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Article

The Convergence of Discrete Models for Nonlinear Schrödinger Equation in Dark Solitons Motion

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Abstract: We firstly present two popular space discretization models of the Nonlinear Schrödinger Equation in dark solitons motion : the Direct-Discrete model and the Ablowitz-Ladik model. Applying the midpoint scheme to the space discretization models, we get two time-space discretization models: the Crank-Nicolson method and the New-Difference method. Secondly, we demonstrate that the solutions of the two space discretization models converge to the solution of the Nonlinear Schrödinger Equation. Also, we prove that the convergence order of the two time-space discretization models are $O(h^2 + \tau^2)$ in discrete L_2-norm error estimates. Finally, the numerical experiments agree well with the proven theoretical results.

Keywords: Nonlinear schrödinger equation; Space discretization models; Time-space discretization models; Crank-Nicolson method; New-Difference method.

1. Introduction

The Nonlinear Schrödinger Equation (NLSE) is one of the most widely used and completely integrable models in nonlinear physics. It plays a crucial role in many physical fields [1–3], such as nonlinear optics, solid state physics, quantum mechanics, optical fiber communication, etc. Therefore, the study of such equations has a profound influence on the development of modern science.

Consider the original NLSE with the initial condition :

$$\begin{cases} iw_t + w_{xx} + a|w|^2 w = 0, \\ w(x,0) = w_0(x), \end{cases} \tag{1}$$

where a is a real constant, $w(x,t)$ is a complex-valued function; $t \in [0,\infty)$, $x \in R$. NLSE is a class of nonlinear partial differential equations, which produces a special form of solution – soliton solution. When $a > 0$ and $|w_0(\infty)| = 0$, NLSE has bright solitons solution[4]; when $a < 0$ and $|w_0(\infty)| = \rho$, NLSE has dark solitons solution [5,6]. The original NLSE has infinite conserved quantities, including:

$$Q = \int_{-\infty}^{+\infty} \left(|w|^2 - \rho^2\right) dx, \qquad P = \int_{-\infty}^{+\infty} \{w\bar{w}_x - \bar{w}w_x\}\, dx$$

where Q, P are *charge* and *momentum*, respectively. Utilizing central difference, we can approximate the conserved quantities Q, P as follows:

$$S_1 = h\sum_j (w_j\bar{w}_j - \rho^2), \qquad S_2 = \sum_j (w_j\bar{w}_{j+1} - w_{j+1}\bar{w}_j)$$

Zakharov and Shabat et al. obtained the exact solution of the original NLSE (1) by using the inverse scattering transformation method [6]. Here, we need to note that the above equation is idealized. But, the actual physical systems has to consider the influence of dissipation and other conditions, making it difficult to get an analytical solution. Consequently, many numerical methods have been found to simulate such equations and study the properties of NLSE according to numerical results[7–14], such as the finite difference, the finite element, or the polynomial approximation.

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As is well known, the solitons for the original NLSE maintain their original state after collision with each other. Based on the above special properties, many people have devoted themselves to studying conservative schemes for simulation. Zhu You-lan considered an implicit scheme and gave its convergence [15]. Guo Ben-yu[16] gave the convergence of the Crank-Nicolson method and the prediction-correction method under the error estimations. In [17–21], compact finite difference schemes are proved to be convergent both in the discrete L_2-norm , and in discrete L_∞-norm . For the important space discretization models of NLSE, the Direct-Discrete model (D-D model) and the Ablowitz-Ladik model (A-L model) can be transformed into the Hamiltonian form, respectively. In [22,23], Tang et al. used the symplectic methods to simulate Hamiltonian system and proved that the solution of the D-D model and the A-L model converge to the original NLSE .

The previous proofs of convergence were almostly about bright solitons motion. Given the different parameters and conditions, it is difficult to directly apply the above convergence to dark solitons motion. There is only a little literature about proving the convergence of dark solitons motion $(a < 0, |W_0(\infty)| = \rho)$. Hence, we give proof of convergence for the space discretization models of the original NLSE in dark solitons motion, which provide theoretical support for numerical simulation. The Crank-Nicolson method is actually obtained by applying the midpoint scheme in time to solve the D-D model. Similarly, we will apply the midpoint scheme to the A-L model , then propose a new difference scheme (called as New-Difference method) of the original NLSE . We will show that the New-Difference method in the dark solitons motion is convergent and of high accuracy by numerical experiment.

This paper is organized as follows. In Section 2, we present the space discretization models and the time-space discretization models for the original NLSE in dark solitons motion, and give some conservation invariants of these models. We confirm the convergence of the space discretization models and the time-space discretization models in Section 3 and Section 4, respectively. In Section 5, we obtain the error order of the space discretization models and the time-space discretization models to test the convergence. In order to further demonstrate the convergence of these models, we get the numerical solutions of these models and check the preservation of the invariants. Finally, we give some conclusions in Section 6.

2. Different Discretization Models

In this section, we present the space discretization models and the time-space discretization models for the original NLSE. The Direct-Discrete model and the Ablowitz-Ladik model discretize the original NLSE in space, while the Crank-Nicolson method and the New-Difference method discretize in time and space simultaneously.

2.1. The space discretization models

Give two classical models of space discretization:
(1) Direct-Discrete model (D-D model) :

$$\begin{cases} i\dfrac{dW^{(l)}}{dt} + \dfrac{W^{(l+1)} - 2W^{(l)} + W^{(l-1)}}{h^2} + a|W^{(l)}|^2 W^{(l)} = 0, \\ W^{(l)}(0) = W_0(lh), \end{cases} \tag{2}$$

By setting $W^{(l)} = p^{(l)} + iq^{(l)}$, D-D model can be directly rewritten as a Hamiltonian system, and has two invariants, that is, the energy and the charge:

$$\widetilde{Q} = \frac{1}{2}\sum_l \left[|W^{(l)}|^2 - \rho^2\right] = Q_1,$$

$$\widetilde{E} = \frac{1}{2h^2}\sum_l [p^{(l)}(p^{(l+1)} - 2p^{(l)} + p^{(l-1)}) + q^{(l)}(q^{(l+1)} - 2q^{(l)}$$

$$+q^{(l-1)}] + \frac{a}{4}\sum_l \left[\left(p^{(l)}\right)^2 + \left(q^{(l)}\right)^2 - \rho^2\right] = E_1$$

(2) Ablowitz-Ladik model (A-L model):

$$\begin{cases} i\dfrac{dW^{(l)}}{dt} + \dfrac{W^{(l+1)} - 2W^{(l)} + W^{(l-1)}}{h^2} + \dfrac{a}{2}|W^{(l)}|^2(W^{(l+1)} + W^{(l-1)}) = 0, \\ W^{(l)}(0) = W_0(lh) \end{cases} \tag{3}$$

where h is the space step-size and $W^{(l)}(t) = W(lh,t)$ for $l = \cdots, -1, 0, 1, \cdots$. A-L model has infinite invariants [6], and the first two invariants are [24]:

$$F_1 = \sum_l W^{(l+1)}\bar{W}^{(l)}$$

$$F_2 = \frac{-ah^2}{4}\sum_l (W^{(l+1)})^2(\bar{W}^{(l)})^2 + 2\sum_l W^{(l+1)}\bar{W}^{(l-1)}U^{(l)}$$

where $U^{(l)} = 1 - \frac{ah^2}{2}|W^{(l)}|^2$. The above models can be converted to Hamiltonian system, then simulated by symplectic method [22,24].

2.2. The time-space discretization models

Appling the midpoint scheme to the D-D model and the A-L model in time, we get the following two models: the Crank-Nicolson method and the New-Difference method. Before introducing the two models, we give some definitions: the time step-size and space step-size of these models are τ, h respectively, and $x_j = jh(j = \cdots -1, 0, 1, \cdots), t_n = n\tau(n = 0, 1 \cdots N)$.

We write the exact solution of the original NLSE as $w_j^n = w(x_j, t_n)$, the numerical solution as $W_j^n = W(x_j, t_n)$, and define:

$$\delta_t V_j^n = \frac{V_j^{n+1} - V_j^n}{\tau}, \quad \delta_x V_j^n = \frac{V_{j+1}^n - V_j^n}{h}, \quad \delta_{\bar{x}} V_j^n = \frac{V_j^n - V_{j-1}^n}{h},$$

$$V_j^{n+\frac{1}{2}} = \frac{V_j^{n+1} + V_j^n}{2}, \quad \delta_x^2 V_j^n = \delta_x \delta_{\bar{x}} V_j^n = \frac{1}{h^2}(V_{j+1}^n - 2V_j^n + V_{j-1}^n)$$

Let define that:

$$(\mathbf{U}^n, \mathbf{V}^n) = h\sum_j U_j^n \bar{V}_j^n, \quad ||\mathbf{V}^n||_{L_2}^2 = (V^n, V^n), \quad ||\mathbf{V}^n||_{L_\infty} = \max_j |V_j^n|,$$
$$j = \cdots -1, 0, 1, \cdots$$

Then, the two difference schemes for the original NLSE are as follow:
(1) Crank-Nicolson method

$$\begin{cases} i\delta_t W_j^n + \dfrac{1}{2}\delta_x^2(W_j^{n+1} + W_j^n) + \dfrac{a}{8}|W_j^{n+1} + W_j^n|^2(W_j^{n+1} + W_j^n)) = 0 \\ j = \cdots -1, 0, 1, \cdots, n = 0, 1 \cdots N \\ W_j^0 = W_0(x_j) \end{cases} \tag{4}$$

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(2) New-Difference method

$$\begin{cases} i\delta_t W_j^n + \frac{1}{2}\delta_x^2(W_j^{n+1} + W_j^n) + \frac{a}{2}|W_j^{n+\frac{1}{2}}|^2(W_{j+1}^{n+\frac{1}{2}} + W_{j-1}^{n+\frac{1}{2}}) = 0 \\ j = \cdots - 1, 0, 1, \cdots, n = 0, 1 \cdots N \\ W_j^0 = W_0(x_j) \end{cases} \tag{5}$$

Note that $\mathbf{W}^n = (\cdots, W_{-1}^n, W_0^n, W_1^n, \cdots)^T$, $|\mathbf{W}^n|^2 = diag(\cdots, |W_{-1}^n|^2, |W_0^n|^2, |W_1^n|^2, \cdots)$, then equation (5) can be rewritten as:

$$i\delta_t \mathbf{W}^n + \frac{1}{2}\delta_x^2(\mathbf{W}^{n+1} + \mathbf{W}^n) + \frac{a}{2}|\mathbf{W}^{n+\frac{1}{2}}|^2 \mathbf{M}\mathbf{W}^{n+\frac{1}{2}} = 0, n = 1 \cdots N \tag{6}$$

where:

$$\mathbf{M} = \begin{pmatrix} \ddots & \ddots & & & & & \\ \ddots & 0 & 1 & & & & \\ & 1 & 0 & 1 & & & \\ & & \cdots & \cdots & \cdots & & \\ & & & 1 & 0 & 1 & \\ & & & & 1 & 0 & \ddots \\ & & & & & \ddots & \ddots \end{pmatrix} \tag{7}$$

In numerical experiments section, in order to test the convergence of the numerical solutions of the above models , we will give the preservation of the conserved quantities'approximation given in Section 1.

3. The Convergence of the Space Discretization Models

In this section, we give the proof of convergence for the two space discretization models in dark solitons motion. Suppose that local item $M_l(t) = M(lh, t)(l = \cdots, -1, 0, 1, \cdots)$,

$$M_l(t) = i\frac{dw^{(l)}}{dt} + \frac{w^{(l+1)} - 2w^{(l)} + w^{(l-1)}}{h^2} + a|w^{(l)}|^2 w^{(l)} \tag{8}$$

Lemma 1. *Suppose that $w(x,t)$ is the solution of the original NLSE, the local item $M_l(t)(l = \cdots, -1, 0, 1, \cdots)$ of D-D model is $O(h^2)$.*

Proof of Lemma 1. $w^{(l)}(t)$ satisfies the original NLSE, so

$$iw_t^{(l)} + w_{xx}^{(l)} + a|w^{(l)}|^2 w^{(l)} = 0$$

Substituting into equation (8), we obtain that

$$M_l(t) = h^2\Big(\frac{w^{(l+1)} - 2w^{(l)} + w^{(l-1)}}{h^4} - \frac{w_{xx}^{(l)}}{h^2}\Big) = h^2 B_l(t)$$

According to Taylor's expansion, $w^{(l+1)} - 2w^{(l)} + w^{(l-1)} = h^2 w_{xx}^{(l)} + \frac{h^4}{12} w_{xxxx}^{(l)} + O(h^6)$, and

$$B_l = \frac{w_{xx}^{(l)} + \frac{h^2}{12} w_{xxxx}^{(l)} + O(h^4)}{h^2} - \frac{w_{xx}^{(l)}}{h^2} = \frac{w_{xxxx}^{(l)}}{12} + O(h^2) \tag{9}$$

So, the local item $M_l(t)$ is of order $O(h^2)$. □

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Theorem 1. *Assume that $W_0(x)$ is the initial condition of D-D model $(a<0)$, and all derivative of the initial condition with respect to x satisfies:*
(1) $W_0(-\infty)$ and $W_0(+\infty)$ exist, and $|W_0(\pm\infty)|=\rho$,
(2) $\int_{-\infty}^{+\infty}||W_0(x)|^2-\rho^2|dx<+\infty$ and $\int_{-\infty}^{+\infty}|\frac{\partial^k}{\partial x^k}W_0(x)|^2dx<+\infty$
when $h\to 0$, the solution of D-D model converges to the solution of the original NLSE $(a<0)$.

Proof of Theorem 1. Suppose that $w(x,t)$ is the solution of the original NLSE, $W^{(l)}(t)=W(lh,t)$ is the solution of D-D model, and $w^{(l)}(t)=w(lh,t)$.

Subtracting equation (2) from equation (8), we get

$$\begin{cases} i\frac{d}{dt}(w^{(l)}-W^{(l)})+\frac{w^{(l+1)}-W^{(l+1)}-2(w^{(l)}-W^{(l)})+w^{(l-1)}-W^{(l-1)}}{h^2} \\ +a|w^{(l)}|^2w^{(l)}-a|W^{(l)}|^2W^{(l)}=M_l \end{cases} \tag{10}$$

Let error term $\varepsilon_l=w^{(l)}-W^{(l)}(l=\cdots,-1,0,1,\cdots)$, then

$$i\frac{d\varepsilon_l}{dt}+\frac{\varepsilon_{l+1}-2\varepsilon_l+\varepsilon_{l-1}}{h^2}+a[(|w^{(l)}|^2+|W^{(l)}|^2)\varepsilon_l+w^{(l)}W^{(l)}\bar{\varepsilon}_l]=M_l \tag{11}$$

Multiplying equation (11) by $\bar{\varepsilon}_l$ (the complex conjugate of ε_l) summing up for all l, then taking the equations' imaginary part, we can obtain that

$$Im\Big[i\sum_l\bar{\varepsilon}_l(\frac{d\varepsilon_l}{dt})\Big]+Im\Big[\sum_l\bar{\varepsilon}_l\cdot\frac{\varepsilon_{l+1}-2\varepsilon_l+\varepsilon_{l-1}}{h^2}\Big]+aIm\Big[\sum_l(|w^{(l)}|^2+|W^{(l)}|^2)\varepsilon_l\bar{\varepsilon}_l$$
$$+\sum_l w^{(l)}W^{(l)}\bar{\varepsilon}_l^2\Big]=Im\Big[\sum_l M_l\bar{\varepsilon}_l\Big]$$

Simplify above equation, then

$$\frac{1}{2}\frac{d}{dt}\Big(\sum_l|\varepsilon_l|^2\Big)=Im\Big(\sum_l M_l\bar{\varepsilon}_l\Big)-aIm\sum_l w^{(l)}W^{(l)}\bar{\varepsilon}_l^2 \tag{12}$$

Scaling equation (12):
(1)$Im\Big(\sum_l M_l\bar{\varepsilon}_l\Big)\le\sum_l|M_l||\bar{\varepsilon}_l|\le\frac{1}{2}\Big(\sum_l h^4|B_l|^2+\sum_l|\varepsilon_l|^2\Big)$.
(2)Suppose that $||w(t)||_\infty=\max_l|w^{(l)}(t)|<C$, $||W(t)||_\infty=\max_l|W^{(l)}(t)|<C$, we get $Im\sum_l w^{(l)}W^{(l)}\bar{\varepsilon}_l^2\le C^2\sum_l|\varepsilon_l|^2$.
Then, we have(C,C_1 are constants)

$$\frac{d}{dt}\Big(\sum_l|\varepsilon_l|^2\Big)\le C_1^2h^3+(1-2aC^2)\sum_l|\varepsilon_l|^2 \tag{13}$$

Multiplying both sides of inequality (13) by space step-size $h>0$, and defining $||.||$ as $||\varepsilon(t)||^2=(\varepsilon(t),\varepsilon(t))=h\sum_l\varepsilon_l(t)\cdot\bar{\eta}_l(t)$, then, we get

$$\frac{d}{dt}(||\varepsilon||^2)\le C_1^2h^4+(1-2aC^2)||\varepsilon||^2 \tag{14}$$

We can obtain that

$$||\varepsilon(T)||^2\le\frac{h^4C_1^2}{1-2aC^2}exp\left\{T(1-2aC^2)\right\}\ (a<0) \tag{15}$$

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where $0 \leq t \leq T$. So, given a simulation time T, the solution of D-D model converges to the solution of original NLSE when $h \rightarrow 0$. □

Remark: Instead of using condition $||W||^2 < C$ to prove convergence in bright solitons motion in [22], we use condition $||W||_\infty < C$ to prove the above conclusion in dark solitons motion.

Theorem 2. *Suppose that $w(x,t)$ is the solution of the original NLSE in dark solitons motion $(a < 0$ and $|w_0(\infty)| = \rho)$, $W^{(l)}(t) = W(lh,t)$ is the solution of A-L model and $w^{(l)}(t) = w(lh,t)$. One can get that*

$$||\varepsilon(T)||^2 \leq exp(CT)Dh^4. \tag{16}$$

Therefore, given a simulation time T, the solution of A-L model converges to the solution of the original NLSE $(h \rightarrow 0)$.

Proof of Theorem 2. Through the similar method in [23], we can deduce

$$||\varepsilon(T)||^2 = ||w(T) - W(T)||^2 \leq exp(CT)Dh^4. \tag{17}$$

Then, the above conclusion can be obtained. □

4. The Convergence of the Time-space Discretization Model

In this section, we give the proof of convergence for the time-space discretization models in dark solitons motion. Let the truncation error be φ_j^n, then:

$$\begin{gathered} i\delta_t w_j^n + \frac{1}{2}\delta_x^2(w_j^{n+1} + w_j^n) + \frac{a}{2}|w_j^{n+\frac{1}{2}}|^2(w_{j+1}^{n+\frac{1}{2}} + w_{j-1}^{n+\frac{1}{2}}) = \varphi_j^n, \\ j = \cdots - 1, 0, 1, \cdots, n = 1 \cdots N \end{gathered} \tag{18}$$

Lemma 2. *Set U^n, V^n, the following equalities hold:*
(1)$(\delta^2 U^n, V^n) = -(\delta U^n, \delta V^n)$ [20] ;
(2)$(\boldsymbol{M\varepsilon}^{n+\frac{1}{2}}, \boldsymbol{\varepsilon}^{n+\frac{1}{2}}) - (\boldsymbol{M\bar{\varepsilon}}^{n+\frac{1}{2}}, \boldsymbol{\bar{\varepsilon}}^{n+\frac{1}{2}}) = 0$;

Proof of Lemma 2.

$$\begin{aligned} &(\mathbf{M}\varepsilon^{n+\frac{1}{2}}, \varepsilon^{n+\frac{1}{2}}) - (\mathbf{M}\bar{\varepsilon}^{n+\frac{1}{2}}, \bar{\varepsilon}^{n+\frac{1}{2}}) \\ &= \sum_j (\varepsilon_{j+1}^{n+\frac{1}{2}} + \varepsilon_{j-1}^{n+\frac{1}{2}})\bar{\varepsilon}_j^{n+\frac{1}{2}} - \sum_j (\bar{\varepsilon}_{j+1}^{n+\frac{1}{2}} + \bar{\varepsilon}_{j-1}^{n+\frac{1}{2}})\varepsilon_j^{n+\frac{1}{2}} \\ &= \sum_j (\varepsilon_{j+1}^{n+\frac{1}{2}} + \varepsilon_{j-1}^{n+\frac{1}{2}})\bar{\varepsilon}_j^{n+\frac{1}{2}} - \sum_j (\bar{\varepsilon}_j^{n+\frac{1}{2}}\varepsilon_{j-1}^{n+\frac{1}{2}} + \bar{\varepsilon}_j^{n+\frac{1}{2}}\varepsilon_{j+1}^{n+\frac{1}{2}}) \\ &= 0 \end{aligned}$$

□

Lemma 3. *The convergence order of the truncation error $||\boldsymbol{\varphi}^n||_{L_2}^2$ is $O(h^4 + \tau^4)$.*

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Proof of Lemma 3. For equation (18), according to Taylor's expansion:

$$\begin{aligned}
&i\delta_t w_j^n = i[\frac{\partial}{\partial t}w_j^n + \frac{\tau}{2!}\frac{\partial^2}{\partial t^2}w_j^n + O(\tau^2)];\\
&\frac{1}{2}\delta_x^2(w_j^{n+1}+w_j^n) = \frac{\partial^2}{\partial x^2}w_j^n + \frac{1}{2}\tau\frac{\partial^3}{\partial t\partial x^2}w_j^n + \frac{1}{12}h^2\frac{\partial^4}{\partial x^4}w_j^n + \frac{1}{4}\tau^2\frac{\partial^4}{\partial t^2\partial x^2}w_j^n\\
&\qquad + O(h^2\tau) + O(\tau^3);\\
&|w_j^{n+\frac{1}{2}}|^2 = |w_j^n|^2 + \tau Im(w_j^n)\frac{\partial}{\partial t}Im(w_j^n) + \tau Re(w_j^n)\frac{\partial}{\partial t}Re(w_j^n) + O(\tau^2);\\
&(w_{j+1}^{n+\frac{1}{2}} + w_{j-1}^{n+\frac{1}{2}}) = 2w_j^n + \tau\frac{\partial}{\partial t}w_j^n + h^2\frac{\partial^2}{\partial x^2}w_j^n + \frac{1}{2}\tau^2\frac{\partial^2}{\partial t^2}w_j^n + O(h^2\tau) + O(\tau^3);
\end{aligned}$$

we get,

$$\begin{aligned}
\varphi_j^n = {}& i[\frac{\partial}{\partial t}w_j^n + \frac{\tau}{2!}\frac{\partial^2}{\partial t^2}w_j^n + O(\tau^2)] + [\frac{\partial^2}{\partial x^2}w_j^n + \frac{1}{2}\tau\frac{\partial^3}{\partial t\partial x^2}w_j^n + \frac{1}{12}h^2\frac{\partial^4}{\partial x^4}w_j^n + \frac{1}{4}\cdot\\
&\tau^2\frac{\partial^4}{\partial t^2\partial x^2}w_j^n + O(h^2\tau) + O(\tau^3)] + \frac{a}{2}[|w_j^n|^2 + \tau Im(w_j^n)\frac{\partial}{\partial t}Im(w_j^n) + \tau Re(w_j^n)\\
&\frac{\partial}{\partial t}Re(w_j^n) + O(\tau^2)][2w_j^n + \tau\frac{\partial}{\partial t}w_j^n h^2\frac{\partial^2}{\partial x^2}w_j^n + \frac{1}{2}\tau^2\frac{\partial^2}{\partial t^2}w_j^n + O(h^2\tau) + O(\tau^3)]
\end{aligned} \tag{19}$$

From the original NLSE, we can obtain:

$$\begin{aligned}
&i\frac{\partial}{\partial t}w_j^n + \frac{\partial^2}{\partial x^2}w_j^n + a|w_j^n|^2 w_j^n = 0;\\
&i\frac{\partial^2}{\partial t^2}w_j^n + \frac{\partial^3}{\partial t\partial x^2}w_j^n + a\frac{\partial}{\partial t}(w_j^n|w_j^n|^2) = 0;
\end{aligned}$$

Substituting them into equation (19), we get that φ_j^n is of order $O(h^2+\tau^2)$ or $||\boldsymbol{\varphi}^n||_{L_2}^2$ is of order $O(h^4+\tau^4)$. □

Lemma 4. *(Gronwall's inequality [25]) Suppose that $\{e_j\}_{j=0}^{\infty}$ is a sequence of nonnegative real numbers satisfying*

$$e_{n+1} \le \alpha + \beta\sum_{j=0}^{n} e_j\tau, \quad n \ge 0 \tag{20}$$

where $\alpha \ge, \beta$ and τ are positive constants. We then have the inequality

$$e_{n+1} \le (\alpha + \tau\beta e_0)e^{\beta(n+1)\tau} \tag{21}$$

Theorem 3. *Suppose that w_j^n is the solution of the original NLSE in dark solitons motion ($a < 0$ and $|w_0(\infty)| = \rho$), W_j^n is the solution of the Crank-Nicolson method. If the time step τ is sufficiently small, we can get*

$$||\boldsymbol{\varepsilon}^n||_{L_2}^2 \le O(h^4+\tau^4) \tag{22}$$

Then the Crank-Nicolson method is of order $O(h^2+\tau^2)$ in discrete L_2-norm error estimates.

Proof of Theorem 3. Using the similar method in [16] , We can prove

$$||\boldsymbol{\varepsilon}^n||_{L_2}^2 \le O(h^4+\tau^4)$$

Then this theorem holds. □

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Theorem 4. *Suppose that w_j^n is the solution of the original NLSE in dark solitons motion ($a < 0$ and $|w_0(\infty)| = \rho$), W_j^n is the solution of the New-Difference method. If τ is sufficiently small, we can get*

$$||\varepsilon^n||_{L_2}^2 \leq O(h^4 + \tau^4) \tag{23}$$

so the New-Difference method's convergence order is $O(h^2 + \tau^2)$ in discrete L_2-norm.

Proof of Theorem 4. Let $\varepsilon^n = \mathbf{w}^n - \mathbf{W}^n$, and $\mathbf{w}^n$ satisfy:

$$\boldsymbol{\varphi}^n = i\delta_t \mathbf{w}_j^n + \frac{1}{2}\delta_x^2(\mathbf{w}_j^{n+1} + \mathbf{w}_j^n) + \frac{a}{2}|\mathbf{w}^{n+\frac{1}{2}}|^2\mathbf{M}\mathbf{w}^{n+\frac{1}{2}}, n = 1\cdots N \tag{24}$$

Subtracting equation (6) from equation (24), we obtain ($n = 1\cdots N$):

$$\boldsymbol{\varphi}^n = i\delta_t \boldsymbol{\varepsilon}^n + \frac{1}{2}\delta_x^2(\boldsymbol{\varepsilon}^{n+1} + \boldsymbol{\varepsilon}^n) + \frac{a}{2}[|\mathbf{w}^{n+\frac{1}{2}}|^2\mathbf{M}\mathbf{w}^{n+\frac{1}{2}} - |\mathbf{W}^{n+\frac{1}{2}}|^2\mathbf{M}\mathbf{W}^{n+\frac{1}{2}}] \tag{25}$$

Taking the inner product of equation (25) with $\boldsymbol{\varepsilon}^{n+\frac{1}{2}}$, and taking the inner product of equation (25)'s conjugate with $\bar{\boldsymbol{\varepsilon}}^{n+\frac{1}{2}}$, then subtracting the obtained two equations, we obtain

$$(\boldsymbol{\varphi}^n, \boldsymbol{\varepsilon}^{n+\frac{1}{2}}) - (\bar{\boldsymbol{\varphi}}^n, \bar{\boldsymbol{\varepsilon}}^{n+\frac{1}{2}}) = II_1 + II_2 + II_3$$

where

$$\begin{aligned}
II_1 =&(i\delta_t\boldsymbol{\varepsilon}^n, \boldsymbol{\varepsilon}^{n+\frac{1}{2}}) - (-i\delta_t\bar{\boldsymbol{\varepsilon}}^n, \bar{\boldsymbol{\varepsilon}}^{n+\frac{1}{2}})\\
=&ih\sum\frac{|\varepsilon_j^{n+1}|^2 - |\varepsilon_j^n|^2}{\tau} = i\frac{||\boldsymbol{\varepsilon}^{n+1}||_{L_2}^2 - ||\boldsymbol{\varepsilon}^n||_{L_2}^2}{\tau}\\
II_2 =&\frac{1}{2}[(\delta_x^2(\boldsymbol{\varepsilon}^{n+1} + \boldsymbol{\varepsilon}^n), \boldsymbol{\varepsilon}^{n+\frac{1}{2}}) - (\delta_x^2(\bar{\boldsymbol{\varepsilon}}^{n+1} + \bar{\boldsymbol{\varepsilon}}^n), \bar{\boldsymbol{\varepsilon}}^{n+\frac{1}{2}})] = 0(\text{ From Lemma 2})\\
II_3 =&\frac{a}{2}[(|\mathbf{w}^{n+\frac{1}{2}}|^2\mathbf{M}\mathbf{w}^{n+\frac{1}{2}}, \boldsymbol{\varepsilon}^{n+\frac{1}{2}}) - (|\mathbf{w}^{n+\frac{1}{2}}|^2\mathbf{M}\,\bar{\mathbf{w}}^{n+\frac{1}{2}}, \bar{\boldsymbol{\varepsilon}}^{n+\frac{1}{2}}) - (|\mathbf{W}^{n+\frac{1}{2}}|^2\mathbf{M}\mathbf{W}^{n+\frac{1}{2}},\\
&\boldsymbol{\varepsilon}^{n+\frac{1}{2}}) - (|\mathbf{W}^{n+\frac{1}{2}}|^2\mathbf{M}\bar{\mathbf{W}}^{n+\frac{1}{2}}, \bar{\boldsymbol{\varepsilon}}^{n+\frac{1}{2}})(\text{where } \boldsymbol{\varepsilon}^n = \mathbf{w}^n - \mathbf{W}^n)]\\
=&\frac{a}{2}[((|\mathbf{w}^{n+\frac{1}{2}}|^2 - |\mathbf{W}^{n+\frac{1}{2}}|^2)\mathbf{M}\mathbf{w}^{n+\frac{1}{2}}, \boldsymbol{\varepsilon}^{n+\frac{1}{2}}) - ((|\mathbf{w}^{n+\frac{1}{2}}|^2 - |\mathbf{W}^{n+\frac{1}{2}}|^2)\mathbf{M}\,\bar{\mathbf{w}}^{n+\frac{1}{2}},\\
&\bar{\boldsymbol{\varepsilon}}^{n+\frac{1}{2}}) + (|\mathbf{W}^{n+\frac{1}{2}}|^2\mathbf{M}\boldsymbol{\varepsilon}^{n+\frac{1}{2}}, \boldsymbol{\varepsilon}^{n+\frac{1}{2}}) - (|\mathbf{W}^{n+\frac{1}{2}}|^2\mathbf{M}\bar{\boldsymbol{\varepsilon}}^{n+\frac{1}{2}}, \bar{\boldsymbol{\varepsilon}}^{n+\frac{1}{2}})]\\
=&aiIm((|\mathbf{w}^{n+\frac{1}{2}}|^2 - |\mathbf{W}^{n+\frac{1}{2}}|^2)\mathbf{M}\mathbf{w}^{n+\frac{1}{2}}, \boldsymbol{\varepsilon}^{n+\frac{1}{2}})
\end{aligned}$$

According to $(\boldsymbol{\varphi}^n, \boldsymbol{\varepsilon}^{n+\frac{1}{2}}) - (\bar{\boldsymbol{\varphi}}^n, \bar{\boldsymbol{\varepsilon}}^{n+\frac{1}{2}}) = 2iIm(\boldsymbol{\varphi}^n, \boldsymbol{\varepsilon}^{n+\frac{1}{2}})$, it follows that:

$$\begin{aligned}
\frac{||\boldsymbol{\varepsilon}^{n+1}||_{L_2}^2 - ||\boldsymbol{\varepsilon}^n||_{L_2}^2}{\tau} =& 2Im(\boldsymbol{\varphi}^n, \boldsymbol{\varepsilon}^{n+\frac{1}{2}})\\
&- aIm((|\mathbf{w}^{n+\frac{1}{2}}|^2 - |\mathbf{W}^{n+\frac{1}{2}}|^2)\mathbf{M}\mathbf{w}^{n+\frac{1}{2}}, \boldsymbol{\varepsilon}^{n+\frac{1}{2}})
\end{aligned} \tag{26}$$

For the first term in the right side of Equation (26), using the Cauchy-Schwardz inequality , we get

$$2Im(\boldsymbol{\varphi}^n, \boldsymbol{\varepsilon}^{n+\frac{1}{2}}) \leq ||\boldsymbol{\varphi}^n||_{L_2}^2 + \frac{1}{2}(||\boldsymbol{\varepsilon}^{n+1}||_{L_2}^2 + ||\boldsymbol{\varepsilon}^n||_{L_2}^2) \tag{27}$$

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For the second term in the right side of Equation (26), we assume that there's a constant C, making the exact solution of the original NLSE to meet:

$$||\mathbf{w}^n||_{L_\infty} \leq C, 0 \leq n \leq N \tag{28}$$

We can get that

$$\begin{aligned}
&Im((|\mathbf{w}^{n+\frac{1}{2}}|^2 - |\mathbf{W}^{n+\frac{1}{2}}|^2)\mathbf{M}\mathbf{w}^{n+\frac{1}{2}}, \boldsymbol{\varepsilon}^{n+\frac{1}{2}}) \\
&= \frac{1}{16}h\sum_{j=1}^{J}[2Re(w_j^{n+1} + w_j^n)(\bar{\varepsilon}_j^{n+1} + \bar{\varepsilon}_j^n) - |\varepsilon_j^{n+1} + \varepsilon_j^n|^2]Im(w_{j+1}^{n+1} + w_{j+1}^n \\
&+ w_{j-1}^{n+1} + w_{j-1}^n)(\varepsilon_j^{n+1} + \varepsilon_j^n) \leq 2h\sum_{j=1}^{J} C_0^2(|\varepsilon_j^{n+1}|^2 + |\varepsilon_j^n|^2) \\
&\leq 2C_0^2(||\boldsymbol{\varepsilon}^{n+1}||_{L_2}^2 + ||\boldsymbol{\varepsilon}^n||_{L_2}^2)
\end{aligned} \tag{29}$$

From Equations (26), (27) and (29), we can obtain that:

$$||\boldsymbol{\varepsilon}^{n+1}||_{L_2}^2 - ||\boldsymbol{\varepsilon}^n||_{L_2}^2 \leq \tau||\boldsymbol{\varphi}^n||_{L_2}^2 + C\tau(||\boldsymbol{\varepsilon}^{n+1}||_{L_2}^2 + ||\boldsymbol{\varepsilon}^n||_{L_2}^2) \tag{30}$$

where $C = -2aC_0^2 + \frac{1}{2} \geq 0$. Since $||\boldsymbol{\varepsilon}^0||_{L_2}^2 = 0$, then ,

$$(1 - C\tau)||\boldsymbol{\varepsilon}^{n+1}||_{L_2}^2 \leq \tau\sum_{m=0}^{n}||\boldsymbol{\varphi}^m||_{L_2}^2 + 2C\tau\sum_{m=0}^{n}||\boldsymbol{\varepsilon}^m||_{L_2}^2$$

As $\tau \to 0$, so $C\tau < \frac{1}{2}$, according to Lemma 4, we have:

$$||\boldsymbol{\varepsilon}^n||_{L_2}^2 \leq O(h^4 + \tau^4) \tag{31}$$

□

5. Numerical Experiments

In this section, we will present the numerical experiment results to test the proved theorems. Consider the initial condition of the original NLSE for one-dark soliton

$$w(x,0) = \rho\frac{1 + e^{i2\theta}e^{\lambda(x-x_0)}}{1 + e^{\lambda(x-x_0)}}, \tag{32}$$

where the exact solution is obtained

$$w(x,t) = \rho e^{ia\rho^2 t}\frac{1 + e^{i2\theta}e^{\lambda(x-x_0+\eta\cdot t)}}{1 + e^{\lambda(x-x_0+\eta\cdot t)}}. \tag{33}$$

and $\lambda = \sqrt{-2a}\rho sin\theta, \eta = \sqrt{-2a}\rho cos\theta, a = -2, \rho = 0.72, \theta = 0.75, x_0 = 0.0$.

5.1. Errors and convengence order

In the subsection, we give the convergence order of the space discretization models and the time-space discretization models by **Experiment 1** and **Experiment 2**.
Experiment 1: We use the midpoint scheme to simulate D-D model and A-L model, and choose a fixed minimum time step-size $\tau = 0.0005$ in order to reduce error caused by difference in time as much as possible. Then, comparing the solution of the space discretization models with the exact solution (33) of the original NLSE, we can obtain error $||\varepsilon(T)||^2$ and the corresponding convergence order at

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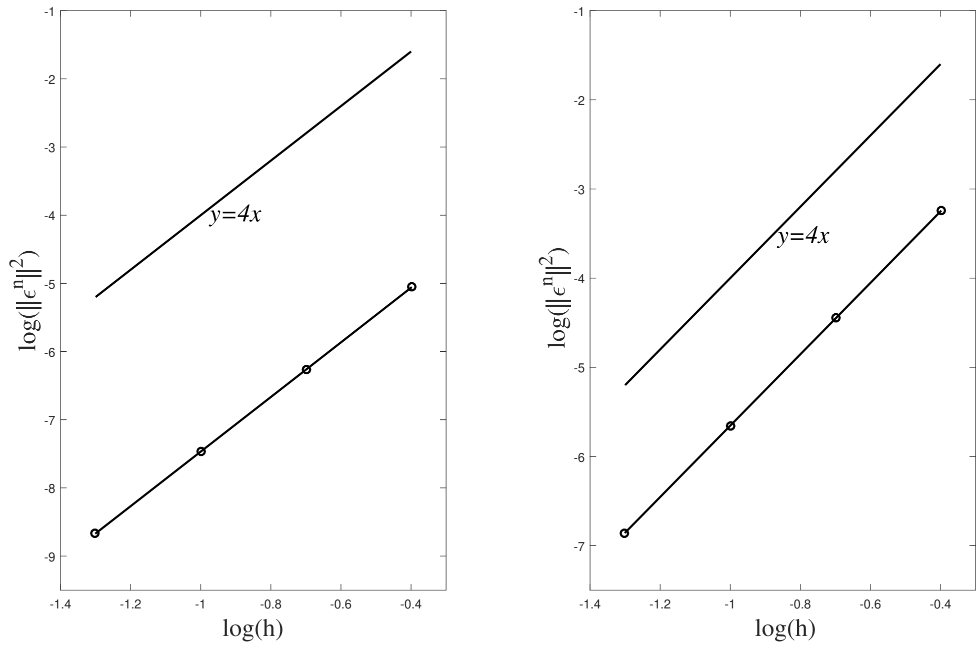
time $t = 1.6$ with different space step-size $h = 0.4, 0.2, 0.1, 0.05$. Finally, we plot "$log(||\varepsilon(T)||^2)$" with respect to "$log(h)$" in Figure 1. Table 1 and 2 and Figure 1 indicate that the order of $||\varepsilon(T)||^2$ is $O(h^4)$. So, we come to the conclution that the convergence of the D-D model and the A-L model is $O(h^2)$ in the defined norm , which fits the results of Theorem 1 and Theorem 2 very well.

Table 1. Errors and convengence order of D-D model at time $t = 1.6$.

h	τ	$\|\|\varepsilon(T)\|\|^2 \times 10^6$	Order
0.4	0.00005	8.804684	
0.2	0.00005	0.546416	4.01416
0.1	0.00005	0.034090	4.00359
0.05	0.00005	0.002130	4.00086

Table 2. Errors and convengence order of A-L model at time $t = 1.6$.

h	τ	$\|\|\varepsilon(T)\|\|^2 \times 10^4$	Order
0.4	0.00005	5.709740	
0.2	0.00005	0.354637	4.01251
0.1	0.00005	0.022128	4.00330
0.05	0.00005	0.001382	4.00083



(**a**) D-D model (**b**) A-L model

Figure 1. Errors and convengence order at time $t = 1.6$.

Experiment 2: In order to determine the convergence order of Crank-Nicolson method and New-Difference method , we choose the space step-size $h = 0.4, 0.2, 0.1, 0.05$ and the time step-size $t = 0.008, 0.004, 0.002, 0.001$. Then, we can calculate the truncation error$||\varepsilon^n||^2_{L_2}$, where $||\varepsilon^n||^2_{L_2} = ||\mathbf{w}^n||^2_{L_2} - ||\mathbf{W}^n||^2_{L_2}$. Due to $h : \tau = K$ (K is fixed), we choose to plot "$log(||\varepsilon^n||^2_{L_2})$" with respect to "$log(h)$" in Figure 2. Tabel 3-4 and Figure 2 indicate that the convergence order of the Crank-Nicolson method and the New-Difference method is $O(h^2 + \tau^2)$ in L_2-norm, which is also in good agreement with the results of Theorem Tables 3 and 4.

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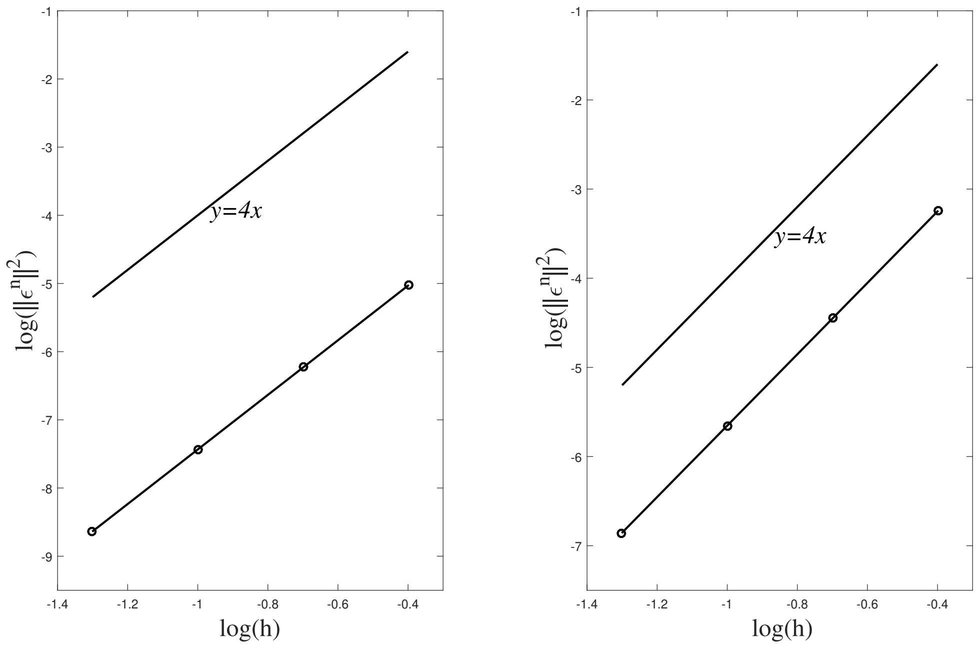
Table 3. Errors and convengence order of Crank-Nicolson method at time $t = 1.6$.

h	τ	$\|\|\varepsilon^n\|\|^2_{L_2} \times 10^6$	Order
0.4	0.008	9.490216	
0.2	0.004	0.589315	4.01295
0.1	0.002	0.036771	4.00331
0.05	0.001	0.002297	4.00083

Table 4. Errors and convengence order of New-Difference method at time $t = 1.6$.

h	τ	$\|\|\varepsilon^n\|\|^2_{L_2} \times 10^4$	Order
0.4	0.008	5.730127	
0.2	0.004	0.355952	4.01223
0.1	0.002	0.022211	4.00323
0.05	0.001	0.001388	4.00082



(**a**) Crank-Nicolson method (**b**) New-Difference method

Figure 2. Errors and convengence order at time $t = 1.6$.

5.2. Numerical simulation of dark solitons motion

Experiment 3: We take the spatial interval $x \in [-125, 75]$ and temporal interval from $t = 0$ to $t = 40$ with two different pairs of integration parameters:

$$h = 0.4, \qquad \tau = 0.02. \tag{34}$$

The numerical solutions for the Crank-Nicolson method and the New-Difference method are drawn in Figures 3 and 4. From the figure, we can see that the two methods simulate the motion of the one-dark soliton very well .

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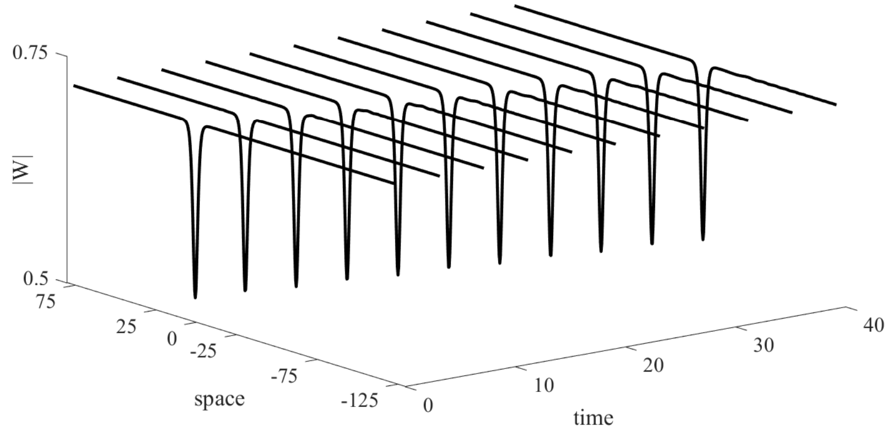

Figure 3. The numerical solutions for the Crank-Nicolson method.

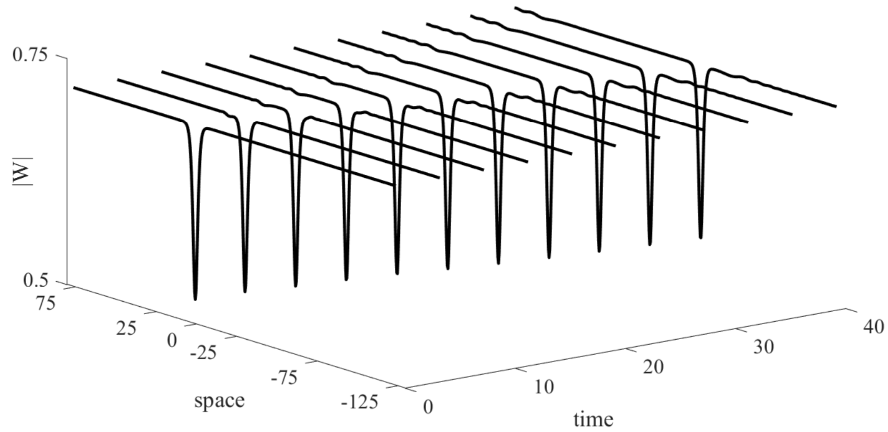

Figure 4. The numerical solutions for the New-Difference method.

5.3. Preservation of invariants

In order to further demonstrate convergence, we check the preservation of the invariants of these models. Here, we take $h = 0.4, \tau = 0.02, 0 \leq t = n\tau \leq 40$, and set $err(A)(t) = A(t) - A(0)$ for any invariant A.

Experiment 4: For D-D model, we give the preservation of the invariants E_1 and Q_1. For A-L model, the invariants F_1 and F_2 have both the real part and the imaginary part, so we persent the real and imaginary part of invariants F_1 and F_2 ($F_m = FR_m + iFI_m$), respectively. Figure 6 shows that the space discretization models have a good simulation effect from the trend of the invariants'error , which further confirms the convergence of these models.

Experiment 5: We use the conserved quantities' approximation S_1 and S_2 ($S_m = SR_m + iSI_m$) of the original NLSE to test the convergence of the time-space discretization models. As the imaginary part SI_1 of S_1 and the real part SR_2 of S_2 are zero, we only present the evolutions of the rest SR_1 and SI_2. Figure 5 shows that the time-space discretization models can maintain the conserved quantities'approximation S_1 and S_2 well, which further illustrates the convergence of these models .

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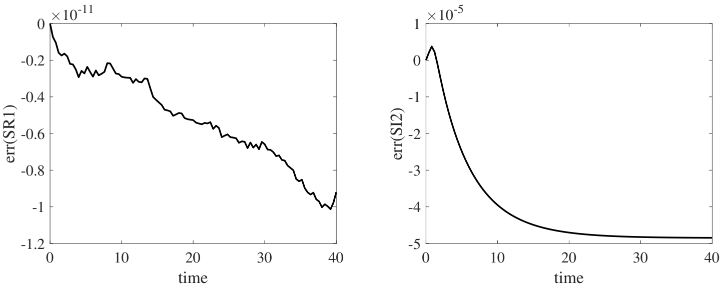
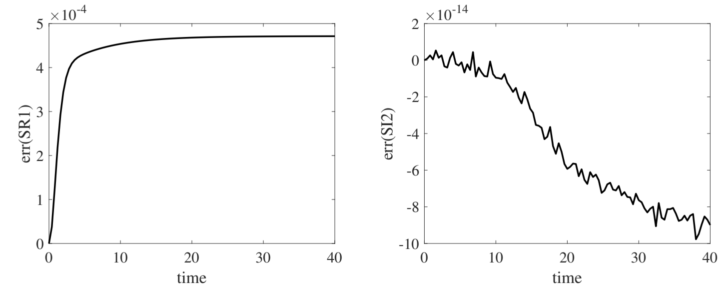
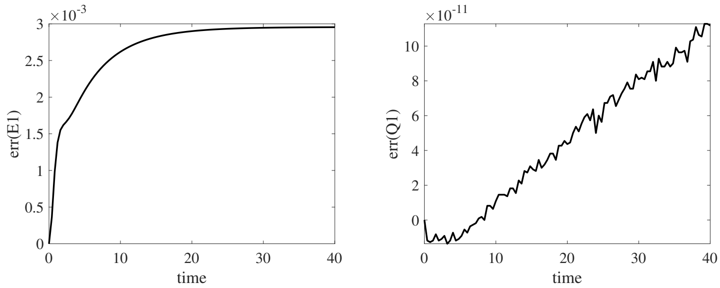


(**a**) Crank-Nicolson method

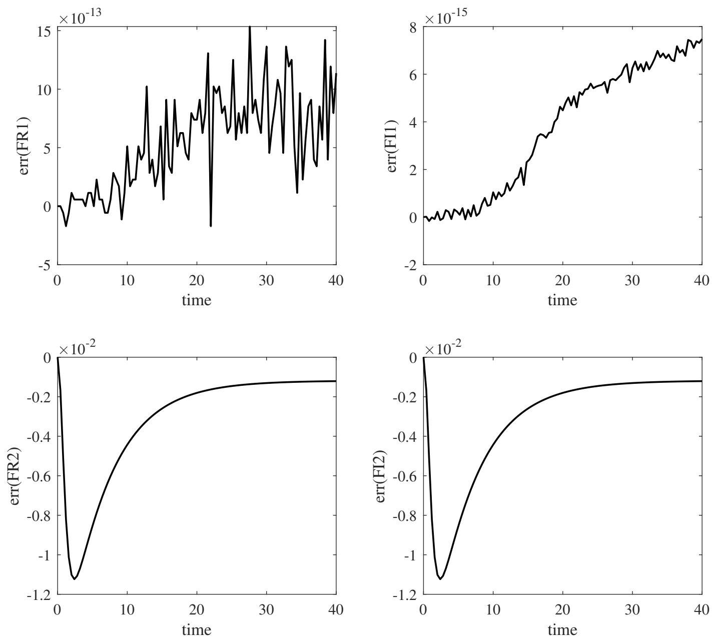


(**b**) New-Difference method

Figure 5. Evolution of invariants by the time-space discretization models.



(**a**) D-D model



(**b**) A-L model

Figure 6. Evolution of invariants by the space discretization models.

6. Conclusions

In dark solitons motion $(a < 0, |W_0(\infty)| = \rho)$, we have proved that the solutions of the D-D model and the A-L model converge to the solution of the original NLSE when $h \to 0$, and their convergence order are $O(h^2)$ in the defined norm. Our results of numerical experiments are in good agreement with the ones of theory. We use the midpoint scheme to solve the D-D model and the A-L model , then get the Crank-Nicolson method and the New-Different method. Through theoretical proof, we show that the two schemes are of order $O(h^2 + \tau^2)$ in discrete L_2-norm error estimates. The corresponding numerical experiments also fit the proven theories well.

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References

1. Ablowitz, M.J.; Segur, H. *Solitons and the inverse scattering transform*; SIAM, 1981. [Crossref]
2. Dodd, R.K.; Eilbeck, J.C.; Gibbon, J.D.; Morris, H.C. Solitons and nonlinear wave equations. *Academic Press,New York* **1982**. [Crossref]
3. Hasegawa, A. Optical solitons in fibers. In *Optical solitons in fibers*; Springer, 1989; pp. 1–74. [Crossref]
4. Konotop, V.V.; et al. *Nonlinear random waves*; World Scientific, Singapore, 1994.
5. Konotop, V.; Vekslerchik, V. Randomly modulated dark soliton. *Journal of Physics A: Mathematical and General* **1991**, *24*, 767. [Crossref]
6. Zakharov, V.E.; Shabat, A.B. Interaction between solitons in a stable medium. *Sov. Phys. JETP* **1973**, *37*, 823–828.
7. Sanz-Serna, J. Methods for the numerical solution of the nonlinear Schrödinger equation. *mathematics of computation* **1984**, *43*, 21–27. [Crossref]
8. Zhang, L. A High Accurate and Conservative Finite Difference Scheme for Nonlinear Schrodinger Equation. *Acta Math. Appl. Sin.* **2005**. [Crossref]
9. Fei, Z.; Pérez-García, V.M.; Vázquez, L. Numerical simulation of nonlinear Schrödinger systems: a new conservative scheme. *Applied Mathematics and Computation* **1995**, *71*, 165–177. [Crossref]
10. Xu, Y.; Shu, C.W. Local discontinuous Galerkin methods for nonlinear Schrödinger equations. *Journal of Computational Physics* **2005**, *205*, 72–97. [Crossref]
11. Bratsos, A.; Ehrhardt, M.; Famelis, I.T. A discrete Adomian decomposition method for discrete nonlinear Schrödinger equations. *Applied mathematics and computation* **2008**, *197*, 190–205. [Crossref]
12. He, J.H. Homotopy perturbation method: a new nonlinear analytical technique. *Applied Mathematics and computation* **2003**, *135*, 73–79. [Crossref]
13. Akrivis, G.D. Finite difference discretization of the cubic Schrödinger equation. *IMA Journal of Numerical Analysis* **1993**, *13*, 115–124. [Crossref]
14. Borhanifar, A.; Abazari, R. Numerical study of nonlinear Schrödinger and coupled Schrödinger equations by differential transformation method. *Optics Communications* **2010**, *283*, 2026–2031. [Crossref]
15. Zhu, Y.L. Implicit difference schemes for the generalized non-linear Schrödinger system. *Journal of Computational Mathematics* **1983**, pp. 116–129. [Crossref]
16. Guo, B.y. The convergence of numerical method for nonlinear Schrodinger equation. *Journal of Computational Mathematics* **1986**, *4*, 121–130.

17. Zhang, L.; Chang, Q. A conservative numerical scheme for a class of nonlinear Schrödinger equation with wave operator. *Applied mathematics and computation* **2003**, *145*, 603–612. [Crossref]
18. Xie, S.S.; Li, G.X.; Yi, S. Compact finite difference schemes with high accuracy for one-dimensional nonlinear Schrödinger equation. *Computer Methods in Applied Mechanics and Engineering* **2009**, *198*, 1052–1060. [Crossref]
19. TingChun WANG, B.G. Unconditional convergence of two conservative compact difference schemes for non-linear Schrödinger equation in one dimension. *SCIENTIA SINICA Mathematica* **2011**, *41*, 207–233. [Crossref]
20. Li, X.; Zhang, L.; Wang, S. A compact finite difference scheme for the nonlinear Schrödinger equation with wave operator. *Applied Mathematics and Computation* **2012**, *219*, 3187–3197. [Crossref]
21. Li, X.; Zhang, L.; Zhang, T. A new numerical scheme for the nonlinear Schrödinger equation with wave operator. *Journal of Applied Mathematics and Computing* **2017**, *54*, 109–125. [Crossref]
22. Tang, Y.F.; Vázquez, L.; Zhang, F.; Pérez-García, V. Symplectic methods for the nonlinear Schrödinger equation. *Comput. Math. with Appl.* **1996**, *32*, 73–83. [Crossref]
23. Tang, Y.F.; Pérez-García, V.M.; Vázquez, L. Symplectic methods for the Ablowitz-Ladik model. *Applied mathematics and computation* **1997**, *82*, 17–38. [Crossref]
24. Tang, Y.F.; Cao, J.; Liu, X.; Sun, Y. Symplectic methods for the Ablowitz–Ladik discrete nonlinear Schrödinger equation. *Journal of Physics A: Mathematical and Theoretical* **2007**, *40*, 24–25. [Crossref]
25. Ben-Yu, G.; Pascual, P.J.; Rodriguez, M.J.; Vázquez, L. Numerical solution of the sine-Gordon equation. *Applied Mathematics and Computation* **1986**, *18*, 1–14. [Crossref]