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Article

A Unified Representation of *q*- and *h*-Integrals and Consequences in Inequalities

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Abstract: This paper aims to unify q-derivative/q-integral and h-derivative/h-integral in a single definition, this will be called $q-h$-derivative/$q-h$-integral. These notions are further extended on finite interval $[a,b]$ in the form of left and right $q-h$-derivatives and $q-h$-integrals. Some inequalities are studied for $q-h$-integrals which are directly connected with well known results. In diverse fields of science and engineering the theory based on q-derivatives/q-integrals and h-derivatives/h-integrals can be unified by using the concept of $q-h$-derivative/$q-h$-integral.

Keywords: q-derivative; q-integral; h-derivative; h-integral; $q-h$-derivative; $q-h$-integral; inequalities

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1. Introduction

The h-derivative and the q-derivative of a function ν have been defined by the quotients

$$\frac{\nu(\gamma+h)-\nu(\gamma)}{h} \text{ and } \frac{\nu(q\gamma)-\nu(\gamma)}{(q-1)\gamma}$$

respectively. The h-derivative is usually denoted by the quotient $\mathcal{D}_h\nu(\gamma) = \frac{d_h\nu(\gamma)}{d_h\gamma}$ while the q-derivative is denoted by the quotient $\mathcal{D}_q\nu(\gamma) = \frac{d_q\nu(\gamma)}{d_q\gamma}$, where $d_h\nu(\gamma) = \nu(\gamma+h)-\nu(\gamma)$ is called the h-differential and $d_q\nu(\gamma) = \nu(q\gamma)-\nu(\gamma)$ is called the q-differential for the function ν. As an example the h-derivative and the q-derivative of γ^n can be computed in the forms $\frac{(\gamma+h)^n-\gamma^n}{h} = n\gamma^{n-1} + \frac{n(n-1)}{2}\gamma^{n-2}h + ... + h^{n-1}$ and $\frac{q^n-1}{q-1}\gamma^{n-1} = (q^{n-1}+...+1)\gamma^{n-1}$ respectively. For the sake of simplicity, the notation $[n]_q$ is used instead of $\frac{q^n-1}{q-1}$, and then $\mathcal{D}_q\gamma^n = [n]_q\gamma^{n-1}$. Since $\lim_{q\to1}\mathcal{D}_q\nu(\gamma) = \lim_{h\to0}\mathcal{D}_h\nu(\gamma) = \frac{d\nu(\gamma)}{d\gamma}$, the h-derivative and the q-derivative are generalizations of ordinary derivative. The q-derivative leads to the subject of q-calculus, see [5] for details.

The sum and product formula of q-derivatives for functions ν_1 and ν_2 are given by;

$$\mathcal{D}_q\{\nu_1(\gamma)+\nu_2(\gamma)\} = \mathcal{D}_q\nu_1(\gamma)+\mathcal{D}_q\nu_2(\gamma) \tag{1}$$

and

$$\mathcal{D}_q\{\nu_1(\gamma)\nu_2(\gamma)\} = \nu_1(q\gamma)\mathcal{D}_q\nu_2(\gamma)+\nu_2(\gamma)\mathcal{D}_q\nu_1(\gamma), \tag{2}$$

respectively. Since $\nu_1(\gamma)\nu_2(\gamma) = \nu_2(\gamma)\nu_1(\gamma)$, (2) is equivalent to the upcoming formula

$$\mathcal{D}_q\{\nu_1(\gamma)\nu_2(\gamma)\} = \nu_1(\gamma)\mathcal{D}_q\nu_2(\gamma)+\nu_2(q\gamma)\mathcal{D}_q\nu_1(\gamma). \tag{3}$$

In view of (2), the quotient formula of q-derivatives is given by;

$$\mathcal{D}_q\left(\frac{\nu_1(\gamma)}{\nu_2(\gamma)}\right)=\frac{\nu_2(\gamma)\mathcal{D}_q\nu_1(\gamma)-\nu_1(\gamma)\mathcal{D}_q\nu_2(\gamma)}{\nu_2(\gamma)\nu_2(q\gamma)}. \quad (4)$$

In view of (3), the quotient formula of q-derivatives is given by;

$$\mathcal{D}_q\left(\frac{\nu_1(\gamma)}{\nu_2(\gamma)}\right)=\frac{\nu_2(q\gamma)\mathcal{D}_q\nu_1(\gamma)-\nu_1(q\gamma)\mathcal{D}_q\nu_2(\gamma)}{\nu_2(\gamma)\nu_2(q\gamma)}. \quad (5)$$

The formulae of h-derivatives are as follows:

$$\mathcal{D}_h\{\nu_1(\gamma)+\nu_2(\gamma)\}=\mathcal{D}_h\nu_1(\gamma)+\mathcal{D}_h\nu_2(\gamma), \quad (6)$$

$$\mathcal{D}_h\{\nu_1(\gamma)\nu_2(\gamma)\}=\nu_1(\gamma)\mathcal{D}_h\nu_2(\gamma)+\nu_2(\gamma+h)\mathcal{D}_h\nu_1(\gamma), \quad (7)$$

and

$$\mathcal{D}_h\left(\frac{\nu_1(\gamma)}{\nu_2(\gamma)}\right)=\frac{\nu_2(\gamma)\mathcal{D}_h\nu_1(\gamma)-\nu_1(\gamma)\mathcal{D}_h\nu_2(\gamma)}{\nu_2(\gamma)\nu_2(\gamma+h)}. \quad (8)$$

Next, we give the definition of q-derivative on a finite interval.

Definition 1 ([2])**.** *Let $\mu : I = [a,b] \to \mathbb{R}$ be a continuous function. For $0 < q < 1$ the q-derivative ${}_a\mathcal{D}_q\mu$ on I, is given by;*

$${}_a\mathcal{D}_q\mu(\xi):=\frac{\mu(q\xi+(1-q)a)-\mu(\xi)}{(q-1)(\xi-a)},\ \xi\neq a,\ {}_a\mathcal{D}_q\mu(a)=\lim_{\xi\to a}{}_a\mathcal{D}_q\mu(\xi). \quad (9)$$

Function μ is called q-differentiable on $[a,b]$ if ${}_a\mathcal{D}_q\mu(\xi)$ exists for all $\xi \in [a,b]$. For $a = 0$, we have ${}_0\mathcal{D}_q\mu(\xi) = \mathcal{D}_q\mu(\xi)$ and $\mathcal{D}_q\mu(\xi)$ is the q-derivative of μ at $\xi \in [a,b]$ defined as follows:

$$\mathcal{D}_q\mu(\xi):=\frac{\mu(q\xi)-\mu(\xi)}{(q-1)\xi},\ \xi\neq 0. \quad (10)$$

The q-integral of function μ on interval $[a,b]$ is defined as follows:

Definition 2 ([2])**.** *Let $\mu : I = [a,b] \to \mathbb{R}$ be a function. For $0 < q < 1$ the q-definite integral on I is given by;*

$$\int_a^\xi \mu(\gamma)\,{}_ad_q\gamma=(1-q)(\xi-a)\sum_{n=0}^{\infty}q^n\mu(q^n\xi+(1-q^n)a),\ \xi\in[a,b]. \quad (11)$$

In (11), by setting $a = 0$, the Jackson q-definite integral given in [5], is deduced as follows:

$$\int_0^\xi \mu(\gamma)\,{}_0d_q\gamma=\int_0^\xi \mu(\gamma)d_q\gamma=(1-q)\xi\sum_{n=0}^{\infty}q^n\mu(q^n\xi),\ \xi\in[a,b]. \quad (12)$$

If $c \in (a,\xi)$, then the q-definite integral on $[c,\xi]$ is calculated as follows:

$$\int_c^\xi \mu(\gamma)\,{}_ad_q\gamma=\int_a^\xi \mu(\gamma)\,{}_ad_q\gamma-\int_a^c \mu(\gamma)\,{}_ad_q\gamma. \quad (13)$$

We are intent to unify the q-derivative and h-derivative in a single notion which will be named $q-h$-derivative. We give sum/difference, product and quotient formulas for $q-h$-derivatives, also the definition of $q-h$-integral is given. Further, we will define $q-h$-derivative and $q-h$-integral on finite interval. The composite derivatives and integrals will provide the opportunity to study theoretical

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and practical concepts and problems of different fields related to q-derivative and h-derivative simultaneously. This paper will be interesting and productive for scientists and engineers.

2. A Generalization of *q*- and *h*-Derivatives

The $(q-h)$-differential of a real valued function μ is defined by;

$$ {}_hd_q\mu(\xi) = \mu(q(\xi+h)) - \mu(\xi). \tag{14}$$

Then for $h=0$, and $q \to 1$ in (14), we have

$$ {}_0d_q\mu(\xi) = \mu(q\xi) - \mu(\xi) = d_q\mu(\xi)$$

and

$$ {}_hd_1\mu(\xi) = \mu(\xi+h) - \mu(\xi) = {}_hd\mu(\xi).$$

In particular,

$$ {}_hd_q(\xi) = q\xi + qh - \xi = (q-1)\xi + qh. \tag{15}$$

Then for $h=0$, and $q \to 1$ in (15), we have

$$ {}_0d_q(\xi) = (q-1)\xi = d_q(\xi) \quad and \quad {}_hd_1(\xi) = h = d_h(\xi). \tag{16}$$

For $u(\xi) = \mu(\xi) + \nu(\xi)$ the $(q-h)$-differential of u is given by;

$$ {}_hd_q(u(\xi)) = {}_hd_q(\mu(\xi)+\nu(\xi)) = (\mu+\nu)(q(\xi+h)) - (\mu+\nu)(\xi) = {}_hd_q\mu(\xi) + {}_hd_q\nu(\xi). \tag{17}$$

For $\alpha \in \mathbb{R}$ the $(q-h)$-differential of $\alpha\mu$ is given by;

$$ {}_hd_q(\alpha\mu)(\xi) = {}_hd_q(\alpha\mu)(\xi) = (\alpha\mu)(q(\xi+h)) - (\alpha\mu)(\xi) = \alpha\ {}_hd_q\mu(\xi). \tag{18}$$

From (17) and (18) one can see that $(q-h)$-differential is linear. Here we see that if $p(\xi) = \mu(\xi)\nu(\xi)$, then $(q-h)$-differential is calculated as follows:

$$\begin{aligned} {}_hd_q(p(\xi)) = {}_hd_q(\mu(\xi)\nu(\xi)) &= \mu(q(\xi+h))\nu(q(\xi+h)) - \mu(\xi)\nu(\xi) \\ &= \mu(q(\xi+h))\nu(q(\xi+h)) + \mu(q(\xi+h))\nu(\xi) \\ &\quad - \mu(q(\xi+h))\nu(\xi) - \mu(\xi)\nu(\xi) \\ &= \mu(q(\xi+h))[\nu(q(\xi+h)) - \nu(\xi)] \\ &\quad + \nu(\xi)[\mu(q(\xi+h)) - \mu(\xi)]. \end{aligned}$$

Hence we get

$$ {}_hd_q(\mu(\xi)\nu(\xi)) = \mu(q(\xi+h))_hd_q\nu(\xi) + \nu(\xi)_hd_q\mu(\xi). \tag{19}$$

For $h=0$, and $q \to 1$ in (19), we have

$$\begin{aligned} {}_0d_q(\mu(\xi)\nu(\xi)) = d_q(\mu(\xi)\nu(\xi)) &= \mu(q\xi)_0d_q\nu(\xi) + \nu(\xi)\,{}_0d_q\mu(\xi) \\ &= \mu(q\xi)d_q\nu(\xi) + \nu(\xi)d_q\mu(\xi) \end{aligned}$$

and

$$\begin{aligned} {}_hd_1(\mu(\xi)\nu(\xi)) = d_h(\mu(\xi)\nu(\xi)) &= \mu(\xi+h)_hd_1\nu(\xi) + \nu(\xi)_hd_1\mu(\xi) \\ &= \mu(\xi+h)d_h\nu(\xi) + \nu(\xi)d_h\mu(\xi), \end{aligned}$$

respectively. Next, we define the $q-h$-derivative as follows:

Definition 3. *Let $0 < q < 1$, $h \in \mathbb{R}$ and $\mu : I \to \mathbb{R}$ be a continuous function. Then the $q-h$-derivative of μ is defined by*

$$\mathcal{C}_h\mathcal{D}_q\mu(\xi) = \frac{{}_hd_q\mu(\xi)}{{}_hd_q\xi} = \frac{\mu(q(\xi+h)) - \mu(\xi)}{(q-1)\xi + qh}, \ \xi \neq \frac{qh}{1-q} := \xi_\circ \quad (20)$$
$$\mathcal{C}_h\mathcal{D}_q\mu(\xi_\circ) = \lim_{\xi\to\xi_\circ} \mathcal{C}_h\mathcal{D}_q\mu(\xi).$$

Provided $q(\xi+h) \in I$.

For $h = 0$ and $q \to 1$ in (20), we have

$$\mathcal{C}_0\mathcal{D}_q\mu(\xi) = \mathcal{D}_q\mu(\xi) = \frac{d_q\mu(\xi)}{d_q\xi} = \frac{\mu(q\xi) - \mu(\xi)}{(q-1)\xi} \quad (21)$$

and

$$\mathcal{C}_h\mathcal{D}_1\mu(\xi) = \mathcal{D}_h\mu(\xi) = \frac{d_h\mu(\xi)}{d_h\xi} = \frac{\mu(\xi+h) - \mu(\xi)}{h}. \quad (22)$$

By setting $h = 0$, $q \to 1$ in (20), we get the ordinary derivative of μ, provided the limit exists.

Example 1. *Consider $P(x) = \xi^n$, $n \in \mathbb{N}$. Then*

$$\mathcal{C}_h\mathcal{D}_q(P(x)) = \frac{q^n(\xi+h)^n - \xi^n}{(q-1)\xi + qh} = \frac{(q^n-1)\xi^n}{(q-1)\xi + qh} + \frac{q^n(n\xi^{n-1}h + \dots + h^n)}{(q-1)\xi + qh}. \quad (23)$$

For $h = 0$ and $q \to 1$ in (23), we have

$$\mathcal{C}_0\mathcal{D}_q(\xi^n) = \frac{q^n\xi^n - \xi^n}{(q-1)\xi} = \frac{q^n-1}{q-1}\xi^{n-1} = [n]_q\xi^{n-1} = \mathcal{D}_q(\xi^n), \quad (24)$$

and

$$\mathcal{C}_h\mathcal{D}_1(\xi^n) = \frac{(\xi+h)^n - \xi^n}{h} = n\xi^{n-1} + \frac{n(n-1)}{2}\xi^{n-2}h + \dots\dots + h^{n-1}. \quad (25)$$

In particular we have $\lim_{h\to 0} \mathcal{C}_h\mathcal{D}_1(\xi^n) = n\xi^{n-1}$.

2.1. Linearity

The $q-h$-derivative is linear i.e. for $\alpha, \beta \in \mathbb{R}$ and using the linearity of $(q-h)$-differentials we have:

$$\mathcal{C}_h\mathcal{D}_q(\alpha\mu(\xi) + \beta\nu(\xi)) = \alpha\,\mathcal{C}_h\mathcal{D}_q\mu(\xi) + \beta\,\mathcal{C}_h\mathcal{D}_q\nu(\xi).$$

2.2. Product formula

The following formula for product of functions by using (19), can be obtained:

$$\mathcal{C}_h\mathcal{D}_q(\mu(\xi)\nu(\xi)) = \frac{{}_hd_q(\mu(\xi)\nu(\xi))}{{}_hd_q\xi} = \frac{\mu(q(\xi+h))\,{}_hd_q\nu(\xi) + {}_hd_q\mu(\xi)\nu(\xi)}{{}_hd_q\xi} \quad (26)$$
$$= \mu(q(\xi+h))\mathcal{C}_h\mathcal{D}_q\nu(\xi) + \nu(\xi)\mathcal{C}_h\mathcal{D}_q\mu(\xi).$$

The product formula for q-derivatives and h-derivatives can be obtained as follows:

By setting $h = 0$ in (26), the q-derivative formula for products of functions is yielded:

$$\begin{aligned} \mathcal{C}_0\mathcal{D}_q(\mu(\xi)\nu(\xi)) = \frac{d_q(\mu(\xi)\nu(\xi))}{d_q\xi} &= \mathcal{D}_q(\mu(\xi)\nu(\xi)) \\ &= \mu(q\xi)\mathcal{C}_0\mathcal{D}_q\nu(\xi) + \nu(\xi)\mathcal{C}_0\mathcal{D}_q\mu(\xi) \\ &= \mu(q\xi)\mathcal{D}_q\nu(\xi) + \nu(\xi)\mathcal{D}_q\mu(\xi). \end{aligned} \tag{27}$$

By taking $q \to 1$ in (26), the h-derivative formula for products of functions is yielded:

$$\begin{aligned} \mathcal{C}_h\mathcal{D}_1(\mu(\xi)\nu(\xi)) = \frac{d_h(\mu(\xi)\nu(\xi))}{d_h\xi} &= \mathcal{D}_h(\mu(\xi)\nu(\xi)) \\ &= \mu(\xi+h)\mathcal{C}_h\mathcal{D}_1\nu(\xi) + \nu(\xi)\mathcal{C}_h\mathcal{D}_1\mu(\xi) \\ &= \mu(\xi+h)\mathcal{D}_h\nu(\xi) + \nu(\xi)\mathcal{D}_h\mu(\xi). \end{aligned} \tag{28}$$

By using symmetry, we can have from (26):

$$\mathcal{C}_h\mathcal{D}_q(\nu(\xi)\mu(\xi)) = \nu(q(\xi+h))\mathcal{C}_h\mathcal{D}_q\mu(\xi) + \mu(\xi)\mathcal{C}_h\mathcal{D}_q\nu(\xi). \tag{29}$$

Both (26) and (29) are equivalent.

2.3. Quotient formula

By using (26) and (29), the quotient formula of $q-h$-derivatives is calculated as follows: We have for $\nu(\xi) \neq 0$

$$\nu(\xi)\frac{\mu(\xi)}{\nu(\xi)} = \mu(\xi). \tag{30}$$

By using definition of $q-h$-derivative and (26), we have

$$\mathcal{C}_h\mathcal{D}_q\left(\nu(\xi)\frac{\mu(\xi)}{\nu(\xi)}\right) = \mathcal{C}_h\mathcal{D}_q(\mu(\xi)). \tag{31}$$

$$\nu(q(\xi+h))\mathcal{C}_h\mathcal{D}_q\left(\frac{\mu(\xi)}{\nu(\xi)}\right) + \frac{\mu(\xi)}{\nu(\xi)}\mathcal{C}_h\mathcal{D}_q\nu(\xi) = \mathcal{C}_h\mathcal{D}_q(\mu(\xi)). \tag{32}$$

Now

$$\begin{aligned} \mathcal{C}_h\mathcal{D}_q\left(\frac{\mu(\xi)}{\nu(\xi)}\right) &= \frac{\mathcal{C}_h\mathcal{D}_q(\mu(\xi)) - \frac{\mu(\xi)}{\nu(\xi)}\mathcal{C}_h\mathcal{D}_q(\nu(\xi))}{\nu(q(\xi+h))} \\ &= \frac{\nu(\xi)\mathcal{C}_h\mathcal{D}_q(\mu(\xi)) - \mu(\xi)\mathcal{C}_h\mathcal{D}_q(\nu(\xi))}{\nu(q(\xi+h))\nu(\xi)}. \end{aligned} \tag{33}$$

By using (29), one can get

$$\frac{\mu(q(\xi+h))}{\nu(q(\xi+h))}\mathcal{C}_h\mathcal{D}_q\Big(\nu(\xi)\Big) + \nu(\xi)\mathcal{C}_h\mathcal{D}_q\left(\frac{\mu(\xi)}{\nu(\xi)}\right) = \mathcal{C}_h\mathcal{D}_q\Big(\mu(\xi)\Big),$$

that is:

$$\mathcal{C}_h\mathcal{D}_q\left(\frac{\mu(\xi)}{\nu(\xi)}\right) = \frac{\mathcal{C}_h\mathcal{D}_q(\mu(\xi))\nu(q(\xi+h)) - \mu(q(\xi+h))\mathcal{C}_h\mathcal{D}_q(\nu(\xi))}{\nu(q(\xi+h))\nu(\xi)}. \tag{34}$$

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Remark 1. *By putting $h = \frac{\omega}{q}$ for $\omega > 0$, equation (26) produces product and (33) produces quotient formulas for (q, ω)-derivatives given in [3].*

Next, let us define the $q-h-$binomial $(\xi - a)^n_{h,q}$ analogue to $(\xi - a)^n$ as follows:

$$(\xi - a)^n_{h,q} = \begin{cases} 1, & n = 0, \\ (\xi - a)(\xi - q(a+h))(\xi - q^2(a+2h))...(\xi - q^{n-1}(a+(n-1)h), & n \geq 1. \end{cases} \tag{35}$$

Then it is clear that for $h = 0$ we have $(\xi - a)^n_{0,q} = (\xi - a)^n_q$ i.e. the q-analogue of $(\xi - a)^n$ is obtained which is defined in [5, Page 8, Definition] as follows:

$$(\xi - a)^n_q = \begin{cases} 1, & n = 0, \\ (\xi - a)(\xi - qa)...(\xi - q^{n-1}a, & n \geq 1. \end{cases} \tag{36}$$

Also, from (35), for $q \to 1$ we have $(\xi - a)^n_{h,1} = (\xi - a)^n_h$ i.e. the h-analogue of $(\xi - a)^n$ is obtained, it is defined in [5, Page 80, Definition] as follows:

$$(\xi - a)^n_h = \begin{cases} 1, & n = 0, \\ (\xi - a)(\xi - a - h)...(\xi - a - (n-1)h), & n \geq 1. \end{cases} \tag{37}$$

In the next, we find the $q-h-$derivative of $q-h-$binomial $(\xi - a)^n_{h,q}$ as follows:
For $n = 1$, we have

$${}_h\mathcal{D}_q((\xi - a)^1_{h,q}) = {}_h\mathcal{D}_q(\xi - a) = 1.$$

For $n = 2$, we have

$$\begin{aligned} &{}_h\mathcal{D}_q((\xi - a)^2_{h,q}) = {}_h\mathcal{D}_q((\xi - a)(\xi - q(a+h))) = (q(\xi + h) - q(a+h)).1 + (\xi - a) \\ &= (\xi - a)(1+q) = [2]_q(\xi - a)^1_{h,q}. \end{aligned}$$

As $h \to 0$ we have ${}_0\mathcal{D}_q((\xi - a)^2_{0,q}) = \mathcal{D}_q((\xi - a)^2_q) = [2]_q(\xi - a)^1_q$. While as $q \to 1$ we have ${}_h\mathcal{D}_1((\xi - a)^2_{h,1}) = \mathcal{D}_h((\xi - a)^2_h) = 2(\xi - a)^1_h$.

For $n = 3$, we have

$$\begin{aligned} &{}_h\mathcal{D}_q((\xi - a)^3_{h,q}) = {}_h\mathcal{D}_q((\xi - a)^2_{h,q}(\xi - q^2(a+2h))) \\ &= (q(\xi + h) - q^2(a+2h))\{(q+1)(\xi - a)\} + (\xi - a)^2_{h,q}.1 \\ &= q(q+1)(\xi - a)(\xi - q(a+h)) + q(1-q^2)(\xi - a)h + (\xi - a)^2_{h,q} \\ &= q(q+1)(\xi - a)^2_{h,q} + (\xi - a)^2_{h,q} + q(1-q^2)(\xi - a)h \\ &= (q^2+q+1)(\xi - a)^2_{h,q} + q(1-q^2)(\xi - a)h = [3]_q(\xi - a)^2_{h,q} + q(1-q^2)h(\xi - a)^1_{h,q}. \end{aligned}$$

As $h \to 0$ we have ${}_0\mathcal{D}_q((\xi - a)^3_{0,q}) = \mathcal{D}_q((\xi - a)^3_q) = [3]_q(\xi - a)^2_q$. While as $q \to 1$ we have ${}_h\mathcal{D}_1((\xi - a)^3_{h,1}) = \mathcal{D}_h((\xi - a)^3_h) = 3(\xi - a)^2_h$.
For $n = 4$, we have

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$$
\begin{aligned}
&{}_h\mathcal{D}_q((\xi-a)^4_{h,q}) = {}_h\mathcal{D}_q((\xi-a)^3_{h,q}(\xi-q^3(a+3h)))\\
&= (q(\xi+h)-q^3(a+3h))\Big\{[3]_q(\xi-a)^2_{h,q}+q(1-q^2)h(\xi-a)^1_{h,q}\Big\}+(\xi-a)^3_{h,q}.1\\
&= [3]_q q(\xi-a)^2_{h,q}(\xi-q^2(a+2h))+hq^2(1-q^2)(\xi-a)(\xi-q^2(a+2h))\\
&+[3]_q qh(1-q^2)(\xi-a)^2_{h,q}+q^2(1-q^2)^2h^2(\xi-a)+(\xi-a)^3_{h,q}\\
&= (1+[3]_q q)(\xi-a)^3_{h,q}+[3]_q q(1-q^2)h(\xi-a)^2_{h,q}+hq^2(1-q^2)(\xi-a)\\
&\{x-q^2(a+3h)+h\}\\
&= [4]_q(\xi-a)^3_{h,q}+[3]_q qh(1-q^2)(\xi-a^2)_{h,q}+hq^2(1-q^2)(\xi-a)(\xi-q(a+h))\\
&+hq^2(1-q^2)(q(a+h)-q^2(a+3h)+h)(\xi-a)\\
&= [4]_q(\xi-a)^3_{h,q}+q(1+q)^2(1-q^2)h(\xi-a)^2_{h,q}\\
&+hq^2(1-q^2)(q(a+h)-q^2(a+3h)+h)(\xi-a).
\end{aligned}
$$

As $h\to 0$ we have ${}_0\mathcal{D}_q((\xi-a)^4_{0,q}) = \mathcal{D}_q((\xi-a)^4_q) = [4]_q(\xi-a)^3_q$. While as $q\to 1$ we have ${}_h\mathcal{D}_1((\xi-a)^4_{h,1}) = \mathcal{D}_h((\xi-a)^4_h) = 4(\xi-a)^3_h$.

Inductively,one can see that:

As $h\to 0$ we have ${}_0\mathcal{D}_q((\xi-a)^n_{0,q}) = \mathcal{D}_q((\xi-a)^n_q) = [n]_q(\xi-a)^{n-1}_q$.

As $q\to 1$ we have ${}_h\mathcal{D}_1((\xi-a)^n_{h,1}) = \mathcal{D}_h((\xi-a)^n_h) = n(\xi-a)^{n-1}_h$.

If μ is $q-h$-derivative of μ i.e. $\mu(\xi) = \mathcal{C}_h\mathcal{D}_q\mu(\xi)$, then μ is called $q-h$-anti-derivative of μ. The $q-h$-anti-derivative is denoted by $\int \mu(\xi)\,{}_hd_qx$.

3. $q-h$-Derivative on a Finite Interval

Here throughout the section, $I := [a,b]$ for $a,b\in\mathbb{R}$. The $q-h-$derivative on I is given in the upcoming definition.

Definition 4. *Let $0<q<1$, $h\in\mathbb{R}$, $\xi\in I$ and $\mu: I\to\mathbb{R}$ be a continuous function. Then left $q-h-$derivative $\mathcal{C}_h\mathcal{D}^{a^+}_q\mu$ and right $q-h-$derivative $\mathcal{C}_h\mathcal{D}^{b-}_q\mu$ on I are defined by;*

$$
\mathcal{C}_h\mathcal{D}^{a^+}_q\mu(\xi) := \frac{\mu((1-q)a+q(\xi+h))-\mu(\xi)}{(1-q)(a-\xi)+qh};\ \xi\neq\frac{qh+(1-q)a}{1-q} := u, \tag{38}
$$

$$
\mathcal{C}_h\mathcal{D}^{b-}_q\mu(\xi) := \frac{\mu((1-q)\xi+q(b+h))-\mu(b)}{(1-q)(\xi-b)+qh};\ \xi\neq\frac{-qh+(1-q)b}{1-q} := v, \tag{39}
$$

provided that $(1-q)a+q(\xi+h)\in[a,\xi]$ and $(1-q)\xi+q(b+h)\in[\xi,b]$. Also, $\mathcal{C}_h\mathcal{D}^{a^+}_q\mu(u) = \lim\limits_{\xi\to u}\mathcal{C}_h\mathcal{D}^{a^+}_q\mu(\xi)$ and $\mathcal{C}_h\mathcal{D}^{b-}_q\mu(v) = \lim\limits_{\xi\to v}\mathcal{C}_h\mathcal{D}^{b-}_q\mu(\xi)$.

We say μ is left $q-h$-differentiable on $(a,x+h)$, if for each of its point $\mathcal{C}_h\mathcal{D}^{a^+}_q\mu(\xi)$ exists, and μ is called right $q-h$-differentiable on $(\xi+h,b)$, if at each of its point $\mathcal{C}_h\mathcal{D}^{b-}_q\mu(\xi)$ exists. One can see that $\mathcal{C}_h\mathcal{D}^{a^+}_q\mu(b) = \mathcal{C}_h\mathcal{D}^{b-}_q\mu(a)$. In (38), by setting $h=0$ one can get the q-derivative defined in Definition 1, i.e. $\mathcal{C}_0\mathcal{D}^{a^+}_q\mu(\xi) = {}_a\mathcal{D}_q\mu(\xi)$. Also for $a=0$ one can have $\mathcal{C}_h\mathcal{D}^{0^+}_q\mu(\xi) = \mathcal{C}_h\mathcal{D}_q\mu(\xi)$, i.e. the $q-h$-derivative given in (20) is deduced; for $h=0=a$ one can have $\mathcal{C}_0\mathcal{D}^{0^+}_q\mu(\xi) = \mathcal{D}_q\mu(\xi)$, i.e. the q-derivative is deduced; for $a=0$, $q=1$ one can have $\mathcal{C}_h\mathcal{D}^{0^+}_1\mu(\xi) = \mathcal{D}_h\mu(\xi)$ i.e. the h-derivative is deduced; for $h=0=a$ and taking limit $q\to 1$ one can get the usual derivative for a differentiable function μ i.e. $\lim\limits_{q\to 1}\mathcal{C}_0\mathcal{D}^{0^+}_q\mu(\xi) = \frac{d}{d\xi}\mu(\xi)$. One can get similar results from equation (39). The definition of left and right $q-$derivatives defined on I can be obtained from (39) by setting $h=0$ as follows:

Definition 5. *Let $0 < q < 1$, $h \in \mathbb{R}$, $\xi \in I$ and $\mu : I \to \mathbb{R}$ be a continuous function. Then left $q-$derivative $\mathcal{D}_q^{a^+}\mu$ and right $q-$derivatives $\mathcal{D}_q^{b-}\mu$ on I are defined as follows:*

$$\mathcal{D}_q^{a^+}\mu(\xi) := \frac{\mu(q\xi + (1-q)a) - \mu(\xi)}{(1-q)(a-\xi)};\ \xi > a, \tag{40}$$

$$\mathcal{D}_q^{b-}\mu(\xi) := \frac{\mu(qb + (1-q)\xi) - \mu(b)}{(1-q)(\xi - b)};\ \xi < b. \tag{41}$$

It is notable that from (40), we have $\mathcal{D}_q^{0^+}\mu(\xi) = \mathcal{D}_q\mu(\xi)$ i.e. the left $q-$derivative coincides with $q-$derivative defined in Definition 1.

Definition 6. *Let $0 < q < 1$ and $\mu : I = [a, b] \to \mathbb{R}$ be a continuous function. Then left $q-h$-integral $I_{q,h}^{a+}\mu$ and right $q-h$-integral $I_{q-h}^{b}\mu$ on I are defined as follows:*

$$\begin{aligned} I_{q,h}^{a+}\mu(\xi) &:= \int_a^{\xi} \mu(\gamma)\, {}_h d_q\gamma \qquad (42)\\ &= ((1-q)(\xi - a) + qh)\sum_{n=0}^{\infty} q^n \mu(q^n a + (1-q^n)\xi + nq^n h),\ \xi > a,\\ I_{q,h}^{b-}\mu(\xi) &:= \int_{\xi}^{b} \mu(\gamma)\, {}_h d_q\gamma \qquad (43)\\ &= ((1-q)(b-\xi) + qh)\sum_{n=0}^{\infty} q^n \mu(q^n \xi + (1-q^n)b + nq^n h),\ \xi < b. \end{aligned}$$

Example 2. *Consider $\mu(\gamma) = \gamma - a$ and $\nu(\gamma) = b - \gamma$. The left and right $q-h$-integrals are calculated as follows:*

$$I_{q,h}^{a+}\mu(\xi) = \int_a^{\xi} (\gamma - a)\, {}_h d_q\gamma = \frac{(1-q)(\xi - a) + qh}{1-q}\left(\frac{q(\xi - a)}{1+q} + (1-q)h\sum_{n=0}^{\infty} nq^{2n}\right) \tag{44}$$

and

$$I_{q,h}^{b-}\nu(\xi) = \int_{\xi}^{b} (b - \gamma)\, {}_h d_q\gamma = \frac{(1-q)(b-\xi) + qh}{1-q}\left(\frac{b-\xi}{1+q} + (1-q)h\sum_{n=0}^{\infty} nq^{2n}\right), \tag{45}$$

where 1 is the radius of convergence of the series involved in above integrals.

Example 3. *Let $\mu(\gamma) = \xi - \gamma$ and $\nu(\gamma) = \gamma - \xi$. Then we have*

$$I_{q,h}^{a+}\mu(\xi) = \int_a^{\xi} (\xi - \gamma)\, {}_h d_q\gamma = \frac{(1-q)(\xi - a) + qh}{1-q}\left(\frac{\xi - a}{1+q} - (1-q)h\sum_{n=0}^{\infty} nq^{2n}\right) \tag{46}$$

and

$$I_{q,h}^{b-}\nu(\xi) = \int_{\xi}^{b} (\gamma - \xi)\, {}_h d_q\gamma = \frac{(1-q)(b-\xi) + qh}{1-q}\left(\frac{q(b-\xi)}{1+q} + (1-q)h\sum_{n=0}^{\infty} nq^{2n}\right), \tag{47}$$

where 1 is the radius of convergence of the series involved in above integrals.

By setting $h = 0$, left and right q-integrals can be obtained and defined as follows:

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Definition 7. *Let $0 < q < 1$ and $\mu : I = [a, b] \to \mathbb{R}$ be a continuous function. Then the left q-integral $I_q^{a+}\mu$ and right q-integral $I_q^b\mu$ on I are given by;*

$$I_{q-0}^{a+}\mu(\xi) = I_q^{a+}\mu(\xi) = \int_a^\xi \mu(\gamma)d_q\gamma = (1-q)(\xi - a)\sum_{n=0}^{\infty} q^n \mu(q^n a + (1-q^n)\xi),\ \xi > a, \tag{48}$$

$$I_{q-0}^{b-}\mu(\xi) = I_q^{b-}\mu(\xi) = \int_\xi^b \mu(\gamma)d_q\gamma = (1-q)(b-\xi)\sum_{n=0}^{\infty} q^n \mu(q^n \xi + (1-q^n)b),\ \xi < b. \tag{49}$$

Left q-integral is same as q_a-definite integral, and right q-integral is same as q^b-definite integral defined in [2] and [1] respectively.

Example 4. *Consider $\mu(\gamma) = \gamma - a$ and $\nu(\gamma) = b - \gamma$. By setting $h = 0$ in Example 2, one can have $I_{q-0}^{a+}\mu(\xi) = I_q^{a+}\mu(\xi) = \int_a^\xi (\gamma - a)d_q\gamma = \frac{q(\xi-a)^2}{1+q}$ and $I_{q-0}^{b-}\nu(\xi) = I_q^{b-}\mu(\xi) = \int_\xi^b (b-\gamma)d_q\gamma = \frac{(b-\xi)^2}{1+q}$.*

By considering $q \to 1$, one can have left and right h-integrals defined in upcoming definition.

Definition 8. *Let $\mu : I = [a, b] \to \mathbb{R}$ be a continuous function. Then the left h-integral $I_h^{a+}\mu$ and right h-integral $I_h^b\mu$ on I are defined as follows:*

$$I_h^{a+}\mu(\xi) = \lim_{q\to 1} I_{q,h}^{a+}\mu(\xi),\ \xi > a, \tag{50}$$

$$I_h^{b-}\mu(\xi) = \lim_{q\to 1} I_{q,h}^{b-}\mu(\xi),\ \xi < b. \tag{51}$$

It is noted from Definition 6 that $I_{q,h}^{a+}\mu(b) = I_{q,h}^{b-}\mu(a) = \int_a^b \mu(\gamma)\,{}_h d_q t$.

4. Some $q - h$-integral inequalities for convex functions

In this section we give inequalities for $q - h$-integrals of convex functions. A function $\mu : [a, b] \to \mathbb{R}$ is called convex if the following inequality holds for all $u, v \in [a, b]$ and $\lambda \in [0, 1]$:

$$\mu(\lambda u + (1-\lambda)v) \le \lambda\mu(u) + (1-\lambda)\mu(v) \tag{52}$$

Theorem 1. *Let $\mu : J \to \mathbb{R}$ be a convex function. Also, let $a, b \in J^\circ$, the interior of J. The left and right $q - h$-integrals satisfy the following inequalities:*

$$I_{q,h}^{a+}\mu(\xi) \le \frac{(1-q)(\xi-a)+qh}{(1-q)(\xi-a)}\left\{\mu(a)\left(\frac{\xi-a}{1+q} - (1-q)hS\right) + \mu(\xi)\left(\frac{q(\xi-a)}{1+q} + (1-q)hS\right)\right\}, \tag{53}$$

and

$$I_{q,h}^{b-}\mu(\xi) \le \frac{(1-q)(b-\xi)+qh}{(1-q)(b-\xi)}\left\{\mu(\xi)\left(\frac{b-\xi}{1+q} + (1-q)hS\right) + \mu(b)\left(\frac{q(b-\xi)}{1+q} + (1-q)hS\right)\right\}, \tag{54}$$

where $S = \sum_{n=0}^{\infty} nq^{2n}$.

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Proof. For $\gamma \in [a, \xi]$, we have $\frac{\xi-\gamma}{\xi-a} \in [0,1]$. By selecting $\lambda = \frac{\xi-\gamma}{\xi-a}$, $u = a$, $v = \xi$ in (52) we get the following inequality:

$$\mu(\gamma) \leq \frac{\xi-\gamma}{\xi-a}\mu(a) + \frac{\gamma-a}{\xi-a}\mu(\xi).$$

By taking $q-h$-integral over $[a, \xi]$ we have

$$\in \gamma_a^{\xi}\mu(\gamma)\,{}_h d_q\gamma \leq \frac{\mu(a)}{\xi-a} \in \gamma_a^{\xi}(\xi-\gamma)\,{}_h d_q\gamma + \frac{\mu(\xi)}{\xi-a}\mu \in \gamma_a^{\xi}(\gamma-a)\,{}_h d_q\gamma.$$

By using values of integrals involved in above inequality from (44) and (46), one can obtain the required inequality (53). On the other hand for $\gamma \in [\xi, b]$, we have $\frac{b-\gamma}{b-\xi} \in [0,1]$. By selecting $\lambda = \frac{b-\gamma}{b-\xi}$, $u = \xi$, $v = b$ in (52) we get the following inequality:

$$\mu(\gamma) \leq \frac{b-\gamma}{b-\xi}\mu(\xi) + \frac{\gamma-\xi}{b-\xi}\mu(b).$$

By taking $q-h$-integral over $[\xi, b]$ we have

$$\in \gamma_{\xi}^{b}\mu(\gamma)\,{}_h d_q\gamma \leq \frac{\mu(\xi)}{b-\xi} \in \gamma_{\xi}^{b}(b-\gamma)\,{}_h d_q\gamma + \frac{\mu(b)}{b-\xi} \in \gamma_{\xi}^{b}(\gamma-\xi)\,{}_h d_q\gamma.$$

By using values of integrals involved in above inequality from (45) and (47), one can obtain the required inequality (54). □

Corollary 1. *As an application of the above theorem, the following inequalities for left and right q-integrals hold:*

$$I_q^{a+}\mu(\xi) \leq \mu(a)\left(\frac{\xi-a}{1+q}\right) + \mu(\xi)\left(\frac{q(\xi-a)}{1+q}\right), \tag{55}$$

and

$$I_q^{b-}\mu(\xi) \leq \mu(\xi)\left(\frac{b-\xi}{1+q}\right) + \mu(b)\left(\frac{q(b-\xi)}{1+q}\right). \tag{56}$$

Remark 2. *By taking $\xi = b$ in (55) or $\xi = a$ in (56), one can obtain the following inequality:*

$$\frac{1}{b-a} \in \gamma_a^b\mu(\gamma)\,{}_a d_q\gamma \leq \frac{\mu(a)+q\mu(b)}{1+q}. \tag{57}$$

The above inequality (57) is independently proved in [2, Theorem 12].

The following lemma is required to prove the next result.

Lemma 1 ([4]). *Let $\mu : [a, b] \to \mathbb{R}$ be a convex function. If μ is symmetric about $\frac{a+b}{2}$, then the following inequality holds:*

$$\mu\left(\frac{a+b}{2}\right) \leq \mu(\xi), \tag{58}$$

for all $\xi \in [a, b]$.

Theorem 2. *If μ is symmetric about $\frac{a+b}{2}$ along with the assumptions of Theorem 1, then the following inequality holds:*

$$\mu\left(\frac{a+b}{2}\right) \leq \frac{1-q}{(1-q)(\xi-a)+qh}\int_a^{\xi}\mu(\gamma)\,{}_hd_q\gamma + \frac{1-q}{(1-q)(b-\xi)+qh}\int_{\xi}^{b}\mu(\gamma)\,{}_hd_q\gamma, \quad \xi \in [a,b]. \tag{59}$$

Proof. A convex function symmetric about $\frac{a+b}{2}$ satisfies the inequality (58). Therefore by taking $q-h$-integration of (58) over $[a,\xi]$ we have

$$\mu\left(\frac{a+b}{2}\right)\frac{(1-q)(\xi-a)+qh}{1-q} \leq \int_a^{\xi}\mu(\gamma)\,{}_hd_q\gamma. \tag{60}$$

On the other hand by taking $q-h$-integration of (58) over $[\xi,b]$ we have

$$\mu\left(\frac{a+b}{2}\right)\frac{(1-q)(b-\xi)+qh}{1-q} \leq \int_{\xi}^{b}\mu(\gamma)\,{}_hd_q\gamma. \tag{61}$$

By adding (60) and (61), one can get the inequality (59). □

Remark 3. *By taking $x=b$ in (60) or $x=a$ along with $h=0$ in (61), one can obtain the following inequality:*

$$\mu\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^{b}\mu(\gamma)\,d_q\gamma. \tag{62}$$

The above inequality (62) is independently proved in [2, Theorem 3.2], but unfortunately the proof is not correct, see [6, Example 5]. But here we have imposed an additional condition of symmetric function to get the result. Hence if we impose a condition of symmetry in addition to assumptions of [2, Theorem 3.2], we get the correct result.

Conclusions

This article aims to provide a base in unifying the theory of $q-$ and h-derivatives given in [5] by Kac and Cheung. In this effort, the notion of $q-h$-derivative is introduced which generates q-derivative and h-derivative. The $q-h-$binomial $(\xi-a)^n_{h,q}$ analogue to $(\xi-a)^n$ is defined, which generates $q-$binomial $(\xi-a)^n_q$ and $h-$binomial $(\xi-a)^n_h$ in particular. The $q-h$-derivatives of $q-h-$binomial $(\xi-a)^n_{h,q}$ are found which generate q-derivative of $q-$binomial $(\xi-a)^n_q$ and h-derivative of $h-$binomial $(\xi-a)^n_h$ in particular. Rest of the theory in [5] needs attention of researchers, it may be unified. Also, $q-h$-derivatives and integrals are defined on an interval $[a,b]$, which are used to establish some inequalities which are linked with recent research and provide correct proof of an inequality of [2].

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