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A Sustainable Closed-Loop Supply Chains Inventory Model Considering Optimal Number of Remanufacturing Times

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Abstract: The mathematical modelling of reverse logistics inventory systems ignores that returned items may arrive out of sequence, i.e., with different number of remanufacturing times. Moreover, such modelling assumes that the retuned items may retain the same quality upon recovery regardless of how many times they have been previously remanufactured. This paper develops a new mathematical expression of the percentage of retuned items that can be remanufactured a finite number of times. The proposed expression is modelled as a function of the expected number of times an item can be remanufactured on its life cycle and the number of times an item can be technologically (or optimally) remanufactured based on its quality upon recovery. The model developed in this paper considers joint production and remanufacturing options. The return rate is a varying demand dependent rate, which is a decision variable with demand, product deterioration, manufacturing and remanufacturing rates being arbitrary functions of time. The model considers the initial inventory of returned items in the mathematical formulation, which enables decision makers to adjust all functions and input parameters for subsequent cycles. Illustrative examples indicate that dependent purchasing price of recovery items and the incorporation of remanufacturing investment cost significantly impact on the optimal remanufacturing policy.

Keywords: reverse logistics; number of remanufacturing times; first remanufacturing cycle; time-varying parameters; demand dependent return rate

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1. Introduction

Reverse logistics emerges as an opportunity beyond traditional logistics role with the main purpose being product returns from end customers for recapturing value or proper disposal (Fleischmann et al., 1997). Further, reverse logistics has been implemented to address economic drivers, government pressure/legislation, social interests and environmental consciousness. The goal of reverse logistic is to effectively manage and control the flow of products returned from end customers to extend their useable lives, reduce solid waste disposal and conserve natural resource consumption (De Brito and Dekker, 2004; Rogers and Tibben-Lembke, 2001). The importance of reverse logistics may vary from one industry to another due to relevant costs and the dynamic nature of production and remanufacturing processes of the products. The reverse logistics process is a merroir image of the traditional forward supply chain one. It comprises activities such as the collection of returned items from end users, their inspection, their processing, their disassembly and finally their distribution for recovery purposes (Bei and Linyan, 2005; De Brito and Dekker, 2004). Whereas a closed-loop supply chain is categorised by the combination of forward and reverse supply chain activities (Bei and Linyan, 2005; Guide et al., 2003).

Supply chain management in reverse logistics has growing attention in recent years. Moreover, due to global competitiveness, there has been more focus among large companies to adopt joint production and remanufacturing options in their businesses (Andrade et al., 2013; Rubio and Jiménez-Parra, 2017). For example, in Germany, about 10% of engines and starter engines are remanufactured (Bras 2007). In this regard, the remanufactured products save 80% of raw materials, require 33% of the labour force, consume 50% of the energy and up to 50%–70% less cost when compared with the newly manufactured products (Cao et al. 2020; Liu et al. 2020; Van Nguyen et al. 2020; Wang et al. 2020). Several companies including BMW and Volkswagen focus on accelerating the upgrading process of older cars and offer fully warrantied service for remanufactured engines and other parts (Flapper et al. 2005; Liu et al. 2020). Therefore, reverse logistics can enhance productivity, reduce costs, improve profitability, meet total product demand and avoid the tarnished reputation associated with customer loyalty (Fleischmann, 2001).

Beyond the economic benefits, there exists a plethora of factors such as social and environmental consciousness and government legislation that may force manufacturers to include such product recovery systems in their businesses (Montabon et al., 2016).

2. Literature review

The first inventory model with returned items was conducted by Schrady (1967). He developed a deterministic Economic Order Quantity (EOQ) model for repaired items with the assumption that manufacturing, and recovery (repair) rates are instantaneous with no disposal cost. Nahmias and Rivera (1979) generalised the model of Schrady (1967) for the case of finite repair rate. Richter (1996a, 1996b, 1997), Richter and Dobos (1999) and Dobos and Richter (2000) carried out several investigations into the EOQ repair and waste disposal model, where the return rate is considered as a decision variable. Richter (1996a, 1996b) considered a modified version of the model of Schrady (1967) by investigating multiple repair and production cycles within a time interval. Dobos and Richter (2003) investigated a manufacturing/recycling system for non-instantaneous manufacturing and recycling rates.

Richter (1997) examined the optimal inventory holding policy when the waste disposal (return) rate is a decision variable. The result of his paper is that the optimal policy is governed by a pure (bang–bang) strategy of either no repair (total waste disposal) or no waste disposal (total repair). Dobos and Richter (2004) generalised their previous work (Dobos and Richter, 2003) for multiple repair and production cycles. Results indicated that a pure strategy is optimal compared to a mixed strategy. Dobos and Richter (2006) assumed that some collected returned items are not always suitable for further recycling. There are numerous studies that relax different assumptions made so far. Examples of these works are cited in (Bazan et al. 2016).

El Saadany and Jaber (2010) considered the collection rate of returned items to be dependent on the purchasing price and the use proportion of these returns. Their results showed that a mixed (production + remanufacturing) strategy is optimal, when compared to a pure strategy as suggested by Dobos and Richter (2003, 2004). Alamri (2011) generalised the first model of El Saadany and Jaber (2010) and verified the examples given in Dobos and Richter (2003, 2004) and El Saadany and Jaber (2010). He showed that a mixed strategy dominates a pure strategy.

El Saadany and Jaber (2008) pointed out that previous studies assumed an infinite planning horizon and did not account for the first cycle as there are no returned items to be remanufactured. They rectified a minor error in the work of Richter (1996a, 1996b) and, consequently, their model produces a lower cost because of the residual inventory assumed in Richter's model. Kozlovskaya et al. (2017) generalised the work of El Saadany and Jaber (2008) and corrected the optimal solution for their model. They showed that the optimal policy depends on the disposal rate. Although El Saadany and Jaber (2008) have provided a closed form formula for the first cycle, their mathematical formulation as well as the other studies in the literature are alike. Alamri (2021) discussed this issue in detail and addressed this limitation by incorporating the initial inventory of returned items in the mathematical formulation. He showed that the optimal policy implies that the cumulative inventory for returned items vary for each cycle before the system plateaus. This is a key consideration that allows the decision maker to change the values of the input parameters for subsequent cycles.

In this section, we have cited research that are directly relevant to this paper. For more details about inventory models related to reverse logistics systems, see (Govindan et al., 2015 Modak. et al. 2023).

3. Research background and contribution

In this section, we address some issues that are related to the number of times a product can be remanufactured as advocated in El Saadany et al. (2013), followed by some discussion that elaborates on our research contributions. Meanwhile, the work of Alamri (2021) constitutes the base model of this research.

3.1. Theoretical background and motivation

El Saadany et al. (2013) developed a mathematical expression that indicates the number of times a product can be remanufactured. They attempted to relax the general assumption that a product can be remanufactured for an indefinite number of times. They assumed that an item can be recovered for a limited (ξ) number of times. When the system plateaus, then for any ξ, a fraction β_ξ of a constant demand rate d is remanufactured and $(1-\beta_\xi)$ is produced, where $\beta_\xi = 1 - \frac{1-\beta}{1-\beta^{\xi+1}}$ and $\beta(0 < \beta < 1)$ is the proportion of used items returned for remanufacturing when an item is recovered an indefinite number of times. It is worth noting here that the mathematical expression used to derive β_ξ focused on the returns of what was produced on previous period and ignored the rest of cumulative produced quantities that have been left or previously being remanufactured. Interested readers are referred to Table 1 in El Saadany et al. (2013). They stated that as $\xi \longrightarrow \infty$, $\beta_\xi = 1 - \frac{1-\beta}{1-\beta^{\infty}} = \beta$, which is what has been suggested in the existing literature. For a pure production case, i.e., $\xi = 0$, $\beta_\xi = 1 - \frac{1-\beta}{1-\beta^{0+1}} = 0$, though a pure production strategy implies that $\beta = 0$, i.e., there are no items returned for recovery purposes. Moreover, the mathematical expression used to derive β_ξ assumes that no waste disposal (total repair) of the proportion β upon recovery. Then, they modified the work of Richter (1997) and Teunter (2001) by replacing β with β_ξ in Richter's and Teunter's models.

Table 1. The actual quality level of an item that recovers ξ number of times when $\tau = 1,2,\ldots 8$.

1	2	3	4	5	6	7	8
0.368	0.607	0.717	0.779	0.819	0.846	0.867	0.882
	0.368	0.513	0.607	0.670	0.717	0.751	0.779
		0.368	0.472	0.549	0.607	0.651	0.687
			0.368	0.449	0.513	0.565	0.607
				0.368	0.435	0.490	0.535
					0.368	0.424	0.472
						0.368	0.417
							0.368

In their model, the produced quantity $(1-\beta_\xi)d$ is also disposed outside the system (e.g., Bazan et al., 2015) since they have defined α as the disposal rate, where $\alpha(\alpha = 1-\beta_\xi)$. Moreover, the role of β in their model is somewhat ambiguous. Therefore, we can distinguish three cases: (1) As can be seen from Figure 1 in El Saadany et al. (2013), $(\alpha+\beta_\xi)d$ entering the repairable stock from which $\beta_\xi d$ is remanufactured and $\alpha d = (1-\beta_\xi)d$ is disposed. This implies that the return rate is d, however, this contradicts what the existing literature suggests, i.e., the return rate is less than demand rate; (2) $\beta_\xi = 1 - \frac{1-\beta}{1-\beta^{\xi+1}}$, which is a function of β, and therefore, the value of β is used to compute β_ξ. In their examples, β is defined as the collection of used items and β_ξ is the effective proportion. In this case (case 2), one can deduce that βd enters the repairable stock from which $\beta_\xi \beta d$ is remanufactured and $(1-\beta_\xi)\beta d$ is disposed. However, β represents the proportion of used items returned for remanufacturing when an item is recovered an indefinite number of times and only β_ξ of β is remanufactured; (3) The return rate is $\beta_\xi d$, which enters the repairable stock and flows in the serviceable stock to be remanufactured with no waste disposal (total repair). That is, the purpose of β, which represents (the proportion of used items returned for remanufacturing when an item is recovered an indefinite number of times) is to compute β_ξ. In this case (case 3), the system should collect β_ξ instead of β, which is reflected in their modified version of the work of Richter (1997) and Teunter (2001). Hence, we can conclude that in all cases, β is used to compute β_ξ, with case 2 being the most appropriate scenario. However, β_ξ in all these cases, has no relation with an item being recovered for a limited (ξ) number of times. In fact, ξ is an arbitrary integer value, which implanted in β_ξ to minimise the total cost. Therefore, considering a fraction $\gamma(0 \leq \gamma \leq 1)$ of the return rate that meets the acceptance quality level to be remanufactured and $(1-\gamma)$ is disposed outside the system is more practicable (Dobos and Richter, 2004; El Saadany and Jaber, 2010, Alamri, 2011; Alamri, 2021).

The difference between β and β_ξ represents about 50% when $\xi = 1$ and $\beta = 0.9$ (see Figure 1). Moreover, for a fixed value of β, this difference decreases with ξ (see Figure 2 in El Saadany et al. (2013) page 600 and Figure 1 in this paper). This seems logical in their expression, since as $\xi \longrightarrow \infty$, $\beta_\xi \longrightarrow \beta$ because they assumed that all returned items have been remanufactured ξ number of times. On the contrary, however, this difference should increase as a returned item with a greater number of remanufacturing times recovers with inferior quality. Furthermore, implementing β_ξ as suggested by El Saadany et al. (2013) would result in a large disposal quantity especially for products that are associated with relatively small number of recovery times since β_ξ increases with ξ (see Figure 2). Finally, incorporating such β_ξ in the mathematical formulation entails that all returned items have been remanufactured ξ number of times. However, fact remains that returned items may arrive out of sequence, i.e., with different number of remanufacturing times assuming also that previous number of remanufacturing times is labelled. It is true to say that considering such classification of returned items in the mathematical expression is not an easy task, however, this limitation will be discussed in the next section.

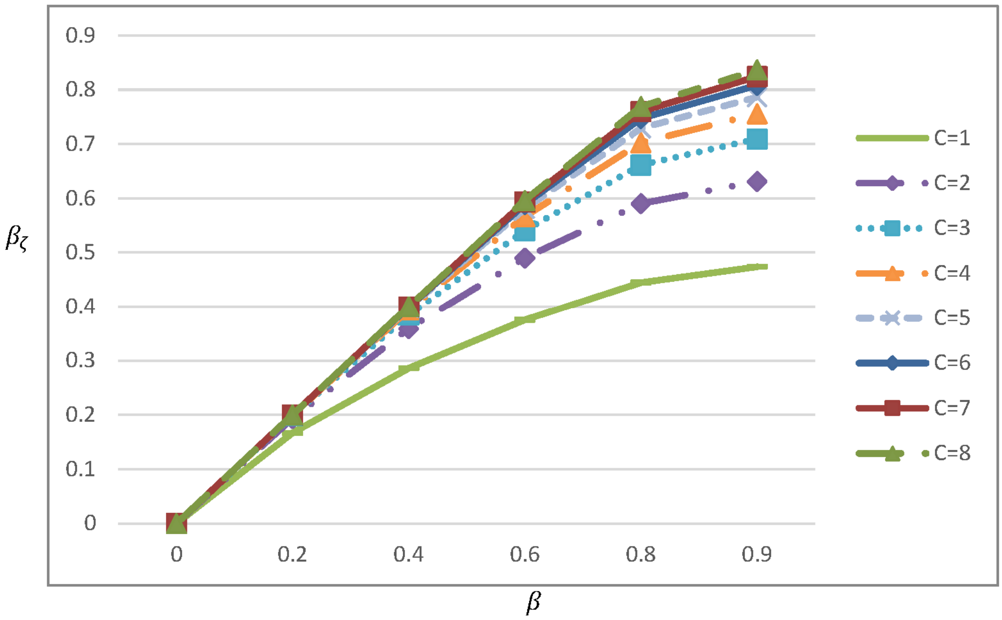

Figure 1. The behaviour of β_ξ (a reproduction of β_ξ as in El Saadany et al. (2013)).

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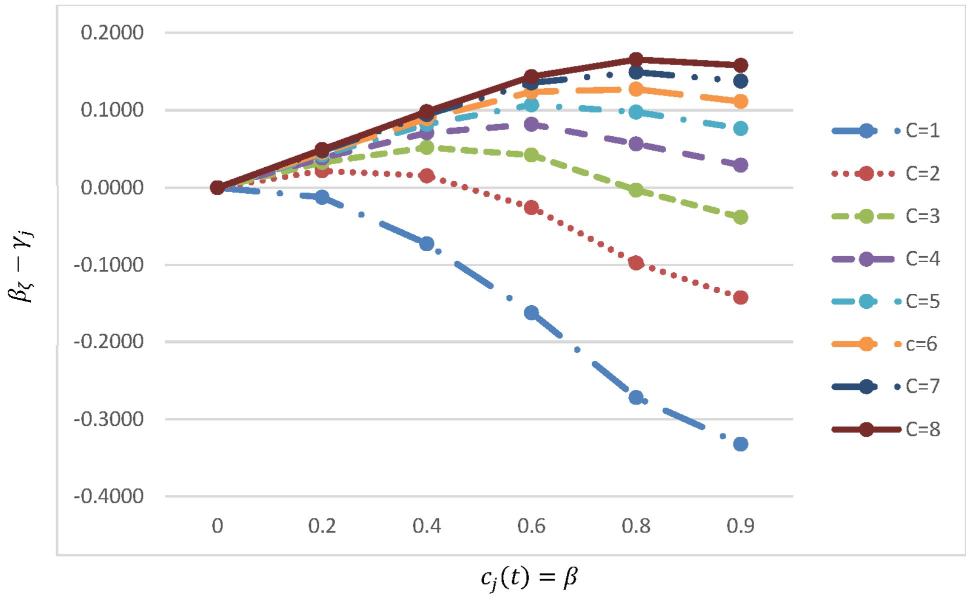

Figure 2. The difference between β_ξ and γ_j.

3.2. Mathematical formulation of the recovery times

The comprehensive discussion in the previous section is necessary to position our study in the existing literature and highlights its research contribution. This paper aims to enhance this line of research by developing a new mathematical expression that models the percentage of returns as a function of the number of times an item is recovered, the corresponding quality for the recovery item and the expected number of times an item can be remanufactured on its life cycle. In this paper, we assume that returned items are collected at a rate of $c_j(t)$ (decision variable) where j denotes the cycle index. Note that a pure production strategy occurs when $c_j(t) = 0$.

Only a fraction $\gamma_{\xi j}$ of these retuned items can be remanufactured. Namely, $\gamma_{\xi j} = e^{\frac{-\xi q_\xi}{\tau}}$, where $q_{\xi j}$ $(0 < q_{\xi j} < 1)$ denotes the quality level of an item that has been recovered ξ number of times. We assume that $q_{\xi j} = e^{\frac{-\xi}{\tau}}$, where $q_{0j} = 1$, i.e., it refers to the quality of a newly manufactured item. In this paper, ξ refers to the maximum number of times an item can be technologically (or optimally) remanufactured and $\tau \geq \xi$ denotes the expected number of times an item can be remanufactured on its life cycle. Note that $q_{\xi j}$ decreases as ξ increseas and it attains a minimum value as $\xi \longrightarrow \tau$, in this case, $q_{\xi j} = q_\tau = q_{min} = e^{-1}$ (Table 1). Accordingly, a recovery item with a quality less than the minimum acceptance quality level q_{min} is considered defective and incurs a disposal cost. Note that $\gamma_{\xi j}' < 0$ and $\gamma_{\xi j}'' > 0$ $\forall\, \xi > 0$, i.e., $\gamma_{\xi j}$ is a monotonically decreasing function over ξ and, as $\tau \longrightarrow \infty, \gamma_{\xi j} \longrightarrow 1$. The same arguments hold true for $q_{\xi j}$. This implies that $\gamma_{\xi j}$ is modelled as a function of the expected number of times an item can be remanufactured on its life cycle and the number of times an item can be technologically remanufactured based on its quality upon recovery. Moreover, $\gamma_{\xi j}$ and $q_{\xi j}$ are free from any judgmental measurements.

In real life settings, returned items may recover out of sequence. This can be attributed to random number of times these items have been remanufactured. Let us define the returned amount for cycle j as R_j, where this amount undergoes a 100 per cent inspection. Assuming an automated remanufacturing system, the observation of R_j seems realistic since all returned items are inspected. Therefore, returned items that are subjected to a 100 per cent screening upon recovery to the repairable stock would imply that $\underline{R_j} = \left(r_{\xi j}, r_{\xi-1 j}, \dots, r_{0j}\right)$. That is, r_{kj} is the collected used/returned items with $k(k = \xi, \xi - 1, \dots, 0)$ is the number of times these items have been previously remanufactured. Here, r_{0j} denotes returned items that have not yet being remanufactured, and $r_{\xi j}$ refers to defective (disposed) returned items that have been remanufactured ξ number of times or items that do not meet the minimum acceptance quality level, q_{min}.

It is worth noting here, that the above-mentioned classification seams realistic because the system can deal with items based on such classification upon recovery. Therefore, as the number of times an item can be recovered increases, its corresponding use proportion decreases. This finding, however, contradicts that of El Saadany et al. (2013). To justify this, suppose that among the returned quantity that can be remanufactured say, 5 times there exists a sub-quantity arrived at the repairable stock with its first-time recovery. In this case, implementing $\gamma_{\xi j}$ for this sub-quantity would result in disposing an equal fraction as that of items with a greater recover time, though these items have not yet been remanufactured. Table 2 depicts the corresponding use proportion where $\underline{\gamma_{ij}} = (\gamma_{1j}, \gamma_{2j}, \dots \gamma_{\xi j})$. For instance, if a quantity of returned items that can be remanufactured say, 5 times, then each sub-quantity is associated with its corresponding accepted fraction, i.e., $\gamma_{5j} = 0.692, \gamma_{4j} = 0.698, \gamma_{3j} = 0.719, \gamma_{2j} = 0.765$ *and* $\gamma_{1j} = 0.849$. That is, γ_{ij} represents the use proportion of the sub-quantity of the retuned items that can be remanufactured for its i^{th} remanufacturing time. However, considering the above-mentioned classification in the mathematical formulation emerges as a challenge that affects the tractability problems in modelling. Therefore, to tackle this issue, we suggest that $\gamma_j = \frac{\sum_{i=1}^{\xi} \gamma_{ij}}{\xi}$ which constitutes an approximation of the average fraction (cumulative average up to ξ) that can be remanufactured in cycle j (Figure 3).

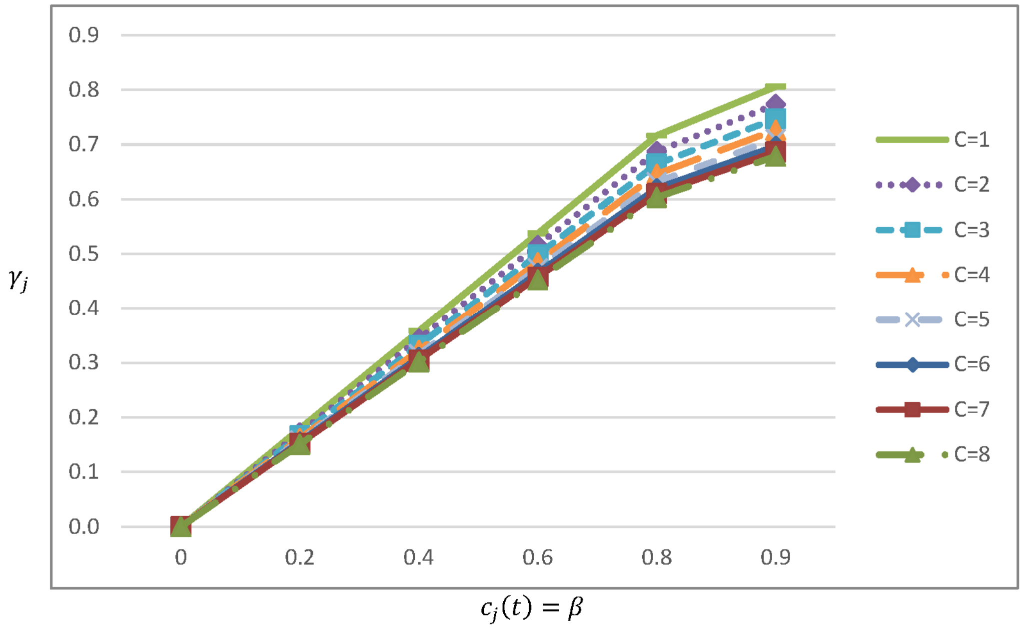

Figure 3. The behaviour of γ_j that flows to be remanufactured in cycle j.

Table 2. The actual proportion of returned items that can be remanufactured ξ number of times when $\tau = 1,2, \dots 8$.

1	2	3	4	5	6	7	8
0.692	0.738	0.788	0.823	0.849	0.868	0.884	0.896
	0.692	0.710	0.738	0.765	0.788	0.807	0.823
		0.692	0.702	0.719	0.738	0.756	0.773
			0.692	0.698	0.710	0.724	0.738
				0.692	0.696	0.705	0.716
					0.692	0.695	0.702
						0.692	0.694
							0.692

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As can be seen from Table 2, assuming $\gamma_{\xi j}$ to be remanufactured in cycle j implies that the proportion $(1-\gamma_{\xi j})c_j(t)$ is disposed outside the system. Conversely, $(1-\gamma_j)c_j(t) \leq (1-\gamma_{\xi j})c_j(t)$, i.e., γ_j considers the cumulative average up to time ξ of returned items (Table 4). This is a key consideration because it governs the behaviour of returned items and ensures reducing the disposal of unnecessary amount. In addition, returned items are coupled with distinct purchasing price $c_{prj} = c_{pm}\, e^{\frac{-1}{q_j}}$, where c_{pm} denotes unit purchasing price for new items.

Table 3. The average fraction (cumulative average up to ξ) of the quality level of items that recover for their i^{th} time when $\tau = 1,2,\ldots 8$.

1	2	3	4	5	6	7	8
0.368	0.607	0.717	0.779	0.819	0.846	0.867	0.882
	0.487	0.615	0.693	0.745	0.782	0.809	0.831
		0.533	0.619	0.679	0.723	0.757	0.783
			0.556	0.622	0.671	0.709	0.739
				0.571	0.624	0.665	0.698
					0.581	0.625	0.660
						0.588	0.626
							0.593

Table 4. The average fraction (cumulative average up to ξ) of returned items that can be remanufactured for their i^{th} time when $\tau = 1,2,\ldots 8$.

1	2	3	4	5	6	7	8
0.692	0.738	0.788	0.823	0.849	0.868	0.884	0.896
	0.715	0.749	0.781	0.807	0.828	0.845	0.859
		0.730	0.754	0.778	0.798	0.816	0.830
			0.739	0.758	0.776	0.793	0.807
				0.745	0.760	0.775	0.789
					0.749	0.762	0.775
						0.752	0.763
							0.754

3.3. Contribution and organisation of the paper

This paper develops a new mathematical expression that specifies the number of times a product can be remanufactured. In particular, the proposed expression is modelled as a function of the expected number of times an item can be remanufactured on its life cycle and the number of times an item can be technologically (or optimally) remanufactured based on its quality upon recovery.

In this paper, we present a general reverse logistics inventory model with a single manufacturing cycle and a single remanufacturing cycle. Demand, deterioration, manufacturing, remanufacturing and return rates are arbitrary functions of time. Therefore, a diverse range of time-varying forms can be disseminated from the general model. The mathematical formulation consists of serviceable and reparable stocks. The serviceable stock is for new and remanufactured items and the reparable stock is for collecting returned items to be remanufactured in the serviceable stock as good as new. Therefore, different holding costs and deterioration rates are considered for manufactured, remanufactured and returned items (e.g., Alamri, 2011; Alamri et al., 2016; Jaber and El Saadany, 2009; Teunter, 2001).

Only a proportion of the returned items that specifies the number of times a product can be remanufactured flows in the serviceable stock. In the first remanufacturing cycle, the initial inventory of retuned items is zero since there are no returned items to be remanufactured. Therefore, the accumulated amount of returned items (during the time gap of non-production and non-remanufacturing processes) represents the initial inventory of returns for the second cycle. This amount, indeed, should differ from that accumulated for subsequent cycles. This is key in our formulation, and therefore, ensures that all optimal values vary for each cycle before the system plateaus. The proposed model accounts for setup changeover costs when switching from manufacturing phase to remanufacturing phase. The proposed model also considers an investment cost associated with the number of times a product is recovered. We assume that returned, manufactured and remanufactured items deteriorate while they are effectively in stock. The return rate of the returned items is a decision variable, which is a function of the demand rate. The purchasing price of returned items is a function of the purchasing price of new items and the quality of items upon recovery. All functions and input parameters can be adjusted for subsequent cycles.

The remainder of the paper is organised as follows: In Section 4, we present our joint manufacturing and remanufacturing model and the solution procedure. Illustrative examples, and special cases are offered in Section 5. Managerial insights and concluding remarks are provided in Sections 6 and 7 respectively.

4. Mathematical formulation of the general model

4.1. Assumptions and notations

Our model is developed under the following notations:

j The cycle index;

z $(z = gm, gr, r)$ gm is for manufactured items, gr is for remanufactured items and r is for returned items;

$I_{zj}(t)$ The inventory level at time t;

$P_{mj}(t)$ The manufactured rate per unit time for new items;

$P_{rj}(t)$ The remanufactured rate per unit time for returned items;

$D_j(t)$ The demand rate per unit time;

$c_j(t)$ The return rate per unit time for returned items (decision variable), where $c_j(t) = \emptyset_j D_j(t)$, and $0 \le \emptyset_j < 1$;

$\delta_{zj}(t)$ The deterioration rate per unit time;

d_{zj} The deteriorated quantity for cycle j;

Q_{mj} The manufactured quantity for cycle j;

Q_{rj} The remanufactured quantity for cycle j;

R_j The returned quantity for cycle j;

Δ_j The accumulated quantity of returned items (during the time gap of non-production and non-remanufacturing processes);

ξ The maximum number of times an item can be remanufactured;

τ The expected number of times an item can be remanufactured on its life cycle, where $\tau \ge \xi$;

$q_{\xi j}$ The actual quality level of an item that has been recovered ξ number of times in cycle j, where $q_{\xi j} = e^{\frac{-\xi}{\tau}}$ (Table 1);

q_j The average fraction (cumulative average up to ξ) of the quality level of items that have been recovered for their i^{th} time in cycle j, where $q_j = \frac{\sum_{i=1}^{\xi} q_{ij}}{\xi}$ (Table 3);

$\gamma_{\xi j}$ The actual proportion of returned items that can be remanufactured in cycle j, where $\gamma_{\xi j} = e^{\frac{-\xi q_\xi}{\tau}}$ (Table 2);

γ_j The average fraction (cumulative average up to ξ) of returned items that can be remanufactured for their i^{th} time in cycle j, where $\gamma_j = \frac{\sum_{i=1}^{\xi} \gamma_{ij}}{\xi}$ (Table 4);

c_{pm} The unit purchasing cost for new items;

c_{prj} The unit purchasing price for retuned items in cycle j, where $c_{prj} = c_{pm} e^{\frac{-1}{q_j}}$;

c_{inv} The remanufacturing investment cost in the design process of an item to, technologically, be able to remanufacture it τ number of times;

c_{invj} The remanufacturing investment cost in cycle j in the design process of an item to, technologically, be able to remanufacture it ξ number of times, where $c_{invj} = c_{inv}\left(1 - e^{\frac{-\xi}{q_j}}\right)$;

c_m The unit manufacturing cost;

c_r The unit remanufacturing cost;

c_s The unit screening cost;

c_w The unit disposal cost for deteriorated and scrap items;

h_z The holding cost per unit per unit time;

S_z The set-up/order cost per cycle;

w_m The switching cost from remanufacturing phase to manufacturing phase;

w_r The switching cost from manufacturing phase to remanufacturing phase;

Below is a list of all assumptions used in the paper:

1. Returned items are collected throughout the time interval at a rate $c_j(t)$.
2. Only a proportion $\gamma_j (0 \le \gamma_j \le 1)$ of the returned items can be remanufactured and the amount $(1-\gamma_j)c_j(t)$ is disposed as waste outside the system.
3. New items are manufactured at a rate $P_{mj}(t)$ and the accepted returned items are remanufactured at a rate $P_{rj}(t)$ as good as new.
4. The demand rate $D_j(t)$ is satisfied from produced and remanufactured items.
5. Items deteriorate at a rate $\delta_{zj}(t)$ while they are effectively in stock, and there is no repair or replacement of deteriorated items.
6. The demand, product deterioration, manufacturing, and remanufacturing rates are arbitrary functions of time.
7. The return rate is a varying demand dependent rate, which is a decision variable.
8. The values of all functions and input parameters can be adjusted for subsequent cycles.
9. Shortages are not allowed, i.e., we require that

$$P_{mj}(t) > D_j(t), P_{rj}(t) > \gamma_j c_j(t) \text{ and } P_{rj}(t) > D_j(t)\ \forall\ t \ge 0.$$

4.2. The general model

In our model, demand in the first cycle is satisfies from production only (see Figure 5), as the inventory of returned items in the first cycle is zero (there are no returned items to be remanufactured). The process is repeated until inventory of product returns can be technologically attainable. Then, at the beginning of each cycle j, the system starts the production prosses until time T_{1j}, by which point in time Q_{mj} units have been produced and stored in the serviceable stock. At time T_{2j}, the inventory level of new items becomes zero and d_{gmj} units have deteriorated, which refers to the difference between the satisfied demand during production cycle and Q_{mj} units that have been manufactured in cycle j. The remanufacturing process starts at time T_{2j} until time T_{3j}, by which point in time the remanufactured quantity Q_{rj} units have been accumulated and stored in the serviceable stock. The returned items are collected throughout the time interval at a rate $c_j(t)$, in which a fraction $\gamma_j c_j(t)$ has been remanufactured and the remaining quantity $(1-\gamma_j)c_j(t)$ is disposed as waste outside the system. The remaining quantity $(1-\gamma_j)c_j(t)$ refers to returned items that have been remanufactured ξ number of times or items that do not meet the minimum acceptance quality level, q_{min}. The inventory level of remanufacturing items becomes zero by time T_{4j} and d_{grj} units have deteriorated, which refers to the difference between the satisfied demand during remanufacturing cycle and Q_{rj} units that have been remanufactured in cycle j. At time

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T_{4j}(the end of cycle j), Δ_j units have been accumulated and stored in the repairable stock, which constitutes the initial inventory of returned items for the next cycle. In our model, the term Δ_{j-1} governs the behaviour of each cycle and at the beginning of the first remanufacturing cycle, Δ_{j-1}= Δ_0= 0. That is, the initial inventory of returned items in the first remanufacturing cycle is set equal to zero. The deteriorated quantity in the repairable stock is d_{rj}, which denotes the difference between the returned quantity that have been accepted to be remanufactured and Q_{rj} units that have been remanufactured in cycle j. The process is repeated. Figure 4 depicts a general framework of production and remanufacturing unified system, and Figure 5 depicts the behaviour of such a unified system.

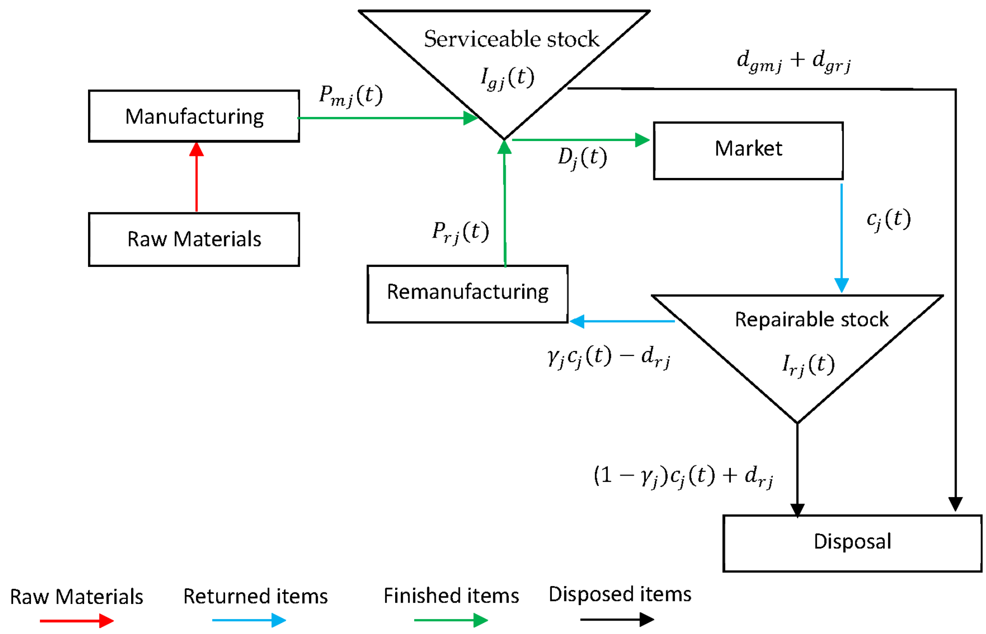

Figure 4. Products flow for production and remanufacturing system in one cycle.

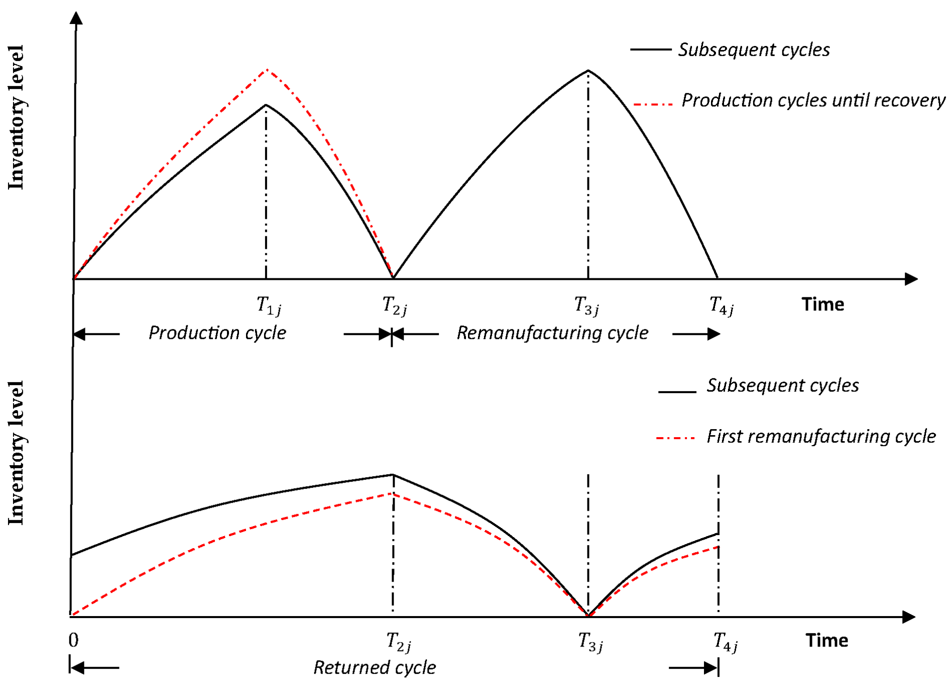

Figure 5. Inventory variation of manufactured, remanufactured and returned items for one cycle.

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The variations in the inventory levels depicted in Figure 5 are given by the following differential equations:

$$\frac{dI_{gmj}(t)}{dt} = P_{mj}(t) - D_j(t) - \delta_{gmj} I_{gmj}(t), \qquad 0 \le t < T_{1j} \quad (1)$$

$$\frac{dI_{gmj}(t)}{dt} = -D_j(t) - \delta_{gmj} I_{gmj}(t), \qquad T_{1j} \le t \le T_{2j} \quad (2)$$

$$\frac{dI_{grj}(t)}{dt} = P_{rj}(t) - D_j(t) - \delta_{grj} I_{grj}(t), \qquad T_{2j} \le t < T_{3j} \quad (3)$$

$$\frac{dI_{grj}(t)}{dt} = -D_j(t) - \delta_{grj} I_{grj}(t), \qquad T_{3j} \le t \le T_{4j} \quad (4)$$

$$\frac{dI_{rj}(t)}{dt} = \gamma_j c_j(t) - \delta_{rj} I_{rj}(t), \qquad 0 \le t < T_{2j} \quad (5)$$

$$\frac{dI_{rj}(t)}{dt} = \gamma_j c_j(t) - P_{rj}(t) - \delta_{rj} I_{rj}(t), \qquad T_{2j} \le t < T_{3j} \quad (6)$$

$$\frac{dI_{rj}(t)}{dt} = \gamma_j c_j(t) - \delta_{rj} I_{rj}(t), \qquad T_{3j} \le t \le T_{4j} \quad (7)$$

with the boundary conditions:

$$I_{gmj}(0) = 0, I_{gmj}(T_{2j}) = 0, I_{grj}(T_{2j}) = 0, I_{grj}(T_{4j}) = 0, I_{rj}(0) = \Delta_{j-1} \text{ and } I_{rj}(T_{3j}) = 0.$$

Considering the boundary conditions, the solutions of the above differential equations are:

$$I_{gmj}(t) = e^{-h_{gmj}(t)} \int_0^t [P_{mj}(u) - D_j(u)] e^{h_{gmj}(u)} du, \qquad 0 \le t < T_{1j} \quad (8)$$

$$I_{gmj}(t) = e^{-h_{gmj}(t)} \int_t^{T_{2j}} D_j(u) e^{h_{gmj}(u)} du, \quad T_{1j} \le t \le T_{2j} \quad (9)$$

$$I_{grj}(t) = e^{-h_{grj}(t)} \int_{T_{2j}}^t [P_{rj}(u) - D_j(u)] e^{h_{grj}(u)} du, \qquad T_{2j} \le t < T_{3j} \quad (10)$$

$$I_{grj}(t) = e^{-h_{grj}(t)} \int_t^{T_{4j}} D_j(u) e^{h_{grj}(u)} du, \qquad T_{3j} \le t \le T_{4j} \quad (11)$$

$$I_{rj}(t) = e^{-\left(h_{rj}(t) - h_{rj}(0)\right)} \Delta_{j-1} + e^{-h_{rj}(t)} \int_0^t [\gamma_j c_j(u)] e^{h_{rj}(u)} du, \quad 0 \le t < T_{2j} \quad (12)$$

$$I_{rj}(t) = e^{-h_{rj}(t)} \int_t^{T_{3j}} [P_{rj}(u) - \gamma_j c_j(u)] e^{h_{rj}(u)} du, \quad T_{2j} \le t < T_{3j} \quad (13)$$

$$I_{rj}(t) = e^{-h_{rj}(t)} \int_{T_{3j}}^t \gamma_j c_j(u) e^{h_{rj}(u)} du, \qquad T_{3j} \le t \le T_{4j} \quad (14)$$

respectively, where

$$h_{zj}(t) = \int \delta_{zj}(t)dt. \quad (15)$$

From Equations (1)-(14), we note that each function is solely modelled and, therefore, functions may or may not be related to each other.

The per cycle total cost components for the underlying inventory model are given as:

Purchase price for returned items (c_{prj}) + Inspection cost (c_s) + Disposal cost for waste and deteriorated items (c_w) + Material cost for new items (c_{pm}) + Manufacturing cost (c_m) + Remanufacturing cost (c_r) + Holding costs (h_z)+ Switching cost for manufacturing (w_m) + Switching costs for remanufacturing (w_r) + Investment cost (c_{invj}) + Set-up and order costs (S_z)=

$$\left(c_{pm}\, e^{\frac{-1}{q_j}} + c_s + c_w(1-\gamma_j)\right)\int_0^{T_{4j}} c_j(u)du + c_w\left(d_{gmj} + d_{grj} + d_{rj}\right) + \left(c_{pm} + c_m\right)\int_0^{T_{1j}} P_{mj}(u)du + c_r \int_{T_{2j}}^{T_{3j}} P_{rj}(u)du + h_z + w_m + w_r + c_{inv}\left(1 - e^{\frac{-\xi}{q_j}}\right) + S_{pm} + S_{pr} + S_r.$$

Now, denote $K = w_m + w_r + S_{pm} + S_{pr} + S_r$, and $d_j = d_{gmj} + d_{grj} + d_{rj}$.

As can be seen, the returned, manufacturing and remanufacturing costs cover and include deteriorated items. This is very well recognised in the literature because items deteriorate while they are effectively in stock (e.g., Inderfurth et al. 2005; Jaggi et al. 2015; Alamri et al. 2016; Polotski et al. 2019).

From Equations (8)-(14), the holding costs are as follows:

Holding costs for produced and remanufactured items at the serviceable stock:

$$h_{gm}\left[I_{gmj}(0, T_{1j}) + I_{gmj}(T_{1j}, T_{2j})\right] + h_{gr}\left[I_{grj}(T_{2j}, T_{3j}) + I_{grj}(T_{3j}, T_{4j})\right].$$

Holding cost for returned items at the repairable stock:

$$h_r\left[I_{rj}(0, T_{2j}) + I_{rj}(T_{2j}, T_{3j}) + I_{rj}(T_{3j}, T_{4j})\right]$$

Therefore, the per unit time total cost function of the unified inventory model during the cycle $[0, T_{4j}]$, as a function of T_{1j}, T_{2j}, T_{3j} and T_{4j} denoted by $L(T_{1j}, T_{2j}, T_{3j}, T_{4j})$ is given by

$$L(T_{1j}, T_{2j}, T_{3j}, T_{4j}) = \frac{1}{T_{4j}}\Bigg\{\left(c_{pm}\, e^{\frac{-1}{q_j}} + c_s + c_w(1-\gamma_j)\right)\int_0^{T_{4j}} c_j(u)du + \left(c_{pm} + c_m\right)\int_0^{T_{1j}} P_{mj}(u)du + c_r \int_{T_{2j}}^{T_{3j}} P_{rj}(u)du + h_{gm}\left[-\int_0^{T_{1j}} H_{gmj}(u)\left[P_{mj}(u) - D_j(u)\right]e^{h_{gmj}(u)}du + H_{gmj}(T_{1j})\int_0^{T_{1j}}\left[P_{mj}(u) - D_j(u)\right]e^{h_{gmj}(u)}du + \int_{T_{1j}}^{T_{2j}}\left[H_{gmj}(u) - H_{gmj}(T_{1j})\right]D_j(u)e^{h_{gmj}(u)}du\right] + h_{gr}\left[\int_{T_{2j}}^{T_{3j}}\left[H_{grj}(T_{3j}) - H_{grj}(u)\right]\left[P_{rj}(u) - D_j(u)\right]e^{h_{grj}(u)}du + \int_{T_{3j}}^{T_{4j}}\left[H_{grj}(u) - H_{grj}(T_{3j})\right]D_j(u)e^{h_{grj}(u)}du\right] + h_r\Bigg[\left[H_{rj}(T_{2j}) - H_{rj}(0)\right]e^{h_{rj}(0)}\Delta_{j-1} + H_{rj}(T_{2j})\int_0^{T_{3j}}\gamma_j c_j(u)e^{h_{rj}(u)}du + \int_{T_{2j}}^{T_{3j}}\left[H_{rj}(u) - H_{rj}(T_{2j})\right]P_{rj}(u)e^{h_{rj}(u)}du - \int_0^{T_{4j}} H_{2j}(u)\gamma_j c_j(u)e^{h_{2j}(u)}du + H_{rj}(T_{4j})\int_{T_{3j}}^{T_{4j}}\gamma_j c_j(u)e^{h_{rj}(u)}du\Bigg] + c_w d_j + c_{inv}\left(1 - e^{\frac{-\xi}{q_j}}\right) + K\Bigg\}, \quad (16)$$

where

$$H_{zj}(t) = \int e^{-h_{zj}(t)}dt. \quad (17)$$

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Note that Equation (16) is a modified version of that of Alamri (2021). Therefore, and to avoid repetition, the existence, uniqueness and global optimality of the solution can be obtained by a quite similar way. Interested readers are referred to (Alamri, 2011; 2021).

The variables $T_{ij}, i(i = 1,2,3,4)$ that minimise $L(T_{ij})$ given by Equation (16) are governed by the following relations:

$$T_{1j} < T_{2j} < T_{3j} < T_{4j}, \tag{18}$$

$$\int_0^{T_{1j}} P_{mj}(u)e^{h_{gmj}(u)}du = \int_0^{T_{2j}} D_j(u)e^{h_{gmj}(u)}du, \tag{19}$$

$$\int_{T_{2j}}^{T_{3j}} P_{rj}(u)e^{h_{rj}(u)}du = e^{h_{rj}(0)}\Delta_{j-1} + \int_0^{T_{3j}} \gamma_j c_j(u)e^{h_{rj}(u)}du, \tag{20}$$

$$\int_{T_{2j}}^{T_{3j}} P_{rj}(u)e^{h_{grj}(u)}du = \int_{T_{2j}}^{T_{4j}} D_j(u)e^{h_{grj}(u)}du, \tag{21}$$

$$R_j = \int_0^{T_{4j}} c_j(u)du, \tag{22}$$

$$\Delta_{j-1} = e^{-h_{rj-1}(T_{4j-1})} \int_{T_{3j-1}}^{T_{4j-1}} \gamma_{j-1}c_{j-1}(u)e^{h_{rj-1}(u)}du. \tag{23}$$

For example, relations 19 and 20 guarantee that the inventory levels for the production and the remanufacturing phases have equal values for $t = T_1$ and for $t = T_3$. Note that the term Δ_{j-1} is modelled as a deterministic value, i.e., it impacts the behaviour of each cycle until the system plateaus. This is a key in the mathematical formulation and, consequently, it ensures that the model is a viable solution for each cycle, whether the input parameters change their values or remain static (Alamri (2021)).

It can be seen from Eq. (19) that $P_{mj}(t) > D_j(t) \Longrightarrow$ Eq. (19) $\Leftrightarrow T_{1j} < T_{2j}$. Also, from Equation (19), $T_{1j} = 0 \Longrightarrow T_{2j} = 0 \Longrightarrow$ a pure strategy of no manufacturing option. In this case, $P_{rj}(t) > D_j(t) \Longrightarrow$ Eq. (21) $\Leftrightarrow T_{3j} < T_{4j}$. Conversely, $T_{3j} = 0 \Longrightarrow T_{4j} = 0$. Thus, from Equations (21) and (22), $T_{2j} = T_{3j} \Longrightarrow T_{3j} = T_{4j} \Longrightarrow T_{1j} < T_{2j} \Longrightarrow$ a pure strategy of no remanufacturing option. Conversely, $T_{1j} = 0 \Longrightarrow T_{2j} = 0 \Longrightarrow T_{3j} = 0 \Longrightarrow T_{4j} = 0$. Thus, $T_{1j} > 0 \Longrightarrow T_{1j} < T_{2j}$ and $T_{2j} < T_{3j} \Longrightarrow < T_{3j} < T_{4j}$. Hence, Equations (19)-(22) implies constraint (18), and, therefore, constraint (18) can be ignored. Thus, our goal is to solve the following objective function:

$$(Z) = \left\{ \begin{array}{c} \text{minimise } L(T_{1j}, T_{2j}, T_{3j}, T_{4j}) \text{ given by Eq. (16)} \\ \text{subject to} \\ \text{Equations}(19-22) \\ 0 \leq \emptyset_j < 1 \text{ (24)} \end{array} \right\}.$$

4.2.1. Solution procedure

As can be seen from Equations (19)-(22) that T_{ij} can be obtained as functions of R_j, where

$$T_{ij} = f_{ij}(R_j). \tag{25}$$

Taking also into account Equations (19)-(22), the objective function (Z) is reduced to the following function with the variable R_j (say (Z_1)) subject to $0 \leq \emptyset_j < 1$.

$$L(R_j) = \frac{1}{f_{4j}}\Bigg\{\bigg(c_{pm}\, e^{\frac{-1}{q_j}} + c_s + c_w(1-\gamma_j)\bigg)\int_0^{f_{4j}} c_j(u)du + (c_{pm} + c_m)\int_0^{f_{1j}} P_{mj}(u)du + c_r \int_{f_{2j}}^{f_{3j}} P_{rj}(u)du +$$

$$h_{gm}\left[-\int_0^{f_{1j}} H_{gmj}(u)P_{mj}(u)e^{h_{gmj}(u)}du + \int_0^{f_{2j}} H_{gmj}(u)D_j(u)e^{h_{gmj}(u)}du\right] +$$

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$$h_{gr}\left[-\int_{f_{2j}}^{f_{3j}} H_{grj}(u)P_{rj}(u)e^{h_{grj}(u)}du + \int_{f_{2j}}^{f_{4j}} H_{grj}(u)D_j(u)e^{h_{grj}(u)}du\right] + h_r\left[-H_{rj}(0)e^{h_{rj}(0)}\Delta_{j-1} + \right.$$

$$\left.\int_{f_{2j}}^{f_{3j}} H_{rj}(u)P_{rj}(u)e^{h_{rj}(u)}du + \int_{f_{3j}}^{f_{4j}} H_{rj}(f_{4j})\gamma_j c_j(u)e^{h_{rj}(u)}du - \int_0^{f_{4j}} H_{rj}(u)\gamma_j c_j(u)e^{h_{rj}(u)}du\right] + c_w d_j +$$

$$c_{inv}\left(1 - e^{\frac{-\xi}{q_j}}\right) + K\Bigg\}, \tag{26}$$

where $h_{zj}(t)$ is given by Equation (15) and $H_{zj}(t)$ is given by Equation (17).

Hence, if $L = \frac{l}{f_{4j}}$, then the necessary condition for having a minimum for (Z_1) is

$$l'_{R_j} f_{4j} = f'_{4j,R_j} l, \tag{27}$$

where l'_{R_j} and f'_{4j,R_j} represent, respectively, the derivatives of l and f_{4j} with respect to R_j.

Now considering Equation (26), then we obtain

$$l'_{R_j} = \left(c_{pm}\, e^{\frac{-1}{q_j}} + c_s + c_w(1-\gamma_j)\right) + (c_{pm} + c_m)f'_{1j,R_j}P_{mj}(f_{1j}) + c_r P_{rj}\left(f'_{3j,R_j}(f_{3j}) - f'_{2j,R_j}(f_{2j})\right) +$$

$$h_{gm}\left[-H_{gmj}(f_{1j})f'_{1j,R_j}P_{mj}(f_{1j})e^{h_{gmj}(f_{1j})} + H_{gmj}(f_{2j})f'_{2j,R_j}D_j(f_{2j})e^{h_{gmj}(f_{2j})}\right] +$$

$$h_{gr}\left[-H_{grj}(f_{3j})f'_{3j,R_j}P_{rj}(f_{3j})e^{h_{grj}(f_{3j})} + H_{grj}(f_{2j})f'_{2j,R_j}P_{rj}(f_{2j})e^{h_{grj}(f_{2j})} + \right.$$

$$\left.H_{grj}(f_{4j})f'_{4j,R_j}D_j(f_{4j})e^{h_{grj}(f_{4j})} - H_{grj}(f_{2j})f'_{2j,R_j}D_j(f_{2j})e^{h_{grj}(f_{2j})}\right] + h_r\left[\Delta_j + \right.$$

$$\left.f'_{3j,R_j}H_{rj}(f_{3j})P_{rj}(f_{3j})e^{h_{rj}(f_{3j})} - f'_{2j,R_j}H_{rj}(f_{2j})P_{rj}(f_{2j})e^{h_{rj}(f_{2j})} - H_{rj}(f_{4j})f'_{3j,R_j}\gamma_j c_j(f_{3j})e^{h_{rj}(f_{3j})}\right]. \tag{28}$$

From which, Equation (27) ⇔ $L = \frac{l}{f_{4j}} = \frac{l'_{R_j}}{f'_{4j,R_j}}$. (29)

Equation (29) can, now, be used to obtain the optimal value of R_j and its corresponding total minimum cost. Then the optimal values of $T_{ij}, i(i = 1, 2,3,4)$ can be obtained from Equations (19)-(22), respectively.

To find the optimal ξ for a given τ, the following steps are required:

1. In the first remanufacturing cycle, start by setting $\xi = 1$, $c_{invj} = c_{inv\xi}, c_{prj} = c_{pr\xi}, \lambda_j = \lambda_\xi$ *and* $\Delta_{j-1} = \Delta_0 = 0$ in Equation (29) and compute L_1
2. Repeat step 1 for Δ_{j-1}(obtained from step 1) to compute $L_{2,1}$.
3. Set $\xi = 2$, $c_{invj} = c_{inv\xi}, c_{prj} = c_{pr\xi}, \lambda_j = \lambda_\xi$ and Δ_{j-1} (obtained from step 1) in Equation (29) and compute L_2
4. Repeat step 3 for Δ_{j-1}(obtained from step 3) to compute $L_{3,2}$.
5. Set $\xi = 3$, $c_{invj} = c_{inv\xi}, c_{prj} = c_{pr\xi}, \lambda_j = \lambda_\xi$ and Δ_{j-1} (obtained from step 3) in Equation (29) and compute L_3
6. Repeat step 5 for Δ_{j-1}(obtained from step 5) to compute $L_{4,3}$.
7. Repeat steps 5 and 6 for $\xi = 4, 5, \ldots, \tau$ and Δ_{j-1}(obtained to find L_{j-1}) to compute $L_{j,\xi}$
8. Set $\xi^* = \xi$ when $(L_{j,\xi})$ at its minimum and continue to insert Δ_{j-1} in Equation (29) until the system plateaus.

Remark: For a mature system, applying the above steps will generate the optimal remanufacturing policy, where Δ_{j-1} represents the current on hand inventory of returned items.

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5. Illustrative examples for different sittings

In this section, we present numerical examples and special cases that reflect different realistic situations. Products that may encounter remanufacturing include tyres, motor vehicle parts, electric motors, computers, air-conditioning units, photocopiers, telecommunication equipment, aerospace devices, aircraft parts, gaming machines, medical equipment, vending machines, automotive parts, industrial equipment, televisions, etc. (Statham, 2006). In real life settings, manufacturing, remanufacturing, demand, return and deterioration rates may vary with time or with any other factors (Alamri and Balkhi, 2007; Alamri and Syntetos 2018; Benkherouf et al., 2014; Datta et al., 1998; Grosse et al., 2013; Hariga and Benkherouf, 1994; Karmarkar and Pitbladdo, 1997; Omar and Yeo, 2009; Sana, 2010). Accordingly, the proposed model allows the incorporation of different forms of time-varying functions. Let us now consider the following functions for time-varying rates:

$P_{mj}(t) = \pi_{mj}t + \phi_{mj}, P_{rj}(t) = \pi_{rj}t + \phi_{rj}, D_j(t) = \alpha_j t + r_j, c_j(t) = \emptyset_j D_j(t)$ and $\delta_{zj}(t) = \frac{l_{zj}}{\vartheta_{zj}-\beta_{zj}t}$, where $\phi_{mj}, \phi_{rj}, r_j, \emptyset_j, \vartheta_{zj} > 0, \pi_{mj}, \pi_{rj}, \alpha_j, l_{zj}, \beta_{zj} \geq 0$ and $\beta_{zj}t < \vartheta_{zj}$.

Note that $\delta_{zj}(t)$ is an increasing function of time.

In real life sitting, all function or input parameters are subject to adjustment due to external competitiveness and/or internal challenges or due to price fluctuations. Therefore, our model is viable if all values are adjusted for subsequent cycles.

The objective function (Z_1) has been coded in *MATLAB* for the input parameters that are presented in Table 5 below and solutions were obtained using Equation (29) subject to $0 \leq \emptyset_j < 1$. Note that each of the return, manufacturing and remanufacturing rates is solely modelled. This is so because they may or may not be considered as functions of the demand rate. Now, let us consider the following functions for varying return, manufacturing and remanufacturing rates as functions of the demand rate:

$$c_j(t) = \emptyset_j D_j(t),\ P_{mj}(t) = \frac{D_j(t)}{0.6} \text{ and } P_{rj}(t) = \frac{D_j(t)}{0.3}.$$

Table 5. Input parameters for time-varying rates.

h_{gm}	h_{gr}	h_r	r_j	α_j	ϕ_{1j}
1.6 Dollars/unit/month	1.6 Dollars/unit/month	1.2 Dollars/unit/month	1000 Unit/month	130 Unit/month	1666.7 Unit/month
π_{1j} 216.7 Unit/month	ϕ_{2j} 3333.3 Unit/month	π_{2j} 433.3 Unit/month	l_{gm} 1 Unit/month	l_{gr} 1 Unit/month	l_r 1 Unit/month
ϑ_{gm} 50 Unit/month	ϑ_{gr} 50 Unit/month	ϑ_r 40 Unit/month	β_{gm} 0.25 Unit/month	β_{gr} 0.25 Unit/month	β_r 0.25 Unit/month
c_{inv} 4000 Dollars/cycle	w_m 100 Dollars/cycle	w_r 100 Dollars/cycle	S_{pm} 2400 Dollars/cycle	S_{pr} 1600 Dollars/cycle	S_r 1200 Dollars/cycle
c_w 0.2 Dollars/unit	c_m 2 Dollars/unit	c_r 1.2 Dollars/unit	c_{pm} 5 Dollars/unit	c_s 0.5 Dollars/unit	

5.1. Example 1

In this example, we investigate the effect of the first remanufacturing cycle on the behaviour of the model with respect to the parameters that are listed in Table 5. In this example, we consider $\tau = 5$, i.e., the expected number of times an item can be remanufactured on its life cycle is 5. In this case,

$c_{invj} = c_{inv}\,(1 - e^{\frac{-\xi}{q_j}})$ and $c_{prj} = c_{pm}\,e^{\frac{-1}{q_j}}$. The optimal values of $\phi_j^*, f_{4j}^*, Q_{mj}^*, Q_{rj}^*, R_j^*, \Delta_j^*, d_j^*, L_j^*$ and l_j^* until the system plateaus are obtained and the results are shown in Table 6. In the first remanufacturing cycle, we have taken $\gamma_1 = 0.849$ (recall Table 4) resulting in a total number of $R_1^* = 2406$ units. This retuned quantity has been accumulated by time $T_{41}^* = 2.954$ months ≈ 90 days at a return rate of $\phi_1^* = 0.683$ or 68.3% of the demand rate. At time T_{4j}, $\Delta_1^* = 571$ units, which constitutes the initial inventory of returned items for the second cycle. The deteriorated quantity in the serviceable stock is $d_{g1} = 27\ (d_{gm1} = 16$ and $d_{gr1} = 11)$ units and $d_{r1} = 38$ units have deteriorated in the repairable stock, i.e., $d_1^* = 16 + 11 + 38 = 65$ units. This deteriorated quantity can be sold at a salvage price or (as in this example) incur a disposal charge. The optimal produced quantity is $Q_{m1}^* = 2113$ units, which has been accumulated by time $T_{11}^* = 1.178$ months ≈ 36 days to satisfy demand until time $T_{21}^* = 1.87$ months ≈ 57 days (the time by which the remanufacturing process started). The optimal remanufactured quantity is $Q_{r1}^* = 1434$ units, which has been accumulated by time $T_{31}^* = 2.21$ months ≈ 67 days to satisfy demand until time $T_{41}^* = 2.954$ months ≈ 90 days. The total minimum cost per month is $L_1^* = 11332$ dollars and the total minimum cost for the first remanufacturing cycle is $l_1^* = 33475$ dollars.

It is worth noting here that in the first remanufacturing cycle, the initial inventory of returned items is zero as there are no returned items to be remanufactured. Accordingly, $Q_{rj}^*(Q_{mj}^*)$ attain their minimum (maximum) values in this cycle resulting in a dramatic decrease in the manufactured quantity in the second cycle. Note that, ϕ_j^*, Δ_j^* and R_j^* attain their minimum values in the second cycle because of the effect of the first cycle. Moreover, cycles $j = 2, 3, 4$ and 5 are influenced by $c_{invj}, c_{prj}, \Delta_{j-1}$ and λ_j and, consequently, the optimal values vary from cycle to cycle and $\phi_j^*, \Delta_j^*, Q_{rj}^*$ and R_j^* reach their maximum values in the fifth cycle. As a result, f_{4j}^*, L_j^*, l_j^* and Q_{mj}^* approach their minimum values in the sixth cycle before the system plateaus in the eighth cycle (Table 6). Therefore, when the system plateaus, the buyback proportion is set equal to $\phi_8^* = 0.776$ and the use proportion is set equal to $\lambda_5 = 0.745$, which is equivalent to a reusable proportion $\phi_8^*\lambda_5 = 0.776 \times 0.745 = 0.5781$ or 57.8% of demand rate. Figure 7 depicts the effect of $c_{invj}, c_{prj}, \Delta_{j-1}$ and λ_j on the behaviour of the optimal values until the system plateaus. As can be seen, in cycles $j = 1, 2, .., 5$, all returned items have been remanufactured less than or equal to $j - 1$ number of times upon recovery and less than or equal to $\zeta_j^* - 1$ when recovered for subsequent cycles. This implies that the number of times an item can be remanufactured is tangible and tractable. Finally, in cycles $j = 1, 2, .., 5$, $c_{invj}, c_{prj}, \Delta_{j-1}$ and λ_j vary from cycle to cycle. Unlike previous works excluding the work of Alamri (2021), this, indeed, constitutes evidence that our model is viable if the values of the input parameters are distinct for subsequent cycles.

As illustrated in example 1, other forms of varying functions can be disseminated from the general formulation to assess the consequences of distinct strategies.

Table 6. Optimal results for varying rates when $\tau = 5$ and $c_{inv} = 4000\$$.

j	ζ_j^*	c_{invj}	c_{prj}	λ_j^*	ϕ_j^*	f_{4j}^*	Q_{mj}^*	Q_{rj}^*	R_j^*	Δ_j^*	d_j^*	L_j^*	l_j^*
1	1	2821	1.474	0.849	0.683	2.954	2113	1434	2406	571	65	11332	33475
2	2	3727	1.305	0.807	0.614	2.692	1624	1562	1944	530	69	11155	30031
3	3	3952	1.147	0.778	0.688	2.773	1663	1634	2251	598	74	11206	31077
4	4	3994	1.001	0.758	0.736	2.733	1547	1697	2369	646	75	11081	30287
5	5	3999	0.868	0.745	0.791	2.702	1442	1760	2512	707	76	10948	29582
6	5	3999	0.868	0.745	0.771	2.652	1378	1755	2397	688	75	10895	28891
7	5	3999	0.868	0.745	0.778	2.668	1399	1757	2435	694	75	10912	29117
8	5	3999	0.868	0.745	0.776	2.663	1392	1756	2423	692	75	10907	29046
9	5	3999	0.868	0.745	0.776	2.663	1392	1756	2423	692	75	10907	29046

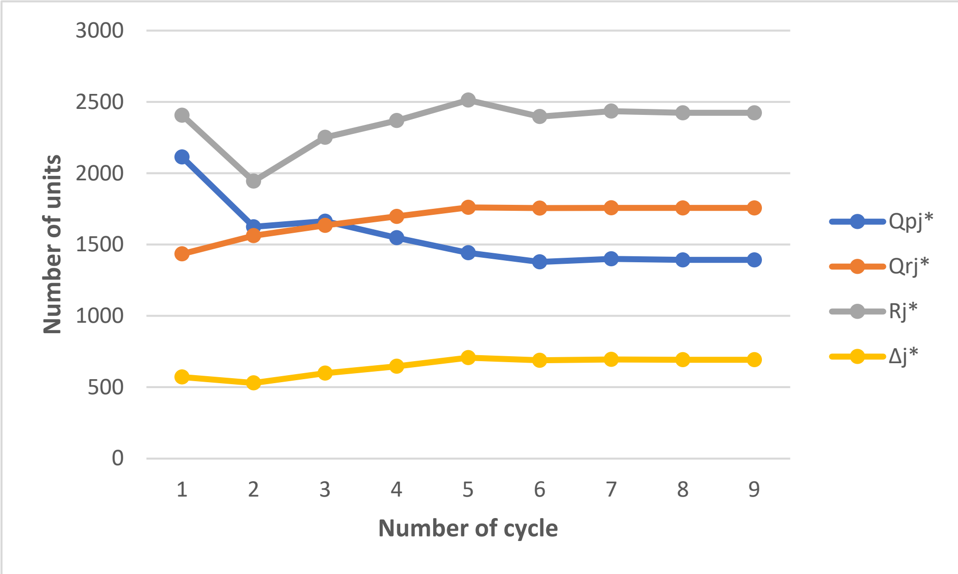

Figure 7. The effect of model parameters on the optimal values when $\tau = 5$ and $c_{inv} = 4000\$$.

5.2. Example 2

In this example, we replicate example 1 to observe the behaviour of the optimal values when $\tau = 3$. As can be seen from Table 7, the optimal values behave similarly when the expected number of times an item can be remanufactured on its life cycle decreases from 3 to 5. The only exception is that the value of R_j^* in the third cycle experiences a slight decrease by 10 units from that accumulated in the first cycle. This can be justified by the fact that the value of Δ_j^* in the second cycle is greater than that accumulated in the first cycle (see Table 6). Note that ϕ_j^*, Δ_j^* and Q_{rj}^* reach their maximum values in the third cycle and f_{4j}^*, L_j^*, l_j^* and Q_{mj}^* approach their minimum values in the fourth cycle before the system plateaus in the eighth cycle (Table 7).

Table 7. Optimal results for varying rates when $\tau = 3$ and $c_{inv} = 4000\$$.

j	ζ_j^*	c_{invj}	c_{prj}	λ_j^*	ϕ_j^*	f_{4j}^*	Q_{mj}^*	Q_{rj}^*	R_j^*	Δ_j^*	d_j^*	L_j^*	l_j^*
1	1	3009	1.238	0.788	0.770	2.981	2089	1498	2741	623	66	11324	33761
2	2	3845	0.983	0.749	0.736	2.684	1497	1679	2320	632	73	11006	29544
3	3	3986	0.765	0.730	0.855	2.716	1415	1806	2731	768	77	10885	29565
4	3	3986	0.765	0.730	0.808	2.604	1277	1793	2460	721	74	10770	28049
5	3	3986	0.765	0.730	0.825	2.645	1325	1798	2558	738	75	10811	28592
6	3	3986	0.765	0.730	0.819	2.630	1308	1796	2522	732	75	10796	28398
7	3	3986	0.765	0.730	0.821	2.636	1315	1797	2535	734	75	10801	28473
8	3	3986	0.765	0.730	0.820	2.634	1312	1797	2530	733	75	10800	28444
9	3	3986	0.765	0.730	0.820	2.634	1312	1797	2530	733	75	10800	28444

5.3. Example 3

In this example, we repeat example 2 by increasing the investment cost, c_{inv} from 4000 dollars to 6000 dollars to investigate the behaviour of the optimal values. Table 8 reveals that the optimal situation in this case is to remanufacture once ($\zeta = 1$). Note that c_{invj}, c_{prj} and λ_j remain static until the system plateaus and, consequently, the only factor affecting the optimal values is Δ_{j-1}. As can be seen from Table 8, the model behaves similarly with respect to ϕ_j^*, Δ_j^* and R_j^* that reach their maximum values in the first cycle and f_{4j}^*, L_j^* and l_j^* attain their minimum values in the second cycle before the system plateaus in the sixth cycle (Table 8). Note that $Q_{rj}^*(Q_{mj}^*)$ reach their minimum (maximum) values in the first cycle since the inventory of returned items is zero (see also Tables 6 and 7). It is worth noting here that $L_{4,3}^* = 11441$ dollars $> L_{2,1}^* = 11351$ dollars (recall solution steps).

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However, when the system plateaus for $\zeta = 3$, the difference between the total minimum cost per month is negligible, i.e., $L^{\zeta=3} = 11464$ dollars > $L^{\zeta=1} = 11428$.

Table 8. Optimal results for varying rates when $\tau = 3$ and $c_{inv} = 6000\$$.

j	ζ_j^*	c_{invj}	c_{prj}	λ_j^*	ϕ_j^*	f_{4j}^*	Q_{mj}^*	Q_{rj}^*	R_j^*	Δ_j^*	d_j^*	L_j^*	l_j^*
1	1	4514	1.238	0.788	0.754	3.224	2318	1614	2940	656	79	11809	38073
2	1	4514	1.238	0.788	0.615	2.768	1647	1644	2010	545	77	11351	31421
3	1	4514	1.238	0.788	0.645	2.859	1772	1645	2187	571	78	11446	32730
4	1	4514	1.238	0.788	0.638	2.838	1743	1645	2145	565	78	11424	32425
5	1	4514	1.238	0.788	0.640	2.843	1749	1644	2154	567	78	11429	32489
6	1	4514	1.238	0.788	0.639	2.843	1749	1645	2153	567	78	11428	32487
7	1	4514	1.238	0.788	0.639	2.843	1749	1645	2153	567	78	11428	32487

5.4. Special cases

Case 1: In this case (Case1), we replicate Tables 6 and 7 to investigate the work of Alamri (2021) for the set of input parameters as listed in Table 5. In Case 1, we let $w_r = w_m = c_{invj} = c_s = 0, c_w = 0.1, c_{prj} = 1, \emptyset_j = 0.231$ and $\lambda_j = 0.875$, which are identical with that of Alamri (2021). Note that $c_j(t) = \emptyset_j D_j(t)$ and an item is recovered an indefinite number of times. By substituting the above values in Equation (29) until the system plateaus, the results are obtained as shown in Table 9. As can be seen, Table 9 is identical with Table 3 page 529 in Alamri (2021). This constitutes evidence that ensures the validity and robustness of our general model.

Table 9. Optimal results for varying rates as in Alamri (2021) with $\emptyset_j = 0.231$ and $\lambda_j = 0.875$.

j	f_{4j}^*	Q_{mj}^*	Q_{rj}^*	R_j^*	Δ_j^*	d_j^*	L_j^*	l_j^*
1	2.454	2373	493	657	69	33	10317	25314
2	2.371	2223	533	632	75	34	10220	24231
3	2.364	2210	536	630	75	34	10211	24140
4	2.364	2210	536	630	75	34	10211	24140

Case 2: In this case (Case2), we investigate the behaviour of the model when the demand rate is adjusted within cycles. In real life sitting, all function or input parameters are subject to adjustment due to external competitiveness and/or internal challenges or due to price fluctuations. Let us now support our finding in Example 1 and show the validity of our model if the input parameters change their values for subsequent cycles. In Case 2, we will illustrate how the system would behave if the decision maker wished to increase the demand rate in the eighth cycle to evaluate the consequences of such increase. In Case 2, we assume that $D_j(t) = \alpha_j t + r_j$, where $\alpha_j = 156$ and $r_j = 1200$. Note that row one of Table 10 represents the results derived for the eighth cycle for example 2 (see Table 7). A comparison between Tables 7 and 10 reveals that in the first cycle of the adjustment, all optimal values increase except f_{4j}^* that encounters a slight decrease. Such increase can be justified by the increase of ϕ_j^* and Δ_j^*. Note that all decision variables attain their maximum (minimum) values in the ninth (tenth) cycle, i.e., in the first (second) cycle of the adjustment of the demand rate.

Table 10. Optimal results for varying rates when $\tau = 3, c_{inv} = 4000\$, \alpha_j = 156$ units and $r_j = 1200$ units.

j	ζ_j^*	c_{invj}	c_{prj}	λ_j^*	ϕ_j^*	f_{4j}^*	Q_{mj}^*	Q_{rj}^*	R_j^*	Δ_j^*	d_j^*	L_j^*	l_j^*
8*	3	3986	0.765	0.730	0.820	2.634	1312	1797	2530	733	75	10800	28444
9	3	3986	0.765	0.730	0.936	2.488	1353	2139	3248	912	76	12099	30106
10	3	3986	0.765	0.730	0.884	2.340	1154	2104	2860	847	71	11925	27908
11	3	3986	0.765	0.730	0.905	2.396	1228	2118	3007	873	74	11992	28737
12	3	3986	0.765	0.730	0.897	2.375	1200	2113	2950	863	73	11966	28419
13	3	3986	0.765	0.730	0.900	2.383	1210	2115	2971	866	73	11976	28538

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14	3	3986	0.765	0.730	0.899	2.380	1205	2115	2965	866	73	11972	28489
15	3	3986	0.765	0.730	0.899	2.380	1205	2115	2965	866	73	11972	28489

Cycle 8*, which represents the steady state situation of Table 7 when $\alpha_j = 130$ units and $r_j = 1000$ units.

Case 3: In this case (Case3), we replicate example 2 to investigate the behaviour of the optimal values in different settings. Row one of Table 11 (base model) denotes the results of the first cycle for example 2 (see Table 7). Table 11 illustrates the effect of distinct model parameter on the behaviour of the optimal values to compare the results with that derived for example 2. Note that in all cases, the model behaves as expected. For example, when the holding costs are equal, i.e., $h_{gm} = h_{rm} = h_r = 1.2$, all optimal values are higher than those of the base model, except the total minimum cost per unit time that experiences a lower cost. This can be justified by the fact that the system reduces the holding cost at the serviceable stock. Note that similar behaviour is also observed for $S_{gm} = S_{gr} = S_r = 2000$ except the fraction of returned items that is associated with a slight decrease. This can be attributed to the increase of the order cost for returned items. Similarly, when $c_{pm} = 6$, which also affecting c_{pr}, all optimal values are higher than those of the base model except the cycle length and the produced quantity that are associated with lower values. This can be attributed to the fact that the fraction of returned items increased by 7.8% $\left(\frac{0.835-0.770}{0.835} = 0.0778\right)$. For $c_w = 0.3$, we note that f_{4j}^*, L_j^*, l_j^* and Q_{mj}^* are associated with greater values than those of the base model, and $\phi_j^*, \Delta_j^*, Q_{rj}^*$ and R_j^* are associated with lower values. This can be justified by the fact that the system reaps the benefit of not disposing more items. Finally, when the deterioration rates are equal, i.e., $\vartheta_{gm} = \vartheta_{gr} = \vartheta_r = 30$, all optimal values are less than those of the base model, except the total minimum cost per unit time and the fraction of returned items that are associated with greater values. As expected, more items (97 units) are deteriorated and disposed outside the system due to the increase of the deterioration rates.

Table 11. Sensitivity analysis of the optimal results for varying rates when $\tau = 3$ and $c_{inv} = 4000\$$.

Parameters	j	ζ_j^*	c_{invj}	c_{prj}	λ_j^*	ϕ_j^*	f_{4j}^*	Q_{mj}^*	Q_{rj}^*	R_j^*	Δ_j^*	d_j^*	L_j^*	l_j^*
Base model*	1	1	3009	1.238	0.788	0.770	2.981	2089	1498	2741	623	66	11324	33761
$h_z = 1.2$	1	1	3009	1.238	0.788	0.772	3.090	2178	1564	2866	652	72	11139	34420
$S_z = 2000$	1	1	3009	1.238	0.788	0.761	3.113	2212	1561	2849	641	73	11586	36072
$c_{pm} = 6$	1	1	3009	1.486	0.788	0.835	2.888	1926	1530	2866	690	65	12244	35364
$c_w = 0.3$	1	1	3009	1.238	0.788	0.760	2.982	2104	1484	2705	609	66	11345	33832
$\vartheta_z = 30$	1	1	3009	1.238	0.788	0.778	2.950	2074	1486	2737	618	97	11377	33564

* Row one of Table 7.

Case 4: In this case (Case 4), we replicate example 2 with respect to constant rates without deterioration. As can be seen from Table 12, the model behaves in a similar way as that observed in Table 7. In particular, the behaviour of R_j^* in the third cycle and Δ_j^* in the second cycle (recall the justification in example 2). Note that ϕ_j^* attains its maximum value when the system plateaus, i.e., it differs from that observed in Table 7. A comparison between Tables 7 and 12 shows that for each cycle j, all optimal values are higher than those of example 2 (Table 7), except L_j^* and ϕ_j^* that are associated with lower values.

Table 12. Optimal results for constant rates without deterioration when $\tau = 3$ and $c_{inv} = 4000\$$.

j	ζ_j^*	c_{invj}	c_{prj}	λ_j^*	ϕ_j^*	f_{4j}^*	Q_{mj}^*	Q_{rj}^*	R_j^*	Δ_j^*	L_j^*	l_j^*
1	1	3009	1.238	0.788	0.635	4.808	3207	1781	3051	623	9479	45577
2	2	3845	0.983	0.749	0.736	4.257	2277	1980	2685	655	9362	39851
3	3	3986	0.765	0.730	0.717	4.225	2128	2097	3028	768	9264	39138
4	3	3986	0.765	0.730	0.695	4.071	1980	2091	2829	743	9218	37532
5	3	3986	0.765	0.730	0.700	4.107	2014	2093	2875	749	9229	37906
6	3	3986	0.765	0.730	0.699	4.099	2006	2092	2864	747	9227	37816

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7	3	3986	0.765	0.730	0.699	4.101	2008	2093	2867	748	9227	37837
8	3	3986	0.765	0.730	0.820	4.100	2008	2093	2866	748	9227	37832
9	3	3986	0.765	0.730	0.820	4.100	2008	2093	2866	748	9227	37832

Case 5: In this case (Case 5), we replicate example 3 with respect to constant rates without deterioration. As Table 13 shows, the model behaves similarly with respect to constant rates without deterioration (see Table 8). As can be seen from Table 13, ϕ_j^*, Δ_j^* and R_j^* reach their maximum values in the first cycle and f_{4j}^*, L_j^* and l_j^* attain their minimum values in the second cycle before the system plateaus in the fifth cycle (Table 13). Similarly, Q_{rj}^* (Q_{mj}^*) reach their minimum (maximum) values in the first cycle since the inventory of returned items is zero. A comparison between Tables 8 and 13 shows that for each cycle j, all optimal values are higher than those of example 3 (Table 8), except L_j^* and ϕ_j^* that are associated with lower values. Note that this finding is also observed in Case 4. In addition, $L_{4,3}^* = 9662$ dollars $> L_{2,1}^* = 9603$ dollars (recall solution steps). However, when the system plateaus for $\zeta = 3$, the difference between the total minimum cost per month is negligible, i.e., $L^{\zeta=3} = 9667$ dollars $> L^{\zeta=1} = 9625$ (see also example 3).

Table 13. Optimal results for constant rates without deterioration when $\tau = 3$ and $c_{inv} = 6000\$$.

j	ζ_j^*	c_{invj}	c_{prj}	λ_j^*	ϕ_j^*	f_{4j}^*	Q_{mj}^*	Q_{rj}^*	R_j^*	Δ_j^*	L_j^*	l_j^*
1	1	4514	1.238	0.788	0.614	5.243	3348	1895	3219	642	9779	51268
2	1	4514	1.238	0.788	0.534	4.476	2526	1950	2390	574	9603	42983
3	1	4514	1.238	0.788	0.544	4.568	2620	1949	2486	585	9628	43984
4	1	4514	1.238	0.788	0.543	4.554	2605	1949	2471	583	9624	43828
5	1	4514	1.238	0.788	0.543	4.556	2607	1949	2474	584	9625	43852
6	1	4514	1.238	0.788	0.543	4.556	2607	1949	2474	584	9625	43852

6. Implications and managerial insights

Considering that returned items may arrive with different number of remanufacturing times reduces the total system cost as well as ensures reducing the disposal of unnecessary amount.

The optimal policy is either to remanufactured once or remanufactured up to the expected number of times an item can be remanufactured on its life cycle.

All function may or may not be related to each other and, therefore, each is solely modelled.

The number of times an item can be remanufactured is definite, tractable and modelled.

The purchasing price of recovery items, remanufacturing investment cost, return rate and the percentage of returns vary until the number of cycles reaches the expected number of times an item can be remanufactured on its life cycle. Such variation implies further reduction in the total cost and ensures a positive environmental impact.

The return rate is a varying demand dependent rate, which is a decision variable. This consideration reduces the total cost and solid waste disposal and, consequently, the system emphasises sustainability because it reflects the influence of economic, social and environmental interests.

The initial inventory of returned items in the first remanufacturing cycle is zero and it differs from cycle to cycle, which in turn implies that the optimal values also vary until the system plateaus. This consideration is key in that it allows for the adjustment of all functions and input parameters for subsequent cycles.

The proposed model is a viable solution for different forms of time-varying functions that can be disseminated from the general formulation as well as for systems encountering periodic review applications.

The solution quality of the special cases is identical with that of published sources, i.e., the validity and robustness of the general model are ascertained.

7. Summary and conclusion

In this paper, we have been concerned with the implications of the number of times an item can be remanufactured. The mathematical modelling of reverse logistics inventory systems assumes that all returned items have been remanufactured with an equal number of times. Nevertheless, this assumption ignores the fact that returned items may arrive out of sequence. The present paper developed a new mathematical expression of the percentage of returns that can be remanufactured a finite number of times. The proposed expression has been modelled as a function of the expected number of times an item can be remanufactured on its life cycle and the number of times an item can be technologically (or optimally) remanufactured based on its quality upon recovery. The mathematical expression has been incorporated in a general joint model for production and remanufacturing options.

In the proposed model, demand, product deterioration, production and remanufacturing rates are arbitrary functions of time so as to reflect a diverse range of time-varying forms. The return rate is a varying demand dependent rate, which is a decision variable. The model considers the initial inventory of returned items in the mathematical formulation, which enables decision makers to adjust all functions and input parameters for subsequent cycles.

We evaluated the impact of varying rates on the optimal quantities subject to the expected number of times an item can be remanufactured on its life cycle. We found that the effect of varying purchasing price of recovery items, remanufacturing investment cost, return rate, the percentage of returns and the initial inventory of returned items significantly impact on the behaviour of the model. Consequently, the optimal policy is either to remanufactured once or remanufactured up to the expected number of times an item can be remanufactured on its life cycle. We tested and observed the behaviour of the model in different realistic situations and discussed some important managerial insights for decision makers. The versatile nature of the proposed model has been emphasised, where the viability, validity and robustness of the proposed model are ascertained.

Further research may include extensions such as allowing for shortages, incorporating learning and forgetting curves in the manufacturing and remanufacturing rates. In addition, the formulation of greenhouse gas (GHG) emissions from manufacturing, remanufacturing and transportation, as well as energy consumption during manufacturing and remanufacturing processes can also be addressed. In parallel, it seems plausible to extend the general model for multiple manufacturing and remanufacturing cycles while accounting for different emission trading schemes.

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