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# A Perturbed Milne's Quadrature Rule for $n$-Times Differentiable Functions with $L^p$-Error Estimates

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**Abstract:** In this work, in spite of Milne's recommendation using the three-point Newton–Cotes open formula (Milne's rule) as a predictor rule and three-point Newton–Cotes closed formula (Simpson's rule) as a corrector rule for 4-th differentiable functions with bounded derivatives. There is still a great need to introduce such formulas in other $L^p$ spaces. Often, we need to approximate real integrals under the assumptions of the function involved. Because of that, the aim of this work is to introduce several $L^p$ error estimates for the proposed perturbed Milne's quadrature rule. Numerical experiments showing that our proposed quadrature rule is better than the classical Milne rule for certain types of functions are provided as well.

**Keywords:** Milne's rule, Simpson's rule, Quadrature rule, Newton–Cotes formulae, Numerical integration, Error estimation.

**MSC:** 65D30, 65D32, 26D10, 26D15.

## 1. Introduction

There are many attractive methods that are used to approximate real integrals. One of the oldest and most well-known is the Newton-Cotes closed and open formulas. Particularly, among other famous formulas; Simpson's rule and Milen's rule are very interesting and close to each other. Since each formula involves a bounded error of the fourth degree. However, it is well-known that Simpson's rule is of closed Newton-type formula, while Milne's formula is of open type. Accordingly, it's very interesting to test both quadrature rules in many situations. In the last decades, the modern theory of inequalities are used at large to verify these quadrature rules (and others) using the Peano-kernel approach.

In terms of Newton–Cotes formulas, Milne's formula is of open type is parallel to Simpson's formula which is of closed type, since they are held under the same conditions. Suppose $g \in C^4([c,d])$, and

$$\left\| g^{(4)} \right\|_\infty := \sup_{s\in(c,d)} \left| g^{(4)}(s) \right| < \infty.$$

In terms of inequalities Simpson's and Milne's inequalities are read, respectively [1]:

$$\left| \frac{d-c}{3} \left[ \frac{g(c)+g(d)}{2} + 2g\left(\frac{c+d}{2}\right) \right] - \int_c^d g(s)ds \right| \le \frac{(d-c)^5}{2880} \left\| g^{(4)} \right\|_\infty, \tag{1}$$

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and

$$\left|\frac{d-c}{3}\left[2g\left(\frac{3c+d}{4}\right)-g\left(\frac{c+d}{2}\right)+2g\left(\frac{c+3d}{4}\right)\right]-\int_c^d g(s)ds\right|\leq\frac{7(d-c)^5}{23040}\left\|g^{(4)}\right\|_\infty. \quad (2)$$

Attempting to apply the Simpson and/or Milne quadrature rules using lower-order derivatives is very promising (especially; for certain types of functions) as we obtained see in this work. Even the Simpson rule is more popular than Milne's due to several reasons, however, the Milne quadrature rule has not attracted many researchers. Because of that, we focus this work on studying the error of the Milne quadrature rule for n-times differentiable functions by obtaining several $L^p$ bounds of this quadrature. At the same time, the considered approach allows us to see how adding derivatives of the used nodes of this rule in oscillating the error term. In other words, how the Milne quadrature rule behaves as a predictor for higher or lower derivatives. In fact, the oscillation of the proposed quadrature rule raises in general. On the other hand, it is shown numerically and practically that, for certain types of functions, the error descends sharply, which means that our approach could be very effective for certain types of functions. Moreover, one of the most important advantages of our result is that it is verified for $p$-variation and Lipschitz functions. Also, since the classical Milne's quadrature rule (2) cannot be applied either when the fourth derivative is unbounded or doesn't exist, therefore the proposed quadrature could be used alternatively.

For more about Simpson's quadrature rule and other related results, the reader is recommended to refer to [2]–[26]. For other types of quadrature rules see [27]–[30] and the references therein. The book [31], is also recommended for recent and classical methods of numerical integration.

In this work, despite Milne recommends using the three-point Newton–Cotes open formula as a predictor rule and three-point Newton–Cotes closed formula (Simpson's rule) as a corrector rule for 4-th differentiable functions with bounded derivatives. There is still a great need to introduce such formulas in other $L^p$ spaces. Often, we need to approximate real integrals under the assumptions of the function involved. Because of that, this work is concentrate to introduce several $L^p$ error estimates for the proposed perturbed Milne's quadrature rule. Numerical experiments showing that our proposed quadrature rule is better than the classical Milne rule for certain types of functions are provided as well.

## 2. Perturbed Milne's quadrature formula

In order to establish our results we need to recall the following two lemmas.

**Lemma 1.** *[16]* Fix $1 \leq p < \infty$. Assume that $g$ is continuous function on $[c,d]$ and $w$ is of bounded $p$–variation on $[c,d]$. Then $\int_c^d g(s)dw(s)$, exists and the inequality:

$$\left|\int_c^d g(s)dw(s)\right| \leq \|g\|_\infty \cdot \bigvee_c^d (w;p), \quad (3)$$

holds, where $\bigvee_c^d(w,p)$, denotes to total $p$-variation of $w$ over $[c,d]$.

**Lemma 2.** *[16] Let $1 \leq p < \infty$. Assume that $g \in L^p[c,d]$ and $w$ has a Lipschitz property on $[c,d]$. Then*

$$\left|\int_c^d g(s)dw(s)\right| \leq \mathrm{Lip}_M(w)(d-c)^{1-\frac{1}{p}} \cdot \|g\|_p, \quad (4)$$

*holds.*

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From now on, $I$ is a real interval and $c, d \in \mathbb{R}$ with $c, d \in I^{\circ}$ the interior of $I$ with $c < d$. Define the set $\mathfrak{V}_p^{(m)}$ to be the set of all $m$-times continuously differentiable function $g$ whose $m$-derivative $(m \geq 1)$ is absolutely continuous with $g^{(m)} \in L^p[c, d]$ $(1 \leq p \leq \infty)$.

In what follows, we present a primary result involving the expansion of Milne's rule for higher-order derivatives using the Peano-kernel approach.

**Lemma 3.** If $g \in \mathfrak{V}_1^{(m)}$, then we have

$$\int_c^d g(s)ds = \sum_{\ell=0}^{m-1} \frac{(d-c)^{\ell+1}}{(\ell+1)!} \left\{ \left[ \left(\frac{5}{12}\right)^{\ell+1} + \left(\frac{1}{4}\right)^{\ell+1} \right] \left[ g^{(\ell)}\left(\frac{3c+d}{4}\right) + (-1)^{\ell} g^{(\ell)}\left(\frac{c+3d}{4}\right) \right] - \frac{\left((-1)^{\ell}+1\right)}{6^{\ell+1}} g^{(\ell)}\left(\frac{c+d}{2}\right) \right\} + (-1)^m \int_c^d \mathcal{K}_m(s) g^{(m)}(s)ds \tag{5}$$

where

$$\mathcal{K}_m(s) = \begin{cases} \frac{1}{m!}(s-c)^m, & \text{if} \quad s \in \left[c, \frac{3c+d}{4}\right] \\ \frac{1}{m!}\left(s - \frac{c+2d}{3}\right)^m, & \text{if} \quad s \in \left(\frac{3c+d}{4}, \frac{c+d}{2}\right] \\ \frac{1}{m!}\left(s - \frac{2c+d}{3}\right)^m, & \text{if} \quad s \in \left(\frac{c+d}{2}, \frac{c+3d}{4}\right] \\ \frac{1}{m!}(s-d)^m, & \text{if} \quad s \in \left(\frac{c+3d}{4}, d\right] \end{cases},$$

for all $m \geq 1$.

**Proof.** We carry out our proof using mathematical induction. For $m = 1$ we have

$$\mathcal{K}_1(t) = \begin{cases} s-c, & s \in \left[c, \frac{3c+d}{4}\right] \\ s - \frac{c+2d}{3}, & s \in \left(\frac{3c+d}{4}, \frac{c+d}{2}\right] \\ s - \frac{2c+d}{3}, & s \in \left(\frac{c+d}{2}, \frac{c+3d}{4}\right] \\ s-d, & s \in \left(\frac{c+3d}{4}, d\right] \end{cases}.$$

Applying the integration by parts, we get

$$\int_c^{\frac{3c+d}{4}} \mathcal{K}_1(s)dg(s) = \left(\frac{d-c}{4}\right) g\left(\frac{3c+d}{4}\right) - \int_c^{\frac{3c+d}{4}} g(s)ds,$$

$$\int_{\frac{3c+d}{4}}^{\frac{c+d}{2}} \mathcal{K}_1(t)dg(s) = \left(\frac{c+d}{2} - \frac{c+2d}{3}\right) g\left(\frac{c+d}{2}\right) - \left(\frac{3c+d}{4} - \frac{c+2d}{3}\right) g\left(\frac{3c+d}{4}\right) - \int_{\frac{3a+b}{4}}^{\frac{c+d}{2}} g(s)ds,$$

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$$\int_{\frac{c+d}{2}}^{\frac{c+3d}{4}} \mathcal{K}_1(t)dg(s) = \left(\frac{c+3d}{4} - \frac{2c+d}{3}\right)g\left(\frac{c+3d}{4}\right) - \left(\frac{c+d}{2} - \frac{2c+d}{3}\right)g\left(\frac{a+b}{2}\right) - \int_{\frac{c+d}{2}}^{\frac{c+3d}{4}} g(s)ds,$$

and

$$\int_{\frac{c+3d}{4}}^{b} \mathcal{K}_1(s)dg(s) = \left(\frac{d-c}{4}\right)g\left(\frac{c+3d}{4}\right) - \int_{\frac{c+3d}{4}}^{b} g(s)ds.$$

Adding the above equalities and arranging the resulting terms, simple calculations yield that

$$\int_c^d g(s)ds = \frac{(d-c)}{3}\left[2g\left(\frac{3c+d}{4}\right) - g\left(\frac{c+d}{2}\right) + 2g\left(\frac{c+3d}{4}\right)\right] - \int_c^d \mathcal{K}_1(s)dg(s).$$

Now, assume that (5) holds for $m = l$. We need to show that it holds for $m = l+1$, i.e.,

$$\begin{aligned}\int_c^d g(s)ds = \frac{(d-c)}{3}\left[2g\left(\frac{3c+d}{4}\right) - g\left(\frac{c+d}{2}\right) + 2g\left(\frac{c+3d}{4}\right)\right] \\ + \sum_{\ell=1}^{l} \frac{(d-c)^{\ell+1}}{(\ell+1)!}\left\{\left[\left(\frac{5}{12}\right)^{\ell+1} + \left(\frac{1}{4}\right)^{\ell+1}\right]\left[g^{(\ell)}\left(\frac{3c+d}{4}\right) + (-1)^{\ell} g^{(\ell)}\left(\frac{c+3d}{4}\right)\right]\right. \\ \left. - \frac{\left((-1)^{\ell}+1\right)}{6^{\ell+1}} g^{(\ell)}\left(\frac{c+d}{2}\right)\right\} + (-1)^{m+1}\int_c^d \mathcal{K}_{l+1}(s)g^{(l+1)}(s)ds\end{aligned} \tag{6}$$

where

$$\mathcal{K}_{l+1}(s) = \begin{cases} \frac{1}{(l+1)!}(s-c)^{l+1}, & \text{if } s \in \left[c, \frac{3c+d}{4}\right] \\ \frac{1}{(l+1)!}\left(s - \frac{c+2d}{3}\right)^{l+1}, & \text{if } s \in \left(\frac{3c+d}{4}, \frac{c+d}{2}\right] \\ \frac{1}{(l+1)!}\left(s - \frac{2c+d}{3}\right)^{l+1}, & \text{if } s \in \left(\frac{c+d}{2}, \frac{c+3d}{4}\right] \\ \frac{1}{(l+1)!}(s-d)^{l+1}, & \text{if } s \in \left(\frac{c+3d}{4}, d\right] \end{cases},$$

for all $l \geq 1$. Again, using integration by parts, we have

$$\int_c^{\frac{3c+d}{4}} \mathcal{K}_{m+1}(s)g^{(l+1)}(s)ds = \frac{1}{(l+1)!}\left(\frac{d-c}{4}\right)^{l+1} g^{(l)}\left(\frac{3c+d}{4}\right) - \frac{1}{l!}\int_c^{\frac{3c+d}{4}} (s-c)^l g^{(l)}(s)ds,$$

$$\begin{aligned}\int_{\frac{3c+d}{4}}^{\frac{c+d}{2}} \mathcal{K}_{l+1}(s)g^{(l+1)}(s)ds = (-1)^{l+1}\frac{(d-c)^{l+1}}{(l+1)!}\left[\left(\frac{1}{6}\right)^{l+1} g^{(l)}\left(\frac{c+d}{2}\right) - \left(\frac{5}{12}\right)^{l+1} g^{(l)}\left(\frac{3c+d}{4}\right)\right] \\ - \frac{1}{l!}\int_{\frac{3c+d}{4}}^{\frac{c+d}{2}} \left(s - \frac{a+2b}{3}\right)^m g^{(l)}(s)ds,\end{aligned}$$

$$\int_{\frac{c+d}{2}}^{\frac{c+3d}{4}} \mathcal{K}_{l+1}(s)g^{(l+1)}(s)ds = \frac{(d-c)^{l+1}}{(l+1)!}\left[\left(\frac{5}{12}\right)^{l+1} g^{(l)}\left(\frac{c+3d}{4}\right) - \left(\frac{1}{6}\right)^{l+1} g^{(l)}\left(\frac{c+d}{2}\right)\right] - \frac{1}{l!}\int_{\frac{c+d}{2}}^{\frac{c+3d}{4}} \left(s - \frac{2c+d}{3}\right)^{l} g^{(l)}(s)ds,$$

and

$$\int_{\frac{c+3d}{4}}^{d} \mathcal{K}_{l+1}(s)g^{(l+1)}(s)ds = \frac{(-1)^{l}}{(l+1)!}\left(\frac{d-c}{4}\right)^{l+1} g^{(l)}\left(\frac{c+3d}{4}\right) - \frac{1}{l!}\int_{\frac{c+3d}{4}}^{d} (s-d)^{l} g^{(l)}(s)ds,$$

Adding the above identities, we get

$$\int_c^d \mathcal{K}_{l+1}(s)g^{(l)}(s)ds = \frac{(d-c)^{l+1}}{(l+1)!}\left[\left(\frac{5}{12}\right)^{m+1} + \left(\frac{1}{4}\right)^{l+1}\right]\left[g^{(l)}\left(\frac{3c+d}{4}\right) + (-1)^{l} g^{(l)}\left(\frac{c+3d}{4}\right)\right] - \frac{\left[(-1)^{l}+1\right]}{6^{l+1}(l+1)!}(d-c)^{l+1}g^{(l)}\left(\frac{c+d}{2}\right) + (-1)^{l}\int_c^d \mathcal{K}_{l+1}(s)g^{(l+1)}(s)ds.$$

Using the mathematical induction hypothesis, we get

$$\begin{aligned}
\int_c^d g(s)ds &= \frac{(d-c)^{l+1}}{(l+1)!}\left[\left(\frac{5}{12}\right)^{l+1} + \left(\frac{1}{4}\right)^{l+1}\right]\left[g^{(l)}\left(\frac{3c+d}{4}\right) + (-1)^{l} g^{(l)}\left(\frac{c+3d}{4}\right)\right] \\
&\quad + \sum_{\ell=0}^{l-1} \frac{(d-c)^{\ell+1}}{(\ell+1)!}\Bigg\{\left[\left(\frac{5}{12}\right)^{\ell+1} + \left(\frac{1}{4}\right)^{\ell+1}\right]\left[g^{(\ell)}\left(\frac{3c+d}{4}\right) + (-1)^{\ell} g^{(\ell)}\left(\frac{c+3d}{4}\right)\right] \\
&\qquad - \frac{\left((-1)^{\ell}+1\right)}{6^{\ell+1}} g^{(\ell)}\left(\frac{c+d}{2}\right)\Bigg\} - \frac{\left[(-1)^{l}+1\right]}{6^{l+1}(l+1)!}(d-c)^{l+1}g^{(l)}\left(\frac{c+d}{2}\right) \\
&\quad - (-1)^{l}\int_c^d \mathcal{K}_{l+1}(s)g^{(l+1)}(s)ds \\
&= \sum_{\ell=0}^{l} \frac{(d-c)^{\ell+1}}{(\ell+1)!}\Bigg\{\left[\left(\frac{5}{12}\right)^{\ell+1} + \left(\frac{1}{4}\right)^{\ell+1}\right]\left[g^{(\ell)}\left(\frac{3c+d}{4}\right) + (-1)^{\ell} g^{(\ell)}\left(\frac{c+3d}{4}\right)\right] \\
&\qquad - \frac{\left((-1)^{\ell}+1\right)}{6^{\ell+1}} g^{(\ell)}\left(\frac{c+d}{2}\right)\Bigg\} + (-1)^{l+1}\int_c^d \mathcal{K}_{l+1}(s)g^{(l+1)}(s)ds
\end{aligned}$$

which gives the representation (6). Hence, by mathematical induction (5) holds for all $l \geq 1$. □

For convenient representation, we may rewrite (5) such as:

$$\begin{aligned}
\int_c^d g(s)ds &= \frac{d-c}{3}\left[2g\left(\frac{3c+d}{4}\right) - g\left(\frac{c+d}{2}\right) + 2g\left(\frac{c+3d}{4}\right)\right] \\
&\quad + \sum_{\ell=1}^{l-1} \frac{(d-c)^{\ell+1}}{(\ell+1)!}\Bigg\{\left[\left(\frac{5}{12}\right)^{\ell+1} + \left(\frac{1}{4}\right)^{\ell+1}\right]\left[g^{(\ell)}\left(\frac{3c+d}{4}\right) + (-1)^{\ell} g^{(\ell)}\left(\frac{c+3d}{4}\right)\right] \\
&\qquad - \frac{\left((-1)^{\ell}+1\right)}{6^{\ell+1}} g^{(\ell)}\left(\frac{c+d}{2}\right)\Bigg\} + (-1)^{m}\int_c^d \mathcal{K}_{m}(s)g^{(m)}(s)ds
\end{aligned} \tag{7}$$

Therefore, we can compute $\int_c^d g(s)ds$ using a perturbed Milne's quadrature formula

$$\int_c^d g(s)ds = \mathcal{M}_m(g) + \mathcal{E}_m(g) \tag{8}$$

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for all $m \geq 1$, where $\mathcal{M}_m(g)$ is the perturbed Milne's rule given by

$$\begin{aligned}\mathcal{M}_m(g) := \frac{d-c}{3}\left[2g\left(\frac{3c+d}{4}\right) - g\left(\frac{c+d}{2}\right) + 2g\left(\frac{c+3d}{4}\right)\right] \\ + \sum_{\ell=1}^{m-1}\frac{(d-c)^{\ell+1}}{(\ell+1)!}\left\{\left[\left(\frac{5}{12}\right)^{\ell+1} + \left(\frac{1}{4}\right)^{\ell+1}\right]\left[g^{(\ell)}\left(\frac{3c+d}{4}\right) + (-1)^{\ell} g^{(\ell)}\left(\frac{c+3d}{4}\right)\right] \right. \\ \left. - \frac{\left((-1)^{\ell}+1\right)}{6^{\ell+1}} g^{(\ell)}\left(\frac{c+d}{2}\right)\right\}\end{aligned} \tag{9}$$

and $\mathcal{E}_m(g)$ is the error term given by

$$\mathcal{E}_m(g) = \int_c^d g(s)ds - \mathcal{M}_m(g) = (-1)^m \int_c^d \mathcal{K}_m(s) g^{(m)}(s)ds \tag{10}$$

for all $m \geq 1$.

**Theorem 1.** *If $g^{(2m)}$ $(m \geq 1)$ is continuous on $[c,d]$, such that $g^{(2m)}(s)$ does not change sign on $[c,d]$. Then there exists $\eta \in (c,d)$ such that*

$$\mathcal{E}_{2m}(g) = \frac{2(d-c)^{2m+1}}{(2m+1)!}\left(\frac{1}{4^{2m+1}} + \frac{5^{2m+1}}{12^{2m+1}} - \frac{1}{6^{2m+1}}\right) g^{(2m)}(\eta). \tag{11}$$

**Proof.** Since $g^{(2m)}(s)$ does not change sign on $[c,d]$, then there exists $\eta \in (c,d)$

$$\begin{aligned}\mathcal{E}_{2m}(g) &= (-1)^{2m}\int_c^d \mathcal{K}_{2m}(s) g^{(2m)}(s)ds \\ &= g^{(2m)}(\eta)\int_c^d \mathcal{K}_{2m}(s)ds \\ &= g^{(2m)}(\eta)\cdot\frac{2(d-c)^{2m+1}}{(2m+1)!}\left(\frac{1}{4^{2m+1}} + \frac{5^{2m+1}}{12^{2m+1}} - \frac{1}{6^{2m+1}}\right),\end{aligned}$$

which completes the proof of the result. □

## 3. Error estimation(s)

We begin with the following result:

**Theorem 2.** If $g^{(l-1)}$ $(l \geq 1)$ is a function of bounded $p$-variation on $I$. Then, we have the inequality

$$|\mathcal{E}_l(g)| \leq \frac{1}{l!}\left[\frac{5(d-c)}{12}\right]^l \cdot \bigvee_c^d \left(g^{(l-1)}, p\right), \tag{12}$$

where $\bigvee_c^d\left(g^{(l-1)}, p\right)$, denotes to total $p$-variation of $g^{(l-1)}$ over $[c,d]$.

**Proof.** From (7), we get

$$
\begin{aligned}
|\mathcal{E}_l(g)| &\le \left|(-1)^l\int_c^d \mathcal{K}_l(s)dg^{(l-1)}(s)\right| \\
&\le \frac{1}{l!}\left|\int_c^{\frac{3c+d}{4}}(s-c)^l dg^{(l-1)}(s)\right| + \frac{1}{l!}\left|\int_{\frac{3c+d}{4}}^{\frac{c+d}{2}}\left(s-\frac{2c+d}{3}\right)^l dg^{(l-1)}(s)\right| \\
&\quad + \frac{1}{l!}\left|\int_{\frac{c+d}{2}}^{\frac{c+3d}{4}}\left(s-\frac{c+2d}{3}\right)^l dg^{(l-1)}(s)\right| + \frac{1}{L!}\left|\int_{\frac{c+3d}{4}}^{d}(s-d)^l dg^{(l-1)}(s)\right| \\
&\le \frac{1}{l!}\sup_{s\in\left[c,\frac{3c+d}{4}\right]}|s-c|^l\cdot\bigvee_c^{\frac{3c+d}{4}}\left(g^{(l-1)},p\right) + \frac{1}{l!}\sup_{s\in\left[\frac{3c+d}{4},\frac{c+d}{2}\right]}\left|s-\frac{2c+d}{3}\right|^l\cdot\bigvee_{\frac{3c+d}{4}}^{\frac{c+d}{2}}\left(g^{(l-1)},p\right) \\
&\quad + \frac{1}{l!}\sup_{s\in\left[\frac{c+d}{2},\frac{c+3d}{4}\right]}\left|s-\frac{c+2d}{3}\right|^l\cdot\bigvee_{\frac{c+d}{2}}^{\frac{c+3d}{4}}\left(g^{(l-1)},p\right) + \frac{1}{l!}\sup_{s\in\left[\frac{c+3d}{4},d\right]}|s-d|^l\cdot\bigvee_{\frac{c+3d}{4}}^{d}\left(g^{(l-1)},p\right) \\
&\le \frac{1}{l!}\left[\frac{5(d-c)}{12}\right]^l\cdot\bigvee_c^d\left(g^{(l-1)},p\right),
\end{aligned}
$$

which proves (12). □

**Theorem 3.** If $g\in\mathfrak{V}_p^{(l-1)}$ $(l\ge 1)$, then

$$
|\mathcal{E}_l(g)| \le \begin{cases} \frac{2(d-c)^{l+1}}{(l+1)!}\left[\frac{1}{4^{l+1}}+\frac{5^{l+1}}{12^{l+1}}-\frac{1}{6^{l+1}}\right]\left\|g^{(l)}\right\|_{\infty,[c,d]}, & \text{if } g^{(l)}\in L^\infty[c,d] \\[2ex] \frac{2^{1/q}(d-c)^{l+\frac{1}{q}}}{m!(lq+1)^{\frac{1}{q}}}\left[\frac{1}{4^{lq+1}}+\frac{5^{lq+1}}{12^{lq+1}}-\frac{1}{6^{lq+1}}\right]^{\frac{1}{q}}\left\|g^{(l)}\right\|_{p,[c,d]}, & \text{if } g^{(l)}\in L^p[c,d] \end{cases} \tag{13}
$$

where $q=\frac{p}{p-1}$, $p>1$.

**Proof.** From (7), we get

$$
\begin{aligned}
|\mathcal{E}_l(g)| &= \left|(-1)^l\int_c^d \mathcal{K}_l(s)g^{(l)}(s)ds\right| \\
&\le \int_c^d |\mathcal{K}_l(s)|\left|g^{(l)}(s)\right|ds \\
&\le \left\|g^{(l)}\right\|_\infty\cdot\int_c^d|\mathcal{K}_l(s)|ds \\
&= \left\|g^{(l)}\right\|_\infty\cdot\left[\frac{1}{l!}\int_c^{\frac{3c+d}{4}}|s-c|^l ds + \frac{1}{l!}\int_{\frac{3c+d}{4}}^{\frac{c+d}{2}}\left|s-\frac{c+2d}{3}\right|^l ds\right. \\
&\quad \left.+\frac{1}{l!}\int_{\frac{c+d}{2}}^{\frac{c+3d}{4}}\left|s-\frac{2c+d}{3}\right|^l ds + \frac{1}{l!}\int_{\frac{c+3d}{4}}^{d}|s-d|^l ds\right] \\
&\le \frac{2(d-c)^{l+1}}{(l+1)!}\left[\frac{1}{4^{l+1}}+\frac{5^{l+1}}{12^{l+1}}-\frac{1}{6^{l+1}}\right]\cdot\left\|g^{(l)}\right\|_\infty,
\end{aligned}
$$

and this proves the first inequality in (13).

The second inequality in (13) can be obtained since $g^{(l)} \in L^p[a,b]$ $(1 \le p < \infty)$, then

$$\begin{aligned}
|\mathcal{E}_l(g)| &\le \int_c^d |\mathcal{K}_l(s)| \left| g^{(l)}(s) \right| ds \\
&\le \left( \int_c^d |\mathcal{K}_l(s)|^q ds \right)^{\frac{1}{q}} \left( \int_c^d \left| g^{(l)}(s) \right|^p ds \right)^{\frac{1}{p}} \\
&\le \left\| g^{(l)} \right\|_p \cdot \frac{1}{l!} \left[ \int_c^{\frac{3c+d}{4}} (s-c)^{lq} ds + \int_{\frac{3c+d}{4}}^{\frac{c+d}{2}} \left( \frac{c+2d}{3} - s \right)^{lq} ds \right. \\
&\quad \left. + \int_{\frac{c+d}{2}}^{\frac{c+3d}{4}} \left( s - \frac{2c+d}{3} \right)^{lq} ds + \int_{\frac{c+3d}{4}}^{d} (d-s)^{lq} ds \right]^{\frac{1}{q}} \\
&= \frac{2^{1/q}(d-c)^{l+\frac{1}{q}}}{l!(lq+1)^{\frac{1}{q}}} \left[ \frac{1}{4^{lq+1}} + \frac{5^{lq+1}}{12^{lq+1}} - \frac{1}{6^{lq+1}} \right]^{\frac{1}{q}} \left\| g^{(l)} \right\|_p
\end{aligned}$$

which proves the last inequality in (13), and thus the proof is established. □

**Theorem 4.** Let $1 \le p < \infty$. If $g^{(l-1)}$ $(l \ge 1)$ has Lipschitz property with constant $\mathrm{Lip}_M\left(g^{(l-1)}\right)$, then

$$|\mathcal{E}_l(g)| \le \frac{4^{\frac{1}{p}} \mathrm{Lip}_M\left(g^{(l-1)}\right)}{2 \cdot m!} (d-c)^{l+1} \left[ \frac{1}{4^{l+\frac{1}{p}}(lp+1)^{\frac{1}{p}}} + \left( \frac{5^{lp+1}}{12^{lp+1}(lp+1)} - \frac{1}{6^{lp+1}(lp+1)} \right)^{\frac{1}{p}} \right]$$

**Proof.** Applying Lemma 2, by setting $w(s) = \mathcal{K}_l(s)$, then we have by triangle inequality, from (7) we have

$$\begin{aligned}
&\left| (-1)^l \int_c^d \mathcal{K}_l(s) dg^{(l-1)}(s) \right| \\
&\le \frac{1}{l!} \left| \int_c^{\frac{3c+d}{4}} (s-c)^l dg^{(l-1)}(s) \right| + \frac{1}{l!} \left| \int_{\frac{3c+d}{4}}^{\frac{c+d}{2}} \left( s - \frac{c+2d}{3} \right)^l dg^{(l-1)}(s) \right| \\
&\quad + \frac{1}{l!} \left| \int_{\frac{c+d}{2}}^{\frac{c+3d}{4}} \left( s - \frac{2c+d}{3} \right)^l dg^{(l-1)}(s) \right| + \frac{1}{l!} \left| \int_{\frac{c+3d}{4}}^{d} (s-d)^l dg^{(l-1)}(s) \right| \\
&\le \frac{\mathrm{Lip}_M\left(g^{(l-1)}\right)}{l!} \left( \frac{d-c}{4} \right)^{1-\frac{1}{p}} \left\{ \left( \int_c^{\frac{3c+d}{4}} (s-c)^{lp} ds \right)^{\frac{1}{p}} + \frac{1}{l!} \left( \int_{\frac{3c+d}{4}}^{\frac{c+d}{2}} \left| s - \frac{c+2d}{3} \right|^{lp} ds \right)^{\frac{1}{p}} \right. \\
&\quad \left. + \frac{1}{l!} \left( \int_{\frac{c+d}{2}}^{\frac{c+3d}{4}} \left| s - \frac{2c+d}{3} \right|^{lp} ds \right)^{\frac{1}{p}} + (d-c)^{\frac{1}{p}} \frac{1}{l!} \left( \int_{\frac{c+3d}{4}}^{d} (d-s)^{lp} ds \right)^{\frac{1}{p}} \right\} \\
&= \frac{2\,\mathrm{Lip}_M\left(g^{(l-1)}\right)}{l!} \left( \frac{d-c}{4} \right)^{1-\frac{1}{p}} \left\{ \frac{(d-c)^{l+\frac{1}{p}}}{4^{l+\frac{1}{p}}(lp+1)^{\frac{1}{p}}} + \left[ \frac{5^{lp+1}(d-c)^{lp+1}}{12^{lp+1}(lp+1)} - \frac{(d-c)^{lp+1}}{6^{lp+1}(lp+1)} \right]^{\frac{1}{p}} \right\},
\end{aligned}$$

and this proves the desired result. □

## 4. Other estimations involving norms

In this section, we improve some of the previous inequalities, e.g., the first inequality in (13) involving $L^\infty$ can be improved by replacing this assumption by $\mathcal{S}_l \le g^{(l)}(s) \le \mathcal{T}_m$, where $\mathcal{S}_l := \inf_{s \in [c,d]} g^{(l)}(s)$ and $\mathcal{T}_l := \sup_{s \in [c,d]} g^{(l)}(s)$. In this case, $\frac{\mathcal{T}_l - \mathcal{S}_l}{2} \le \left\| g^{(l)} \right\|_\infty$ which means

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the bounds involving $\frac{\mathcal{T}_l-\mathcal{S}_l}{2}$ is better than $\left\|g^{(l)}\right\|_{\infty}$. If $\mathcal{T}_l=-\mathcal{S}_l$ then both assumptions are equivalent.

To see how this is efficient, let us consider the following result(s):

**Theorem 5.** If $g\in\mathfrak{V}_1^{(l)}$, then

$$|\mathcal{E}_{2l-1}(g)|\leq\frac{2(d-c)^{2l}}{(2l)!}\left(\frac{1}{4^{2l}}+\frac{5^{2l}}{12^{2l}}-\frac{1}{6^{2l}}\right)\left\|g^{(2l-1)}-C\right\|_{\infty},\tag{14}$$

*and*

$$\left|\mathcal{E}_{2l}(g)-C\cdot\frac{2(d-c)^{2l+1}}{(2l+1)!}\left(\frac{1}{4^{2l+1}}+\frac{5^{2l+1}}{12^{2l+1}}-\frac{1}{6^{2l+1}}\right)\right|\\ \leq\frac{2(d-c)^{2l+1}}{(2l+1)!}\left(\frac{1}{4^{2l+1}}+\frac{5^{2l+1}}{12^{2l+1}}-\frac{1}{6^{2l+1}}\right)\left\|g^{(2l)}-C\right\|_{\infty},\tag{15}$$

for all $l\geq1$ and any constant $C\in\mathbb{R}$.

**Proof.** Since

$$\int_c^d\mathcal{K}_l(s)ds=\frac{(d-c)^{l+1}}{(l+1)!}\left(1+(-1)^l\right)\left(\frac{1}{4^{l+1}}+\frac{5^{l+1}}{12^{l+1}}-\frac{1}{6^{l+1}}\right)\\ =\begin{cases}0, & \text{if } l=\text{odd}\\ \frac{2(d-c)^{l+1}}{(l+1)!}\left(\frac{1}{4^{l+1}}+\frac{5^{l+1}}{12^{l+1}}-\frac{1}{6^{l+1}}\right), & \text{if } l=\text{even}\end{cases}.\tag{16}$$

Assume $l$ is odd and setting $l=2\nu-1$, $\nu\geq1$. If $C\in\mathbb{R}$ is any constant then from (7) and (16) we get

$$\int_c^d\mathcal{K}_{2\nu-1}(s)\left[g^{(2\nu-1)}(s)-C\right]ds=\mathcal{E}_{2\nu-1}(g).$$

Applying the triangle integral inequality, we get

$$\left|\int_c^d\mathcal{K}_{2\nu-1}(s)\left[g^{(2\nu-1)}(s)-C\right]ds\right|\\ \leq\sup_{s\in[c,d]}\left|g^{(2\nu-1)}(s)-C\right|\cdot\int_c^d|\mathcal{K}_{2\nu-1}(s)|ds\\ =\frac{2(d-c)^{2\nu}}{(2\nu)!}\left(\frac{1}{4^{2l}}+\frac{5^{2\nu}}{12^{2\nu}}-\frac{1}{6^{2\nu}}\right)\cdot\left\|g^{(2\nu-1)}-C\right\|_{\infty}$$

for all $\nu\geq1$, where

$$\int_c^d|\mathcal{K}_{2\nu-1}(s)|ds=\frac{2(d-c)^{2\nu}}{(2\nu)!}\left(\frac{1}{4^{2\nu}}+\frac{5^{2\nu}}{12^{2\nu}}-\frac{1}{6^{2\nu}}\right)$$

which gives the desired result (14). The proof of the second inequality (15) follows similarly by considering $l=2\nu$ $(\nu\geq1)$ and omitting the details. □

Next, we improve the first inequality in (13) in the case that $g$ has odd derivatives.

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**Corollary 1.** Let $g \in \mathfrak{V}_1^{(m)}$. Then there exist constants $\mathcal{S}_m, \mathcal{T}_m > 0$ such that $\mathcal{S}_m \leq g^{(m)}(s) \leq \mathcal{T}_m$ $(m \geq 1)$, $\forall s \in [c,d]$, such that

$$|\mathcal{E}_{2l-1}(g)| \leq \frac{(d-c)^{2l}}{(2l)!}\left(\frac{1}{4^{2l}} + \frac{5^{2l}}{12^{2l}} - \frac{1}{6^{2l}}\right)(\mathcal{T}_{2l-1} - \mathcal{S}_{2l-1}), \tag{17}$$

if $m$ is odd; $m = 2l-1$ $(l \geq 1)$,

$$\left|\mathcal{E}_{2l}(f) - \frac{(d-c)^{2l+1}}{(2l+1)!}\left(\frac{1}{4^{2l+1}} + \frac{5^{2l+1}}{12^{2l+1}} - \frac{1}{6^{2l+1}}\right) \cdot (\mathcal{T}_{2l} - \mathcal{S}_{2l})\right| \leq \frac{(d-c)^{2l+1}}{(2l+1)!}\left(\frac{1}{4^{2l+1}} + \frac{5^{2l+1}}{12^{2l+1}} - \frac{1}{6^{2l+1}}\right) \cdot (\mathcal{T}_{2l} - \mathcal{S}_{2l}), \tag{18}$$

and if $m$ is even; $m = 2\ell$ $(\ell \geq 1)$.

**Proof.** We give the proof when $m$ is odd. In the proof of the Theorem 5, set

$$C = \frac{\mathcal{T}_{2l-1} - \mathcal{S}_{2l-1}}{2}, \quad \forall l \geq 1$$

then

$$\int_c^d \mathcal{K}_{2l-1}(s)\left[g^{(2l-1)}(s) - \frac{\mathcal{T}_{2l-1} - \mathcal{S}_{2l-1}}{2}\right]ds = \mathcal{E}_{2l-1}(g).$$

Taking the modulus and applying the triangle inequality, we have

$$\begin{aligned}
&\left|\int_c^d \mathcal{K}_{2l-1}(s)\left[g^{(2l-1)}(s) - \frac{\mathcal{T}_{2l-1} - \mathcal{S}_{2l-1}}{2}\right]ds\right| \\
&\leq \sup_{s\in[c,d]}\left|g^{(2l-1)}(x) - \frac{\mathcal{T}_{2l-1} - \mathcal{S}_{2l-1}}{2}\right| \cdot \int_c^d |\mathcal{K}_{2l-1}(s)|ds \\
&= \frac{(d-c)^{2l}}{(2l)!}\left(\frac{1}{4^{2l}} + \frac{5^{2l}}{12^{2l}} - \frac{1}{6^{2l}}\right)(\mathcal{T}_{2l-1} - \mathcal{S}_{2L-1})
\end{aligned}$$

which holds for all $l \geq 1$, since

$$\sup_{s\in[c,d]}\left|g^{(2l-1)}(s) - \frac{\mathcal{T}_{2l-1} - \mathcal{S}_{2l-1}}{2}\right| \leq \frac{\mathcal{T}_{2l-1} - \mathcal{S}_{2l-1}}{2}$$

and

$$\int_c^d |\mathcal{K}_{2l-1}(s)|ds = \frac{2(d-c)^{2l}}{(2l)!}\left(\frac{1}{4^{2l}} + \frac{5^{2l}}{12^{2l}} - \frac{1}{6^{2l}}\right),$$

and this proves (17). The inequality (18) holds trivially in a similar fashion. □

**Remark 1.** Clearly, the estimation (14) improves the first estimation in (13) by $\frac{1}{2}$ when $m$ is odd and thus (14) is better than (13).

## 5. More on $L^p$–Bounds

In this section, we introduce more $L^p$–bounds of the perturbed Milne's quadrature rule.

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*5.1. Bounds in $L^2[c,d]$*

**Theorem 6.** If $g^{(2l-1)}$ is absolutely continuous on $I$ and $g^{(2l-1)} \in L^2[c,d]$ $(l \geq 1)$. Then,

$$|\mathcal{E}_{2l-1}(g)| \leq \frac{2^{1/2}(d-c)^{2l-\frac{1}{2}}}{(2l)!(4l-1)^{\frac{1}{2}}}\left[\frac{1}{4^{4l-1}}+\frac{5^{4l-1}}{12^{4l-1}}-\frac{1}{6^{4l-1}}\right]^{\frac{1}{2}} \cdot \sqrt{\tau\left(g^{(2l-1)}\right)}, \tag{19}$$

where

$$\tau\left(g^{(2l-1)}\right) = \left\|g^{(2l-1)}\right\|_2^2 - \frac{\left(g^{(2l-2)}(d)-g^{(2l-2)}(c)\right)^2}{d-c},$$

for all $l \geq 1$.

**Proof.** Using the identity

$$\int_c^d \mathcal{K}_l(s)\left[g^{(l)}(s)-\frac{1}{d-c}\int_c^d g^{(l)}(t)dt\right]ds$$
$$= \int_c^d \mathcal{K}_l(s)g^{(l)}(s)ds - \frac{g^{(l-1)}(d)-g^{(l-1)}(c)}{d-c}\cdot\int_c^d \mathcal{K}_l(s)ds, \tag{20}$$

and since $l$ is odd i.e., $l = 2v-1$, $\forall v \geq 1$, then by (16) we have $\int_c^d \mathcal{K}_{2v-1}(s)ds = 0$, so that (20) reduces to

$$\int_c^d \mathcal{K}_{2v-1}(s)\left[g^{(2v-1)}(s)-\frac{1}{d-c}\int_c^d g^{(2v-1)}(t)dt\right]ds = \int_c^d \mathcal{K}_{2v-1}(s)g^{(2v-1)}(s)ds.$$

Employing the triangle inequality, we get

$$\left|\int_c^d \mathcal{K}_{2v-1}(s)\left[g^{(2v-1)}(s)-\frac{1}{d-c}\int_c^d g^{(2v-1)}(t)dt\right]ds\right|$$
$$\leq \|\mathcal{K}_{2v-1}\|_2\left\|g^{(2v-1)}-\frac{1}{d-c}\int_c^d g^{(2v-1)}(t)dt\right\|_2$$

where,

$$\|\mathcal{K}_{2v-1}\|_2 := \frac{2^{1/2}(d-c)^{2v-\frac{1}{2}}}{(2v)!(4v-1)^{\frac{1}{2}}}\left[\frac{1}{4^{4v-1}}+\frac{5^{4v-1}}{12^{4v-1}}-\frac{1}{6^{4v-1}}\right]^{\frac{1}{2}}$$

and

$$\left\|g^{(2v-1)}-\frac{1}{d-c}\int_c^d g^{(2v-1)}(t)dt\right\|_2 = \left[\left\|g^{(2v-1)}\right\|_2^2 - \frac{\left(g^{(2v-2)}(d)-g^{(2v-2)}(c)\right)^2}{d-c}\right]^{1/2}$$
$$= \sqrt{\tau\left(g^{(2v-1)}\right)},$$

which gives the required result. □

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*5.2. Bounds in $L^p[c,d]$*

**Theorem 7.** If $g^{(2l)}$ is absolutely continuous on $I$ and $g^{(2l)} \in L^p[c,d]$ $(l \geq 1)$. Then,

$$\begin{aligned} &|\mathcal{E}_{2l+1}(g)| \\ &\leq \frac{p\sin\left(\frac{\pi}{p}\right)}{\pi\sqrt[p]{p-1}} \cdot \frac{2^{1/q}(d-c)^{2l+\frac{1}{q}}}{(2l-1)!((2l-1)q+1)^{\frac{1}{q}}} \left[\frac{1}{4^{(2l-1)q+1}} + \frac{5^{(2l-1)q+1}}{12^{(2l-1)q+1}} - \frac{1}{6^{(2l-1)q+1}}\right]^{\frac{1}{q}} \cdot \left\|g^{(2l)}\right\|_p, \end{aligned} \tag{21}$$

for all $p,q$ such that $\frac{1}{p}+\frac{1}{q}=1$, $p>1$.

**Proof.** Repeating the proof of Theorem 6, so that from (20) we can conclude

$$\begin{aligned} &\left|\int_c^d \mathcal{K}_{2l-1}(s)\left[g^{(2l-1)}(x) - \frac{1}{d-c}\int_c^d g^{(2l-1)}(s)ds\right]dx\right| \\ &\qquad\qquad \leq \|\mathcal{K}_{2l-1}\|_q \left\|g^{(2l-1)} - \frac{1}{d-c}\int_c^d g^{(2l-1)}(s)ds\right\|_p. \end{aligned} \tag{22}$$

Now,

$$\begin{aligned} \left\|g^{(2l-1)} - \frac{1}{d-c}\int_c^d g^{(2l-1)}(s)ds\right\|_p^p &= \int_c^d \left|g^{(2l-1)}(x) - \frac{1}{d-c}\int_c^d g^{(2l+1)}(s)ds\right|^p ds \\ &= \int_c^d \left|\frac{1}{d-c}\int_c^d \left[g^{(2l-1)}(x) - g^{(2l-1)}(s)\right]ds\right|^p dx \\ &\leq \frac{1}{d-c}\int_c^d\int_c^d \left|g^{(2l-1)}(x) - g^{(2l-1)}(s)\right|^p ds\,dx \\ &\leq \sup_{s\in[c,d]} \int_c^d \left|g^{(2l-1)}(x) - g^{(2l-1)}(s)\right|^p dx. \end{aligned}$$

Applying [17, Theorem 4] to $g^{(2l-1)}$, then we have

$$\int_c^d \left|g^{(2l-1)}(x) - g^{(2l-1)}(s)\right|^p dx \leq \frac{p^p\sin^p\left(\frac{\pi}{p}\right)}{\pi^p(p-1)}\left[\frac{d-c}{2} + \left|s - \frac{c+d}{2}\right|\right]^p \cdot \int_c^d \left(g^{(2l)}(x)\right)^p dx.$$

Therefore,

$$\sup_{s\in[c,d]} \int_c^d \left|g^{(2l-1)}(x) - g^{(2m-1)}(s)\right|^p dx \leq \frac{p^p\sin^p\left(\frac{\pi}{p}\right)}{\pi^p(p-1)}(d-c)^p \cdot \int_c^d \left(g^{(2l)}(x)\right)^p dx,$$

which gives by (22), that

$$\begin{aligned} &\left|\int_c^d \mathcal{K}_{2l-1}(x)\left[g^{(2l-1)}(x) - \frac{1}{d-c}\int_c^d g^{(2l-1)}(s)ds\right]dx\right| \\ &\leq \|\mathcal{K}_{2l-1}\|_q \left\|g^{(2l-1)} - \frac{1}{d-c}\int_c^d g^{(2l-1)}(s)ds\right\|_p \\ &\leq \frac{p\sin\left(\frac{\pi}{p}\right)}{\pi\sqrt[p]{p-1}} \cdot \frac{2^{1/q}(d-c)^{2l+\frac{1}{q}}}{(2l-1)!((2l-1)q+1)^{\frac{1}{q}}} \left[\frac{1}{4^{(2l-1)q+1}} + \frac{5^{(2l-1)q+1}}{12^{(2l-1)q+1}} - \frac{1}{6^{(2l-1)q+1}}\right]^{\frac{1}{q}} \cdot \left\|g^{(2l)}\right\|_p, \end{aligned}$$

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where,

$$\|\mathcal{K}_{2l-1}\|_q := \frac{2^{1/q}(d-c)^{2l-1+\frac{1}{q}}}{(2l-1)!((2l-1)q+1)^{\frac{1}{q}}}\left[\frac{1}{4^{(2l-1)q+1}}+\frac{5^{(2l-1)q+1}}{12^{(2l-1)q+1}}-\frac{1}{6^{(2l-1)q+1}}\right]^{\frac{1}{q}},$$

and this proves (21). □

**Example 1.** *In the following numerical experiment, we apply our quadrature rule* (13) *for the listed functions on the interval* $[0,1]$.

| $g(t)$ | Milne Rule (2) | $\mathcal{M}_m(g)$ (13) | E. V. | A. E. (13) | A. E. (2) |
|---|---|---|---|---|---|
| $\sqrt{\frac{4+t^7}{5+4t^2}}$ | 0.81286 | 0.81517 | 0.81533 | 0.00015 | 0.00246 |
| $\frac{\sqrt{25+t^7}}{1+2t^2+t^4}$ | 3.25485 | 3.21786 | 3.21744 | 0.00041 | 0.03740 |
| $\frac{144+t^6}{1+2t^2+t^4}$ | 93.68507 | 92.67250 | 92.59549 | 0.07700 | 1.08957 |

E. V. := *The exact value of* $\int_0^1 g(t)dt$.
A. E. (13) := *The absolute error of our proposed quadrature rule* (13) *relative to the exact value.*
A. E. (2) := *The absolute error of the classical Milne's quadrature rule* (2) *relative to the exact value.*

*As we can see the quadrature rule* (13) *gives better approximations than the classical Milne's rule* (2). *Moreover, comparing the absolute error of these quadrature rules relative to the exact value shows that* (13) *is much better than* (2).

## 6. Conclusions

In this work, a perturbed Milne's quadrature formula is established. Namely, we have

$$\int_c^d g(s)ds = \mathcal{M}_m(g) + \mathcal{E}_m(g)$$

for all $m \geq 1$, where $\mathcal{M}_m(g)$ is the perturbed Milne's rule given by

$$\begin{aligned}\mathcal{M}_m(g) := \frac{d-c}{3}&\left[2g\left(\frac{3c+d}{4}\right)-g\left(\frac{c+d}{2}\right)+2g\left(\frac{c+3d}{4}\right)\right]\\&+\sum_{\ell=1}^{m-1}\frac{(d-c)^{\ell+1}}{(\ell+1)!}\left\{\left[\left(\frac{5}{12}\right)^{\ell+1}+\left(\frac{1}{4}\right)^{\ell+1}\right]\left[g^{(\ell)}\left(\frac{3c+d}{4}\right)+(-1)^{\ell}g^{(\ell)}\left(\frac{c+3d}{4}\right)\right]\right.\\&\left.\qquad-\frac{\left((-1)^{\ell}+1\right)}{6^{\ell+1}}g^{(\ell)}\left(\frac{c+d}{2}\right)\right\}\end{aligned}$$

and $\mathcal{E}_m(g)$ is the error term given by

$$\mathcal{E}_m(g) = \int_c^d g(s)ds - \mathcal{M}_m(g) = (-1)^m\int_c^d \mathcal{K}_m(s)g^{(m)}(s)ds$$

for all $m \geq 1$.

Furthermore, several error estimates involving $L^p$-bounds are proved. One of the most important advantages of our result is that it is verified for $p$-variation and Lipschitz functions (non-differentiable functions). Also, since the classical Milne's quadrature rule (2) cannot be applied either when the fourth derivative is unbounded or doesn't exist, therefore the proposed quadrature (13) could be used alternatively. For example, the second, third, and fourth derivatives of the function $g(t) = t^3\sin\left(\frac{1}{t}\right)$, $t \in [0,1]$, don't exist. While $g$ is continuous and differentiable on $[0,1]$, and $g'(0) = 0$ and the exact value of $\int_0^1 g(s)ds = 0.22384$. However, by applying the above formula for $m = 1$, we get

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$\int_0^1 g(s)ds \approx 0.22758$, with absolute error $= 0.00373$. Keeping in mind, that we didn't use derivatives in our formula; i.e., since $m = 1$ then $\sum_{\ell=1}^{0}$ is conventionally $= 0$. This is a very powerful indication that ensures that our result is better (in some cases) than (2).

Finally, it is convenient to note that other $L^p$-error estimates have been established. This will be very useful in case of the fourth derivative is unbounded in $L^\infty$-norm. However, it could be possible to use other $L^p$-norms and this gives more advantage and strength to our other obtained results involving $L^p$-norms. Indeed, our approaches cover both cases of differentiable and non-differentiable functions, for example, (12) is a good example of this assertion.

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