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Tzong-Mo Tsai *

Posted Date: 24 May 2023

doi: 10.20944/preprints202305.1665.v1

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Higher Monotonicity Properties for Zeros of Certain Sturm-Liouville Functions

Tzong-Mo Tsai

General Education Center, Ming Chi University of Technology, New Taipei City, 24301, Taiwan; tsaitm@mail.mcut.edu.tw

Abstract: We consider the differential equation $y'' + \omega^2\rho(x)y = 0$ where ω is a positive parameter. The principal concern here is to find conditions on the function $\rho^{-1/2}(x)$, which ensure that the consecutive differences of sequences constructed from the zeros of a nontrivial solution of the equation are regular in sign for ω sufficiently large. In particular, if $c_{\nu k}(\alpha)$ denotes the kth positive zero of the general Bessel (cylinder) function $C_\nu(x;\alpha) = J_\nu(x)\cos\alpha - Y_\nu(x)\sin\alpha$ of order ν, and if $|\nu| < 1/2$, we prove that

$$(-1)^m\Delta^{m+2}c_{\nu k}(\alpha) > 0 \quad (m = 0, 1, 2, ...; k = 1, 2, ...),$$

where $\Delta a_k = a_{k+1} - a_k$. This type of inequalities was conjectured by Lorch and Szego in 1963. We also show that the differences of the zeros of various orthogonal polynomials with higher degrees possess the sign-regularity.

Keywords: Sturm-Liouville equations; differences; zeros; higher monotonicity; Bessel functions; orthogonal polynomials

MSC: 34B24; 33C10

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1. Introduction

We consider the differential equation

$$y'' + \omega^2\rho(x)y = 0, \quad a \le x \le b, \tag{1.1}$$

associated with a positive parameter ω. By a Sturm-Liouville function, we mean a nontrivial real solution of (1.1). Let $\{x_k(\omega)\}$ denote the ascending sequence of the zeros of a Sturm-Liouville function in the interval $[a, b]$. The Sturm comparison theorem (see e.g., [1, p.314] or [3, p.56]) states that the second differences of the sequence $\{x_k(\omega)\}$ are all positive if $\rho'(x) < 0$, and are all negative if $\rho'(x) > 0$. Our main purpose here is to go beyond the second differences and to show that higher consecutive differences of sequences constructed from $\{x_k(\omega)\}$ are regular in sign. Lorch and Szego [3] initiated the study of the sign-regularity of higher differences of the sequences associated with Sturm-Liouville functions. In particular, if $c_{\nu k}(\alpha)$ denotes the kth positive zero of the general Bessel (cylinder) function

$$C_\nu(x;\alpha) = J_\nu(x)\cos\alpha - Y_\nu(x)\sin\alpha,$$

they proved that, for $|\nu| > 1/2$,

$$(-1)^m\Delta^{m+1}c_{\nu k}(\alpha) > 0 \quad (m = 0, 1, 2, ...; k = 1, 2, ...), \tag{1.2$_m$}$$

and conjectured [3, p.71] on the basis of numerical evidence that, for $|\nu| < 1/2$,

$$(-1)^m\Delta^{m+2}c_{\nu k}(\alpha) > 0 \quad (m = 0, 1, 2, ...; k = 1, 2, ...). \tag{1.3$_m$}$$

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The symbol $\Delta^m a_k$ means, as usual, the mth (forward) difference of the sequence $\{a_k\}$:

$$\Delta^0 a_k = a_k, \quad \Delta^m a_k = \Delta^{m-1} a_{k+1} - \Delta^{m-1} a_k \quad (m = 1, 2, ...; k = 1, 2, ...).$$

Note that $C_\nu(x;\alpha)$ is a solution of the equation

$$y'' + q(x)y = 0, \quad x \in (0, \infty), \tag{1.4}$$

with $q(x) = 1 - (\nu^2 - (1/4))x^{-2}$. Since $q'(x) = 2(\nu^2 - (1/4))x^{-3}$, we see that the Sturm comparison theorem gives the results $(1.2)_1$ and $(1.3)_0$. They also mentioned in [3] that the signs of the first M differences of zeros of a Sturm-Liouville function of (1.4) could be inferred from the signs of $q^{(m)}(x)$, $m = 1, 2, ..., M$. Muldoon [7] made some progress in $(1.3)_m$. He proved that $(1.3)_m$ holds when $1/3 \le |\nu| < 1/2$ ([7, Corollary 4.2]).

Our approach here is based on the ideas and results of [10], where the string equation $y'' + \lambda\rho(x)y = 0$ with $y(0) = y(1) = 0$ was considered. Using the eigenvalues and the nodal points, we constructed a sequence of piecewise continuous linear functions which converges to $\rho^{-1/2}(x)$ uniformly on $[0, 1]$. We also obtained a formula for derivatives of $\rho^{-1/2}(x)$ in terms of the eigenvalues and the differences of the nodal points.

This paper is organized as follows. In Section 2, we use the zeros $x_k(\omega)$ of a Sturm-Liouville function as nodes to obtain a difference-derivative theorem (Lemma 2.1). We also give asymptotic estimates for $\rho^{-1/2}(x_k(\omega))$ as $\omega \to \infty$ (Lemma 2.3). Then we are able to express the higher differences $\Delta^{m+1}x_k(\omega)$ in terms of the derivatives of $\rho^{-1/2}(x)$ at those zeros. Moreover, the expression can be used to determine the regular manner of these differences (Theorems 2.4 and 2.5). Besides, we construct sequences from $x_k(\omega)$, whose all mth differences have the same sign (Corollary 2.6). The proofs of Lemmas 2.1 and 2.3 rely on a system of interlaced inductions, which will be given in Section 5. In Section 3, we use an approximation process for the zeros of the general Bessel function to prove the conjecture of Lorh and Szego (Theorem 3.1). In Section 4, the zeros of various orthogonal polynomials with higher degrees are shown to share similar sign-regularity (Theorems 4.1 and 4.2).

The notation used throughout is standard. A function $\varphi(x)$ is said to be M-monotonic (resp., absolutely M-monotonic) on an interval I if

$$(-1)^m \varphi^m(x) \ge 0 \quad (resp., \varphi^m(x) \ge 0), \quad (x \in I; m = 0, 1, ..., M). \tag*{$(1.5)_M$}$$

If $(1.5)_M$ holds for $M = \infty$, then $\varphi(x)$ is said to be completely (resp., absolutely) monotonic on I. A sequence $\{a_k(\omega)\}$, depending on a positive parameter ω, is said to be asymptotically M-monotonic (resp., asymptotically absolutely M-monotonic) if

$$(-1)^m \Delta^m a_k(\omega) \ge 0 \quad (resp., \Delta^m a_k(\omega) \ge 0), \quad (m = 0, 1, 2, ...M; k = 1, 2, ...)$$

for ω sufficiently large.

2. Main Results

In this section we consider the differential equation

$$y'' + \omega^2 \rho(x) y = 0, \quad a \le x \le b, \tag{2.1}$$

where ω is a positive parameter. We shall assume throughout that $\rho(x)$ is a positive C^∞-function on the interval $[a, b]$. The notation $f(x)$ is reserved for the function $\rho^{-1/2}(x)$. Let $y(x;\omega)$ be a nontrival real solution of (2.1), and let $x_1(\omega) < x_2(\omega) < \cdots$ be the zeros of $y(x;\omega)$ in the interval $[a, b]$. For

$a \le x < b$, we denote by $k(x;\omega)$ the smallest positive interge k such that $x \le x_k(\omega)$. It is known(see e.g., [9, 10]) that

$$\min_{[x_k(\omega),x_{k+1}(\omega)]} f \le \frac{\omega}{\pi}\Delta x_k(\omega) \le \max_{[x_k(\omega),x_{k+1}(\omega)]} f. \tag{2.2}$$

It follows that $\pi \min_{[a,b]} f \le \omega\Delta x_k(\omega) \le \pi \max_{[a,b]} f$. In particular, we have

$$\Delta x_k(\omega) = O(\omega^{-1}) \quad as \quad \omega \to \infty. \tag{2.3}$$

Thus, by (2.2) and the continuity of f, we obtain $f(x) = \lim_{\omega\to\infty} \frac{\omega}{\pi}\Delta x_{k(x;\omega)}(\omega)$ and, for any fixed l,

$$\lim_{\omega\to\infty} \frac{\Delta x_{k(x;\omega)+l}(\omega)}{\Delta x_{k(x;\omega)}(\omega)} = 1. \tag{2.4}$$

Note that (2.4) means that, as $\omega \to \infty$, the sequence $x_k(\omega)$ behaves as equally distributed.

If φ is m-times differentiable in $(t, t+md)$ and the lower derivatives of φ are continuous on $[t, t+md]$, a mean-value theorem [8, p. 52, no. 98] for differences and derivatives states that there exists a δ, such that

$$\Delta_d^m \varphi(t) = d^m \varphi^{(m)}(t + \delta md),$$

where $\Delta_d \varphi(t) = \varphi(t+d) - \varphi(t)$. It is interesting to look for a difference-derivative theorem which can express the differences of a smooth function on the sequence $\{x_k(\omega)\}$ in terms of its derivatives at this sequence. The following lemma provides such a result.

Lemma 2.1. *Let $x_k = x_k(\omega)$. If φ is a C^∞-function on $[a,b]$, then, for $m = 1, 2, ...,$*

$$\Delta^m \varphi(x_k) = O(\omega^{-m}). \tag{2.5$_m$}$$

Moreover,

$$\Delta^m \varphi(x_k) = \sum_{q=1}^{m} A_{q,k}^{(m)} \varphi^{(q)}(x_{k+m-q}) + O(\omega^{-m-1}), \tag{2.6$_m$}$$

where the coefficients $A_{q,k}^{(m)}$ satisfy the recurrence relation:

$$A_{1,k}^{(m)} = \Delta^m x_k, \quad A_{q,k}^{(m)} = \sum_{r=q-1}^{m-1} \binom{m-1}{r} A_{q-1,k+m-1-r}^{(r)} \Delta^{m-r} x_k, \tag{2.7$_m$}$$

for $q = 2, 3, ..., m$.

To prove Lemma 2.1, we need a more detailed investigation on the behaviour of $x_k(\omega)$. We use the Prüfer method to achieve our purpose. For each nontrivial solution $y(x;\omega)$ of (2.1), we define the Prüfer angle $\theta(x;\omega)$ as follows:

$$\omega\rho^{1/2}(x)\cot\theta(x;\omega) = \frac{y'(x;\omega)}{y(x;\omega)}.$$

Then $\theta(x;\omega)$ satisfies the differential equation

$$\theta'(x;\omega) = \omega\rho^{1/2}(x) + \frac{\rho'(x)}{4\rho(x)}\sin 2\theta(x;\omega). \tag{2.8}$$

 doi:10.20944/preprints202305.1665.v1

If we specify the initial condition for $\theta(x;\omega)$ to be $\theta(a;\omega) = \theta_a(\omega)$ with $0 \le \theta_a(\omega) < \pi$, then, by the standard results (see e.g., [1, p. 315]), we have

$$\theta(x_k(\omega);\omega) = k\pi, \tag{2.9}$$

and $k\pi \le \theta(x;\omega) \le (k+1)\pi$, $x \in [x_k(\omega), x_{k+1}(\omega)]$. Let $x_k = x_k(\omega)$. Integrating both sides of (2.8) from x_k to x_{k+1}, and using (2.9), we find

$$\pi = \omega \int_{x_k}^{x_{k+1}} \rho^{1/2}(x)dx + \int_{x_k}^{x_{k+1}} \frac{\rho'(x)}{4\rho(x)} \sin 2\theta(x;\omega)dx. \tag{2.10}$$

Taking the Taylor expansion of $(1/f)(x)$ at x_k and using (2.3), we obtain

$$\int_{x_k}^{x_{k+1}} \rho^{1/2}(x)dx = \sum_{r=0}^{m} \frac{(1/f)^{(r)}(x_k)}{(r+1)!}(\Delta x_k)^{r+1} + O(\omega^{-m-2}). \tag{2.11}$$

The estimate of the second integral in (2.10) is stated as the following lemma. Its proof consists of a reducible system of integrals which will be given in Appendix.

Lemma 2.2. *Let $x_k = x_k(\omega)$. Then, for $m = 2, 3, \ldots$, we have*

$$\int_{x_k}^{x_{k+1}} \frac{\rho'(x)}{4\rho(x)} \sin 2\theta(x;\omega)dx = \sum_{r=0}^{m-2} \Delta F_r(x_k)\omega^{-r-1} + R_{m-2}(x_k), \qquad (2.12)_m$$

where the functions F_r depend on $f = \rho^{-1/2}$, and

$$R_{m-2}(x_k) = O(\omega^{-m-1}). \qquad (2.13)_m$$

Note that the first two functions F_r appeared in $(2.12)_m$ are of the forms

$$F_0 = \frac{f'}{4} - \int \frac{(f')^2}{8f}dx \quad and \quad F_1 = 0. \tag{2.14}$$

For $m = 2, 3, \ldots$, using the estimates (2.11), $(2.12)_m$ and $(2.13)_m$ and multiplying (2.10) by $f(x_k)/\pi$, we find the estimate for $f(x_k)$:

$$f(x_k) = \frac{\omega}{\pi} \sum_{r=0}^{m} \frac{g_r(x_k)}{(r+1)!}(\Delta x_k)^{r+1} + \frac{1}{\pi} \sum_{r=0}^{m-2} (f\Delta F_r)(x_k)\omega^{-r-1} + O(\omega^{-m-1}), \qquad (2.15)_m$$

where the functions $g_r = f(1/f)^{(r)}$, $r = 0, 1, 2, \ldots, m$. Note that $g_0 = 1$. Moreover, if we apply the mth order difference operator to $(2.15)_m$, then we can find the estimates for differences of the function $f(x)$ at those zeros. Indeed, we have

Lemma 2.3. *Let $f(x)$ and $x_k = x_k(\omega)$ as above. Then, for $m = 1, 2, 3, \ldots$, we have*

$$\Delta^m x_k = O(\omega^{-m}). \qquad (2.16)_m$$

Moreover,

$$\Delta^m f(x_k) = \frac{\omega}{\pi}\Delta^{m+1} x_k + O(\omega^{-m-1}). \qquad (2.17)_m$$

The proofs of Lemmas 2.1 and 2.3 will be given in Section 5.

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Now, if we apply Lemma 2.1 to the function $f(x)$, then, by $(2.17)_m$, we have the estimate for the higher differences of $x_k = x_k(\omega)$:

$$\frac{\omega}{\pi}\Delta^{m+1}x_k = \sum_{q=1}^{m} A_{q,k}^{(m)} f^{(q)}(x_{k+m-q}) + O(\omega^{-m-1}). \tag{2.18$_m$}$$

Moreover, using (2.3) and $(2.7)_m$, iterating $(2.18)_m$ for m from 1 to M, and then taking ω sufficiently large, we can ensure the monotonicity of the sequence $\{\Delta x_k(\omega)\}$ by f.

Theorem 2.4. *Let $x_k = x_k(\omega)$ and $f(x) = \rho^{-1/2}(x)$ be as those mentioned above. If $f(x)$ is M-monotonic on the interval $[a,b]$, then the sequence $\{\Delta x_k(\omega)\}$ is asymptotically M-monotonic.*

Proof. Since

$$(-1)^m f^{(m)}(x) \geq 0 \quad (x \in [a,b]; m = 0,1,2,...,M), \tag{2.19}$$

it suffices to show that

$$(-1)^{m-q} A_{q,k}^{(m)} \geq 0 \quad (q = 1,2,...,m; m = 1,2,...,M), \tag{2.20$_M$}$$

as $\omega \to \infty$, to conclude that

$$(-1)^m \Delta^{m+1} x_k(\omega) \geq 0, \quad (m = 0,1,2,...,M). \tag{2.21}$$

We prove $(2.20)_M$ by induction on M. When $M = 1$, $(2.20)_1$ reduces to $A_{1,k}^{(1)} \geq 0$, which is true because $A_{1,k}^{(1)} = \Delta x_k$, by $(2.7)_1$. Now, suppose that $(2.20)_N$, $1 \leq N < M$, is true. By $(2.18)_N$, we have

$$\frac{\omega}{\pi}(-1)^N \Delta^{N+1} x_k = \sum_{q=1}^{N} [(-1)^{N-q} A_{q,k}^{(N)}][(-1)^q f^{(q)}(x_{k+N-q})] + O(\omega^{-N-1}),$$

which is nonnegative as $\omega \to \infty$, by induction hypothesis, (2.19) and $(2.16)_{N+1}$. Thus, by $(2.7)_{N+1}$, $(-1)^N A_{1,k}^{(N+1)} = (-1)^N \Delta^{N+1} x_k \geq 0$, and , for $q = 1,2,...,N+1$,

$$(-1)^{N+1-q} A_{q,k}^{(N+1)} = \sum_{r=q-1}^{N} \binom{N}{r} [(-1)^{r-q+1} A_{q-1,k+N-r}^{(r)}][(-1)^{N-r} \Delta^{N+1-r} x_k] \geq 0,$$

by induction hypothesis again. This prove $(2.20)_{N+1}$ and thus the theorem. □

Note that, if the factors $(-1)^m$ are deleted from the assumptions (2.19), then, by making the obvious changes in the above proof, the conclusion (2.21) remains valid, provided they are amended by eliminating the factors $(-1)^m$. Thus we have

Theorem 2.5. *Let $x_k = x_k(\omega)$ and $f(x) = \rho^{-1/2}(x)$ be as those mentioned above. If $f(x)$ is absolutely M-monotonic on the interval $[a,b]$, then the sequence $\{\Delta x_k(\omega)\}$ is asymptotically absolutely M-monotonic.*

As consequences of Lemma 2.1, Theorems 2.4 and 2.5, we can use the zeros of a solution of (2.1) to construct sequences whose all mth differences have the same sign.

Corollary 2.6. *(a) Let $f(x)$ be M-monotonic on $[a,b]$. If $\varphi(x)$ is also M-monotonic on $[a,b]$, then the sequence $\{\varphi(x_k)\}$ is asymptotically M-monotonic.*
(b) Let $f(x)$ be absolutely M-monotonic on $[a,b]$. If $\varphi(x)$ is also absolutely M-monotonic on $[a,b]$, then the

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sequence $\{\varphi(x_k)\}$ *is asymptotically absolutely M-monotonic.*

Proof. Since $f(x)$ is M-monotonic on $[a,b]$, we see from the proof of Theorem 2.4 that $(2.20)_M$ holds. On the other hand, the M-monotonicity of $\varphi(x)$ on $[a,b]$ means that

$$(-1)^m\varphi^{(m)}(x)\geq 0 \quad (x\in[a,b]; m=0,1,2,...,M). \tag{2.22}$$

It now follows from $(2.6)_m$, $(2.20)_M$, (2.22) and $(2.5)_m$ that

$$(-1)^m\Delta^m\varphi(x_k)=\sum_{q=1}^{m}[(-1)^{m-q}A_{q,k}^{(m)}][(-1)^q\varphi^{(q)}(x_{k+m-q})]+O(\omega^{-m-1})\geq 0,$$

for all k and $m=0,1,2,...,M$, as $\omega\to\infty$. The proof of (b) is similar to that of part (a). □

Note that, by the definition of the function $f(x)=\rho^{-1/2}(x)$, the conclusion of Theorem 2.4 (resp., Theorem 2.5) can be inferred directly from the assumptions on $\rho(x)$. In fact, $(-1)^m\rho^{(m+1)}(x)\geq 0$ (resp., $\rho^{(m+1)}(x)\leq 0$) on $[a,b]$, for $m=0,1,2,...,M-1$, imply $(-1)^m f^{(m)}(x)\geq 0$ (resp., $f^{(m)}(x)\geq 0$) on $[a,b]$, for $m=1,2,...,M$. To examine the assertions, we can proceed by induction on M. For $M=1$, by the facts $f(x)=\rho^{-1/2}(x)$ and $f'(x)=(-1/2)\rho^{-3/2}(x)\rho'(x)$, the assertion is valid. For higher derivatives of $f(x)$, a general term of $f^{(m)}(x)$ would appear as

$$S_m=C[\rho]^{\alpha_0}[\rho']^{\alpha_1}[\rho'']^{\alpha_2}\cdots[\rho^{(m)}]^{\alpha_m}$$

with exponentials α_0 a negative half-integer and $\alpha_1,\alpha_2,...,\alpha_m$, all nonnegative integers. The induction is carried through by differentiating S_m. We have

$$S'_m=C\alpha_0[\rho]^{\alpha_0-1}[\rho']^{\alpha_1+1}[\rho'']^{\alpha_2}\cdots[\rho^{(m)}]^{\alpha_m}+C\alpha_1[\rho]^{\alpha_0}[\rho']^{\alpha_1-1}[\rho'']^{\alpha_2+1}\cdots[\rho^{(m)}]^{\alpha_m}$$
$$+\cdots+C\alpha_m[\rho]^{\alpha_0}[\rho']^{\alpha_1}[\rho'']^{\alpha_2}\cdots[\rho^{(m)}]^{\alpha_m-1}[\rho^{(m+1)}],$$

and under the conditions $(-1)^m\rho^{(m+1)}(x)\geq 0$ (resp., $\rho^{(m+1)}(x)\leq 0$) and the negative α_0, each term in the last sum has opposite sign (resp., the same sign) as S_m. Thus, $f^{(m)}(x)$ and $f^{(m+1)}(x)$ have alternating signs (resp., the same sign), and then the inductions are complete. Hence we obtain

Corollary 2.7. *Let $x_k=x_k(\omega)$ be as above. (a) If $\rho'(x)$ is $(M-1)$-monotonic on $[a,b]$, then the sequence $\{\Delta x_k(\omega)\}$ is asymptotically M-monotonic.*
(b) If $-\rho'(x)$ is absolutely $(M-1)$-monotonic on $[a,b]$, then the sequence $\{\Delta x_k(\omega)\}$ is asymptotically absolutely M-monotonic.

Although Corollary 2.7(a) is a partial result included in [4, Theorem 3.3], the techniques employ in this section are independent of the methods in the series of papers [4, 5, 7] and the results of Hartman [2, Theorems 18.1_n and 20.1_n.] It also gives the connection of the quantities between the differences of the zeros and the coefficient function $\rho(x)$. However, it might have some numerical interest.

One can find similar results concerned with the critical points of a Sturm-Liouville function of (2.1). In fact, by letting $x'_k(\omega)$ denote the kth critical point of a solution $y(x;\omega)$ of (2.1) in the interval $[a,b]$ and noting the definition of the Prüfer angle

$$\theta(x'_k(\omega);\omega)=(k-\frac{1}{2})\pi,$$

the procedures employed in this section are all valid. Thus if we replace $\{x_k(\omega)\}$ in Theorems 2.4 and 2.5, and Corollaries 2.6 and 2.7 by $\{x'_k(\omega)\}$, the conclusions in these Theorems and Corollaries still hold.

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3. Applications to Bessel Functions.

Let $c_{\nu k}(\alpha)$ be the kth positive zero of the general Bessel (cylinder) function

$$C_\nu(x;\alpha) = J_\nu(x)\cos\alpha - Y_\nu(x)\sin\alpha,$$

where $J_\nu(x)$ and $Y_\nu(x)$ denote the Bessel functions with order ν of the first and second kind, respectively. The main results in this section are stated as follows:

Theorem 3.1. *(a) For $|\nu| < 1/2$, we have*

$$(-1)^m \Delta^{m+2} c_{\nu k}(\alpha) > 0 \quad (m = 0, 1, 2, ...; k = 1, 2, 3, ...).$$

(b) For $0 < |\nu| < 1/2$, we have

$$(-1)^m \Delta^{m+1} c_{\nu k}^{2|\nu|}(\alpha) > 0 \quad (m = 0, 1, 2, ...; k = 1, 2, 3, ...).$$

The Airy functions (see e.g., [11, p.18]) satisfy the differential equation $y'' + \frac{x}{3}y = 0$. We consider a broader class of functions, including the Airy functions, which satisfy the differential equation (see e.g., [12, p. 97(9)])

$$z'' + \omega^2 x^\gamma z = 0, \quad x \in (0, \infty), \tag{3.1}$$

where $0 < \gamma < \infty$. These functions are closely related to Bessel functions. Indeed,

$$z(x;\omega) = x^{1/2} C_\nu(2\nu\omega x^{1/2\nu};\alpha), \quad \textit{where} \quad \nu = 1/(\gamma+2),$$

is a nontrivial real solution of (3.1). Note that, for each $\omega > 0$, the kth positive zeros $\xi_k(\omega)$ of $z(x;\omega)$ satisfies the identities

$$2\nu\omega(\xi_k(\omega))^{1/2\nu} = c_{\nu k}(\alpha) \quad \textit{and} \quad (2\nu\omega)^{2\nu}\xi_k(\omega) = c_{\nu k}^{2\nu}(\alpha).$$

Moreover, for each $\omega > 0$, and $m = 0, 1, 2, ...$, we have

$$\Delta^{m+2} c_{\nu k}(\alpha) = 2\nu\omega \Delta^{m+2}(\xi_k(\omega))^{1/2\nu}, \tag{3.2}$$

and

$$\Delta^{m+1} c_{\nu k}^{2\nu}(\alpha) = (2\nu\omega)^{2\nu} \Delta^{m+1} \xi_k(\omega). \tag{3.3}$$

The identities (3.2) and (3.3) are really the key for us to study the regularity behaviour of the Bessel zeros.

To prove Theorem 3.1, we consider the family of differential equations:

$$y'' + \omega^2 (x+a)^\gamma y = 0 \quad (a > 0; 0 < \gamma < \infty), \tag{3.4}$$

on the interval $[0, b]$. Let $y_a(x;\omega)$ be a nontrivial real solution of (3.4) and let the sequence $\{x_k(\omega;a)\}$ be the zeros of $y_a(x;\omega)$ with the ascending order in $[0, b]$. Following Theorem 2.4 with $f(x) = (x+a)^{-\gamma/2}$ and Corollary 2.6(a) with the function $\varphi(x) = (x+a)^{-1/2\nu}$, we have

$$(-1)^m \Delta^{m+1} x_k(\omega;a) \geq 0 \quad (m = 0, 1, 2, ..., M), \tag{3.5}$$

and

$$(-1)^m \Delta^m (x_k(\omega;a)+a)^{-1/2\nu} \geq 0 \quad (m = 0,1,2,...,M), \tag{3.6}$$

as $\omega \to \infty$. If we specify the initial conditions for the solution $y_a(x;\omega)$ of (3.4) to be

$$y_a(0;\omega) = z(a;\omega) \quad and \quad y_a'(0;\omega) = z'(a;\omega),$$

then it is easy to verify that $y_a(x;\omega) = z(x+a;\omega)$ for $x \in [0,b]$, and hence, for each k, $x_k(\omega;a)+a$ converges to ξ_k as $a \to 0^+$. Thus, for each $\omega > 0$, by (3.2) and (3.3), we have

$$\Delta^{m+2} c_{\nu k}(\alpha) = \lim_{a\to 0^+} 2\nu\omega \Delta^{m+2} (x_k(\omega;a))^{1/2\nu}, \tag{3.7}$$

and

$$\Delta^{m+1} c_{\nu k}^{2\nu}(\alpha) = \lim_{a\to 0^+} (2\nu\omega)^{2\nu} \Delta^{m+1} x_k(\omega;a). \tag{3.8}$$

Recalling (2.10) and $(2.12)_{m+3}$ with the function $\rho(x) = (x+a)^\gamma$ and denoting $x_k = x_k(\omega;a)$, we have

$$\omega \int_{x_k}^{x_{k+1}} (x+a)^{\gamma/2} dx = \pi - \sum_{r=0}^{m+1} \Delta F_r(x_k)\omega^{-r-1} - R_{m+1}(x_k). \tag{3.9}$$

Note that $\nu = 1/(\gamma+2)$ and $f(x) = (x+a)^{(2\nu-1)/2\nu}$. By (2.14), we have

$$\Delta F_0(x_k) = \frac{4\nu^2-1}{16\nu} \Delta (x_k+a)^{-1/2\nu}.$$

Thus, (3.9) becomes

$$2\nu\omega\Delta(x_k+a)^{1/2\nu} = \pi + \frac{1-4\nu^2}{16\nu\omega}\Delta(x_k+a)^{-1/2\nu} - \sum_{r=1}^{m+1} \Delta F_r(x_k)\omega^{-r-1} - R_{m+1}(x_k). \tag{3.10}$$

If we apply the difference operator Δ^{m+1} to (3.10), by $(2.5)_{m+2}$ and $(2.13)_{m+3}$, then we can find

$$2\nu\omega\Delta^{m+2}(x_k+a)^{1/2\nu} = \frac{1-4\nu^2}{16\nu\omega}\Delta^{m+2}(x_k+a)^{-1/2\nu} + O(\omega^{-m-4}). \tag{3.11}$$

Moreover, multiplying (3.11) by $(-1)^m\omega^{m+3}$, we have

$$2\nu\omega^{m+4}(-1)^m\Delta^{m+2}(x_k+a)^{1/2\nu} = \frac{1-4\nu^2}{16\nu}\omega^{m+2}(-1)^m\Delta^{m+2}(x_k+a)^{-1/2\nu} + O(\omega^{-1}). \tag{3.12}$$

By (3.12), (3.6), $(2.5)_{m+2}$ and $0 < \nu < 1/2$, we have

$$(-1)^m\Delta^{m+2}(x_k+a)^{1/2\nu} \geq 0 \quad as \quad \omega \to \infty. \tag{3.13}$$

Now, for each $a > 0$, if we choose $\omega = \omega(a)$ sufficiently large such that (3.13) and (3.5) hold, then, by (3.7) and (3.8), we have

$$(-1)^m\Delta^{m+2}c_{\nu k}(\alpha) \geq 0 \quad (m = 0,1,2,...;k = 1,2,3,...), \tag{3.14}$$

and

$$(-1)^m\Delta^{m+1}c_{\nu k}^{2\nu}(\alpha) \geq 0 \quad (m = 0,1,2,...;k = 1,2,3,...). \tag{3.15}$$

Secondly, according to $Y_\nu(x) = (J_\nu(x)\cos\pi\nu - J_{-\nu}(x))/\sin\pi\nu$ (see e.g., [12, p. 64]), it is easily to verify that $C_{-\nu}(x;\alpha) = C_\nu(x;\alpha+\pi\nu)$, and hence

$$c_{-\nu k}(\alpha) = c_{\nu k}(\alpha+\pi\nu).$$

Thus, for $0 < |\nu| < 1/2$, (3.14) holds and (3.15) holds in the modified form

$$(-1)^m\Delta^{m+1}c_{\nu k}^{2|\nu|}(\alpha) \geq 0 \quad (m = 0,1,2,...;k = 1,2,3,...). \tag{3.16}$$

Thirdly, for $\nu = 0$, any positive zero $c_{\nu k}(\alpha)$ of $C_\nu(x;\alpha)$ is definable as a continuously increasing function of the real variable ν (see e.g., [12, p. 508]), so that, by an approximating process, (3.14) hold for all $|\nu| < 1/2$.

Finally, since neither $\{\Delta^2 c_{\nu k}(\alpha)\}$ nor $\{\Delta c_{\nu k}^{2|\nu|}(\alpha)\}$ are constant sequences, the results of Lorch, Szego and Muldoon for completely monotonic sequences ([3, p. 72] or [6, Theorem 2]) guarantee the strict inequalities of (3.14) and (3.16). This completes the proof of Theorem 3.1.

4. Applications to Classical Orthogonal Polynomials.

Some important classical orthogonal polynomials are related to Sturm-Liouville functions such as Hermite and Jacobi polynomials. In [3, p. 73], Lorch, Szego and their coworkers conjectured, on the basis of numerical evidence, that the θ-zeros of the Legendre polynomials, the special cases of Jacobi polynomials, and the positive zeros of the Hermite polynomials form the sequences whose mth differences have the constant sign. In this section, we shall apply the results in Sections 2 and 3 to obtain some partial answers for these conjectures.

4.1. Positive zeros of Hermite polynomials

Let $H_n(t)$ be the Hermite polynomial (see e.g., [11, p.105(5.5.3)]), defined by

$$H_n(t) = (-1)^n e^{t^2}(\frac{d}{dt})^n e^{-t^2}. \tag{4.1}$$

We consider the Hermite differential equation:

$$H_n'' - 2tH_n' + 2nH_n = 0,$$

and the related equation:

$$u'' + [(2n+1) - t^2]u = 0. \tag{4.2}$$

A simple calculation shows that (see e.g., [11, p.105(5.5.2)])

$$u_n(t) = e^{-t^2/2}H_n(t)$$

is a nontrivial solution of (4.2). From the general theory of orthogonal polynomials, we know that $H_n(t)$ has precisely n real zeros. By (4.1), we see that for n even, $H_n(t)$ is an even function of t, and for n odd, $H_n(t)$ is an odd function of t. Accordingly, all zeros of $H_n(t)$ are placed symmetrically with respect to the origin, and the same phenomenon is clearly true for $u_n(t)$. For each n, the positive zeros of $H_n(t)$ are named by $h_1^{(n)} < h_2^{(n)} < \cdots < h_{[n/2]}^{(n)}$, where $[\cdot]$ is the greatest integer function.

The main result concerned with Hermite polynomials is as follows:

Theorem 4.1. *Let $h_k^{(n)}$ be as above. Then, for each k, we have*

$$\Delta^m h_k^{(n)} \geq 0 \quad (m = 1, 2, ..., M), \tag{4.3}$$

for n sufficiently large.

Proof. For each n, by introducing the variable $x = t/\sqrt{2n+1}$ and letting $z_n(x) = u_n(t)$, equation (4.2) is transformed into

$$z_n'' + (2n+1)^2(1 - x^2)z_n = 0.$$

We denote the kth positive zero of $z_n(x)$ by $\xi_k^{(n)}$, where $\xi_k^{(n)} = h_k^{(n)}/\sqrt{2n+1}$. Thus, we have

$$\Delta^m h_k^{(n)} = \sqrt{2n+1}\Delta^m \xi_k^{(n)}.$$

To prove (4.3), we consider the differential equation:

$$y'' + (2n+1)^2(a - x^2)y = 0 \quad (a > 1; x \in [0, 1]). \tag{4.4}$$

Let $\omega = 2n+1$, $f(x) = (a - x^2)^{-1/2}$, and let $y_n(x; a)$ be a nontrivial real solution of (4.4) and $x_k^{(n)}(a)$ be the kth positive zero of $y_n(x; a)$. Then, by the following fact about $f^{(m)}(x)$:

$$f^{(m)}(x) = \{a\, polynomial\, of\, x\, with\, nonnegative\, coefficients\}(a - x^2)^{-(2m+1)/2},$$

we know that $f^{(m)}(x) \geq 0$ on the interval $[0, 1]$ for $m = 1, 2, 3, ...$ Thus, by Theorem 2.5, we obtain

$$\Delta^m x_k^{(n)}(a) \geq 0 \quad (m = 1, 2, ..., M),$$

for n sufficiently large. If we specify the initial conditions for $y_n(x; a)$ to be

$$y_n(0; a) = z_n(0) \quad and \quad y_n'(0; a) = z_n'(0),$$

then it is easy to verify that $y_n(x; a)$ uniformly converges to $z_n(x)$ on the interval $[0, 1]$ as $a \to 1^+$. Consequently, for $k = 1, 2, ..., [\frac{n}{2}]$, the zero $x_k^{(n)}(a)$ converges to $\xi_k^{(n)}$ as $a \to 1^+$. Therefore, for k fixed,

$$\Delta^m \xi_k^{(n)} = \lim_{a \to 1^+} \Delta^m x_k^{(n)}(a) \geq 0,$$

and thus (4.3) holds. □

4.2. Zeros of Jacobi polynomials

Given $a > -1$ and $b > -1$, the Jacobi polynomial $P_n^{(a,b)}(x)$ (see e.g., [11, p.67(4.3.1)]) is defined by

$$(1-x)^a(1+x)^b P_n^{(a,b)}(x) = \frac{(-1)^n}{2^n n!}(\frac{d}{dx})^n\{(1-x)^{n+a}(1+x)^{n+b}\}.$$

Concerned with the Jacobi polynomials $P_n^{(a,b)}(x)$ on the orthogonal interval $[-1, 1]$, if we denote the zeros $x_k^{(n)} = x_k^{(n)}(a, b)$ of $P_n^{(a,b)}(x)$ with the descending order

$$1 > x_1^{(n)} > x_2^{(n)} > \cdots > x_n^{(n)} > -1,$$

 doi:10.20944/preprints202305.1665.v1

then the θ-zeros $\theta_k^{(n)} = \theta_k^{(n)}(a,b)$, $x_k^{(n)} = cos\theta_k^{(n)}$, of $P_n^{(a,b)}(cos\theta)$ behave the order

$$0 < \theta_1^{(n)} < \theta_2^{(n)} < \cdots < \theta_n^{(n)} < \pi.$$

According to the uniformly convergent theorem [11, Theorem 8.1.1, p.190]:

$$\lim_{n\to\infty} n^{-\alpha} P_n^{(a,b)}(cos\frac{x}{n}) = (\frac{x}{2})^{-a} J_a(x),$$

we know that

$$\lim_{n\to\infty} n\theta_k^{(n)}(a,b) = j_{ak}.$$

Now, by Theorem 3.1(a), for $\nu = a$ and $\alpha = 0$, we have the following theorem.

Theorem 4.2. *For $|a| < 1/2$ and k fixed, we have*

$$(-1)^m \Delta^{m+2}\theta_k^{(n)}(a,b) \geq 0 \quad (m = 0,1,2,...,M),$$

for n sufficiently large.

5. Proofs of Lemmas 2.1 and 2.3.

In this section we shall prove $(2.5)_m$, $(2.6)_m$, $(2.7)_m$, $(2.16)_m$ and $(2.17)_m$ by inductions, simultaneously.

For $m = 1$, taking the Taylor expansion of φ at x_k:

$$\varphi(x_{k+1}) = \varphi(x_k) + \varphi'(x_k)\Delta x_k + \varphi''(\xi_{k,2})\frac{(\Delta x_k)^2}{2},$$

where $x_k \leq \xi_{k,2} \leq x_{k+1}$, and using (2.3), we have

$$\Delta\varphi(x_k) = \varphi'(x_k)\Delta x_k + O(\omega^{-2}),$$

and hence $(2.5)_1$, $(2.6)_1$ and $(2.7)_1$ are valid. If we apply the first order difference operator to $(2.15)_2$ and use $(2.5)_1$ with $\varphi = F_0$, then we have

$$\Delta f(x_k) = \frac{\omega}{\pi}\Delta^2 x_k + \frac{\omega}{2!\pi}\Delta\{g_1(x_k)(\Delta x_k)^2\} + O(\omega^{-2}).$$

By the fact $\Delta\{\alpha_k\beta_k\} = \alpha_{k+1}(\Delta\beta_k) + (\Delta\alpha_k)\beta_k$, we also have

$$\begin{aligned}\Delta f(x_k) &= \tfrac{\omega}{\pi}\Delta^2 x_k + \tfrac{\omega}{2!\pi} g_1(x_{k+1})\{\Delta x_{k+1}\Delta^2 x_k + \Delta^2 x_k \Delta x_k\} + O(\omega^{-2}) \\ &= \tfrac{\omega}{\pi}\Delta^2 x_k(1 + O(\omega^{-1})) + O(\omega^{-2}).\end{aligned}$$

Applying $(2.5)_1$ again to the function $f(x)$, we find that $\Delta f(x_k) = O(\omega^{-1})$, and then we have

$$\Delta^2 x_k = O(\omega^{-2}),$$

and hence

$$\Delta f(x_k) = \frac{\omega}{\pi}\Delta^2 x_k + O(\omega^{-2}).$$

Thus $(2.16)_2$ and $(2.17)_1$ are valid. The validity of $(2.16)_2$ is the impetus of our induction argument.

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Now, suppose $(2.5)_m$, $(2.6)_m$, $(2.7)_m$, $(2.16)_m$ and $(2.17)_m$ are fulfilled for $m = 1, 2, ..., N$. If we apply $(2.5)_N$ with $\varphi(x) = f(x)$ to $(2.17)_N$, then we have $(2.16)_{N+1}$, that is

$$\Delta^{N+1} x_k = O(\omega^{-N-1}).$$

Taking the Taylor expansion of φ at x_k:

$$\varphi(x_{k+1}) = \varphi(x_k) + \sum_{p=1}^{N+1} \frac{\varphi^{(p)}(x_k)}{p!}(\Delta x_k)^p + \frac{\varphi^{(N+2)}(\xi_{k,N+2})}{(N+2)!}(\Delta x_k)^{N+2}, \tag{5.1}$$

where $x_k \le \xi_{k,N+2} \le x_{k+1}$, applying the Nth order difference operator to (5.1) and then using $(2.16)_1$, we have

$$\Delta^{N+1}\varphi(x_k) = \sum_{p=1}^{N+1} \frac{1}{p!}\Delta^N\{\varphi^{(p)}(x_k)(\Delta x_k)^p\} + O(\omega^{-N-2}). \tag{5.2}$$

Following the product rule for higher differences, we know that

$$\Delta^N\{\varphi^{(p)}(x_k)(\Delta x_k)^p\} = \sum_{r=0}^{N} \binom{N}{r} \Delta^r \varphi^{(p)}(x_{k+N-r})\Delta^{N-r}(\Delta x_k)^p.$$

If we replace $\varphi(x_k)$ by $\varphi^{(p)}(x_{k+N-r})$ in $(2.5)_r$ for $r = 1, 2, ..., N$, and use $(2.16)_m$ for $m = 1, 2, ..., N+1$, then we obtain

$$\Delta^N\{\varphi^{(p)}(x_k)(\Delta x_k)^p\} = O(\omega^{-N-p}) \quad (p = 1, 2, ..., N+1). \tag{5.3}$$

Thus (5.2) and (5.3) imply $(2.5)_{N+1}$. Moreover, we have

$$\begin{aligned} \Delta^{N+1}\varphi(x_k) &= \Delta^N\{\varphi'(x_k)\Delta x_k\} + O(\omega^{-N-2}) \\ &= \textstyle\sum_{r=0}^{N} \binom{N}{r} \Delta^r \varphi'(x_{k+N-r})\Delta^{N+1-r}x_k + O(\omega^{-N-2}). \end{aligned} \tag{5.4}$$

Applying $(2.6)_r$ with $\varphi'(x_{k+N-r})$ instead of $\varphi(x_k)$ for $r = 1, 2, ..., N$ to (5.4), we find

$$\begin{aligned} \Delta^{N+1}\varphi(x_k) = \;& \varphi'(x_{k+N})\Delta^{N+1}x_k \\ & + \textstyle\sum_{r=1}^{N} \binom{N}{r} \{\sum_{q=1}^{r} A^{(r)}_{q,k+N-r}\varphi^{(q+1)}(x_{k+N-q})\}\Delta^{N+1-r}x_k + O(\omega^{-N-2}). \end{aligned} \tag{5.5}$$

If we change the order of the summation in (5.5) and shift the index q, then we can find

$$\begin{aligned} \Delta^{N+1}\varphi(x_k) = \;& \varphi'(x_{k+N})\Delta^{N+1}x_k \\ & + \textstyle\sum_{q=2}^{N+1} \varphi^{(q)}(x_{k+N+1-q})\{\sum_{r=q-1}^{N} \binom{N}{r} A^{(r)}_{q-1,k+N-r}\Delta^{N+1-r}x_k\} + O(\omega^{-N-2}). \end{aligned}$$

Thus $(2.6)_{N+1}$ and $(2.7)_{N+1}$ are valid.

Finally, to prove $(2.17)_{N+1}$, applying the $(N+1)$th order difference operator to $(2.15)_{N+1}$, we have

$$\begin{aligned} \Delta^{N+1} f(x_k) = \;& \tfrac{\omega}{\pi} \textstyle\sum_{r=0}^{N+1} \frac{1}{(r+1)!}\Delta^{N+1}\{g_r(x_k)(\Delta x_k)^{r+1}\} \\ & + \tfrac{1}{\pi} \textstyle\sum_{r=0}^{N-1} \Delta^{N+1}\{f(x_k)\Delta F_r(x_k)\}\omega^{-r-1} + O(\omega^{-N-2}). \end{aligned} \tag{5.6}$$

Following the product rule for higher differences again, we have

$$\Delta^{N+1}\{g_r(x_k)(\Delta x_k)^{r+1}\} = \sum_{\beta=0}^{N+1} \binom{N+1}{\beta} \Delta^\beta g_r(x_{k+N+1-\beta})\Delta^{N+1-\beta}(\Delta x_k)^{r+1}.$$

 doi:10.20944/preprints202305.1665.v1

Using $(2.5)_\beta$ with $g_r(x_{k+N+1-\beta})$ replacing $\varphi(x_k)$ for $\beta = 1, 2, ..., N+1$, and using $(2.16)_m$ for $m = 1, 2, ..., N+1$, we obtain

$$\begin{aligned} &\Delta^{N+1}\{g_r(x_k)(\Delta x_k)^{r+1}\} \\ &= g_r(x_{k+N+1})\Delta^{N+1}(\Delta x_k)^{r+1} + \textstyle\sum_{\beta=1}^{N+1}\binom{N+1}{\beta}\Delta^\beta g_r(x_{k+N+1-\beta})\Delta^{N+1-\beta}(\Delta x_k)^{r+1} \\ &= g_r(x_{k+N+1})(\Delta^{N+2}x_k)O(\omega^{-r}) + O(\omega^{-N-r-2}). \end{aligned} \tag{5.7}$$

On the other hand, applying $(2.5)_m$ to the functions $f(x)$ and $F_r(x)$ for $m = 1, 2, ..., N+1$, we also have

$$\begin{aligned} &\Delta^{N+1}\{f(x_k)\Delta F_r(x_k)\} \\ &= f(x_{k+N+1})\Delta^{N+2}F_r(x_k) + \textstyle\sum_{\beta=1}^{N+1}\binom{N+1}{\beta}\Delta^\beta f(x_{k+N+1-\beta})\Delta^{N+2-\beta}F_r(x_k) \\ &= O(\omega^{-N-1}) + O(\omega^{-N-2}). \end{aligned} \tag{5.8}$$

Applying the estimates (5.7) and (5.8) to (5.6), we obtain

$$\begin{aligned} \Delta^{N+1}f(x_k) &= \tfrac{\omega}{\pi}\Delta^{N+2}x_k + (\tfrac{\omega}{\pi}\Delta^{N+2}x_k)O(\omega^{-1}) + O(\omega^{-N-2}) \\ &= \tfrac{\omega}{\pi}\Delta^{N+2}x_k(1 + O(\omega^{-1})) + O(\omega^{-N-2}). \end{aligned} \tag{5.9}$$

If we replace $\varphi(x_k)$ by $f(x_k)$ in $(2.5)_{N+1}$, then we have

$$\Delta^{N+1}f(x_k) = O(\omega^{-N-1}). \tag{5.10}$$

Note that (5.9) and (5.10) imply

$$\frac{\omega}{\pi}\Delta^{N+2}x_k = O(\omega^{-N-1}). \tag{5.11}$$

Then by (5.9) and (5.11), we have $(2.17)_{N+1}$. This completes the proofs of Lemmas 2.1 and 2.3.

Appendix A

Recalling $f(x) = \rho^{-1/2}(x)$ and the differential equation (2.8) for the Prüfer angle $\theta(x;\omega)$, we have

$$\theta'(x;\omega) = \frac{\omega}{f(x)}\{1 - \frac{f'(x)}{2\omega}\sin 2\theta(x;\omega)\}. \tag{6.1}$$

Then

$$\{-\frac{f'}{2f}\sin 2\theta\}\frac{\theta'}{\{1-(f'/2\omega)\sin 2\theta\}\omega/f} = -\sum_{r=0}^{\infty}\{\frac{\omega^{-1}}{2}f'\sin 2\theta\}^{r+1}\theta', \tag{6.2}$$

and hence

$$\int_{x_k}^{x_{k+1}}\frac{\rho'}{4\rho}\sin 2\theta dx = -\sum_{r=0}^{m-1}\frac{\omega^{-r-1}}{2^{r+1}}\int_{x_k}^{x_{k+1}}(f')^{r+1}(sin^{r+1}2\theta)\theta' dx + O(\omega^{-m-1}), \tag{6.3}$$

where $\theta = \theta(x;\omega)$, $\theta' = \theta'(x;\omega)$ and $x_k = x_k(\omega)$.

To prove Lemma 2.2, we introduce the following integrals for a C^∞-function φ which is defined on $[a,b]$:

$$P_r[\varphi] = \int_{x_k}^{x_{k+1}}\varphi\cdot sin^r(2\theta)\cdot\theta' dx,$$

$$Q_r[\varphi] = \int_{x_k}^{x_{k+1}}\varphi\cdot sin^r(2\theta)\cdot cos(2\theta) dx,$$

 doi:10.20944/preprints202305.1665.v1

and

$$R_r[\varphi] = \int_{x_k}^{x_{k+1}} \varphi \cdot sin^{r+1}(2\theta)dx.$$

where $r = 0, 1, 2, ...$ Now, (6.3) can be written as

$$\int_{x_k}^{x_{k+1}} \frac{\rho'}{4\rho} \sin 2\theta dx = -\sum_{r=0}^{m-1} \frac{\omega^{-r-1}}{2^{r+1}} P_{r+1}[(f')^{r+1}] + O(\omega^{-m-1}). \tag{6.4}$$

By integration by parts, we have the reduced formula for $P_{r+1}[\varphi]$ that

$$P_{r+1}[\varphi] = \frac{-\varphi \cdot sin^r 2\theta \cdot cos2\theta}{2(r+1)}|_{x_k}^{x_{k+1}} + \frac{r}{r+1} P_{r-1}[\varphi] + \frac{1}{2(r+1)} Q_r[\varphi']. \tag{6.5}$$

Introducing θ' in the same way as we did in (6.2), and using integration by parts and (2.9), we have the following estimates for $Q_r[\varphi]$ and $R_r[\varphi]$,

$$Q_r[\varphi] = -\sum_{j=0}^{m-r-3} \frac{\omega^{-j-1}}{2^{j+1}(r+j+1)} R_{r+j}[(\varphi_j)'] + O(\omega^{-m+r}), \tag{6.6}$$

and

$$R_r[\varphi] = \sum_{j=0}^{m-r-3} \frac{\omega^{-j-1}}{2^j} P_{r+j+1}[\varphi_j] + O(\omega^{-m+r+1}), \tag{6.7}$$

where $\varphi_j = \varphi f (f')^j$. Applying the estimates (6.6) and (6.7) with suitable integrands to (6.5), and then collecting the terms with the same order of ω in the sum together, we can find

$$\begin{aligned} P_{r+1}[\varphi] = & \tfrac{-\varphi \cdot sin^r 2\theta \cdot cos2\theta}{2(r+1)}|_{x_k}^{x_{k+1}} + \tfrac{r}{r+1} P_{r-1}[\varphi] \\ & -\textstyle\sum_{\beta=0}^{m-r-3} \tfrac{\omega^{-\beta-2}}{2^{\beta+2}(r+1)} \sum_{j=0}^{\beta} \tfrac{1}{r+j+1} P_{r+\beta+1}[(\varphi')_{j,\beta-j}] + O(\omega^{-m+r}), \end{aligned} \tag{6.8}$$

where $\varphi_{j_1,j_2} = [(\varphi_{j_1})']_{j_2}$. By (6.8) and (2.9), we have

$$P_1[\varphi] = \frac{-\Delta\varphi(x_k)}{2} - \sum_{\beta=0}^{m-3} \frac{\omega^{-\beta-2}}{2^{\beta+2}} \sum_{j=0}^{\beta} \frac{1}{j+1} P_{\beta+1}[(\varphi')_{j,\beta-j}] + O(\omega^{-m}), \tag{6.9}$$

and

$$P_2[\varphi] = \frac{P_0[\varphi]}{2} - \sum_{\beta=0}^{m-4} \frac{\omega^{-\beta-2}}{2^{\beta+3}} \sum_{j=0}^{\beta} \frac{1}{j+2} P_{\beta+2}[(\varphi')_{j,\beta-j}] + O(\omega^{-m+1}). \tag{6.10}$$

If we apply (6.1) and (6.7) to the integral $P_0[\varphi]$, then we have

$$P_0[\varphi] = \omega \int_{x_k}^{x_{k+1}} \frac{\varphi}{f} dx - \sum_{j=0}^{m-3} \frac{\omega^{-j-1}}{2^{j+1}} P_{j+1}[(\varphi f'/f)_j] + O(\omega^{-m+1}). \tag{6.11}$$

Applying (6.11) to (6.10), we obtain

$$\begin{aligned} P_2[\varphi] = & \tfrac{\omega}{2} \textstyle\int_{x_k}^{x_{k+1}} \tfrac{\varphi}{f} dx - \sum_{j=0}^{m-3} \tfrac{\omega^{-j-1}}{2^{j+2}} P_{j+1}[(\varphi f'/f)_j] \\ & -\textstyle\sum_{\beta=0}^{m-4} \tfrac{\omega^{-\beta-2}}{2^{\beta+3}} \sum_{j=0}^{\beta} \tfrac{1}{j+2} P_{\beta+2}[(\varphi')_{j,\beta-j}] + O(\omega^{-m-1}). \end{aligned} \tag{6.12}$$

In (6.4), if we apply (6.8) to the function $\varphi = (f')^{r+1}$, and use (6.9) and (6.12) to collect the reductions of those integrals $P_{r-1}[(f')^{r+1}]$ and $P_{r+\beta+1}[((f')^{r+1})'_{j,\beta-j}]$, then all processes of reductions shall be stopped, after a finite steps, while the remainders behave as $O(\omega^{-m-1})$. This completes the proof of Lemma 2.2.

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