \ldots\}$ be a sequence of independent possibly differently distributed random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with distribution functions $\{F_{\xi_1}, F_{\xi_2}, \ldots\}$. Let $\eta$ be a counting random variable independent of sequence $\{\xi_1, \xi_2 \ldots\}$. In this paper, we find conditions under which distribution function of randomly stopped sum $S_\eta = \xi_1 + \xi_2 + \ldots + \xi_\eta$ belongs to the class of generalized subexponential distributions.

**Keywords:** subexponentiality; generalized subexponentiality; heavy tail; randomly stopped sum

**MSC:** 60G50; 60G40; 60E05

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## 1. Introduction

Let $\{\xi_1, \xi_2, \ldots\}$ be a sequence of independent random variables (r.v.s) with distribution functions (d.f.s) $\{F_{\xi_1}, F_{\xi_2}, \ldots\}$, and let $\eta$ be a counting random variable, that is, a nonnegative, nondegenerate at 0, and integer-valued r.v. In addition, we suppose that the r.v. $\eta$ and the sequence $\{\xi_1, \xi_2, \ldots\}$ are independent.

Let $S_0 := 0$, $S_n := \xi_1 + \ldots + \xi_n$ for $n \in \mathbb{N}$, and let

$$S_\eta = \sum_{k=1}^{\eta} \xi_k$$

be the *randomly stopped sum* of the r.v.s $\xi_1, \xi_2, \ldots$.

By $F_{S_\eta}$ we denote the d.f. of $S_\eta$, and by $\overline{F}$ we denote the tail function (t.f.) of a d.f. $F$, that is, $\overline{F}(x) = 1 - F(x)$ for $x \in \mathbb{R}$. It is obvious that the following equalities hold for positive $x$:

$$F_{S_\eta}(x) = \mathbb{P}(\eta = 0) + \sum_{n=1}^{\infty} \mathbb{P}(\eta = n)\mathbb{P}(S_n \leqslant x)$$

$$\overline{F}_{S_\eta}(x) = \sum_{n=1}^{\infty} \mathbb{P}(\eta = n)\mathbb{P}(S_n > x).$$

In this paper, we consider a sequence $\{\xi_1, \xi_2, \ldots\}$ of independent and possibly nonidentically distributed r.v.s. We suppose that some of d.f.s of these r.v.s belong to the class of generalized subexponential distributions $\mathcal{OS}$, and we find conditions under which d.f. $F_{S_\eta}$ remains in this class.

We use the following three notations for the asymptotic relations of arbitrary positive functions $f$ and $g$: $f(x) = o(g(x))$ means that $\lim_{x\to\infty} f(x)/g(x) = 0$; $f(x) \sim cg(x)$, $c > 0$, means that $\lim_{x\to\infty} f(x)/g(x) = c$; and $f(x) \asymp g(x)$ means that

$$0 < \liminf_{x\to\infty} \frac{f(x)}{g(x)} \leqslant \limsup_{x\to\infty} \frac{f(x)}{g(x)} < \infty.$$

The rest of the paper is organized as follows. In Section 2, we desribe class of generalized subexponential distributions. Section 4 consists of some results on closure under randomly stopped

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sums for regularity classes related with generalized subexponential distributions. The main results of the paper are formulated in Section 3. The proofs of the main results are given in sections 5 and 6.

## 2. Generalized subexponentiality

Let $\xi$ be a r.v. defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with d.f. $F_\xi$.
• *A d.f. $F_\xi$ of a real-valued r.v. is said to be generalized subexponential, denoted $F_\xi \in \mathcal{OS}$, if*

$$\limsup_{x\to\infty} \frac{\overline{F_\xi * F_\xi}(x)}{\overline{F}_\xi(x)} < \infty,$$

*where $F_\xi * F_\xi$ denote the convolution of d.f. $F_\xi$ with it self, i.e.*

$$F_\xi * F_\xi(x) = F_\xi^{*2}(x) := \int_{-\infty}^{\infty} F_\xi(x-y)\mathrm{d}F_\xi(y),\ x \in \mathbb{R}.$$

For distributions of non-negative r.v.s class $\mathcal{OS}$ was introduced by Klüppelberg [1] and later for real-valued r.v.s was studied by Shimura and Watanabe [2], Baltrūnas et al. [3], Watanabe and Yamamuro [4], Yu and Wang [5], Cheng and Wang [6], Lin and Wang [7], Konstantinides et al. [8], Mikutavičius and Šiaulys [9] among others.

In [2], the class of distributions $\mathcal{OS}$ is considered together with other distribution regularity classes. In that paper, several closedness properties of the class $\mathcal{OS}$ were proved. For example, it is shown that the class $\mathcal{OS}$ is not closed under convolution roots. This means that there exist r.v. $\xi$ such that $n$-fold convolution $F_\xi^{*n} \in \mathcal{OS}$ for all $n \geqslant 2$, but $F_\xi \notin \mathcal{OS}$. In [3], the simple conditions are provided under which d.f. of the special form

$$F_\xi(x) = 1 - \exp\left\{ -\int\limits_0^x q(u)\mathrm{d}u \right\}$$

belongs to the class $\mathcal{OS}$, where $q$ is some integrable hazard rate function. For distributions of class $\mathcal{OS}$ the closure under tail-equivalence and the closure under convolution are established in [4]. The detailed proofs of these closures for non-negative r.v.s are presented in [1] and for real-valued r.v.s in [5]. The closure under convolution tail equivalence means that in case of independent r.v.s $\xi_1, \xi_2$ conditions $F_{\xi_1} \in \mathcal{OS}$, $F_{\xi_2} \in \mathcal{OS}$ imply that $F_{\xi_1} * F_{\xi_2} = F_{\xi_1+\xi_2} \in \mathcal{OS}$. The closure under tail-equivalence means that conditions $F_{\xi_1} \in \mathcal{OS}$, $\overline{F}_{\xi_1}(x) \asymp \overline{F}_{\xi_2}(x)$ imply $F_{\xi_2} \in \mathcal{OS}$.

A counterexample, showing that $F_{\xi_1}, F_{\xi_2} \in \mathcal{OS}$ for independent r.v.s $\xi_1, \xi_2$ does not imply $F_{\xi_1 \vee \xi_2} \in \mathcal{OS}$ can be found in [7]. Moreover in that paper, the closure under maximum is established which means that $F_{\xi_1}, F_{\xi_2} \in \mathcal{OS}$ for independent r.v.s $\xi_1, \xi_2$ imply $F_{\xi_1 \wedge \xi_2} \in \mathcal{OS}$. The authors of articles [8] and [9] consider when the distribution of the product of two independent random variables $\xi, \theta$ belongs to the class $\mathcal{OS}$. For instance in [9], it is proved that d.f. $F_{\xi\theta}$ is generalized subexponential if $F_\xi \in \mathcal{OS}$ and $\theta$ is non-negative and not-degenerated at zero.

## 3. Main results

In this section, we formulate two theorems which are the main assertions of this paper. The first theorem deals to the case when the counting r.v. has a finite support.

**Theorem 1.** *Let $\{\xi_1, \xi_2, \ldots\}$ be a sequence of independent r.v.s, and $\eta$ be a counting r.v. independent of $\{\xi_1, \xi_2, \ldots\}$. If $\eta$ is bounded, $F_{\xi_1} \in \mathcal{OS}$, and for other indices $k \neq 1$ either $F_{\xi_k} \in \mathcal{OS}$ or $\overline{F}_{\xi_k}(x) = O\big(\overline{F}_{\xi_1}(x)\big)$, then d.f. of randomly stopped sum $F_{S_\eta}$ belongs to the class $\mathcal{OS}$.*

The case of unbounded support of counting r.v. is considered in the second theorem. In such a case, to be $F_{S_\eta} \in \mathcal{OS}$ we need the counting random variable to have a light tail.

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**Theorem 2.** *Let $\{\eta, \xi_1, \xi_2, ...\}$ be independent random variables, where counting r.v. $\eta$ be such that $\mathbb{E}e^{\lambda\eta} < \infty$ for all $\lambda > 0$. Then $F_{S_\eta} \in OS$, if $F_{\xi_1} \in OS$ and one of the conditions below is satisfied:*

$$\text{(i)}\ \mathbb{P}(\eta = 1) > 0 \ \text{ and } \ \limsup_{x\to\infty} \sup_{k\geqslant 1} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} < \infty;$$
$$\text{(ii)}\ 0 < \liminf_{x\to\infty} \inf_{k\geqslant 1} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} \leqslant \limsup_{x\to\infty} \sup_{k\geqslant 1} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} < \infty.$$

We will present the proofs of both theorems in section 6. According to the statements of these theorems, many random variables with generalized subexponential distributions can be constructed. We will demonstrate such constructions in section **??**

## 4. Similar results for related regularity classes

In this section, we will describe several classes of distributions related to the class $\mathcal{OS}$. For the described classes, we will present some results on their closure with respect to a randomly stopped sum. We note that for some classes, the closedness of the randomly stopped sum is studied only in the case where the summands are identically distributed.

The class of generalized subexponential distributions is the direct generalization of

$$\widehat{\mathcal{S}} = \bigcup_{\gamma\geqslant 0} \mathcal{S}(\gamma),$$

where $\mathcal{S}(0) = \mathcal{S}$ is the class of the subexponential distributions and $\{\mathcal{S}(\gamma), \gamma > 0\}$ are the convolution equivalent distributions classes.

• *A d.f. $F_\xi$ of a non-negative r.v. $\xi$ is said to be subexponential, denoted $F_\xi \in \mathcal{S}$, if*

$$\overline{F_\xi * F_\xi}(x) \sim 2\overline{F}_\xi(x).$$

*A d.f. $F_\xi$ of a real-valued r.v. $\xi$ is called subexponential if the positive part of d.f.*

$$F_\xi^+(x) = F_\xi(x)\mathbb{I}_{[0,\infty)}(x)$$

*belongs to the class $\mathcal{S}$.*

The class of subexponential distributions was introduced by Chistyakov [10] and later considered by Athreya and Ney [11], Chover et al. [12,13], Embrechts and Goldie [14], Embrechts and Omey [15], Cline [16] and Cline and Samorodnitsky [17] among others.

• *A d.f. $F_\xi$ of a real-valued r.v. $\xi$ is said to be convolution equivalent with parameter $\gamma > 0$, denoted $F_\xi \in \mathcal{S}(\gamma)$, if the following requirements are satisfyed*

$$\text{(i)}\ \widehat{F}_\xi(\gamma) := \int_{-\infty}^{\infty} e^{\gamma x} dF_\xi(x) < \infty,$$
$$\text{(ii)}\ \lim_{x\to\infty} \frac{\overline{F}_\xi(x-y)}{\overline{F}_\xi(x)} = e^{\gamma y} \ \text{ for all } \ y > 0,$$
$$\text{(iii)}\ \lim_{x\to\infty} \frac{\overline{F_\xi * F_\xi}(x)}{\overline{F}_\xi(x)} = 2c_\xi \ \text{ for some constant } c_\xi$$

The study of class $\mathcal{S}(\gamma)$ goes back to Chover et al. [12,13], Embrechts and Goldie [14], Klüppelberg [18]. It is well known that $F \in \mathcal{S}(\gamma)$ if and only if $F_\xi^+ \in \mathcal{S}(\gamma)$, see Corollary 2.1(i) in [19], and the constant $c_\xi$ in the definition above is equal to $\widehat{F}_\xi(\gamma)$, see [19–21]. For $\gamma > 0$ a standard example of d.f. in $\mathcal{S}(\gamma)$ is d.f. satisfying

$$\widehat{F}(x) \underset{x\to\infty}{\sim} c\,\mathrm{e}^{-\gamma x} x^{-\alpha}$$

with parameters $c > 0, \gamma > 0, \alpha > 1$, see [22,23].

For the class $\mathcal{S}$ the following result is obtained in Theorem 3.37 of [24], see also [25–28].

**Theorem 3.** *Let $\{\xi_1, \xi_2, \dots\}$ be a sequence of independent real-valued r.v.s with common distribution $F_\xi \in \mathcal{S}$, and let $\eta$ be independent counting r.v. with expectation $\mathbb{E}\eta$, such that $\mathbb{E}(1+\varepsilon)^\eta < \infty$ for some $\varepsilon > 0$. Then*

$$\overline{F}_{S_\eta}(x) \sim \mathbb{E}\eta \overline{F}_\eta(x),$$

*and $F_{S_\eta} \in \mathcal{S}$.*

For the class $\mathcal{S}(\gamma)$ with $\gamma > 0$ the following assertion is derived in Theorem C of [29], see also [30–32] for related results.

**Theorem 4.** *Let $\{\xi_1, \xi_2, \dots\}$ independent real-valued r.v.s with common distribution $F_\xi \in \mathcal{S}(\gamma)$, $\gamma > 0$, and let $\eta$ be independent counting r.v. independent of $\{\xi_1, \xi_2, \dots\}$. If*

$$\sum_{n=0}^{\infty} \mathbb{P}(\eta = n) \max\left\{\left(\widehat{F}_\xi(\gamma) + \varepsilon\right)^n, 1\right\} < \infty$$

*for some $\varepsilon > 0$, then $F_{S_\eta} \in \mathcal{S}(\gamma)$.*

We note that in the theorems 3 and 4 r.v.s in the sequences $\{\xi_1, \xi_2, \dots\}$ are identically distributed. However, there are related regularity classes for which similar results can be obtained in cases where r.v.s in $\{\xi_1, \xi_2, \dots\}$ are not necessarily identically distributed. Here we discuss two such classes.

- *A d.f. $F_\xi$ of a real valued r.v. $\xi$ is said to be dominatedly varying, denoted $F_\xi \in \mathcal{D}$, if*

$$\limsup_{x\to\infty} \frac{\overline{F_\xi}(yx)}{\overline{F}_\xi(x)} < \infty$$

*for all (or, equivalently, for some) $y \in (0,1)$.*

- *A d.f. $F_\xi$ of a real valued r.v. $\xi$ is said to be exponential-like-tailed, denoted $F_\xi \in \mathcal{L}(\gamma)$, if*

$$\lim_{x\to\infty} \frac{\overline{F_\xi}(x-y)}{\overline{F}_\xi(x)} = \mathrm{e}^{\gamma y}$$

*for all $y > 0$.*

Class of dominatedly varying d.f.s $\mathcal{D}$ was introduced by Feller [33] and later considered in [4,34–39] among others. The class of long-tailed d.f.s $\mathcal{L}(0)$ was introduced by Chistyakov [10] in the context of branching processes. The class $\mathcal{L}(\gamma)$ with $\gamma > 0$ was introduced by Chover et al. [12,13]. Later the various properties of long-tailed and exponential-like-tailed d.f.s were considered in [1,19,24,29,38,40,41] for instance. Here we recall only that $\mathcal{L}(0) \cap \mathcal{D} \subset \mathcal{S}$ and $\mathcal{S}(\gamma) \subset \mathcal{L}(\gamma)$.

The following assertion on $F_{S_\eta} \in \mathcal{D}$ is presented in Theorem 4 of [42].

**Theorem 5.** *Let $\{\xi_1, \xi_2, \ldots\}$ be a sequence of independent real-valued r.v.s with common d.f. $F_\xi \in \mathcal{D}$, and let $\eta$ be a counting r.v. independent of $\{\xi_1, \xi_2, \ldots\}$. Then $F_{S_\eta} \in \mathcal{D}$ if $\mathbb{E}\eta^{p+1} < \infty$ for some*

$$p > J^+_{F_\xi} := -\lim_{y\to\infty} \frac{1}{\log y} \log \liminf_{x\to\infty} \frac{\overline{F}_\xi(xy)}{\overline{F}_\xi(x)}.$$

In the inhomogeneous case, when sumands are not necessary identically distributed, the following statement is obtained in Theorem 2.1 of [43].

**Theorem 6.** *Let $\{\xi_1, \xi_2, \ldots\}$ be a sequence independent nonnegative r.v.s, and let $\eta$ be a counting r.v. independent of $\{\xi_1, \xi_2, \ldots\}$. Then $F_{S_\eta} \in \mathcal{D}$ if the following three conditions are satisfied:*

(i) $F_{\xi_\varkappa} \in \mathcal{D}$ *for some* $\varkappa \in \operatorname{supp}(\eta) := \{n \in \mathbb{N}_0 : \mathbb{P}(\eta = n) > 0\}$,

(ii) $\limsup\limits_{x\to\infty} \sup\limits_{n>\varkappa} \frac{1}{n\overline{F}_{\xi_\varkappa}(x)} \sum\limits_{i=1}^{n} \overline{F}_{\xi_i}(x) < \infty$,

(iii) $\mathbb{E}\eta^{p+1} < \infty$ *for some* $p > J^+_{F_{\xi_\varkappa}}$.

Examples of conditions for the function $F_{S_\eta}$ to belong to the class $\mathcal{L}(\gamma)$ are given in the theorems below. Theorem 7 proved in [42] present conditions for the homogeneous case for class $\mathcal{L} = \mathcal{L}(0)$, while Theorem 8 proved in [44] gives conditions for the inhomogeneous case for class $\mathcal{L}(\gamma)$ with $\gamma \geqslant 0$.

**Theorem 7.** *Suppose that $\{\xi_1, \xi_2, \ldots\}$ are independent nonnegative r.v.s with common distribution $F_\xi \in \mathcal{L}$, and let $\eta$ be a counting r.v. independent of $\{\xi_1, \xi_2, \ldots\}$. If*

$$\overline{F}_\eta(\delta x) = o\big(\sqrt{x}\,\overline{F}_\xi(x)\big)$$

*for any $\delta \in (0,1)$, then $F_{S_\eta} \in \mathcal{L}$.*

**Theorem 8.** *Let $\{\xi_1, \xi_2, \ldots\}$ be a sequence of independent r.v.s such that for some $\gamma \geqslant 0$*

$$\sup_{k\geqslant 1} \left| \frac{\overline{F}_{\xi_k}(x+y)}{\overline{F}_{\xi_k}(x)} - \mathrm{e}^{-\gamma y} \right| \underset{x\to\infty}{\to} 0$$

*for each fixed $y > 0$, and let $\eta$ be a counting r.v. independent of $\{\xi_1, \xi_2, \ldots\}$. If*

$$\frac{\mathbb{P}(\eta = k+1)}{\mathbb{P}(\eta = k)} \underset{k\to\infty}{\to} 0,$$

*then $F_{S_\eta} \in \mathcal{L}(\gamma)$.*

In the context of the randomly stopped sums the class $OS$ was considered by Shimura and Watanabe [2]. In Proposition 3.1 of that paper the following assertion is presented.

**Theorem 9.** *Let $\{\xi_1, \xi_2, \ldots\}$ be a sequence of nonnegative independent r.v.s with common d.f. $F_\xi$, and let $\eta$ be a counting r.v. such that*

$$\mathbb{P}(\eta > 1) > 0, \;\; \sup\left\{ x \geqslant 1 : \sum_{k=0}^{\infty} \mathbb{P}(\eta = k) x^k < \infty \right\} = \infty.$$

*Then $F_\xi \in \mathcal{OS}$ if and only if $\overline{F}_{S_\eta}(x) \underset{x\to\infty}{\asymp} \overline{F}_\xi(x)$.*

From the information presented, it can be seen that our main theorems 1 and 2 in fact are inhomogeneous versions of the formulated theorem 9.

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## 5. Auxiliary lemmas

In this section, we will present and prove some auxiliary lemmas that will be applied to the derivations of the main theorems 1 and 2.

**Lemma 1.** *Let $X$ and $Y$ be two real valued r.v.s with corresponding d.f.s $F_X$ and $F_Y$. The following statements hold:*

(i) $F_X \in \mathcal{OS}$ *if and only if* $\sup\limits_{x\in\mathbb{R}} \frac{\overline{F_X * F_X}(x)}{\overline{F}_X(x)} < \infty$.

(ii) *If* $F_X \in \mathcal{OS}$ *and* $\overline{F}_Y(x) \underset{x\to\infty}{\asymp} \overline{F}_X(x)$, *then* $F_Y \in \mathcal{OS}$.

(iii) *If* $F_X \in \mathcal{OS}$ *and* $F_Y \in \mathcal{OS}$, *then* $F_X * F_Y \in \mathcal{OS}$.

(iv) *If* $F_X \in \mathcal{OS}$, *then* $F_X \in \mathcal{OL}$ *i.e.* $\limsup\limits_{x\to\infty} \frac{\overline{F}_X(x-1)}{\overline{F}_X(x)} = 1$.

(v) *If* $F_X \in \mathcal{OS}$ *and* $\overline{F}_Y(x) = O(\overline{F}_X(x))$, *then* $F_X * F_Y \in \mathcal{OS}$ *and* $\overline{F_X * F_Y}(x) \underset{x\to\infty}{\asymp} \overline{F}_X(x)$.

*Proof.* A large part of the properties of the class $\mathcal{OS}$ listed in Lemma 1 can be found, for instance, in [1,2,4,5]. However, for the sake of exposition completeness, we present the full proof of the formulated lemma.

*Part* (i). If $F_X \in OS$, then

$$\limsup_{x\to\infty} \frac{\overline{F_X * F_X}(x)}{\overline{F}_X(x)} < \infty \tag{1}$$

according to definition. This estimate implies that $\overline{F}_X(x) > 0$ for each $x \in \mathbb{R}$. In addition, the inequality (1) gives that

$$\frac{\overline{F_X * F_X}(x)}{\overline{F}_X(x)} \leqslant M$$

if $x \geqslant x_M$ for some $M$ and $x_M$.

If $x < x_M$, then, obviously, $\overline{F}_X(x) \geqslant \overline{F}_X(x_M)$ and $\overline{F_X * F_X}(x) \leqslant 1$.

Therefore, for each $x \in \mathbb{R}$ we get that

$$\frac{\overline{F_X * F_X}(x)}{\overline{F}_X(x)} \leqslant \max\left\{M, \frac{1}{\overline{F}_X(x_M)}\right\} < \infty$$

because $\overline{F}_X(x_M) > 0$. The last estimate finishes the proof of the part (i) because the condition

$$\sup_{x\in\mathbb{R}} \frac{\overline{F_X * F_X}(x)}{\overline{F}_X(x)} < \infty$$

implies (1) obviously.

*Part* (ii). The condition $\overline{F}_Y(x) \underset{x\to\infty}{\asymp} \overline{F}_X(x)$ implies

$$\liminf_{x\to\infty} \frac{\overline{F}_Y(x)}{\overline{F}_X(x)} > 0 \text{ and } \limsup_{x\to\infty} \frac{\overline{F}_Y(x)}{\overline{F}_X(x)} < \infty. \tag{2}$$

It follows from this that

$$\frac{\overline{F}_Y(x)}{\overline{F}_X(x)} \leqslant M,\ x \geqslant x_M,$$

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for some $M$ and $x_M$. If $x < x_M$, then

$$\frac{\overline{F}_Y(x)}{\overline{F}_X(x)} \leqslant \frac{1}{\overline{F}_X(x_M)} < \infty$$

because $F_X \in \mathcal{OS}$. According to the derived estimates

$$\sup_{x\in\mathbb{R}} \frac{\overline{F}_Y(x)}{\overline{F}_X(x)} = \max\left\{M, \frac{1}{\overline{F}_X(x_M)}\right\} = C < \infty.$$

Therefore for each $x \in \mathbb{R}$

$$\begin{aligned}
\overline{F_Y * F_Y}(x) &= \int_{-\infty}^{\infty} \frac{\overline{F}_Y(x-y)}{\overline{F}_X(x-y)}\overline{F}_X(x-y)\mathrm{d}F_Y(y) \leqslant C\int_{-\infty}^{\infty} \overline{F}_X(x-y)\mathrm{d}F_Y(y) \\
&= C\int_{-\infty}^{\infty} \overline{F}_Y(x-y)\mathrm{d}F_X(y) = C\int_{-\infty}^{\infty} \frac{\overline{F}_Y(x-y)}{\overline{F}_X(x-y)}\overline{F}_X(x-y)\mathrm{d}F_X(y) \\
&\leqslant C^2\int_{-\infty}^{\infty} \overline{F}_X(x-y)\mathrm{d}F_X(y) = C^2\,\overline{F_X * F_X}(x).
\end{aligned}$$

This estimate implies that

$$\begin{aligned}
\limsup_{x\to\infty} \frac{\overline{F_Y * F_Y}(x)}{\overline{F}_Y(x)} &\leqslant C^2 \limsup_{x\to\infty} \frac{\overline{F_X * F_X}(x)}{\overline{F}_Y(x)} \\
&\leqslant C^2 \limsup_{x\to\infty} \frac{\overline{F_X * F_X}(x)}{\overline{F}_X(x)} \frac{1}{\liminf\limits_{x\to\infty} \frac{\overline{F}_Y(x)}{\overline{F}_X(x)}} < \infty
\end{aligned}$$

due to the assumption $F_X \in \mathcal{OS}$ and the first inequality in (2). The last estimate gives that d.f. $F_Y$ belongs to the class $\mathcal{OS}$. Part (ii) of the lemma is proved.

*Part* (iii). According to part (i) we have that

$$\sup_{x\in\mathbb{R}} \frac{\overline{F_X * F_X}(x)}{\overline{F}_X(x)} = C_1 < \infty \ \text{ and } \ \sup_{x\in\mathbb{R}} \frac{\overline{F_Y * F_Y}(x)}{\overline{F}_Y(x)} = C_2 < \infty$$

Let $X_1, X_2, Y_1, Y_2$ be independent r.v.s. Suppose that $X_1$, $X_2$ are distributed according to the d.f. $F_X$, and $Y_1$, $Y_2$ are distributed according to the d.f. $F_Y$. For each $x \in \mathbb{R}$ we get

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$$
\begin{aligned}
\overline{((F_X * F_Y)^*)^2}(x) &= \overline{(F_X * F_Y) * (F_X * F_Y)}(x) = \mathbb{P}(X_1 + Y_1 + X_2 + Y_2 > x) \\
&= \mathbb{P}(X_1 + X_2 + Y_1 + Y_2) > x) = \int_{-\infty}^{\infty} \mathbb{P}(X_1 + X_2 > x - y) \mathrm{d}\mathbb{P}(Y_1 + Y_2 \leqslant y) \\
&= \int_{-\infty}^{\infty} \frac{\overline{F_X * F_X}(x-y)}{\overline{F}_X(x-y)} \overline{F}_X(x-y) \mathrm{d}\mathbb{P}(Y_1 + Y_2 \leqslant y) \\
&\leqslant C_1 \int_{-\infty}^{\infty} \overline{F}_X(x-y) \mathrm{d}\mathbb{P}(Y_1 + Y_2 \leqslant y) = C_1 \mathbb{P}(X_1 + Y_1 + Y_2 > x) \\
&= C_1 \int_{-\infty}^{\infty} \frac{\overline{F_Y * F_Y}(x-y)}{\overline{F}_Y(x-y)} \overline{F}_Y(x-y) \mathrm{d}\mathbb{P}(X_1 \leqslant y) \\
&\leqslant C_1 C_2 \int_{-\infty}^{\infty} \overline{F}_Y(x-y) dF_X(y) = C_1 C_2 \overline{F_X * F_Y}(x).
\end{aligned}
$$

Hence

$$
\sup_{x \in \mathbb{R}} \frac{\overline{((F_X * F_Y)^*)^2}(x)}{\overline{F_X * F_Y}(x)} \leqslant C_1 C_2
$$

implying that $F_X * F_Y \in \mathcal{OS}$ by part (i). Part (iii) of the lemma is proved.

*Part* (iv). Due to the part (i)

$$
\sup_{x \in \mathbb{R}} \frac{\overline{F_X * F_X}(x)}{\overline{F}_X(x)} = C_3 < \infty.
$$

In addition, for $x > 2$, we obtain

$$
\begin{aligned}
\overline{F_X * F_X}(x) &= \int_{-\infty}^{\infty} \overline{F}_X(x-t) \mathrm{d}F_X(t) \geqslant \int_{(1,x]} \overline{F}_X(x-t) \mathrm{d}F_X(t) \\
&\geqslant \overline{F}_X(x-1)(F_X(x) - F_X(1))
\end{aligned}
$$

When $x$ is large enough we have $F(x) - F(1) > 0$, and, therefore,

$$
\frac{\overline{F}_X(x-1)}{\overline{F}_X(x)} \leqslant \frac{\overline{F_X * F_X}(x)}{\overline{F}_X(x)} \frac{1}{F_X(x) - F_X(1)}.
$$

Hence

$$
\limsup_{x \to \infty} \frac{\overline{F}_X(x-1)}{\overline{F}_X(x)} \leqslant \frac{C_3}{\overline{F}_X(1)} < \infty,
$$

and part (iv) of the lemma is proved.

*Part* (v). Since $\overline{F}_Y(x) = O\big(\overline{F}_X(x)\big)$, we have

$$
\frac{\overline{F}_Y(x)}{\overline{F}_X(x)} \leqslant M, \ x > x_M,
$$

with certain constants $M$ and $x_M$. If $x \leqslant x_M$, then

$$
\frac{\overline{F}_Y(x)}{\overline{F}_X(x)} \leqslant \frac{1}{\overline{F}_X(x_M)} < \infty
$$

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because $F_X \in \mathcal{OS}$ implies $\overline{F}_X(x_M) > 0$. From the both above inequalities it follows that

$$\sup_{x\in\mathbb{R}} \frac{\overline{F}_Y(x)}{\overline{F}_X(x)} \leqslant \max\left\{M, \frac{1}{\overline{F}_X(x_M)}\right\} = C_4$$

Consequently, for $x \in \mathbb{R}$ we get

$$\begin{aligned}\overline{F_X * F_Y}(x) &= \int_{-\infty}^{\infty} \overline{F}_Y(x-y)\mathrm{d}F_X(y) \leqslant C_4 \int_{-\infty}^{\infty} \overline{F}_X(x-y)\mathrm{d}F_X(y) \\ &= C_4\overline{F_X * F_X}(x) \leqslant C_5\overline{F_X}(x) \end{aligned} \tag{3}$$

with some positive constant $C_5$, where the last step in the above derivation follows from part (i) of the lemma.

On the other hand, there exists a real $b \in \mathbb{R}$ for which

$$\overline{F}_Y(b) = 1 - F_Y(b) \geqslant \frac{1}{2}$$

For this $b$, we get

$$\begin{aligned}\overline{F_X * F_Y}(x) &\geqslant \int_{(b,\infty)} \overline{F}_X(x-y)\mathrm{d}F_Y(y) \geqslant \overline{F}_X(x-b) \int_{(b,\infty)} \mathrm{d}F_Y(y) \\ &= \overline{F}_X(x-b)\overline{F}_Y(b) \geqslant \frac{1}{2}\overline{F}_X(x)\frac{\overline{F}_X(x-b)}{\overline{F}_X(x)}\end{aligned}$$

Hence,

$$\liminf_{x\to\infty} \frac{\overline{F_X * F_Y}(x)}{\overline{F}_X(x)} \geqslant \frac{1}{2}\liminf_{x\to\infty}\frac{\overline{F}_X(x-b)}{\overline{F}_X(x)}. \tag{4}$$

In part (iv) of the lemma we proved that $F_X \in \mathcal{OL}$. It is easy to verify that

$$F_X \in \mathcal{OL} \;\Leftrightarrow\; \limsup_{x\to\infty}\frac{\overline{F}_X(x-1)}{\overline{F}_X(x)} < \infty \;\; \overline{F}_X(x-b) \underset{x\to\infty}{\asymp} \overline{F}_X(x) \text{ for each } b \in \mathbb{R}.$$

Therefore, the estimate (4) implies that

$$\liminf_{x\to\infty}\frac{\overline{F_X * F_Y}(x)}{\overline{F}_X(x)} > 0. \tag{5}$$

From (3) and (5) inequalities it follows that $\overline{F_X * F_Y}(x) \underset{x\to\infty}{\asymp} \overline{F_X}(x)$. Moreover by part (ii) of the lemma $F_X * F_Y \in \mathcal{OS}$. This finish the proof of the last part of the lemma. □

**Lemma 2.** *Let $\{\xi_1, \xi_2 ...\}$ be a sequence of independent r.v.s, for which $F_{\xi_1} \in \mathcal{OS}$, and for others indices $k \geqslant 2$ either $F_{\xi_k} \in \mathcal{OS}$ or $\overline{F}_{\xi_k} = O(\overline{F}_{\xi_1}(x))$. Then $F_{S_n} \in \mathcal{OS}$ for all $n \in \mathbb{N}$.*

*Proof.* If $n = 1$,then the statement is obvious because $S_1 = \xi_1$. If $n = 2$, then two options are possible $F_{\xi_2} \in \mathcal{OS}$ or $\overline{F}_{\xi_2} = O(\overline{F}_{\xi_1}(x))$. In the first case $F_{S_2} = F_{\xi_1} * F_{\xi_2} \in \mathcal{OS}$ according to the part (iii) of Lemma 1. In the second case $F_{S_2} \in \mathcal{OS}$ by the part (v) of the same lemma.

Let now $n > 2$. Denote

$$\mathcal{K} = \{k \in \{2, ..., n\} : \overline{F}_{\xi_k}(x) = O(\overline{F}_{\xi_1}(x))\}.$$

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Initially assume that the set $\mathcal{K}$ is empty. In such a case, $F_{\xi_k} \in \mathcal{OS}$ for all indices $k \in \mathcal{K}^c = \{1, 2, 3, \ldots, n\}$. By the part (iii) of Lemma 1 we get that $F_{S_n} \in \mathcal{OS}$.

Let now the index set $\mathcal{K} = \{k_1, k_2, ...k_r\} \subset \{1, ...n\}$ is not empty. Since

$$\overline{F}_{\xi_{k_1}}(x) = O(\overline{F}_{\xi_1}(x)),$$

part (v) of Lemma 1 implies that

$$F_{\xi_1} * F_{\xi_{k_1}} \in \mathcal{OS}, \tag{6}$$

and

$$\overline{F_{\xi_1} * F_{\xi_{k_1}}}(x) \asymp \overline{F}_{\xi_1}(x). \tag{7}$$

According the relation (7)

$$\limsup_{x\to\infty} \frac{\overline{F}_{\xi_{k_2}}(x)}{\overline{F_{\xi_1} * F_{\xi_{k_1}}}(x)} \leqslant \limsup_{x\to\infty} \frac{\overline{F}_{\xi_{k_2}}(x)}{\overline{F}_{\xi_1}(x)} \frac{1}{\liminf\limits_{x\to\infty} \frac{\overline{F_{\xi_1} * F_{\xi_{k_1}}}(x)}{\overline{F}_{\xi_1}(x)}} < \infty$$

because $\overline{F}_{\xi_{k_2}}(x) = O(\overline{F}_{\xi_1}(x))$. This means that

$$\overline{F}_{\xi_{k_2}}(x) = O(\overline{F_{\xi_1} * F_{\xi_{k_1}}}(x)).$$

Hence according to (6) and part (v) of Lemma 1 we get

$$F_{\xi_1} * F_{\xi_{k_1}} * F_{\xi_{k_2}} = (F_{\xi_1} * F_{\xi_{k_1}}) * F_{\xi_{k_2}} \in \mathcal{OS},$$

and

$$\overline{F_{\xi_1} * F_{\xi_{k_1}} * F_{\xi_{k_2}}}(x) \asymp \overline{F_{\xi_1} * F_{\xi_{k_1}}}(x).$$

Continuing the process we obtain

$$F_{\mathcal{K}} := F_{\xi_1} * \coprod_{j=1}^{r} F_{\xi_{k_j}} = F_{\xi_1} * F_{\xi_{k_1}} * F_{\xi_{k_2}} * ... * F_{\xi_{k_r}} \in \mathcal{OS},$$

and

$$\overline{F_{\xi_1} * F_{\xi_{k_1}} * F_{\xi_{k_2}} * ... * F_{\xi_{k_r}}}(x) \asymp \overline{F_{\xi_1} * F_{\xi_{k_1}} * F_{\xi_{k_2}} * ... * F_{\xi_{k_{r-1}}}}(x).$$

For the remaining indices $k \in \mathcal{K}^c = \{2, 3, ..., n\} \setminus \{k_1, k_2, ..., k_r\}$ d.f. $F_{\xi_k} \in \mathcal{OS}$. By the part (iii) of Lemma 1 we get

$$F_{\mathcal{K}^c} := \coprod_{k\in\mathcal{K}^c} F_{\xi_k} \in \mathcal{OS}.$$

Using part (iii) of Lemma 1 again we derive that

$$F_{S_n} = F_{\mathcal{K}} * F_{\mathcal{K}^c} \in \mathcal{OS}.$$

This finish the proof of Lemma 2. □

**Lemma 3.** *Let $\xi_1, \xi_2, ...$ be a sequence of independent random variables,for which $F_{\xi_1} \in OS$ and*

$$\limsup_{x\to\infty} \sup_{k\geqslant 1} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} < \infty \tag{8}$$

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*Then there exists a constant* $\widehat{C}$ *for which*

$$\overline{F}_{S_n}(x) \leqslant \widehat{C}^{\,n-1}\overline{F}_{\xi_1}(x) \tag{9}$$

*for all* $x \in \mathbb{R}$ *and for all* $n \geqslant 2$.

*Proof.* The condition (8) implies that

$$\sup_{k\geqslant 1} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} \leqslant C_6$$

for all $x \geqslant A$ with some positive constants $C_6$ and $A$. If $x < A$,then

$$\sup_{k\geqslant 1} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} \leqslant \frac{1}{\overline{F}_{\xi_1}(x)} \leqslant \frac{1}{\overline{F}_{\xi_1}(A)} < \infty.$$

Therefore, for each $x \in \mathbb{R}$

$$\sup_{k\geqslant 1} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} \leqslant \max\left\{C_6, \frac{1}{\overline{F}_{\xi_1}(A)}\right\} := C_7. \tag{10}$$

In addition, the part (i) of Lemma 1 gives that

$$\overline{F_{\xi_1} * F_{\xi_1}}(x) \leqslant C_8\overline{F}_{\xi_1}(x) \tag{11}$$

for all $x \in \mathbb{R}$ with some positive constant $C_8$.

We will prove the inequality (9) with constant $\widehat{C} = C_7C_8$. If $n = 1$, the inequality (9) holds evidently because $\overline{F}_{S_1}(x) = \overline{F}_{\xi_1}(x)$. If $n = 2$, then by (10) and (11) for $x \in \mathbb{R}$ we have

$$\begin{aligned}\overline{F}_{S_2}(x) &= \int_{-\infty}^{\infty} \overline{F}_{\xi_2}(x-y)\mathrm{d}F_{\xi_1}(y) \leqslant C_7 \int_{-\infty}^{\infty} \overline{F}_{\xi_1}(x-y)\mathrm{d}F_{\xi_1}(y)\\ &= C_7\overline{F_{\xi_1} * F_{\xi_1}}(x) \leqslant \widehat{C}\,\overline{F}_{\xi_1}(x).\end{aligned}$$

Suppose now that the inequality (9) holds for $n = m \geqslant 2$, i.e.

$$\frac{\overline{F}_{S_m}(x)}{\overline{F}_{\xi_1}(x)} \leqslant \widehat{C}^{\,m-1},\ x \in \mathbb{R}.$$

After choosing $n = m + 1$, from this assumption and from (10), (11) we get

$$\begin{aligned}\overline{F}_{S_{m+1}}(x) &= \int_{-\infty}^{\infty} \overline{F}_{S_m}(x-y)\mathrm{d}F_{\xi_{m+1}}(y) \leqslant \widehat{C}^{\,m-1} \int_{-\infty}^{\infty} \overline{F}_{\xi_1}(x-y)\mathrm{d}F_{\xi_{m+1}}(y)\\ &= \widehat{C}^{\,m-1} \int_{-\infty}^{\infty} \overline{F}_{\xi_{m+1}}(x-y)\mathrm{d}F_{\xi_1}(y) \leqslant \widehat{C}^{\,m-1}C_7 \int_{-\infty}^{\infty} \overline{F}_{\xi_1}(x-y)\mathrm{d}F_{\xi_1}(y)\\ &= \widehat{C}^{\,m-1}C_7\,\overline{F_{\xi_1} * F_{\xi_1}}(x) \leqslant \widehat{C}^{\,m}\,\overline{F}_{\xi_1}(x),\ x \in \mathbb{R}.\end{aligned}$$

According to the induction principle, the inequality (9) holds for all $n \in \mathbb{N}$. Lemma 3 is proved. □

## 6. Proofs of the main results

In this section, we present proofs of the main results op the paper.

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*Proof of Theorem 1.* suppose that $\mathbb{P}(\eta \in \{0, 1, \ldots, L\}) = 1$ for some $L \in \mathbb{N}$. We have

$$\overline{F}_{S_\eta}(x) = \sum_{n=1}^{L} \mathbb{P}(\eta = n)\overline{F}_{S_n}(x),\ x \in \mathbb{R}.$$

Let

$$L^* = \max\{k \in \{1, 2, ..., L\} : \mathbb{P}(\eta = k) > 0\}.$$

For each $x \in \mathbb{R}$

$$\frac{\overline{F}_{S_\eta}(x)}{\overline{F}_{S_{L^*}}(x)} \geqslant \frac{\mathbb{P}(\eta = L^*)\overline{F}_{S_{L^*}}(x)}{\overline{F}_{S_{L^*}}(x)} = \mathbb{P}(\eta = L^*) > 0. \tag{12}$$

On the other hand

$$\overline{F}_{S_\eta}(x) = \sum_{k=0}^{L^*-1} \mathbb{P}(\eta = L^* - k)\mathbb{P}(S_{L^*-k} > x) \tag{13}$$

For any random variable $\xi_k$, $k \in \{1, 2, \ldots, L^*\}$, there exists a negative number $-a_k$, for which $\mathbb{P}(\xi_k \geqslant -a_k) \geqslant 1/2$. We have

$$\begin{aligned}\mathbb{P}(S_{L^*-1} > x) &= \mathbb{P}(S_{L^*-1} - a_{L^*} > x - a_{L^*}, \xi_{L^*} \geqslant -a_{L^*}) + \mathbb{P}(S_{L^*-1} > x, \xi_{L^*} < -a_{L^*}) \\ &\leqslant \mathbb{P}(S_{L^*} > x - a_{L^*}) + \mathbb{P}(S_{L^*-1} > x)\mathbb{P}(\xi_{L^*} < -a_{L^*})\end{aligned}$$

From this we derive that

$$\mathbb{P}(S_{L^*-1} > x) \leqslant 2\mathbb{P}(S_{L^*} > x - a_{L^*})$$

for each $x \in \mathbb{R}$. Similarly,

$$\mathbb{P}(S_{L^*-2} > x) \leqslant 2\mathbb{P}(S_{L^*-1} > x - a_{L^*-1}) \leqslant 4\mathbb{P}(S_{L^*} > x - a_{L^*-1} - a_{L^*})$$

also for each real number $x$. Continuing the process we obtain

$$\mathbb{P}(S_{L^*-k} > x) \leqslant 2^k \mathbb{P}\left(S_{L^*} > x - \sum_{j=0}^{k-1} a_{L^*-j}\right)$$

for all $x \in \mathbb{R}$ and for all $k = 1, 2, ..., L^* - 1$. After inserting the obtained estimates into inequality (13), we get that

$$\begin{aligned}\overline{F}_{S_\eta}(x) &\leqslant \sum_{k=0}^{l^*-1} \mathbb{P}(\eta = L^* - k)\, 2^k\, \mathbb{P}(S_{L^*} > x - \sum_{j=0}^{k-1} a_{L^*-j}) \\ &\leqslant \mathbb{P}(S_{L^*} > x - a^\star) \sum_{k=0}^{L^*-1} 2^k\, \mathbb{P}(\eta = L^* - k) \\ &= C^* \overline{F}_{S_{L^*}}(x - a^*),\end{aligned}$$

where

$$C^* = \sum_{k=0}^{L^*-1} 2^k\, \mathbb{P}(\eta = L^* - k), \text{ and } a^* = \sum_{j=2}^{L^*} a_j.$$

Consequently, for all $x$

$$\frac{\overline{F}_{S_\eta}(x)}{\overline{F}_{S_{L^*}}(x)} \leqslant \frac{C^* \overline{F}_{S_{L^*}}(x - a^\star)}{\overline{F}_{S_{L^*}}(x)} \tag{14}$$

By Lemma 2 and part (iv) of Lemma 1 we have that $F_{S_{L^*}} \in OS \subset OL$. Therefore

$$\limsup_{x\to\infty} \frac{\overline{F}_{S_\eta}(x)}{\overline{F}_{S_{L^*}}(x)} < \infty$$

By (12) and (14) we have,that

$$\overline{F}_{S_\eta}(x) \asymp \overline{F}_{S_{L^*}}(x)$$

Therefore $F_{S_\eta} \in OS$ together with $F_{S_{L^*}}$ by the part (ii) of Lemma 1. Theorem 1 is proved. □

*Proof of Theorem 2. Part* (i) Whereas

$$\overline{F}_{S_\eta}(x) = \sum_{n=1}^{\infty} \mathbb{P}(\eta = n)\overline{F}_{S_n}(x),$$

by Lemma 3 for all real numbers $x$ we obtain

$$\frac{\overline{F}_{S_\eta}(x)}{\overline{F}_{\xi_1}(x)} \leqslant \frac{\sum_{n=1}^{\infty} \widehat{C}^{n-1}\mathbb{P}(\eta = n)\overline{F}_{\xi_1}(x)}{\overline{F}_{\xi_1}(x)} \leqslant \mathbb{E}e^{\widehat{C}\eta} < \infty, \tag{15}$$

where $\widehat{C}$ is some positive constant.

On the other hand

$$\overline{F}_{S_\eta}(x) \geqslant \mathbb{P}(\eta = 1)\overline{F}_{\xi_1}(x).$$

Hence under conditions of part (i), we have that $\overline{F}_{S_\eta}(x) \underset{x\to\infty}{\asymp} \overline{F}_{\xi_1}(x)$. Therefore $F_{S_\eta} \in OS$ according to part (ii) of Lemma 1. Part (i) of Theorem 2 is proved.

*Part*(ii). If $\mathbb{P}(\eta = 1) > 0$, then assertion of this part follows from the proved part (i). Since $\mathbb{E}e^{\lambda\eta} < \infty$ for each $\lambda > 0$, the inequality (15) implies that

$$\limsup_{x\to\infty} \frac{\overline{F}_{S_\eta}(x)}{\overline{F}_{\xi_1}(x)} < \infty. \tag{16}$$

In addition, conditions of part (ii) of the theorem give that

$$\inf_{k\geqslant 1} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} \geqslant \Delta$$

for all $x \geqslant x_\Delta$ and some positive $\Delta$. If $x < x_\Delta$, then

$$\inf_{k\geqslant 1} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} \geqslant \inf_{k\geqslant 1} \frac{\overline{F}_{\xi_k}(x_\Delta)}{\overline{F}_{\xi_1}(x_\Delta)} \overline{F}_{\xi_1}(x_\Delta) \geqslant \Delta\,\overline{F}_{\xi_1}(x_\Delta) := \widetilde{C} > 0$$

due to the assumption $F_{\xi_1} \in \mathcal{OS}$. The derived inequalities imply that

$$\overline{F}_{\xi_k}(x) \geqslant \widetilde{C}\,\overline{F}_{\xi_1}(x)$$

for some positive constant $\widetilde{C}$, and for all $x \in \mathbb{R}$, $k \in \{1, 2, \ldots\}$.

Using the last estimate we get

$$\begin{aligned}\overline{F}_{S_2}(x) &= \int_{-\infty}^{\infty} \frac{\overline{F}_{\xi_2}(x-y)}{\overline{F}_{\xi_1}(x-y)}\,\overline{F}_{\xi_1}(x-y)\,\mathrm{d}F_{\xi_1}(y) \\ &\geqslant \widetilde{C}\,\overline{F_{\xi_1} * F_{\xi_1}}(x) \geqslant \widetilde{C}\,\overline{F}_{\xi_1}(0)\overline{F}_{\xi_1}(x),\ x \in \mathbb{R}.\end{aligned}$$

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Similarly,

$$\overline{F}_{S_3}(x) = \int_{-\infty}^{\infty} \frac{\overline{F}_{S_2}(x-y)}{\overline{F}_{\xi_1}(x-y)} \overline{F}_{\xi_1}(x-y)\,\mathrm{d}F_{\xi_1}(y)$$
$$\geqslant \widetilde{C}\,\overline{F}_{\xi_1}(0)\overline{F_{\xi_1} * F_{\xi_1}}(x) \geqslant \widetilde{C}\,\left(\overline{F}_{\xi_1}(0)\right)^2 \overline{F}_{\xi_1}(x),\; x \in \mathbb{R}.$$

Continuing process we obtain

$$\overline{F}_{S_n}(x) \geqslant \widetilde{C}\,\left(\overline{F}_{\xi_1}(0)\right)^{n-1} \overline{F}_{\xi_1}(x)$$

for all $x \in \mathbb{R}$ and $n \in \{2, 3, \ldots\}$.

Therefore,

$$\liminf_{x\to\infty} \frac{\overline{F}_{S_\eta}(x)}{\overline{F}_{\xi_1}(x)} \geqslant \liminf_{x\to\infty} \frac{\mathbb{P}(\eta = \widetilde{L})\overline{F}_{S_{\widetilde{L}}}}{\overline{F}_{\xi_1}(x)}$$
$$\geqslant \mathbb{P}(\eta = \widetilde{L})\,\widetilde{C}\,\left(\overline{F}_{\xi_1}(0)\right)^{\widetilde{L}-1} > 0, \tag{17}$$

where $\widetilde{L} = \min\{n \geqslant 2 : \mathbb{P}(\eta = n) > 0\}$.

The derived inequalities (16) and (17) imply $\overline{F}_{S_\eta}(x) \underset{x\to\infty}{\asymp} \overline{F}_{\xi_1}(x)$. By part (ii) of Lemma 1 we get $F_{S_\eta} \in \mathcal{OS}$. Theorem 2 is proved.

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