Article



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Posted Date: 10 May 2023

doi: 10.20944/preprints202305.0743.v1

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*Preprints* (www.preprints.org) | NOT PEER-REVIEWED | Posted: 10 May 2023 doi:10.20944/preprints202305.0743.v1



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# Identification and Classification of Causes and Risk Factors of Osteoporosis Based on Complex Intuitionistic Fuzzy Frank Aggregation Operators

**Xiaopeng Yang [1], Tahir Mahmood [2], Zeeshan Ali [2] and Khizar Hayat [3]**

[1] School of Mathematics and Statistics, Hanshan Normal University, Chaozhou, China
[2] Department of Mathematics & Statistics, International Islamic University Islamabad, Pakistan
[3] Department of Mathematics, University of Kotli, Kotli 11100, Azad Jammu and Kashmir, Pakistan
happyyangxp@163.com; tahirbakhat@iiu.edu.pk; zeeshanalinsr@gmail.com; khizarhayat@uokajk.edu.pk
* **Corresponding author:** Xiaopeng Yang (happyyangxp@163.com)

**Abstract:** More than fifty million people over the age of fifty have osteoporosis issues due to low bone mass, disease contribution, and not exercising continuously. Certain most important risk factors and causes of osteoporosis contain age, gender, hormones, visits to inexperienced doctors, and certain medical conditions. In this manuscript, we diagnosed certain frank operational laws under the availability of complex intuitionistic fuzzy (CIF) information. Furthermore, using the evaluated operational laws, we pioneered the theory of CIF frank weighted averaging (CIFFWA), CIF frank ordered weighted averaging (CIFFOWA), CIF frank hybrid averaging (CIFFHA), CIF frank weighted geometric (CIFFWG), CIF frank ordered weighted geometric (CIFFOWG), CIF frank hybrid geometric (CIFFHG) operators, and described their beneficial and valuable results with certain useful properties. Moreover, we aim to evaluate the most dangerous type of osteoporosis based on their symptoms, causes, and risk factors using diagnosed approaches. Finally, to enhance the worth of the evaluated operators, we compare the diagnosed result with certain prevailing results and illustrated their geometrical representation with the help of MATLAB.

**Keywords:** complex intuitionistic fuzzy sets; osteoporosis diseases; frank averaging/geometric aggregation operators; decision-making evaluations

## 1. Introduction

The word osteoporosis means "porous bone" and the condition is distinguished by gradual bone weakening. Osteoporosis is a result of discrepancies between new bone growth and old bone resorption. Osteoporosis is one of the most complicated and dangerous types of bone diseases and nowadays certain people have faced especially those who have crossed the age of fifty years. Osteoporosis is a result of discrepancies between new bone growth and old bone resorption. The word osteoporosis means "porous bone" and the condition is distinguished by gradual bone weakening. More than fifty million people over the age of fifty have osteoporosis issues due to low bone mass, disease contribution, and not exercising continuously. Osteoporosis is well-known in non-Hispanic white women and Asian women. African and Hispanic women have very little chance of affection for osteoporosis. Certain most important risk factors and causes of osteoporosis contain age, gender, hormones, visits to inexperienced doctors, and certain medical conditions. Further, age is the main factor in the affection of osteoporosis because as someone gets older it means that the shape of your bones becomes weak day by day and a time comes when the small holes in bones are started and after some time it becomes bigger and due to these reasons, the bones of the aged man brooked automatically without any attractions. As noticed from some research articles, women have faced these issues after sixty-five-year age and men will face these issues after seventy-five years. Moreover, low estrogen, menopause, and gender are also played a very critical role in the causes of osteoporosis. From the research of the national osteoporosis foundation, 80% of women have faced these issues because the bones of mostly women have very tiny than males. Further, estrogen is also one of the most reasonable factors in women, where the hormone in women saves bones. Some important risk factors for women to increase osteoporosis diseases are discussed in the shape: before forty-five doing menopause, going menstrual period for a long time without any complications, irregular periods. Some real pictures of osteoporosis are presented in the shape of Figure 1.

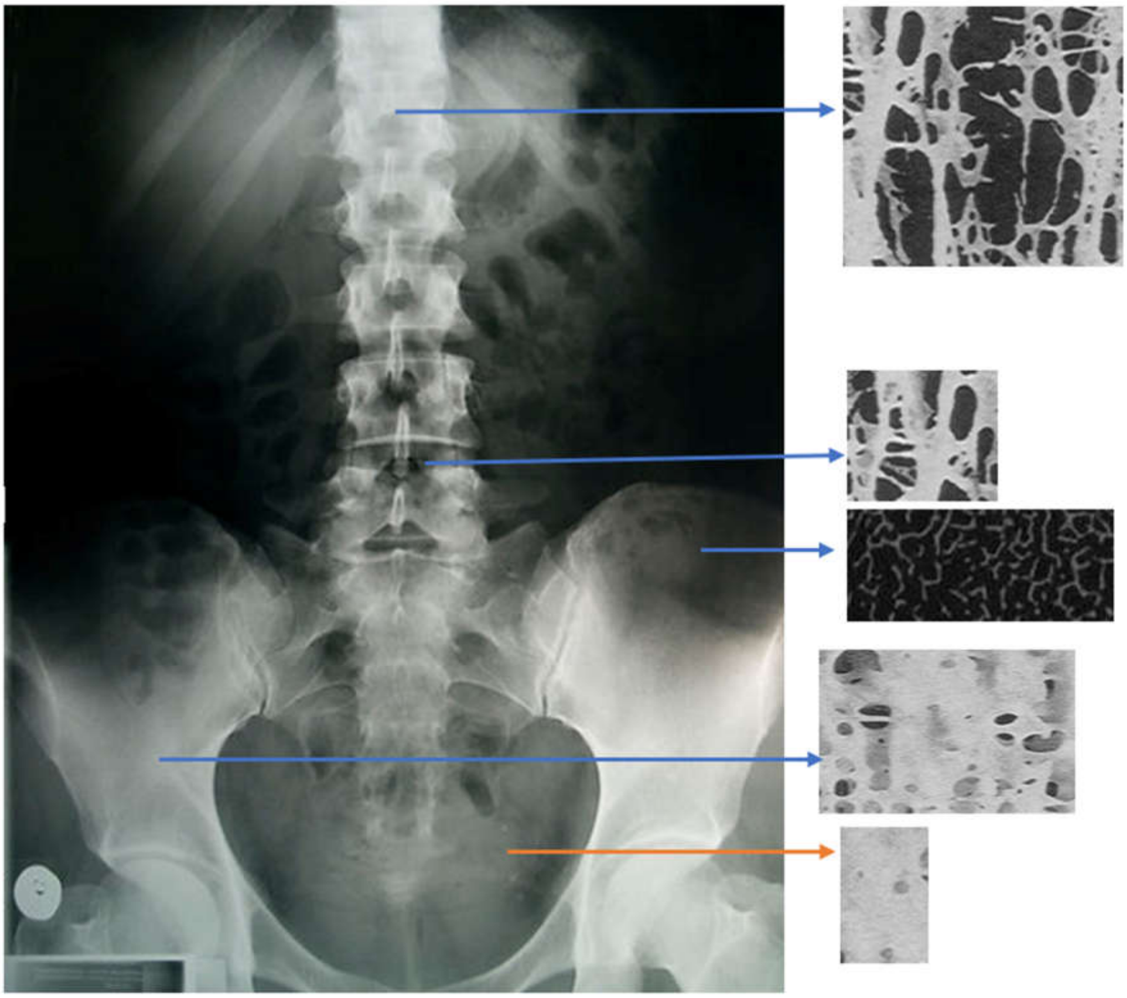

**Figure 1.** Represented the X-ray of bones of human body.

Multi-attribute decision-making (MADM) is a type of procedure used for justifying the best optimal with the help of a decision-making procedure from the finite collection of alternatives. In the last few years, certain investigations of companies have been employed by many scholars. In a genuine decision-making tool, a valuable dilemma is how to represent the criteria term massive feasibly and dominantly. Further, due to vague uncertainty in real life, for every expert, it is very awkward to find the best decision under the consideration and availability of classical information. For this, to find the best optimal based on a fuzzy set instead of classical information, in 1965, Zadeh [1] evaluated the main theory of fuzzy information (FI). FI depends on the value of the supporting function and the final or resultant value has included in [0, 1]. Further, to improve the worth of the FI, certain theories are available in the shape: process based on the MADM technique for FI was pioneered by Jain [2], Akram et al. [3, 4] diagnosed the mathematical structure which is different from FI, called fuzzy N-soft sets and hesitant fuzzy N-soft sets, Ohlan and Ohlan [5] evaluated the problem of bibliometric under the consideration of FI, Abdullah et al. [6] mixed the theory of fractional and FI, and recently Mahmood [7] evaluated the main concept of bipolar soft sets. But certain scholars have mentioned with sold examples that the theory of FI has been uncompleted. To remove these deficiencies, Atanassov [8, 9] diagnosed the intuitionistic FI (IFI), where IFI contained two supporting

 doi:10.20944/preprints202305.0743.v1



functions, called positive and negative grades. The condition of IFI stated that the sum of both values of the IFI is necessarily contained in the unit interval. Moreover, an extended MAIRCA technique based on IFI was diagnosed by Ecer [10], to analyze the quality of the software in different fields based on IFI was developed by Thao and Chou [11], and pattern recognition and decision-making evaluation based on IFI was presented by Gohain et al. [12], analysis of three ways decision using IFI, evaluated by Yang et al. [13], and distance measures based on IFI was diagnosed by Garg and Rani [14].

A brief explanation is available of existing FI and IFI and one thing is clear the prevailing information has received a lot of attention from different individuals, but it is also clear that the prevailing FI can handle only one-dimension information at a time. In every genuine life place, experts have faced two-dimension information, because when we considered the example of medical research, biometrics and facial recognition have changed the information time by time. Thus, to handle such kind of problematic situation, the theory of complex FI (CFI) was pioneered by Ramot et al. [15] by fixing the imaginary or one more extra term in supporting grade and constructing their shape in the shape of polar co-ordinates. From the following work, we clear that the theory of CFI has a lot of applications in different fields, for instance, entropy measures [16], neighborhood operators [17], complex fuzzy logic [18, 19], operation properties [20], complex fuzzy neuro structure [21], distance measures and continuity [22], complex fuzzy N-soft sets [23], and complex fuzzy soft sets [24]. Further, noticed that the theory of CFI has ignored the grade of anti-support which make a source of losing information during the decision-making procedure. To remove these deficiencies, Alkouri and Salleh [25] diagnosed the complex IFI (CIFI), where CIFI contained two supporting functions, called positive and negative grades in the shape of complex values. Conditions in CIFI stated that the sum of both values of the CIFI is necessarily contained in the unit interval (for both real and imaginary parts). Moreover, Ali et al. [26] proposed another shape of complex intuitionistic fuzzy soft sets. Garg and Rani [27, 28] evaluated the information measures and correlation coefficient measures for CIFI. Jan et al. [29] investigated cyber-security and cyber-crime under the consideration of CIF relations.

The main theory of frank t-norm and t-conorm was evaluated by Frank [30]. Further, Seikh and Mandel [31] utilized the frank aggregation operators based on IFI. From these two manuscripts, we noticed that, yet no one can develop frank aggregation operators in the environment of complex fuzzy sets and complex intuitionistic fuzzy sets. So, inspired by the above analysis, we computed the following ideologies:

1. To diagnose certain frank operational laws under the availability of CIF information.
2. To pioneer the theory of CIFFWA, CIFFOWA, CIFFHA, CIFFWG, CIFFOWG, and CIFFHG operators, and described their beneficial and valuable results with certain useful properties.
3. To evaluate the most dangerous type of osteoporosis based on their symptoms, causes, and risk factors using diagnosed approaches.
4. To enhance the worth of the evaluated operators, we compare the diagnosed result with certain prevailing results and illustrated their geometrical representation with the help of MATLAB.

This manuscript is computed or organized in the shape: In section 2, information on the CIF set and their algebraic laws with frank t-norm and t-conorm are revised. In section 3, we diagnosed certain frank operational laws under the availability of CIF information. Furthermore, using the evaluated operational laws, we pioneered the theory of CIFFWA, CIFFOWA, CIFFHA, CIFFWG, CIFFOWG, and CIFFHG operators, and described their beneficial and valuable results with certain useful properties. In section 4, we aim to evaluate the most dangerous type of osteoporosis based on their symptoms, causes, and risk factors using diagnosed approaches. In section 5, to enhance the worth of the evaluated operators, we compare the diagnosed result with certain prevailing results and illustrated their geometrical representation with the help of MATLAB. Final and concluding information is available in section 6.

## 2. Preliminaries

Certain most important risk factors and causes of osteoporosis contain age, gender, hormones, visits to inexperienced doctors, and certain medical conditions. Osteoporosis is well-known in non-Hispanic white women and Asian women. African and Hispanic women have very little chance of affection for osteoporosis. Osteoporosis is one of the most complicated and dangerous types of bone diseases and nowadays certain people have faced especially those who have crossed the age of fifty years. Osteoporosis is a result of discrepancies between new bone growth and old bone resorption. Further, age is the main factor in the affection of osteoporosis because as someone gets older it means that the shape of your bones becomes weak day by day and a time comes when the small holes in bones are started and after some time it becomes bigger and due to these reasons, the bones of the aged man brooked automatically without any attractions. As noticed from some research articles, women have faced these issues after sixty-five-year age and men will face these issues after seventy-five years. Moreover, low estrogen, menopause, and gender are also played a very critical role in the causes of osteoporosis. From the research of the national osteoporosis foundation, 80% of women have faced these issues because the bones of mostly women have very tiny than males. Further, estrogen is also one of the most reasonable factors in women, where the hormone in women saves bones. Some important risk factors for women to increase osteoporosis diseases are discussed in the shape: before forty-five doing menopause, going menstrual period for a long time without any complications, irregular periods. Information on the CIF set and their algebraic laws with frank t-norm and t-conorm are revised in this section, where the universal set is shown by $\mathcal{X}$, and the term $m_{\text{I}_{R-\mathcal{s}}} e^{i2\pi\left(m_{\text{I}_{I-\mathcal{s}}}\right)}$ and $n_{\text{I}_{R-\mathcal{s}}} e^{i2\pi\left(n_{\text{I}_{I-\mathcal{s}}}\right)}$, represented the truth and falsity grades.

**Definition 1: [25]** *Assumed $\mathcal{X}$ as a universal set, then the shape of $\mathfrak{P}_{\mathcal{s}}$, mentioned below:*

$$\text{I}_{\mathcal{s}} = \left\{\left(m_{\text{I}_{\mathcal{s}}}(x), n_{\text{I}_{\mathcal{s}}}(x)\right) : x \in \mathcal{X}\right\} \quad (1)$$

*With a well-known condition:* $0 \leq m_{\text{I}_R}(x) + n_{\text{I}_R}(x) \leq 1$ *and* $0 \leq m_{\text{I}_I}(x) + n_{\text{I}_I}(x) \leq 1$, *represented as CIF information. The mathematical shape:* $\mathcal{R}_{\text{I}_{\mathcal{s}}}(x) = \mathcal{R}_{\text{I}_R}(x) e^{i2\pi\left(\mathcal{R}_{\text{I}_I}(x)\right)} = \left(1 - \left(m_{\text{I}_R}(x) + n_{\text{I}_R}(x)\right)\right) e^{i2\pi\left(1-\left(m_{\text{I}_I}(x)+n_{\text{I}_I}(x)\right)\right)}$, *shown as refusal information. Furthermore, the theory of CIF number (CIFN) is represented by:* $\text{I}_{\mathcal{s}} = \left(m_{\text{I}_{R-\mathcal{s}}} e^{i2\pi\left(m_{\text{I}_{I-\mathcal{s}}}\right)}, n_{\text{I}_{R-\mathcal{s}}} e^{i2\pi\left(n_{\text{I}_{I-\mathcal{s}}}\right)}\right), \mathcal{s} = 1,2, \dots, t.$

**Definition 2: [25]** *We recall some algebraic laws based on CIFNs* $\text{I}_{\mathcal{s}} = \left(m_{\text{I}_{R-\mathcal{s}}} e^{i2\pi\left(m_{\text{I}_{I-\mathcal{s}}}\right)}, n_{\text{I}_{R-\mathcal{s}}} e^{i2\pi\left(n_{\text{I}_{I-\mathcal{s}}}\right)}\right), \mathcal{s} = 1,2,$ *such that*

$$\text{I}_1 \oplus \text{I}_2 = \left(\left(m_{\text{I}_{R-1}} + m_{\text{I}_{R-2}} - m_{\text{I}_{R-1}} m_{\text{I}_{R-2}}\right) e^{i2\pi\left(m_{\text{I}_{I-1}} + m_{\text{I}_{I-2}} - m_{\text{I}_{I-1}} m_{\text{I}_{I-2}}\right)}, \left(n_{\text{I}_{R-1}} n_{\text{I}_{R-2}}\right) e^{i2\pi\left(n_{\text{I}_{I-1}} n_{\text{I}_{I-2}}\right)}\right) \quad (2)$$

$$\text{I}_1 \otimes \text{I}_2 = \left(\left(m_{\text{I}_{R-1}} m_{\text{I}_{R-2}}\right) e^{i2\pi\left(m_{\text{I}_{I-1}} m_{\text{I}_{I-2}}\right)}, \left(n_{\text{I}_{R-1}} + n_{\text{I}_{R-2}} - n_{\text{I}_{R-1}} n_{\text{I}_{R-2}}\right) e^{i2\pi\left(n_{\text{I}_{I-1}} + n_{\text{I}_{I-2}} - n_{\text{I}_{I-1}} n_{\text{I}_{I-2}}\right)}\right) \quad (3)$$

$$\check{\rho}\text{I}_{\mathcal{s}} = \left(\left(1 - \left(1 - m_{\text{I}_{R-\mathcal{s}}}\right)^{\check{\rho}}\right) e^{i2\pi\left(1-\left(1-m_{\text{I}_{I-\mathcal{s}}}\right)^{\check{\rho}}\right)}, n_{\text{I}_{R-\mathcal{s}}}^{\check{\rho}} e^{i2\pi\left(n_{\text{I}_{I-\mathcal{s}}}^{\check{\rho}}\right)}\right) \quad (4)$$

$$\text{I}_{\mathcal{s}}^{\check{\rho}} = \left(m_{\text{I}_{R-\mathcal{s}}}^{\check{\rho}} e^{i2\pi\left(m_{\text{I}_{I-\mathcal{s}}}^{\check{\rho}}\right)}, \left(1 - \left(1 - n_{\text{I}_{R-\mathcal{s}}}\right)^{\check{\rho}}\right) e^{i2\pi\left(1-\left(1-n_{\text{I}_{I-\mathcal{s}}}\right)^{\check{\rho}}\right)}\right) \quad (5)$$

**Definition 3:** *[25] We recall some important ideas based on CIFNs* $\text{I}_{\mathcal{s}} = \left(m_{\text{I}_{R-\mathcal{s}}} e^{i2\pi\left(m_{\text{I}_{I-\mathcal{s}}}\right)}, n_{\text{I}_{R-\mathcal{s}}} e^{i2\pi\left(n_{\text{I}_{I-\mathcal{s}}}\right)}\right), \mathcal{s} = 1,2, \dots, t,$ *such that*

$$\mathcal{C}_{\mathcal{s}\mathcal{s}}(\text{I}_{\mathcal{s}}) = \frac{1}{2}\left(m_{\text{I}_{R-\mathcal{s}}} + m_{\text{I}_{I-\mathcal{s}}} - n_{\text{I}_{R-\mathcal{s}}} - n_{\text{I}_{I-\mathcal{s}}}\right), \mathcal{C}_{\mathcal{s}\mathcal{s}}(\text{I}_{\mathcal{s}}) \in [-1,1] \quad (6)$$

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$$\mathcal{A}_{a\mathscr{s}}(\mathrm{I}_{\mathscr{s}}) = \frac{1}{2}\left(m_{\mathrm{I}_{R-\mathscr{s}}} + m_{\mathrm{I}_{I-\mathscr{s}}} + n_{\mathrm{I}_{R-\mathscr{s}}} + n_{\mathrm{I}_{I-\mathscr{s}}}\right), \mathcal{A}_{a\mathscr{s}}(\mathrm{I}_{\mathscr{s}}) \in [0,1] \quad (7)$$

*Considered as a score value and accuracy value.*

**Definition 4:** *[25] We recall some important rules based on CIFNs* $\mathrm{I}_{\mathscr{s}} = \left(m_{\mathrm{I}_{R-\mathscr{s}}} e^{i2\pi\left(m_{\mathrm{I}_{I-\mathscr{s}}}\right)}, n_{\mathrm{I}_{R-\mathscr{s}}} e^{i2\pi\left(n_{\mathrm{I}_{I-\mathscr{s}}}\right)}\right), \mathscr{s} = 1,2,\dots,t,$ *such that if* $\mathcal{C}_{s\mathscr{s}}(\mathrm{I}_1) > \mathcal{C}_{s\mathscr{s}}(\mathrm{I}_2)$, *then* $\mathrm{I}_1 > \mathrm{I}_2$, *if* $\mathcal{C}_{s\mathscr{s}}(\mathrm{I}_1) = \mathcal{C}_{s\mathscr{s}}(\mathrm{I}_2)$, *then if* $\mathcal{A}_{a\mathscr{s}}(\mathrm{I}_1) > \mathcal{A}_{a\mathscr{s}}(\mathrm{I}_2)$, *then* $\mathrm{I}_1 > \mathrm{I}_2$, *if* $\mathcal{A}_{a\mathscr{s}}(\mathrm{I}_1) = \mathcal{A}_{a\mathscr{s}}(\mathrm{I}_2)$, *then* $\mathrm{I}_1 = \mathrm{I}_2$.

**Definition 5:** *[30] A Frank t-norm and t-conorm are illustrated below:*

$$T(x,y) = log_{\Xi}\left(1 + \frac{(\Xi^{x} - 1)(\Xi^{y} - 1)}{\Xi - 1}\right) \quad (8)$$

$$S(x,y) = 1 - log_{\Xi}\left(1 + \frac{(\Xi^{1-x} - 1)(\Xi^{1-y} - 1)}{\Xi - 1}\right) \quad (9)$$

*Where* $x, y \in [0,1]$ *and* $\Xi \in (1, +\infty)$.

## 3. Frank Aggregation Operators for CIF Information

The word osteoporosis means "porous bone" and the condition is distinguished by gradual bone weakening. Osteoporosis is a result of discrepancies between new bone growth and old bone resorption. More than fifty million people over the age of fifty have osteoporosis issues due to low bone mass, disease contribution, and not exercising continuously. Certain most important risk factors and causes of osteoporosis contain age, gender, hormones, visits to inexperienced doctors, and certain medical conditions. In this manuscript, we diagnosed certain frank operational laws under the availability of CIF information. Furthermore, using the evaluated operational laws, we pioneered the theory of CIFFWA, CIFFOWA, CIFFHA, CIFFWG, CIFFOWG, and CIFFHG operators, and described their beneficial and valuable results with certain useful properties.

**Definition 6:** *We diagnose some frank laws based on CIFNs* $\mathrm{I}_{\mathscr{s}} = \left(m_{\mathrm{I}_{R-\mathscr{s}}} e^{i2\pi\left(m_{\mathrm{I}_{I-\mathscr{s}}}\right)}, n_{\mathrm{I}_{R-\mathscr{s}}} e^{i2\pi\left(n_{\mathrm{I}_{I-\mathscr{s}}}\right)}\right), \mathscr{s} = 1,2,$ *such that*

$$\mathrm{I}_1 \oplus \mathrm{I}_2 = \begin{pmatrix} 1 - \log_{\Xi}\left(1 + \frac{\left(\Xi^{1-m_{\mathrm{I}_{R-1}}} - 1\right)\left(\Xi^{1-m_{\mathrm{I}_{R-2}}} - 1\right)}{\Xi - 1}\right) e^{i2\pi\left(1-\log_{\Xi}\left(1+\frac{\left(\Xi^{1-m_{\mathrm{I}_{I-1}}}-1\right)\left(\Xi^{1-m_{\mathrm{I}_{I-2}}}-1\right)}{\Xi-1}\right)\right)}, \\ \log_{\Xi}\left(1 + \frac{\left(\Xi^{n_{\mathrm{I}_{R-1}}} - 1\right)\left(\Xi^{n_{\mathrm{I}_{R-2}}} - 1\right)}{\Xi - 1}\right) e^{i2\pi\left(\log_{\Xi}\left(1+\frac{\left(\Xi^{n_{\mathrm{I}_{I-1}}}-1\right)\left(\Xi^{n_{\mathrm{I}_{I-2}}}-1\right)}{\Xi-1}\right)\right)} \end{pmatrix} \quad (10)$$

$$\mathrm{I}_1 \otimes \mathrm{I}_2 = \begin{pmatrix} \log_{\Xi}\left(1 + \frac{\left(\Xi^{m_{\mathrm{I}_{R-1}}} - 1\right)\left(\Xi^{m_{\mathrm{I}_{R-2}}} - 1\right)}{\Xi - 1}\right) e^{i2\pi\left(\log_{\Xi}\left(1+\frac{\left(\Xi^{m_{\mathrm{I}_{I-1}}}-1\right)\left(\Xi^{m_{\mathrm{I}_{I-2}}}-1\right)}{\Xi-1}\right)\right)}, \\ 1 - \log_{\Xi}\left(1 + \frac{\left(\Xi^{1-n_{\mathrm{I}_{R-1}}} - 1\right)\left(\Xi^{1-n_{\mathrm{I}_{R-2}}} - 1\right)}{\Xi - 1}\right) e^{i2\pi\left(1-\log_{\Xi}\left(1+\frac{\left(\Xi^{1-n_{\mathrm{I}_{I-1}}}-1\right)\left(\Xi^{1-n_{\mathrm{I}_{I-2}}}-1\right)}{\Xi-1}\right)\right)} \end{pmatrix} \quad (11)$$

$$\breve{\rho}\mathrm{I}_1 = \begin{pmatrix} 1 - \log_{\Xi}\left(1 + \frac{\left(\Xi^{1-m_{\mathrm{I}_{R-1}}} - 1\right)^{\breve{\rho}}}{(\Xi - 1)^{\breve{\rho}-1}}\right) e^{i2\pi\left(1-\log_{\Xi}\left(1+\frac{\left(\Xi^{1-m_{\mathrm{I}_{I-1}}}-1\right)^{\breve{\rho}}}{(\Xi-1)^{\breve{\rho}-1}}\right)\right)}, \\ \log_{\Xi}\left(1 + \frac{\left(\Xi^{n_{\mathrm{I}_{R-1}}} - 1\right)^{\breve{\rho}}}{(\Xi - 1)^{\breve{\rho}-1}}\right) e^{i2\pi\left(\log_{\Xi}\left(1+\frac{\left(\Xi^{n_{\mathrm{I}_{I-1}}}-1\right)^{\breve{\rho}}}{(\Xi-1)^{\breve{\rho}-1}}\right)\right)} \end{pmatrix} \quad (12)$$

$$\text{I}_{\mathcal{s}}^{\breve{p}} = \begin{pmatrix} \log_{\Xi}\left(1 + \frac{(\Xi^{m_{\text{I}_{R-1}}} - 1)^{\breve{p}}}{(\Xi - 1)^{\breve{p}-1}}\right) e^{i2\pi\left(\log_{\Xi}\left(1+\frac{\left(\Xi^{m_{\text{I}_{I-1}}}-1\right)^{\breve{p}}}{(\Xi-1)^{\breve{p}-1}}\right)\right)}, \\ 1 - \log_{\Xi}\left(1 + \frac{\left(\Xi^{1-n_{\text{I}_{R-1}}} - 1\right)^{\breve{p}}}{(\Xi - 1)^{\breve{p}-1}}\right) e^{i2\pi\left(1-\log_{\Xi}\left(1+\frac{\left(\Xi^{1-n_{\text{I}_{I-1}}}-1\right)^{\breve{p}}}{(\Xi-1)^{\breve{p}-1}}\right)\right)} \end{pmatrix} \quad (13)$$

**Definition 7:** *We diagnose the idea of a CIFFWA operator based on CIFNs* $\text{I}_{\mathcal{s}} = \left(m_{\text{I}_{R-\mathcal{s}}} e^{i2\pi\left(m_{\text{I}_{I-\mathcal{s}}}\right)}, n_{\text{I}_{R-\mathcal{s}}} e^{i2\pi\left(n_{\text{I}_{I-\mathcal{s}}}\right)}\right), \mathcal{s} = 1,2, \dots, t,$ *such that*

$$CIFFWA(\text{I}_1, \text{I}_2, \dots, \text{I}_t) = \mu_{w-1}\text{I}_1 \oplus \mu_{w-2}\text{I}_2 \oplus \dots \oplus \mu_{w-t}\text{I}_t = \oplus_{\mathcal{s}=1}^{t} \mu_{w-\mathcal{s}}\text{I}_{\mathcal{s}} \quad (14)$$

*Where* $\sum_{\mathcal{s}=1}^{t} \mu_{w-\mathcal{s}} = 1, \mu_{w-\mathcal{s}} \in [0,1]$, *representing the weight vector. In particular, if we use the value of* $\mu_{w-\mathcal{s}} = \left(\frac{1}{t}, \frac{1}{t}, \dots, \frac{1}{t}\right)$ *in Eq. (14), then we have*

$$CIFFA(\text{I}_1, \text{I}_2, \dots, \text{I}_t) = \frac{1}{t}(\text{I}_1 \oplus \text{I}_2 \oplus \dots \oplus \text{I}_t) = \frac{1}{t} \oplus_{\mathcal{s}=1}^{t} \text{I}_{\mathcal{s}} \quad (15)$$

*Called CIF frank averaging (CIFFA) operator.*

**Theorem 1:** *Verify that the resultant value of Eq. (14) is again in the shape of CIFNs, such that*

$$CIFFWA(\text{I}_1, \text{I}_2, \dots, \text{I}_t)$$
$$= \begin{pmatrix} 1 - \log_{\Xi}\left(1 + \frac{\prod_{\mathcal{s}=1}^{t}\left(\Xi^{1-m_{\text{I}_{R-\mathcal{s}}}} - 1\right)^{\mu_{w-\mathcal{s}}}}{(\Xi - 1)^{\sum_{\mathcal{s}=1}^{t}\mu_{w-\mathcal{s}}-1}}\right) e^{i2\pi\left(1-\log_{\Xi}\left(1+\frac{\prod_{\mathcal{s}=1}^{t}\left(\Xi^{1-m_{\text{I}_{I-\mathcal{s}}}}-1\right)^{\mu_{w-\mathcal{s}}}}{(\Xi-1)^{\sum_{\mathcal{s}=1}^{t}\mu_{w-\mathcal{s}}-1}}\right)\right)}, \\ \log_{\Xi}\left(1 + \frac{\prod_{\mathcal{s}=1}^{t}(\Xi^{n_{\text{I}_{R-\mathcal{s}}}} - 1)^{\mu_{w-\mathcal{s}}}}{(\Xi - 1)^{\sum_{\mathcal{s}=1}^{t}\mu_{w-\mathcal{s}}-1}}\right) e^{i2\pi\left(\log_{\Xi}\left(1+\frac{\prod_{\mathcal{s}=1}^{t}\left(\Xi^{n_{\text{I}_{I-\mathcal{s}}}}-1\right)^{\mu_{w-\mathcal{s}}}}{(\Xi-1)^{\sum_{\mathcal{s}=1}^{t}\mu_{w-\mathcal{s}}-1}}\right)\right)} \end{pmatrix} \quad (16)$$

**Proof:** Mathematical induction is used in the proof of Eq. (16), so, for this, we consider $t = 2$, then

$$\mu_{w-1}\text{I}_1 = \begin{pmatrix} 1 - \log_{\Xi}\left(1 + \frac{\left(\Xi^{1-m_{\text{I}_{R-1}}} - 1\right)^{\mu_{w-1}}}{(\Xi - 1)^{\mu_{w-1}-1}}\right) e^{i2\pi\left(1-\log_{\Xi}\left(1+\frac{\left(\Xi^{1-m_{\text{I}_{I-1}}}-1\right)^{\mu_{w-1}}}{(\Xi-1)^{\mu_{w-1}-1}}\right)\right)}, \\ \log_{\Xi}\left(1 + \frac{(\Xi^{n_{\text{I}_{R-1}}} - 1)^{\mu_{w-1}}}{(\Xi - 1)^{\mu_{w-1}-1}}\right) e^{i2\pi\left(\log_{\Xi}\left(1+\frac{\left(\Xi^{n_{\text{I}_{I-1}}}-1\right)^{\mu_{w-1}}}{(\Xi-1)^{\mu_{w-1}-1}}\right)\right)} \end{pmatrix}$$

$$\mu_{w-2}\text{I}_2 = \begin{pmatrix} 1 - \log_{\Xi}\left(1 + \frac{\left(\Xi^{1-m_{\text{I}_{R-2}}} - 1\right)^{\mu_{w-2}}}{(\Xi - 1)^{\mu_{w-2}-1}}\right) e^{i2\pi\left(1-\log_{\Xi}\left(1+\frac{\left(\Xi^{1-m_{\text{I}_{I-2}}}-1\right)^{\mu_{w-2}}}{(\Xi-1)^{\mu_{w-2}-1}}\right)\right)}, \\ \log_{\Xi}\left(1 + \frac{(\Xi^{n_{\text{I}_{R-2}}} - 1)^{\mu_{w-2}}}{(\Xi - 1)^{\mu_{w-2}-1}}\right) e^{i2\pi\left(\log_{\Xi}\left(1+\frac{\left(\Xi^{n_{\text{I}_{I-2}}}-1\right)^{\mu_{w-2}}}{(\Xi-1)^{\mu_{w-2}-1}}\right)\right)} \end{pmatrix}$$

Then

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$$\mu_{w-1}\mathrm{I}_1 \oplus \mu_{w-2}\mathrm{I}_2 = \begin{pmatrix} 1-\log_{\Xi}\left(1+\frac{\left(\Xi^{1-m_{\mathrm{I}_{R-1}}}-1\right)^{\mu_{w-1}}}{(\Xi-1)^{\mu_{w-1}-1}}\right)e^{i2\pi\left(1-\log_{\Xi}\left(1+\frac{\left(\Xi^{1-m_{\mathrm{I}_{I-1}}}-1\right)^{\mu_{w-1}}}{(\Xi-1)^{\mu_{w-1}-1}}\right)\right)}, \\ \log_{\Xi}\left(1+\frac{\left(\Xi^{n_{\mathrm{I}_{R-1}}}-1\right)^{\mu_{w-1}}}{(\Xi-1)^{\mu_{w-1}-1}}\right)e^{i2\pi\left(\log_{\Xi}\left(1+\frac{\left(\Xi^{n_{\mathrm{I}_{I-1}}}-1\right)^{\mu_{w-1}}}{(\Xi-1)^{\mu_{w-1}-1}}\right)\right)} \end{pmatrix}$$

$$\oplus \begin{pmatrix} 1-\log_{\Xi}\left(1+\frac{\left(\Xi^{1-m_{\mathrm{I}_{R-2}}}-1\right)^{\mu_{w-2}}}{(\Xi-1)^{\mu_{w-2}-1}}\right)e^{i2\pi\left(1-\log_{\Xi}\left(1+\frac{\left(\Xi^{1-m_{\mathrm{I}_{I-2}}}-1\right)^{\mu_{w-2}}}{(\Xi-1)^{\mu_{w-2}-1}}\right)\right)}, \\ \log_{\Xi}\left(1+\frac{\left(\Xi^{n_{\mathrm{I}_{R-2}}}-1\right)^{\mu_{w-2}}}{(\Xi-1)^{\mu_{w-2}-1}}\right)e^{i2\pi\left(\log_{\Xi}\left(1+\frac{\left(\Xi^{n_{\mathrm{I}_{I-2}}}-1\right)^{\mu_{w-2}}}{(\Xi-1)^{\mu_{w-2}-1}}\right)\right)} \end{pmatrix}$$

$$= \begin{pmatrix} 1-\log_{\Xi}\left(1+\frac{\prod_{s=1}^{2}\left(\Xi^{1-m_{\mathrm{I}_{R-s}}}-1\right)^{\mu_{w-s}}}{(\Xi-1)^{\sum_{s=1}^{2}\mu_{w-s}-1}}\right)e^{i2\pi\left(1-\log_{\Xi}\left(1+\frac{\prod_{s=1}^{2}\left(\Xi^{1-m_{\mathrm{I}_{I-s}}}-1\right)^{\mu_{w-s}}}{(\Xi-1)^{\sum_{s=1}^{2}\mu_{w-s}-1}}\right)\right)}, \\ \log_{\Xi}\left(1+\frac{\prod_{s=1}^{2}\left(\Xi^{n_{\mathrm{I}_{R-s}}}-1\right)^{\mu_{w-s}}}{(\Xi-1)^{\sum_{s=1}^{2}\mu_{w-s}-1}}\right)e^{i2\pi\left(\log_{\Xi}\left(1+\frac{\prod_{s=1}^{2}\left(\Xi^{n_{\mathrm{I}_{I-s}}}-1\right)^{\mu_{w-s}}}{(\Xi-1)^{\sum_{s=1}^{2}\mu_{w-s}-1}}\right)\right)} \end{pmatrix}$$

Hence, Eq. (16) fulfills the criteria for the value of $t = 2$, further, we assume for $t = k$, then

$$CIFFWA(\mathrm{I}_1, \mathrm{I}_2, \ldots, \mathrm{I}_k) = \begin{pmatrix} 1-\log_{\Xi}\left(1+\frac{\prod_{s=1}^{k}\left(\Xi^{1-m_{\mathrm{I}_{R-s}}}-1\right)^{\mu_{w-s}}}{(\Xi-1)^{\sum_{s=1}^{k}\mu_{w-s}-1}}\right)e^{i2\pi\left(1-\log_{\Xi}\left(1+\frac{\prod_{s=1}^{k}\left(\Xi^{1-m_{\mathrm{I}_{I-s}}}-1\right)^{\mu_{w-s}}}{(\Xi-1)^{\sum_{s=1}^{k}\mu_{w-s}-1}}\right)\right)}, \\ \log_{\Xi}\left(1+\frac{\prod_{s=1}^{k}\left(\Xi^{n_{\mathrm{I}_{R-s}}}-1\right)^{\mu_{w-s}}}{(\Xi-1)^{\sum_{s=1}^{k}\mu_{w-s}-1}}\right)e^{i2\pi\left(\log_{\Xi}\left(1+\frac{\prod_{s=1}^{k}\left(\Xi^{n_{\mathrm{I}_{I-s}}}-1\right)^{\mu_{w-s}}}{(\Xi-1)^{\sum_{s=1}^{k}\mu_{w-s}-1}}\right)\right)} \end{pmatrix}$$

Then, we prove that Eq. (16) is also fulfilling the criteria for the value of $t = k+1$, such that

$$CIFFWA(\mathrm{I}_1, \mathrm{I}_2, \ldots, \mathrm{I}_{k+1}) = \mu_{w-1}\mathrm{I}_1 \oplus \mu_{w-2}\mathrm{I}_2 \oplus \ldots \oplus \mu_{w-k}\mathrm{I}_k \oplus \mu_{w-k+1}\mathrm{I}_{k+1} = \oplus_{s=1}^{k}\mu_{w-s}\mathrm{I}_s \oplus \mu_{w-k+1}\mathrm{I}_{k+1}$$

$$= \begin{pmatrix} 1-\log_{\Xi}\left(1+\frac{\prod_{s=1}^{k}\left(\Xi^{1-m_{\mathrm{I}_{R-s}}}-1\right)^{\mu_{w-s}}}{(\Xi-1)^{\sum_{s=1}^{k}\mu_{w-s}-1}}\right)e^{i2\pi\left(1-\log_{\Xi}\left(1+\frac{\prod_{s=1}^{k}\left(\Xi^{1-m_{\mathrm{I}_{I-s}}}-1\right)^{\mu_{w-s}}}{(\Xi-1)^{\sum_{s=1}^{k}\mu_{w-s}-1}}\right)\right)}, \\ \log_{\Xi}\left(1+\frac{\prod_{s=1}^{k}\left(\Xi^{n_{\mathrm{I}_{R-s}}}-1\right)^{\mu_{w-s}}}{(\Xi-1)^{\sum_{s=1}^{k}\mu_{w-s}-1}}\right)e^{i2\pi\left(\log_{\Xi}\left(1+\frac{\prod_{s=1}^{k}\left(\Xi^{n_{\mathrm{I}_{I-s}}}-1\right)^{\mu_{w-s}}}{(\Xi-1)^{\sum_{s=1}^{k}\mu_{w-s}-1}}\right)\right)} \end{pmatrix} \oplus \mu_{w-k+1}\mathrm{I}_{k+1}$$

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$$= \begin{pmatrix} 1-\log_{\Xi}\left(1+\frac{\prod_{s=1}^{k}\left(\Xi^{1-m_{I_{R-s}}}-1\right)^{\mu_{w-s}}}{(\Xi-1)^{\sum_{s=1}^{k}\mu_{w-s}-1}}\right)e^{i2\pi\left(1-\log_{\Xi}\left(1+\frac{\prod_{s=1}^{k}\left(\Xi^{1-m_{I_{I-s}}}-1\right)^{\mu_{w-s}}}{(\Xi-1)^{\sum_{s=1}^{k}\mu_{w-s}-1}}\right)\right)}, \\ \log_{\Xi}\left(1+\frac{\prod_{s=1}^{k}\left(\Xi^{n_{I_{R-s}}}-1\right)^{\mu_{w-s}}}{(\Xi-1)^{\sum_{s=1}^{k}\mu_{w-s}-1}}\right)e^{i2\pi\left(\log_{\Xi}\left(1+\frac{\prod_{s=1}^{k}\left(\Xi^{n_{I_{I-s}}}-1\right)^{\mu_{w-s}}}{(\Xi-1)^{\sum_{s=1}^{k}\mu_{w-s}-1}}\right)\right)} \end{pmatrix}$$

$$\oplus \begin{pmatrix} 1-\log_{\Xi}\left(1+\frac{\left(\Xi^{1-m_{I_{R-k+1}}}-1\right)^{\mu_{w-k+1}}}{(\Xi-1)^{\mu_{w-k+1}-1}}\right)e^{i2\pi\left(1-\log_{\Xi}\left(1+\frac{\left(\Xi^{1-m_{I_{I-k+1}}}-1\right)^{\mu_{w-k+1}}}{(\Xi-1)^{\mu_{w-k+1}-1}}\right)\right)}, \\ \log_{\Xi}\left(1+\frac{\left(\Xi^{n_{I_{R-k+1}}}-1\right)^{\mu_{w-k+1}}}{(\Xi-1)^{\mu_{w-k+1}-1}}\right)e^{i2\pi\left(\log_{\Xi}\left(1+\frac{\left(\Xi^{n_{I_{I-k+1}}}-1\right)^{\mu_{w-k+1}}}{(\Xi-1)^{\mu_{w-k+1}-1}}\right)\right)} \end{pmatrix}$$

$$= \begin{pmatrix} 1-\log_{\Xi}\left(1+\frac{\prod_{s=1}^{k+1}\left(\Xi^{1-m_{I_{R-s}}}-1\right)^{\mu_{w-s}}}{(\Xi-1)^{\sum_{s=1}^{k+1}\mu_{w-s}-1}}\right)e^{i2\pi\left(1-\log_{\Xi}\left(1+\frac{\prod_{s=1}^{k+!}\left(\Xi^{1-m_{I_{I-s}}}-1\right)^{\mu_{w-s}}}{(\Xi-1)^{\sum_{s=1}^{k+1}\mu_{w-s}-1}}\right)\right)}, \\ \log_{\Xi}\left(1+\frac{\prod_{s=1}^{k+1}\left(\Xi^{n_{I_{R-s}}}-1\right)^{\mu_{w-s}}}{(\Xi-1)^{\sum_{s=1}^{k+1}\mu_{w-s}-1}}\right)e^{i2\pi\left(\log_{\Xi}\left(1+\frac{\prod_{s=1}^{k+1}\left(\Xi^{n_{I_{I-s}}}-1\right)^{\mu_{w-s}}}{(\Xi-1)^{\sum_{s=1}^{k+!}\mu_{w-s}-1}}\right)\right)} \end{pmatrix}$$

Hence, Eq. (16) holds for all possible values of $t$.

**Property 1:** (*Idempotency*) *When* $I_s = I = \left(m_{I_R}e^{i2\pi\left(m_{I_I}\right)}, n_{I_R}e^{i2\pi\left(n_{I_I}\right)}\right)$, $s = 1,2,\dots,t$, *then*

$$CIFFWA(I_1, I_2, \dots, I_k) = I \quad (17)$$

**Proof:** *Given that* $I_s = I = \left(m_{I_R}e^{i2\pi\left(m_{I_I}\right)}, n_{I_R}e^{i2\pi\left(n_{I_I}\right)}\right)$, *then*

$$CIFFWA(I_1, I_2, \dots, I_t) = \begin{pmatrix} 1-\log_{\Xi}\left(1+\frac{\prod_{s=1}^{t}\left(\Xi^{1-m_{I_{R-s}}}-1\right)^{\mu_{w-s}}}{(\Xi-1)^{\sum_{s=1}^{t}\mu_{w-s}-1}}\right)e^{i2\pi\left(1-\log_{\Xi}\left(1+\frac{\prod_{s=1}^{t}\left(\Xi^{1-m_{I_{I-s}}}-1\right)^{\mu_{w-s}}}{(\Xi-1)^{\sum_{s=1}^{t}\mu_{w-s}-1}}\right)\right)}, \\ \log_{\Xi}\left(1+\frac{\prod_{s=1}^{t}\left(\Xi^{n_{I_{R-s}}}-1\right)^{\mu_{w-s}}}{(\Xi-1)^{\sum_{s=1}^{t}\mu_{w-s}-1}}\right)e^{i2\pi\left(\log_{\Xi}\left(1+\frac{\prod_{s=1}^{t}\left(\Xi^{n_{I_{I-s}}}-1\right)^{\mu_{w-s}}}{(\Xi-1)^{\sum_{s=1}^{t}\mu_{w-s}-1}}\right)\right)} \end{pmatrix}$$

$$= \begin{pmatrix} 1-\log_{\Xi}\left(1+\frac{\prod_{s=1}^{t}\left(\Xi^{1-m_{I_R}}-1\right)^{\mu_{w-s}}}{(\Xi-1)^{\sum_{s=1}^{t}\mu_{w-s}-1}}\right)e^{i2\pi\left(1-\log_{\Xi}\left(1+\frac{\prod_{s=1}^{t}\left(\Xi^{1-m_{I_I}}-1\right)^{\mu_{w-s}}}{(\Xi-1)^{\sum_{s=1}^{t}\mu_{w-s}-1}}\right)\right)}, \\ \log_{\Xi}\left(1+\frac{\prod_{s=1}^{t}\left(\Xi^{n_{I_R}}-1\right)^{\mu_{w-s}}}{(\Xi-1)^{\sum_{s=1}^{t}\mu_{w-s}-1}}\right)e^{i2\pi\left(\log_{\Xi}\left(1+\frac{\prod_{s=1}^{t}\left(\Xi^{n_{I_I}}-1\right)^{\mu_{w-s}}}{(\Xi-1)^{\sum_{s=1}^{t}\mu_{w-s}-1}}\right)\right)} \end{pmatrix}$$

$$= \begin{pmatrix} 1-\log_{\Xi}\left(1+\frac{\left(\Xi^{1-m_{I_R}}-1\right)^{\sum_{s=1}^{t}\mu_{w-s}}}{(\Xi-1)^{\sum_{s=1}^{t}\mu_{w-s}-1}}\right)e^{i2\pi\left(1-\log_{\Xi}\left(1+\frac{\left(\Xi^{1-m_{I_I}}-1\right)^{\sum_{s=1}^{t}\mu_{w-s}}}{(\Xi-1)^{\sum_{s=1}^{t}\mu_{w-s}-1}}\right)\right)}, \\ \log_{\Xi}\left(1+\frac{\left(\Xi^{n_{I_R}}-1\right)^{\sum_{s=1}^{t}\mu_{w-s}}}{(\Xi-1)^{\sum_{s=1}^{t}\mu_{w-s}-1}}\right)e^{i2\pi\left(\log_{\Xi}\left(1+\frac{\left(\Xi^{n_{I_I}}-1\right)^{\sum_{s=1}^{t}\mu_{w-s}}}{(\Xi-1)^{\sum_{s=1}^{t}\mu_{w-s}-1}}\right)\right)} \end{pmatrix}$$

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$$= \begin{pmatrix} 1-\log_{\Xi}\left(1+\frac{\left(\Xi^{1-m_{I_R}}-1\right)^1}{(\Xi-1)^{1-1}}\right) e^{i2\pi\left(1-\log_{\Xi}\left(1+\frac{\left(\Xi^{1-m_{I_I}}-1\right)^1}{(\Xi-1)^{1-1}}\right)\right)}, \\ \log_{\Xi}\left(1+\frac{\left(\Xi^{n_{I_R}}-1\right)^1}{(\Xi-1)^{1-1}}\right) e^{i2\pi\left(\log_{\Xi}\left(1+\frac{\left(\Xi^{n_{I_I}}-1\right)^1}{(\Xi-1)^{1-1}}\right)\right)} \end{pmatrix}$$

$$= \begin{pmatrix} 1-\log_{\Xi}\left(1+\left(\Xi^{1-m_{I_R}}-1\right)^1\right) e^{i2\pi\left(1-\log_{\Xi}\left(1+\left(\Xi^{1-m_{I_I}}-1\right)^1\right)\right)}, \\ \log_{\Xi}\left(1+\left(\Xi^{n_{I_R}}-1\right)\right) e^{i2\pi\left(\log_{\Xi}\left(1+\left(\Xi^{n_{I_I}}-1\right)^1\right)\right)} \end{pmatrix}$$

$$= \begin{pmatrix} 1-\log_{\Xi}\Xi^{1-m_{I_R}}\, e^{i2\pi\left(1-\log_{\Xi}\Xi^{1-m_{I_I}}\right)}, \\ \log_{\Xi}\Xi^{n_{I_R}}\, e^{i2\pi\left(\log_{\Xi}\Xi^{n_{I_I}}\right)} \end{pmatrix} = \left(m_{I_R} e^{i2\pi\left(m_{I_I}\right)}, n_{I_R} e^{i2\pi\left(n_{I_I}\right)}\right) = I.$$

**Property 2:** (*Monotonicity*) *When* $I_s \leq I_s^*$ *where* $I_s = \left(m_{I_{R-s}} e^{i2\pi\left(m_{I_{I-s}}\right)}, n_{I_{R-s}} e^{i2\pi\left(n_{I_{I-s}}\right)}\right)$ *and* $I_s^* = \left(m^*_{I_{R-s}} e^{i2\pi\left(m^*_{I_{I-s}}\right)}, n^*_{I_{R-s}} e^{i2\pi\left(n^*_{I_{I-s}}\right)}\right)$, $s = 1,2,\dots,t$, *then*

$$CIFFWA(I_1, I_2, \dots, I_t) \leq CIFFWA(I_1^*, I_2^*, \dots, I_t^*) \quad (18)$$

**Proof:** Considered that $I_s \leq I_s^*$ it means that $m_{I_{R-s}} \leq m^*_{I_{R-s}}, m_{I_{I-s}} \leq m^*_{I_{I-s}}$ and $n_{I_{R-s}} \geq n^*_{I_{R-s}}, n_{I_{I-s}} \geq n^*_{I_{I-s}}$, then

$$1-\log_{\Xi}\left(1+\frac{\prod_{s=1}^{t}\left(\Xi^{1-m_{I_{R-s}}}-1\right)^{\mu_{w-s}}}{(\Xi-1)^{\sum_{s=1}^{t}\mu_{w-s}-1}}\right) \leq 1-\log_{\Xi}\left(1+\frac{\prod_{s=1}^{t}\left(\Xi^{1-m^*_{I_{R-s}}}-1\right)^{\mu_{w-s}}}{(\Xi-1)^{\sum_{s=1}^{t}\mu_{w-s}-1}}\right)$$

$$1-\log_{\Xi}\left(1+\frac{\prod_{s=1}^{t}\left(\Xi^{1-m_{I_{I-s}}}-1\right)^{\mu_{w-s}}}{(\Xi-1)^{\sum_{s=1}^{t}\mu_{w-s}-1}}\right) \leq 1-\log_{\Xi}\left(1+\frac{\prod_{s=1}^{t}\left(\Xi^{1-m^*_{I_{I-s}}}-1\right)^{\mu_{w-s}}}{(\Xi-1)^{\sum_{s=1}^{t}\mu_{w-s}-1}}\right)$$

and

$$\log_{\Xi}\left(1+\frac{\prod_{s=1}^{t}\left(\Xi^{n_{I_{R-s}}}-1\right)^{\mu_{w-s}}}{(\Xi-1)^{\sum_{s=1}^{t}\mu_{w-s}-1}}\right) \geq \log_{\Xi}\left(1+\frac{\prod_{s=1}^{t}\left(\Xi^{n^*_{I_{R-s}}}-1\right)^{\mu_{w-s}}}{(\Xi-1)^{\sum_{s=1}^{t}\mu_{w-s}-1}}\right)$$

$$\log_{\Xi}\left(1+\frac{\prod_{s=1}^{t}\left(\Xi^{n_{I_{I-s}}}-1\right)^{\mu_{w-s}}}{(\Xi-1)^{\sum_{s=1}^{t}\mu_{w-s}-1}}\right) \geq \log_{\Xi}\left(1+\frac{\prod_{s=1}^{t}\left(\Xi^{n^*_{I_{I-s}}}-1\right)^{\mu_{w-s}}}{(\Xi-1)^{\sum_{s=1}^{t}\mu_{w-s}-1}}\right)$$

Then, by using Eq. (6), we have

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$$\mathcal{C}_{s\delta}\left(CIFFWA(\text{Ɩ}_1,\text{Ɩ}_2,\ldots,\text{Ɩ}_t)\right) = \frac{1}{2}\left(1-\log_{\Xi}\left(1+\frac{\prod_{\delta=1}^{t}\left(\Xi^{1-m_{\text{Ɩ}_{R-\delta}}}-1\right)^{\mu_{w-\delta}}}{(\Xi-1)^{\sum_{\delta=1}^{t}\mu_{w-\delta}-1}}\right)+1 - \log_{\Xi}\left(1+\frac{\prod_{\delta=1}^{t}\left(\Xi^{1-m_{\text{Ɩ}_{I-\delta}}}-1\right)^{\mu_{w-\delta}}}{(\Xi-1)^{\sum_{\delta=1}^{t}\mu_{w-\delta}-1}}\right)-\log_{\Xi}\left(1+\frac{\prod_{\delta=1}^{t}\left(\Xi^{n_{\text{Ɩ}_{R-\delta}}}-1\right)^{\mu_{w-\delta}}}{(\Xi-1)^{\sum_{\delta=1}^{t}\mu_{w-\delta}-1}}\right) - \log_{\Xi}\left(1+\frac{\prod_{\delta=1}^{t}\left(\Xi^{n_{\text{Ɩ}_{I-\delta}}}-1\right)^{\mu_{w-\delta}}}{(\Xi-1)^{\sum_{\delta=1}^{t}\mu_{w-\delta}-1}}\right)\right)$$

$$\leq \frac{1}{2}\left(1-\log_{\Xi}\left(1+\frac{\prod_{\delta=1}^{t}\left(\Xi^{1-m^*_{\text{Ɩ}_{R-\delta}}}-1\right)^{\mu_{w-\delta}}}{(\Xi-1)^{\sum_{\delta=1}^{t}\mu_{w-\delta}-1}}\right)+1 - \log_{\Xi}\left(1+\frac{\prod_{\delta=1}^{t}\left(\Xi^{1-m^*_{\text{Ɩ}_{I-\delta}}}-1\right)^{\mu_{w-\delta}}}{(\Xi-1)^{\sum_{\delta=1}^{t}\mu_{w-\delta}-1}}\right)-\log_{\Xi}\left(1+\frac{\prod_{\delta=1}^{t}\left(\Xi^{n^*_{\text{Ɩ}_{R-\delta}}}-1\right)^{\mu_{w-\delta}}}{(\Xi-1)^{\sum_{\delta=1}^{t}\mu_{w-\delta}-1}}\right) - \log_{\Xi}\left(1+\frac{\prod_{\delta=1}^{t}\left(\Xi^{n^*_{\text{Ɩ}_{I-\delta}}}-1\right)^{\mu_{w-\delta}}}{(\Xi-1)^{\sum_{\delta=1}^{t}\mu_{w-\delta}-1}}\right)\right) = \mathcal{C}_{s\delta}\left(CIFFWA(\text{Ɩ}^*_1,\text{Ɩ}^*_2,\ldots,\text{Ɩ}^*_t)\right)$$

Hence,

$$CIFFWA(\text{Ɩ}_1,\text{Ɩ}_2,\ldots,\text{Ɩ}_t) \leq CIFFWA(\text{Ɩ}^*_1,\text{Ɩ}^*_2,\ldots,\text{Ɩ}^*_t).$$

If $\mathcal{C}_{s\delta}\left(CIFFWA(\text{Ɩ}_1,\text{Ɩ}_2,\ldots,\text{Ɩ}_t)\right) = \mathcal{C}_{s\delta}\left(CIFFWA(\text{Ɩ}^*_1,\text{Ɩ}^*_2,\ldots,\text{Ɩ}^*_t)\right)$, then

$$\mathcal{A}_{a\delta}\left(CIFFWA(\text{Ɩ}_1,\text{Ɩ}_2,\ldots,\text{Ɩ}_t)\right) = \frac{1}{2}\left(1-\log_{\Xi}\left(1+\frac{\prod_{\delta=1}^{t}\left(\Xi^{1-m_{\text{Ɩ}_{R-\delta}}}-1\right)^{\mu_{w-\delta}}}{(\Xi-1)^{\sum_{\delta=1}^{t}\mu_{w-\delta}-1}}\right)+1 - \log_{\Xi}\left(1+\frac{\prod_{\delta=1}^{t}\left(\Xi^{1-m_{\text{Ɩ}_{I-\delta}}}-1\right)^{\mu_{w-\delta}}}{(\Xi-1)^{\sum_{\delta=1}^{t}\mu_{w-\delta}-1}}\right)+\log_{\Xi}\left(1+\frac{\prod_{\delta=1}^{t}\left(\Xi^{n_{\text{Ɩ}_{R-\delta}}}-1\right)^{\mu_{w-\delta}}}{(\Xi-1)^{\sum_{\delta=1}^{t}\mu_{w-\delta}-1}}\right) + \log_{\Xi}\left(1+\frac{\prod_{\delta=1}^{t}\left(\Xi^{n_{\text{Ɩ}_{I-\delta}}}-1\right)^{\mu_{w-\delta}}}{(\Xi-1)^{\sum_{\delta=1}^{t}\mu_{w-\delta}-1}}\right)\right)$$

$$\leq \frac{1}{2}\left(1-\log_{\Xi}\left(1+\frac{\prod_{\delta=1}^{t}\left(\Xi^{1-m^*_{\text{Ɩ}_{R-\delta}}}-1\right)^{\mu_{w-\delta}}}{(\Xi-1)^{\sum_{\delta=1}^{t}\mu_{w-\delta}-1}}\right)+1 - \log_{\Xi}\left(1+\frac{\prod_{\delta=1}^{t}\left(\Xi^{1-m^*_{\text{Ɩ}_{I-\delta}}}-1\right)^{\mu_{w-\delta}}}{(\Xi-1)^{\sum_{\delta=1}^{t}\mu_{w-\delta}-1}}\right)+\log_{\Xi}\left(1+\frac{\prod_{\delta=1}^{t}\left(\Xi^{n^*_{\text{Ɩ}_{R-\delta}}}-1\right)^{\mu_{w-\delta}}}{(\Xi-1)^{\sum_{\delta=1}^{t}\mu_{w-\delta}-1}}\right) + \log_{\Xi}\left(1+\frac{\prod_{\delta=1}^{t}\left(\Xi^{n^*_{\text{Ɩ}_{I-\delta}}}-1\right)^{\mu_{w-\delta}}}{(\Xi-1)^{\sum_{\delta=1}^{t}\mu_{w-\delta}-1}}\right)\right) = \mathcal{A}_{a\delta}\left(CIFFWA(\text{Ɩ}^*_1,\text{Ɩ}^*_2,\ldots,\text{Ɩ}^*_t)\right)$$

Hence,

$$CIFFWA(\text{Ɩ}_1,\text{Ɩ}_2,\ldots,\text{Ɩ}_t) \leq CIFFWA(\text{Ɩ}^*_1,\text{Ɩ}^*_2,\ldots,\text{Ɩ}^*_t).$$

Property 3: (Boundedness) When $\text{Ɩ}^+_{\delta} = \left(\max_{\delta} m_{\text{Ɩ}_{R-\delta}}\, e^{i2\pi\left(\max_{\delta} m_{\text{Ɩ}_{I-\delta}}\right)}, \min_{\delta} n_{\text{Ɩ}_{R-\delta}}\, e^{i2\pi\left(\min_{\delta} n_{\text{Ɩ}_{I-\delta}}\right)}\right)$ and $\text{Ɩ}^-_{\delta} = \left(\min_{\delta} m_{\text{Ɩ}_{R-\delta}}\, e^{i2\pi\left(\min_{\delta} m_{\text{Ɩ}_{I-\delta}}\right)}, \max_{\delta} n_{\text{Ɩ}_{R-\delta}}\, e^{i2\pi\left(\max_{\delta} n_{\text{Ɩ}_{I-\delta}}\right)}\right)$, then

$$\text{Ɩ}^-_{\delta} \leq CIFFWA(\text{Ɩ}_1,\text{Ɩ}_2,\ldots,\text{Ɩ}_t) \leq \text{Ɩ}^+_{\delta} \quad (19)$$

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**Proof:** Assume that $\text{I}_{s}^{+} = \left(\max_{s} m_{\text{I}_{R-s}}\, e^{i2\pi\left(\max_{s} m_{\text{I}_{I-s}}\right)}, \min_{s} n_{\text{I}_{R-s}}\, e^{i2\pi\left(\min_{s} n_{\text{I}_{I-s}}\right)}\right)$ and $\text{I}_{s}^{-} = \left(\min_{s} m_{\text{I}_{R-s}}\, e^{i2\pi\left(\min_{s} m_{\text{I}_{I-s}}\right)}, \max_{s} n_{\text{I}_{R-s}}\, e^{i2\pi\left(\max_{s} n_{\text{I}_{I-s}}\right)}\right)$, then $\min_{s} m_{\text{I}_{R-s}} \leq m_{\text{I}_{R-s}} \leq \max_{s} m_{\text{I}_{R-s}}, \min_{s} m_{\text{I}_{I-s}} \leq m_{\text{I}_{I-s}} \leq \max_{s} m_{\text{I}_{I-s}}, \min_{s} n_{\text{I}_{R-s}} \leq n_{\text{I}_{R-s}} \leq \max_{s} n_{\text{I}_{R-s}}, \min_{s} n_{\text{I}_{I-s}} \leq n_{\text{I}_{I-s}} \leq \max_{s} n_{\text{I}_{I-s}}$, then

$$\min_{s} m_{\text{I}_{R-s}} = 1 - \log_{\Xi}\left(1 + \frac{\prod_{s=1}^{t}\left(\Xi^{1-\min_{s} m_{\text{I}_{R-s}}} - 1\right)^{\mu_{w-s}}}{(\Xi-1)^{\sum_{s=1}^{t}\mu_{w-s}-1}}\right) \leq 1 - \log_{\Xi}\left(1 + \frac{\prod_{s=1}^{t}\left(\Xi^{1-m_{\text{I}_{R-s}}} - 1\right)^{\mu_{w-s}}}{(\Xi-1)^{\sum_{s=1}^{t}\mu_{w-s}-1}}\right)$$

$$= m_{\text{I}_{R-s}} \leq 1 - \log_{\Xi}\left(1 + \frac{\prod_{s=1}^{t}\left(\Xi^{1-\max_{s} m_{\text{I}_{R-s}}} - 1\right)^{\mu_{w-s}}}{(\Xi-1)^{\sum_{s=1}^{t}\mu_{w-s}-1}}\right) \leq \max_{s} m_{\text{I}_{R-s}}$$

$$\min_{s} m_{\text{I}_{I-s}} = 1 - \log_{\Xi}\left(1 + \frac{\prod_{s=1}^{t}\left(\Xi^{1-\min_{s} m_{\text{I}_{I-s}}} - 1\right)^{\mu_{w-s}}}{(\Xi-1)^{\sum_{s=1}^{t}\mu_{w-s}-1}}\right) \leq 1 - \log_{\Xi}\left(1 + \frac{\prod_{s=1}^{t}\left(\Xi^{1-m_{\text{I}_{I-s}}} - 1\right)^{\mu_{w-s}}}{(\Xi-1)^{\sum_{s=1}^{t}\mu_{w-s}-1}}\right)$$

$$= m_{\text{I}_{R-s}} \leq 1 - \log_{\Xi}\left(1 + \frac{\prod_{s=1}^{t}\left(\Xi^{1-\max_{s} m_{\text{I}_{I-s}}} - 1\right)^{\mu_{w-s}}}{(\Xi-1)^{\sum_{s=1}^{t}\mu_{w-s}-1}}\right) \leq \max_{s} m_{\text{I}_{I-s}}$$

$$\min_{s} n_{\text{I}_{R-s}} = \log_{\Xi}\left(1 + \frac{\prod_{s=1}^{t}\left(\Xi^{\min_{s} n_{\text{I}_{R-s}}} - 1\right)^{\mu_{w-s}}}{(\Xi-1)^{\sum_{s=1}^{t}\mu_{w-s}-1}}\right) \leq \log_{\Xi}\left(1 + \frac{\prod_{s=1}^{t}\left(\Xi^{n_{\text{I}_{R-s}}} - 1\right)^{\mu_{w-s}}}{(\Xi-1)^{\sum_{s=1}^{t}\mu_{w-s}-1}}\right) = n_{\text{I}_{R-s}}$$

$$\leq \log_{\Xi}\left(1 + \frac{\prod_{s=1}^{t}\left(\Xi^{\max_{s} n_{\text{I}_{R-s}}} - 1\right)^{\mu_{w-s}}}{(\Xi-1)^{\sum_{s=1}^{t}\mu_{w-s}-1}}\right) \leq \max_{s} n_{\text{I}_{R-s}}$$

$$\min_{s} n_{\text{I}_{I-s}} = \log_{\Xi}\left(1 + \frac{\prod_{s=1}^{t}\left(\Xi^{\min_{s} n_{\text{I}_{I-s}}} - 1\right)^{\mu_{w-s}}}{(\Xi-1)^{\sum_{s=1}^{t}\mu_{w-s}-1}}\right) \leq \log_{\Xi}\left(1 + \frac{\prod_{s=1}^{t}\left(\Xi^{n_{\text{I}_{I-s}}} - 1\right)^{\mu_{w-s}}}{(\Xi-1)^{\sum_{s=1}^{t}\mu_{w-s}-1}}\right) = n_{\text{I}_{I-s}}$$

$$\leq \log_{\Xi}\left(1 + \frac{\prod_{s=1}^{t}\left(\Xi^{\max_{s} n_{\text{I}_{I-s}}} - 1\right)^{\mu_{w-s}}}{(\Xi-1)^{\sum_{s=1}^{t}\mu_{w-s}-1}}\right) \leq \max_{s} n_{\text{I}_{I-s}}$$

Then by using Eq. (6) and Eq. (7), we have

$$\mathcal{C}_{ss}\left(CIFFWA(\text{I}_1, \text{I}_2, \dots, \text{I}_t)\right) = \frac{1}{2}\left(m_{\text{I}_{R-s}} + m_{\text{I}_{I-s}} - n_{\text{I}_{R-s}} - n_{\text{I}_{I-s}}\right)$$

$$\geq \frac{1}{2}\left(\min_{s} m_{\text{I}_{R-s}} + \min_{s} m_{\text{I}_{I-s}} - \max_{s} n_{\text{I}_{R-s}} - \max_{s} n_{\text{I}_{I-s}}\right) = \mathcal{C}_{ss}(\text{I}_{s}^{-})$$

$$\mathcal{C}_{ss}\left(CIFFWA(\text{I}_1, \text{I}_2, \dots, \text{I}_t)\right) = \frac{1}{2}\left(m_{\text{I}_{R-s}} + m_{\text{I}_{I-s}} - n_{\text{I}_{R-s}} - n_{\text{I}_{I-s}}\right)$$

$$\leq \frac{1}{2}\left(\max_{s} m_{\text{I}_{R-s}} + \max_{s} m_{\text{I}_{I-s}} - \min_{s} n_{\text{I}_{R-s}} - \min_{s} n_{\text{I}_{I-s}}\right) = \mathcal{C}_{ss}(\text{I}_{s}^{+})$$

In both situations, we obtained

$$\text{I}_{s}^{-} \leq CIFFWA(\text{I}_1, \text{I}_2, \dots, \text{I}_t) \leq \text{I}_{s}^{+}$$

But if $\mathcal{C}_{ss}\left(CIFFWA(\text{I}_1, \text{I}_2, \dots, \text{I}_t)\right) = \mathcal{C}_{ss}(\text{I}_{s}^{+}) = \mathcal{C}_{ss}(\text{I}_{s}^{-})$, then

$$\mathcal{A}_{as}\left(CIFFWA(\text{I}_1, \text{I}_2, \dots, \text{I}_t)\right) = \frac{1}{2}\left(m_{\text{I}_{R-s}} + m_{\text{I}_{I-s}} - n_{\text{I}_{R-s}} - n_{\text{I}_{I-s}}\right)$$

$$\geq \frac{1}{2}\left(\min_{s} m_{\text{I}_{R-s}} + \min_{s} m_{\text{I}_{I-s}} - \max_{s} n_{\text{I}_{R-s}} - \max_{s} n_{\text{I}_{I-s}}\right) = \mathcal{A}_{as}(\text{I}_{s}^{-})$$

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$$\mathcal{A}_{as}\left(CIFFWA(\mathrm{I}_1,\mathrm{I}_2,\dots,\mathrm{I}_t)\right)=\frac{1}{2}\left(m_{\mathrm{I}_{R-s}}+m_{\mathrm{I}_{I-s}}-n_{\mathrm{I}_{R-s}}-n_{\mathrm{I}_{I-s}}\right)$$

$$\leq\frac{1}{2}\left(\max_{s} m_{\mathrm{I}_{R-s}}+\max_{s} m_{\mathrm{I}_{I-s}}-\min_{s} n_{\mathrm{I}_{R-s}}-\min_{s} n_{\mathrm{I}_{I-s}}\right)=\mathcal{A}_{as}(\mathrm{I}_s^+)$$

In both situations, we obtained

$$\mathrm{I}_s^-\leq CIFFWA(\mathrm{I}_1,\mathrm{I}_2,\dots,\mathrm{I}_t)\leq\mathrm{I}_s^+.$$

**Definition 8:** *We diagnose the idea of a CIFFOWA operator based on CIFNs* $\mathrm{I}_s=\left(m_{\mathrm{I}_{R-s}}e^{i2\pi\left(m_{\mathrm{I}_{I-s}}\right)},n_{\mathrm{I}_{R-s}}e^{i2\pi\left(n_{\mathrm{I}_{I-s}}\right)}\right),s=1,2,\dots,t$, *such that*

$$CIFFOWA(\mathrm{I}_1,\mathrm{I}_2,\dots,\mathrm{I}_t)=\mu_{w-1}\mathrm{I}_{o(1)}\oplus\mu_{w-2}\mathrm{I}_{o(2)}\oplus\dots\oplus\mu_{w-t}\mathrm{I}_{o(t)}=\oplus_{s=1}^{t}\mu_{w-s}\mathrm{I}_{o(s)}\quad(20)$$

*Where* $\sum_{s=1}^{t}\mu_{w-s}=1,\mu_{w-s}\in[0,1]$, *representing the weight vector with* $\mathrm{I}_{o(s)}\leq\mathrm{I}_{o(s-1)}$. *In particular, if we use the value of* $\mu_{w-s}=\left(\frac{1}{t},\frac{1}{t},\dots,\frac{1}{t}\right)$ *in Eq. (20), then we have*

$$CIFFOA(\mathrm{I}_1,\mathrm{I}_2,\dots,\mathrm{I}_t)=\frac{1}{t}\left(\mathrm{I}_{o(1)}\oplus\mathrm{I}_{o(2)}\oplus\dots\oplus\mathrm{I}_{o(t)}\right)=\frac{1}{t}\oplus_{s=1}^{t}\mathrm{I}_{o(s)}\quad(21)$$

*Called CIF frank ordered averaging (CIFFOA) operator.*

**Theorem 2:** *Verify that the resultant value of Eq. (20) is again in the shape of CIFNs, such that*

$$CIFFOWA(\mathrm{I}_1,\mathrm{I}_2,\dots,\mathrm{I}_t)$$
$$=\left(\begin{array}{c}\left(1-\log_{\Xi}\left(1+\frac{\prod_{s=1}^{t}\left(\Xi^{1-m_{\mathrm{I}_{R-o(s)}}}-1\right)^{\mu_{w-s}}}{(\Xi-1)^{\sum_{s=1}^{t}\mu_{w-s}-1}}\right)\right)e^{i2\pi\left(1-\log_{\Xi}\left(1+\frac{\prod_{s=1}^{t}\left(\Xi^{1-m_{\mathrm{I}_{I-o(s)}}}-1\right)^{\mu_{w-s}}}{(\Xi-1)^{\sum_{s=1}^{t}\mu_{w-s}-1}}\right)\right)},\\ \log_{\Xi}\left(1+\frac{\prod_{s=1}^{t}\left(\Xi^{n_{\mathrm{I}_{R-o(s)}}}-1\right)^{\mu_{w-s}}}{(\Xi-1)^{\sum_{s=1}^{t}\mu_{w-s}-1}}\right)e^{i2\pi\left(\log_{\Xi}\left(1+\frac{\prod_{s=1}^{t}\left(\Xi^{n_{\mathrm{I}_{I-o(s)}}}-1\right)^{\mu_{w-s}}}{(\Xi-1)^{\sum_{s=1}^{t}\mu_{w-s}-1}}\right)\right)}\end{array}\right)\quad(22)$$

**Proof:** Omitted.

**Definition 9:** *We diagnose the idea of a CIFFHA operator based on CIFNs* $\mathrm{I}_s=\left(m_{\mathrm{I}_{R-s}}e^{i2\pi\left(m_{\mathrm{I}_{I-s}}\right)},n_{\mathrm{I}_{R-s}}e^{i2\pi\left(n_{\mathrm{I}_{I-s}}\right)}\right),s=1,2,\dots,t$, *such that*

$$CIFFHA(\mathrm{I}_1,\mathrm{I}_2,\dots,\mathrm{I}_t)=\mu_{w-1}\mathrm{I}_{o(1)}^*\oplus\mu_{w-2}\mathrm{I}_{o(2)}^*\oplus\dots\oplus\mu_{w-t}\mathrm{I}_{o(t)}^*=\oplus_{s=1}^{t}\mu_{w-s}\mathrm{I}_{o(s)}^*\quad(23)$$

*Where* $\mathrm{I}_{o(s)}^*=t\mu_{w-s}^*\mathrm{I}_s$ *and* $\sum_{s=1}^{t}\mu_{w-s}^*=1,\sum_{s=1}^{t}\mu_{w-s}=1,\mu_{w-s}^*,\mu_{w-s}\in[0,1]$, *representing the weight vector with* $\mathrm{I}_{o(s)}^*\leq\mathrm{I}_{o(s-1)}^*$. *In particular, if we use the value of* $\mu_{w-s}=\left(\frac{1}{t},\frac{1}{t},\dots,\frac{1}{t}\right)$ *in Eq. (23), then we have*

$$CIFFHA(\mathrm{I}_1,\mathrm{I}_2,\dots,\mathrm{I}_t)=\frac{1}{t}\left(\mathrm{I}_{o(1)}^*\oplus\mathrm{I}_{o(2)}^*\oplus\dots\oplus\mathrm{I}_{o(t)}^*\right)=\frac{1}{t}\oplus_{s=1}^{t}\mathrm{I}_{o(s)}^*\quad(24)$$

*Called CIF frank hybrid averaging (CIFFHA) operator.*

**Theorem 3:** *Verify that the resultant value of Eq. (20) is again in the shape of CIFNs, such that*

$$CIFFHA(\text{Ɩ}_1, \text{Ɩ}_2, \dots, \text{Ɩ}_t)$$

$$= \left( \begin{array}{c} \left(1 - \log_{\Xi}\left(1 + \frac{\prod_{s=1}^{t}\left(\Xi^{1-m^*_{\text{Ɩ}_{R-o(s)}}} - 1\right)^{\mu_{w-s}}}{(\Xi - 1)^{\sum_{s=1}^{t}\mu_{w-s}-1}}\right)\right) e^{i2\pi\left(1-\log_{\Xi}\left(1+\frac{\prod_{s=1}^{t}\left(\Xi^{1-m^*_{\text{Ɩ}_{I-o(s)}}}-1\right)^{\mu_{w-s}}}{(\Xi-1)^{\sum_{s=1}^{t}\mu_{w-s}-1}}\right)\right)}, \\ \log_{\Xi}\left(1 + \frac{\prod_{s=1}^{t}\left(\Xi^{n^*_{\text{Ɩ}_{R-o(s)}}} - 1\right)^{\mu_{w-s}}}{(\Xi - 1)^{\sum_{s=1}^{t}\mu_{w-s}-1}}\right) e^{i2\pi\left(\log_{\Xi}\left(1+\frac{\prod_{s=1}^{t}\left(\Xi^{n^*_{\text{Ɩ}_{I-o(s)}}}-1\right)^{\mu_{w-s}}}{(\Xi-1)^{\sum_{s=1}^{t}\mu_{w-s}-1}}\right)\right)} \end{array} \right) \quad (25)$$

**Proof:** Omitted.

**Definition 10:** *We diagnose the idea of a CIFFWG operator based on CIFNs* $\text{Ɩ}_s = \left(m_{\text{Ɩ}_{R-s}} e^{i2\pi\left(m_{\text{Ɩ}_{I-s}}\right)}, n_{\text{Ɩ}_{R-s}} e^{i2\pi\left(n_{\text{Ɩ}_{I-s}}\right)}\right), s = 1,2,\dots,t,$ *such that*

$$CIFFWG(\text{Ɩ}_1, \text{Ɩ}_2, \dots, \text{Ɩ}_t) = \text{Ɩ}_1^{\mu_{w-1}} \otimes \text{Ɩ}_2^{\mu_{w-2}} \otimes \dots \otimes \text{Ɩ}_t^{\mu_{w-t}} = \otimes_{s=1}^{t} \text{Ɩ}_s^{\mu_{w-s}} \quad (26)$$

*Where* $\sum_{s=1}^{t}\mu_{w-s} = 1, \mu_{w-s} \in [0,1]$, *representing the weight vector. In particular, if we use the value of* $\mu_{w-s} = \left(\frac{1}{t}, \frac{1}{t}, \dots, \frac{1}{t}\right)$ *in Eq. (26), then we have*

$$CIFFA(\text{Ɩ}_1, \text{Ɩ}_2, \dots, \text{Ɩ}_t) = (\text{Ɩ}_1 \otimes \text{Ɩ}_2 \otimes \dots \otimes \text{Ɩ}_t)^{\frac{1}{t}} = (\otimes_{s=1}^{t} \text{Ɩ}_s)^{\frac{1}{t}} \quad (27)$$

*Called CIF frank geometric (CIFFG) operator.*

**Theorem 4:** *Verify that the resultant value of Eq. (26) is again in the shape of CIFNs, such that*

$$CIFFWG(\text{Ɩ}_1, \text{Ɩ}_2, \dots, \text{Ɩ}_t)$$

$$= \left( \begin{array}{c} \log_{\Xi}\left(1 + \frac{\prod_{s=1}^{t}\left(\Xi^{m_{\text{Ɩ}_{R-s}}} - 1\right)^{\mu_{w-s}}}{(\Xi - 1)^{\sum_{s=1}^{t}\mu_{w-s}-1}}\right) e^{i2\pi\left(\log_{\Xi}\left(1+\frac{\prod_{s=1}^{t}\left(\Xi^{m_{\text{Ɩ}_{I-s}}}-1\right)^{\mu_{w-s}}}{(\Xi-1)^{\sum_{s=1}^{t}\mu_{w-s}-1}}\right)\right)}, \\ 1 - \log_{\Xi}\left(1 + \frac{\prod_{s=1}^{t}\left(\Xi^{1-n_{\text{Ɩ}_{R-s}}} - 1\right)^{\mu_{w-s}}}{(\Xi - 1)^{\sum_{s=1}^{t}\mu_{w-s}-1}}\right) e^{i2\pi\left(1-\log_{\Xi}\left(1+\frac{\prod_{s=1}^{t}\left(\Xi^{1-n_{\text{Ɩ}_{I-s}}}-1\right)^{\mu_{w-s}}}{(\Xi-1)^{\sum_{s=1}^{t}\mu_{w-s}-1}}\right)\right)} \end{array} \right) \quad (28)$$

**Proof:** Omitted.

**Definition 11:** *We diagnose the idea of a CIFFOWG operator based on CIFNs* $\text{Ɩ}_s = \left(m_{\text{Ɩ}_{R-s}} e^{i2\pi\left(m_{\text{Ɩ}_{I-s}}\right)}, n_{\text{Ɩ}_{R-s}} e^{i2\pi\left(n_{\text{Ɩ}_{I-s}}\right)}\right), s = 1,2,\dots,t,$ *such that*

$$CIFFOWG(\text{Ɩ}_1, \text{Ɩ}_2, \dots, \text{Ɩ}_t) = \text{Ɩ}_{o(1)}^{\mu_{w-1}} \otimes \text{Ɩ}_{o(2)}^{\mu_{w-2}} \otimes \dots \otimes \text{Ɩ}_{o(t)}^{\mu_{w-t}} = \otimes_{s=1}^{t} \text{Ɩ}_{o(s)}^{\mu_{w-s}} \quad (29)$$

*Where* $\sum_{s=1}^{t}\mu_{w-s} = 1, \mu_{w-s} \in [0,1]$, *representing the weight vector with* $\text{Ɩ}_{o(s)} \leq \text{Ɩ}_{o(s-1)}$. *In particular, if we use the value of* $\mu_{w-s} = \left(\frac{1}{t}, \frac{1}{t}, \dots, \frac{1}{t}\right)$ *in Eq. (29), then we have*

$$CIFFG(\text{Ɩ}_1, \text{Ɩ}_2, \dots, \text{Ɩ}_t) = \left(\text{Ɩ}_{o(1)} \otimes \text{Ɩ}_{o(2)} \otimes \dots \otimes \text{Ɩ}_{o(t)}\right)^{\frac{1}{t}} = \left(\otimes_{s=1}^{t} \text{Ɩ}_{o(s)}\right)^{\frac{1}{t}} \quad (30)$$

*Called CIF frank ordered geometric (CIFFOG) operator.*

**Theorem 5:** *Verify that the resultant value of Eq. (29) is again in the shape of CIFNs, such that*

$$CIFFOWG(\text{I}_1,\text{I}_2,\dots,\text{I}_t) = \left(\begin{array}{c} \log_{\Xi}\left(1+\frac{\prod_{\mathscr{s}=1}^{t}\left(\Xi^{m_{\text{I}_{R-o(\mathscr{s})}}}-1\right)^{\mu_{w-\mathscr{s}}}}{(\Xi-1)^{\sum_{\mathscr{s}=1}^{t}\mu_{w-\mathscr{s}}-1}}\right)e^{i2\pi\left(\log_{\Xi}\left(1+\frac{\prod_{\mathscr{s}=1}^{t}\left(\Xi^{m_{\text{I}_{I-o(\mathscr{s})}}}-1\right)^{\mu_{w-\mathscr{s}}}}{(\Xi-1)^{\sum_{\mathscr{s}=1}^{t}\mu_{w-\mathscr{s}}-1}}\right)\right)}, \\ 1-\log_{\Xi}\left(1+\frac{\prod_{\mathscr{s}=1}^{t}\left(\Xi^{1-n_{\text{I}_{R-o(\mathscr{s})}}}-1\right)^{\mu_{w-\mathscr{s}}}}{(\Xi-1)^{\sum_{\mathscr{s}=1}^{t}\mu_{w-\mathscr{s}}-1}}\right)e^{i2\pi\left(1-\log_{\Xi}\left(1+\frac{\prod_{\mathscr{s}=1}^{t}\left(\Xi^{1-n_{\text{I}_{I-o(\mathscr{s})}}}-1\right)^{\mu_{w-\mathscr{s}}}}{(\Xi-1)^{\sum_{\mathscr{s}=1}^{t}\mu_{w-\mathscr{s}}-1}}\right)\right)} \end{array}\right) \quad (31)$$

**Proof:** Omitted.

**Definition 12:** *We diagnose the idea of a CIFHG operator based on CIFNs* $\text{I}_{\mathscr{s}} = \left(m_{\text{I}_{R-\mathscr{s}}}e^{i2\pi\left(m_{\text{I}_{I-\mathscr{s}}}\right)}, n_{\text{I}_{R-\mathscr{s}}}e^{i2\pi\left(n_{\text{I}_{I-\mathscr{s}}}\right)}\right), \mathscr{s}=1,2,\dots,t$, *such that*

$$CIFFHG(\text{I}_1,\text{I}_2,\dots,\text{I}_t) = \text{I}^{*\mu_{w-1}}_{o(1)} \otimes \text{I}^{*\mu_{w-2}}_{o(2)} \otimes \dots \otimes \text{I}^{*\mu_{w-t}}_{o(t)} = \otimes_{\mathscr{s}=1}^{t} \text{I}^{*\mu_{w-\mathscr{s}}}_{o(\mathscr{s})} \quad (32)$$

*Where* $\text{I}^*_{o(\mathscr{s})} = t\mu^*_{w-\mathscr{s}}\text{I}_{\mathscr{s}}$ *and* $\sum_{\mathscr{s}=1}^{t}\mu^*_{w-\mathscr{s}} = 1, \sum_{\mathscr{s}=1}^{t}\mu_{w-\mathscr{s}} = 1, \mu^*_{w-\mathscr{s}}, \mu_{w-\mathscr{s}} \in [0,1]$, *representing the weight vector with* $\text{I}^*_{o(\mathscr{s})} \leq \text{I}^*_{o(\mathscr{s}-1)}$. *In particular, if we use the value of* $\mu_{w-\mathscr{s}} = \left(\frac{1}{t},\frac{1}{t},\dots,\frac{1}{t}\right)$ *in Eq. (32), then we have*

$$CIFFHG(\text{I}_1,\text{I}_2,\dots,\text{I}_t) = \left(\text{I}^*{}_{o(1)} \otimes \text{I}^*{}_{o(2)} \otimes \dots \otimes \text{I}^*{}_{o(t)}\right)^{\frac{1}{t}} = \left(\otimes_{\mathscr{s}=1}^{t} \text{I}^*{}_{o(\mathscr{s})}\right)^{\frac{1}{t}} \quad (33)$$

*Called CIF frank hybrid geometric (CIFFHG) operator.*

**Theorem 6:** *Verify that the resultant value of Eq. (32) is again in the shape of CIFNs, such that*

$$CIFFHG(\text{I}_1,\text{I}_2,\dots,\text{I}_t) = \left(\begin{array}{c} \log_{\Xi}\left(1+\frac{\prod_{\mathscr{s}=1}^{t}\left(\Xi^{m^*_{\text{I}_{R-o(\mathscr{s})}}}-1\right)^{\mu_{w-\mathscr{s}}}}{(\Xi-1)^{\sum_{\mathscr{s}=1}^{t}\mu_{w-\mathscr{s}}-1}}\right)e^{i2\pi\left(\log_{\Xi}\left(1+\frac{\prod_{\mathscr{s}=1}^{t}\left(\Xi^{m^*_{\text{I}_{I-o(\mathscr{s})}}}-1\right)^{\mu_{w-\mathscr{s}}}}{(\Xi-1)^{\sum_{\mathscr{s}=1}^{t}\mu_{w-\mathscr{s}}-1}}\right)\right)}, \\ 1-\log_{\Xi}\left(1+\frac{\prod_{\mathscr{s}=1}^{t}\left(\Xi^{1-n^*_{\text{I}_{R-o(\mathscr{s})}}}-1\right)^{\mu_{w-\mathscr{s}}}}{(\Xi-1)^{\sum_{\mathscr{s}=1}^{t}\mu_{w-\mathscr{s}}-1}}\right)e^{i2\pi\left(1-\log_{\Xi}\left(1+\frac{\prod_{\mathscr{s}=1}^{t}\left(\Xi^{1-n^*_{\text{I}_{I-o(\mathscr{s})}}}-1\right)^{\mu_{w-\mathscr{s}}}}{(\Xi-1)^{\sum_{\mathscr{s}=1}^{t}\mu_{w-\mathscr{s}}-1}}\right)\right)} \end{array}\right) \quad (34)$$

**Proof:** Omitted.

## 4. Application: Causes and Risk Factors of Osteoporosis based on Diagnosed Operators

Osteoporosis is a constant sickness by which there is a continuous disintegration of the tissue that involves your bones. It is frequently called a "quiet illness" since there are not many to no side effects of osteoporosis. Over the long run, the speed of new bone development can't stay aware of bone misfortune. Thusly, the decrease in bone mass debilitates the skeleton, making bones frail, delicate, more permeable, and more inclined to break. While many variables can add to osteoporosis, bone wellbeing can be upgraded through work out, keeping up with calcium and vitamin D admission, abstaining from smoking, and restricting liquor consumption. Being watching out for signs and side effects of osteoporosis, would it be a good idea for them if they happen, can assist you with seeking a leap in treatment. The word osteoporosis means “porous bone” and the condition is distinguished by gradual bone weakening. Osteoporosis is a result of discrepancies between new bone growth and old bone resorption. More than fifty million people over the age of fifty have osteoporosis issues due to low bone mass, disease contribution, and not exercising continuously. Certain most important risk factors and causes of osteoporosis contain age, gender, hormones, visits to inexperienced doctors, and certain medical conditions. We aim to evaluate the most dangerous type of osteoporosis based on their symptoms, causes, and risk factors using diagnosed approaches.

Finally, to enhance the worth of the evaluated operators, we compare the diagnosed result with certain prevailing results and illustrated their geometrical representation with the help of MATLAB. For this, we use the collection of alternatives $\mathrm{I}_1, \mathrm{I}_2, \dots, \mathrm{I}_m$ and their criteria $\mathrm{I}_{AT-1}, \mathrm{I}_{AT-2}, \dots, \mathrm{I}_{AT-n}$ with weight vector $\sum_{s=1}^{t} \mu_{w-s} = 1, \mu_{w-s} \in [0,1]$. For the information, we need to construct the matrix by putting the value of CIF information: $\mathrm{I}_s = \left(m_{\mathrm{I}_{R-s}} e^{i2\pi\left(m_{\mathrm{I}_{I-s}}\right)}, n_{\mathrm{I}_{R-s}} e^{i2\pi\left(n_{\mathrm{I}_{I-s}}\right)}\right), s = 1,2,\dots,t$, with a well-known condition: $0 \le m_{\mathrm{I}_R}(x) + n_{\mathrm{I}_R}(x) \le 1$ and $0 \le m_{\mathrm{I}_I}(x) + n_{\mathrm{I}_I}(x) \le 1$, represented as CIF information. To formulate the above dilemmas, we organize the procedure of planning, whose stages are available in the shape:

**Stage 1:** Usually, we faced two types of criteria, benefits and cost types. Therefore, the information or computed matrix is normalized by:

$$N = \begin{cases} \left(m_{\mathrm{I}_{R-s}} e^{i2\pi\left(m_{\mathrm{I}_{I-s}}\right)}, n_{\mathrm{I}_{R-s}} e^{i2\pi\left(n_{\mathrm{I}_{I-s}}\right)}\right) & \mathrm{I}_s \in B \\ \left(n_{\mathrm{I}_{R-s}} e^{i2\pi\left(n_{\mathrm{I}_{I-s}}\right)}, m_{\mathrm{I}_{R-s}} e^{i2\pi\left(m_{\mathrm{I}_{I-s}}\right)}\right) & \mathrm{I}_s \in C \end{cases}$$

Where $B$ is used for benefit and $C$ is used for cost types.

**Stage 2:** After normalization, we used the theory of CIFFWA and CIFFWG operators to aggregate the information in the decision matrix.

$$CIFFWA(\mathrm{I}_1, \mathrm{I}_2, \dots, \mathrm{I}_t) = \begin{pmatrix} 1 - \log_{\Xi}\left(1 + \prod_{s=1}^{t}\left(\Xi^{1-m_{\mathrm{I}_{R-s}}} - 1\right)^{\mu_{w-s}}\right) e^{i2\pi\left(1-\log_{\Xi}\left(1+\prod_{s=1}^{t}\left(\Xi^{1-m_{\mathrm{I}_{I-s}}}-1\right)^{\mu_{w-s}}\right)\right)}, \\ \log_{\Xi}\left(1 + \prod_{s=1}^{t}\left(\Xi^{n_{\mathrm{I}_{R-s}}} - 1\right)^{\mu_{w-s}}\right) e^{i2\pi\left(\log_{\Xi}\left(1+\prod_{s=1}^{t}\left(\Xi^{n_{\mathrm{I}_{I-s}}}-1\right)^{\mu_{w-s}}\right)\right)} \end{pmatrix}$$

$$CIFFWG(\mathrm{I}_1, \mathrm{I}_2, \dots, \mathrm{I}_t) = \begin{pmatrix} \log_{\Xi}\left(1 + \prod_{s=1}^{t}\left(\Xi^{m_{\mathrm{I}_{R-s}}} - 1\right)^{\mu_{w-s}}\right) e^{i2\pi\left(\log_{\Xi}\left(1+\prod_{s=1}^{t}\left(\Xi^{m_{\mathrm{I}_{I-s}}}-1\right)^{\mu_{w-s}}\right)\right)}, \\ 1 - \log_{\Xi}\left(1 + \prod_{s=1}^{t}\left(\Xi^{1-n_{\mathrm{I}_{R-s}}} - 1\right)^{\mu_{w-s}}\right) e^{i2\pi\left(1-\log_{\Xi}\left(1+\prod_{s=1}^{t}\left(\Xi^{1-n_{\mathrm{I}_{I-s}}}-1\right)^{\mu_{w-s}}\right)\right)} \end{pmatrix}$$

**Stage 3:** Moreover, we find the score value of the obtained accumulated information.

**Stage 4:** Finally, we find the ordering of the alternatives based on their Score values.

To justify the above problem with the help of an example, we use the issues which are faced by some old and new generations, called osteoporosis disease. We aim to evaluate the most dangerous type of osteoporosis based on their symptoms, causes, and risk factors using diagnosed approaches.

### *4.1. Illustrated Example*

Osteoporosis is a consequence of lopsided characteristics between new bone development and old bone resorption. In bone resorption, osteoclasts separate bone tissues and deliver certain minerals that move calcium from unresolved issues. With osteoporosis, the body might neglect to frame new bone, or a lot of the old bone is consumed. It is likewise workable for the two occasions to happen. Typically, the deficiency of bone requires numerous prior years of osteoporosis creates. More often than not, an individual won't realize they have the condition until they support a crack. By that point, the illness will be progressed and harm from it tends to be very not kidding.

The absolute most normal gamble factors and reasons for osteoporosis incorporate age, orientation, chemicals, utilization of specific drugs, and a few ailments. The main theme of this theory is to use five types of osteoporosis, represented as an alternative, explained in the shape: $\mathrm{I}_1$: Osteopenia; $\mathrm{I}_2$: Primary Osteoporosis; $\mathrm{I}_3$: Secondary Osteoporosis; $\mathrm{I}_4$: Rare type of Osteoporosis; and $\mathrm{I}_5$: Transient migratory Osteoporosis. To find the most affected one from the above ones, we use the value of criteria, represented in the shape of the symptoms of the osteoporosis: $\mathrm{I}_{AT-1}$: Deformity and

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mobility; $\text{I}_{AT-2}$: Pain; $\text{I}_{AT-3}$: Anxiety and sleeping complications, and $\text{I}_{AT-4}$: Chronic illness. To evaluate the problem discussed above, we use weight vectors like 0.4,0.3,0.2, and 0.1. Then, to formulate the above dilemmas, we organize the procedure of planning, whose stages are available in the shape:

**Stage 1:** Usually, we faced two types of criteria, benefits and cost types. Therefore, the information or computed matrix is normalized by:

$$N = \begin{cases} \left(m_{\text{I}_{R-\mathcal{s}}} e^{i2\pi\left(m_{\text{I}_{I-\mathcal{s}}}\right)}, n_{\text{I}_{R-\mathcal{s}}} e^{i2\pi\left(n_{\text{I}_{I-\mathcal{s}}}\right)}\right) & \text{I}_{\mathcal{s}} \in B \\ \left(n_{\text{I}_{R-\mathcal{s}}} e^{i2\pi\left(n_{\text{I}_{I-\mathcal{s}}}\right)}, m_{\text{I}_{R-\mathcal{s}}} e^{i2\pi\left(m_{\text{I}_{I-\mathcal{s}}}\right)}\right) & \text{I}_{\mathcal{s}} \in C \end{cases}$$

Where $B$ is used for benefit and $C$ is used for cost types. But the information given in Table 1 are benefit types, so we cannot normalize it.

**Table 1.** Expressed the CIF matrix.

| | $\text{I}_{AT-1}$ | $\text{I}_{AT-2}$ |
|---|---|---|
| $\text{I}_1$ | $\left(0.7e^{i2\pi(0.6)}, 0.1e^{i2\pi(0.2)}\right)$ | $\left(0.71e^{i2\pi(0.61)}, 0.11e^{i2\pi(0.21)}\right)$ |
| $\text{I}_2$ | $\left(0.3e^{i2\pi(0.4)}, 0.2e^{i2\pi(0.3)}\right)$ | $\left(0.31e^{i2\pi(0.41)}, 0.21e^{i2\pi(0.31)}\right)$ |
| $\text{I}_3$ | $\left(0.4e^{i2\pi(0.5)}, 0.2e^{i2\pi(0.1)}\right)$ | $\left(0.41e^{i2\pi(0.51)}, 0.21e^{i2\pi(0.11)}\right)$ |
| $\text{I}_4$ | $\left(0.6e^{i2\pi(0.5)}, 0.3e^{i2\pi(0.3)}\right)$ | $\left(0.61e^{i2\pi(0.51)}, 0.31e^{i2\pi(0.31)}\right)$ |
| $\text{I}_5$ | $\left(0.8e^{i2\pi(0.7)}, 0.1e^{i2\pi(0.1)}\right)$ | $\left(0.81e^{i2\pi(0.71)}, 0.11e^{i2\pi(0.11)}\right)$ |
| | $\text{I}_{AT-3}$ | $\text{I}_{AT-4}$ |
| $\text{I}_1$ | $\left(0.7e^{i2\pi(0.6)}, 0.1e^{i2\pi(0.2)}\right)$ | $\left(0.7e^{i2\pi(0.6)}, 0.1e^{i2\pi(0.2)}\right)$ |
| $\text{I}_2$ | $\left(0.32e^{i2\pi(0.42)}, 0.22e^{i2\pi(0.32)}\right)$ | $\left(0.33e^{i2\pi(0.43)}, 0.23e^{i2\pi(0.33)}\right)$ |
| $\text{I}_3$ | $\left(0.42e^{i2\pi(0.52)}, 0.22e^{i2\pi(0.12)}\right)$ | $\left(0.43e^{i2\pi(0.53)}, 0.23e^{i2\pi(0.13)}\right)$ |
| $\text{I}_4$ | $\left(0.62e^{i2\pi(0.52)}, 0.32e^{i2\pi(0.32)}\right)$ | $\left(0.63e^{i2\pi(0.53)}, 0.33e^{i2\pi(0.33)}\right)$ |
| $\text{I}_5$ | $\left(0.82e^{i2\pi(0.72)}, 0.12e^{i2\pi(0.12)}\right)$ | $\left(0.83e^{i2\pi(0.73)}, 0.13e^{i2\pi(0.13)}\right)$ |

**Stage 2:** After normalization, we used the theory of CIFFWA and CIFFWG operators to aggregate the information in the decision matrix, see Table 2.

**Table 2.** Expressed the aggregated values.

| | **CIFFWA Operator** | **CIFFWG Operator** |
|---|---|---|
| $\text{I}_1$ | $\left(0.7101e^{i2\pi(0.6101)}, 0.1095e^{i2\pi(0.2097)}\right)$ | $\left(0.7099e^{i2\pi(0.6099)}, 0.11004e^{i2\pi(0.21005)}\right)$ |
| $\text{I}_2$ | $\left(0.31006e^{i2\pi(0.41007)}, 0.2097e^{i2\pi(0.2097)}\right)$ | $\left(0.3098e^{i2\pi(0.4099)}, 0.21005e^{i2\pi(0.21005)}\right)$ |
| $\text{I}_3$ | $\left(0.41007e^{i2\pi(0.51009)}, 0.2097e^{i2\pi(0.1095)}\right)$ | $\left(0.4099e^{i2\pi(0.5099)}, 0.21005e^{i2\pi(0.11004)}\right)$ |
| $\text{I}_4$ | $\left(0.61011e^{i2\pi(0.51009)}, 0.30986e^{i2\pi(0.30986)}\right)$ | $\left(0.60993e^{i2\pi(0.50992)}, 0.31006e^{i2\pi(0.31006)}\right)$ |
| $\text{I}_5$ | $\left(0.81025e^{i2\pi(0.71016)}, 0.10958e^{i2\pi(0.10958)}\right)$ | $\left(0.80995e^{i2\pi(0.70995)}, 0.11004e^{i2\pi(0.11004)}\right)$ |

**Stage 3:** Moreover, we find the score value of the obtained accumulated information, see Table 3. To represent the information in Table 3, we use some MATLAB codes stated in the shape of Figure 2 and their resultant information are given in Figure 3.

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```
close all
clear all
clc

d1=[0.50046 0.15028 0.3004  0.25024 0.65063;
    0.4999  0.14983 0.29986 0.24987 0.64991]


figure
for k=1
subplot(5,2,k);
bar3(eval(strcat('d',num2str(k))));
set(gca,'yTickLabel',{'CIFFWA operator','CIFFWG operator','C','D'})
box on
end
```

**Figure 2.** Running codes in MATLAB for computing the shape of Figure 3.

**Table 3.** Expressed the score values.

| | CIFFWA Operator | CIFFWG Operator |
|---|---|---|
| $I_1$ | 0.50046 | 0.4999 |
| $I_2$ | 0.15028 | 0.14983 |
| $I_3$ | 0.3004 | 0.29986 |
| $I_4$ | 0.25024 | 0.24987 |
| $I_5$ | 0.65063 | 0.64991 |

**Stage 4:** Finally, we find the ordering of the alternatives based on their score values, see Table 4.

**Table 4.** Expressed the ranking values.

| Methods | Ranking values |
|---|---|
| **CIFFWA Operator** | $I_5 \geq I_1 \geq I_3 \geq I_4 \geq I_2$ |
| **CIFFWG Operator** | $I_5 \geq I_1 \geq I_3 \geq I_4 \geq I_2$ |

The considered best optimal is $I_5$, represented the transient migratory Osteoporosis. We noticed the diagnosed operators can easily evaluate the CIF types of data. But instead of CIF information, if we used the theory of intuitionistic fuzzy information, then what happened. For this, we considered the information in Table 1 without imaginary parts. Then using imaginary parts, the evaluated information is discussed. For this, we find the score value of the obtained accumulated information, see Table 5 (using the information in Table 1 without imaginary parts.

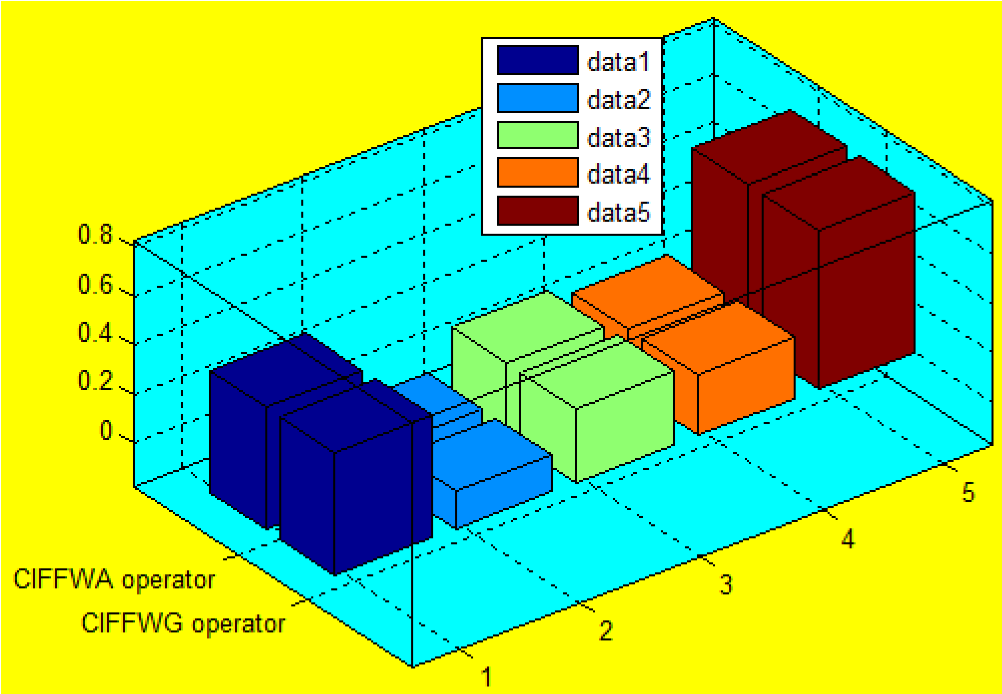


**Figure 3.** Geometrical shape of information in Table 3.

**Table 5.** Expressed the score values (without imaginary parts).

| | CIFFWA Operator | CIFFWG Operator |
|---|---|---|
| $I_1$ | 0.60058 | 0.5999 |
| $I_2$ | 0.1003 | 0.0998 |
| $I_3$ | 0.2003 | 0.1998 |
| $I_4$ | 0.3003 | 0.2999 |
| $I_5$ | 0.7007 | 0.6999 |

Finally, we find the ordering of the alternatives based on their score values, see Table 6.

**Table 6.** Expressed the ranking values.

| Methods | Ranking values |
|---|---|
| **CIFFWA Operator** | $I_5 \geq I_1 \geq I_4 \geq I_3 \geq I_2$ |
| **CIFFWG Operator** | $I_5 \geq I_1 \geq I_4 \geq I_3 \geq I_2$ |

The considered best optimal is $I_5$, represented the transient migratory Osteoporosis. We noticed the diagnosed operators easily evaluated the intuitionistic and CIF types of data very easily. Finally, to enhance the worth of the evaluated operators, we compare the diagnosed result with certain prevailing results and illustrated their geometrical representation with the help of MATLAB.

## 5. Comparative Analysis

Certain individuals have evaluated different types of operators based on FI, IFI, CFI, and CIFI, so motivated by this information, in this case, we compare the diagnosed result with certain prevailing results and illustrated their geometrical representation with the help of MATLAB. To compare the diagnosed operators with some existing operators, we use the following theories, for instance, Aggregation operators (AOs) under the consideration of IFI were pioneered by Xu [32], geometric AOs under the availability of IFI were evaluated by Xu and Yager [33], Novel AOs for CIFI was presented by Garg and Rani [34], generalized geometric AOs for CIFI was explored by Garg and Rani [35], and also considered the evaluated operators in this manuscript, called CIFFWA and CIFFWG operators. Then, we use the information in Table 2, and the comparative analysis of the diagnosed and existing operators is presented in Table 7. To represent the information in Table 5, we use some MATLAB codes stated in the shape of Figure 4 and their resultant information are given in Figure 5. To represent the information in Table 7 and Table 8, we use some MATLAB codes stated in the shape of Figure 6, and their resultant information is given in Figure 7 and Figure 8.

For computing Figure 5, we use information available in Matrix d1.

Where d2 is used for computing Figure 3.

```
close all
clear all
clc

d1=[0.60058 0.1003  0.2003  0.3003  0.7007;
    0.5999  0.0998  0.1998 0.2999   0.6999]

d2=[0.50046 0.15028 0.3004  0.25024 0.65063;
    0.4999  0.14983 0.29986 0.24987 0.64991]


figure
for k=1
subplot(5,2,k);
bar3(eval(strcat('d',num2str(k))));
set(gca,'yTickLabel',{'CIFFWA operator','CIFFWG operator','C','D'})
box on
end
```

**Figure 4.** Running codes in MATLAB for computing the shape of Figure 5.

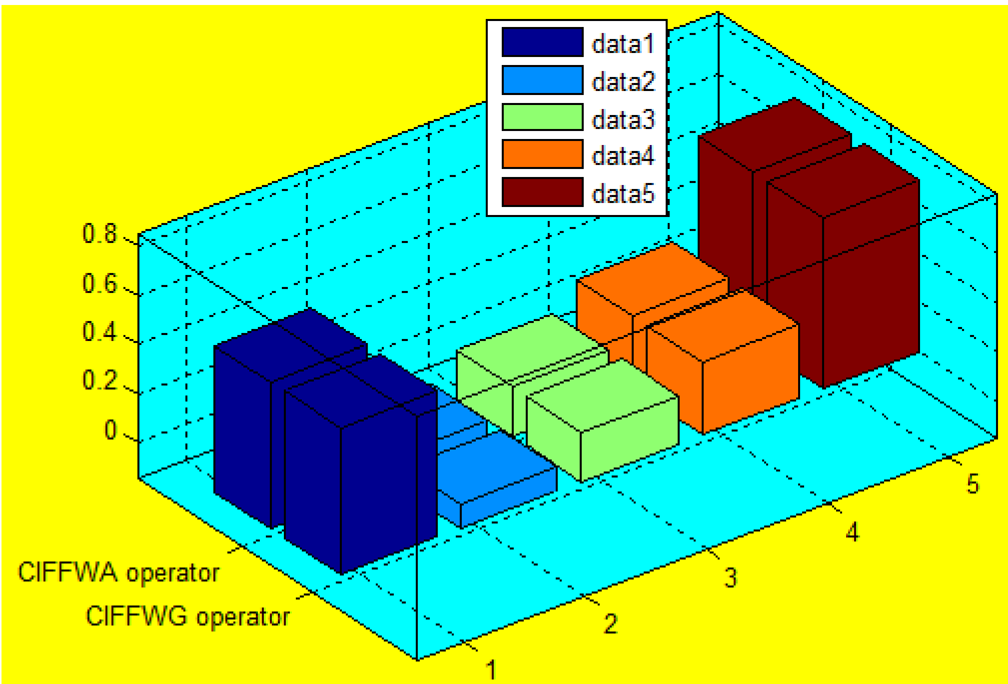


**Figure 5.** Geometrical shape of information in Table 5.

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For computing Figure 8, we use information available in Matrix d1.

For computing Figure 7, we use information available in Matrix d2.

```
close all
clear all
clc

d1=[0.6006  0.1003  0.2003  0.3003  0.7007;
    0.5999  0.0998  0.1998  0.2499  0.6999;
    1.001   0.3006  0.6009  0.5006  1.3013;
    0.9997  0.2996  0.5997  0.4997  1.2998;
    0.60058 0.1003  0.2003  0.3003  0.7007;
    0.5999  0.0998  0.1998  0.2999   0.6999]

d2=[0.0 0.0 0.0 0.0 0.0;
    0.0 0.0 0.0 0.0 0.0;
    1.001   0.3006  0.6009  0.5006  1.3013;
    0.9997  0.2996  0.5997  0.4997  1.2998;
    0.50046 0.15028 0.3004  0.25024 0.65063;
    0.4999  0.14983 0.29986 0.24987 0.64991]

figure
for k=1
subplot(5,2,k);
bar3(eval(strcat('d',num2str(k))));
set(gca,'yTickLabel',{'Xu [32]','Xu and Yager [33]','Garg and Rani [34]
    ','Garg and Rani [35]','CIFFWA operator','CIFFWG operator'})
box on
end
```

**Figure 6.** Running codes in MATLAB for computing the shape of Figure 7 and Figure 8.

**Table 7.** Represented the ranking values, using the information in Table 2.

| Methods | Score values | Ranking values |
|---|---|---|
| **Xu [32]** | *Not evaluated due to some dilemmas* | *Not evaluated due to some dilemmas* |
| **Xu and Yager [33]** | *Not evaluated due to some dilemmas* | *Not evaluated due to some dilemmas* |
| **Garg and Rani [34]** | 1.001,0.3006,0.6009,0.5006,1.3013 | $I_5 \geq I_1 \geq I_3 \geq I_4 \geq I_2$ |
| **Garg and Rani [35]** | 0.9997,0.2996,0.5997,0.4997,1.2998 | $I_5 \geq I_1 \geq I_3 \geq I_4 \geq I_2$ |
| **CIFFWA Operator** | 0.50046,0.15028,0.3004,0.25024,0.65063 | $I_5 \geq I_1 \geq I_3 \geq I_4 \geq I_2$ |
| **CIFFWG Operator** | 0.4999,0.14983,0.29986,0.24987,0.64991 | $I_5 \geq I_1 \geq I_3 \geq I_4 \geq I_2$ |

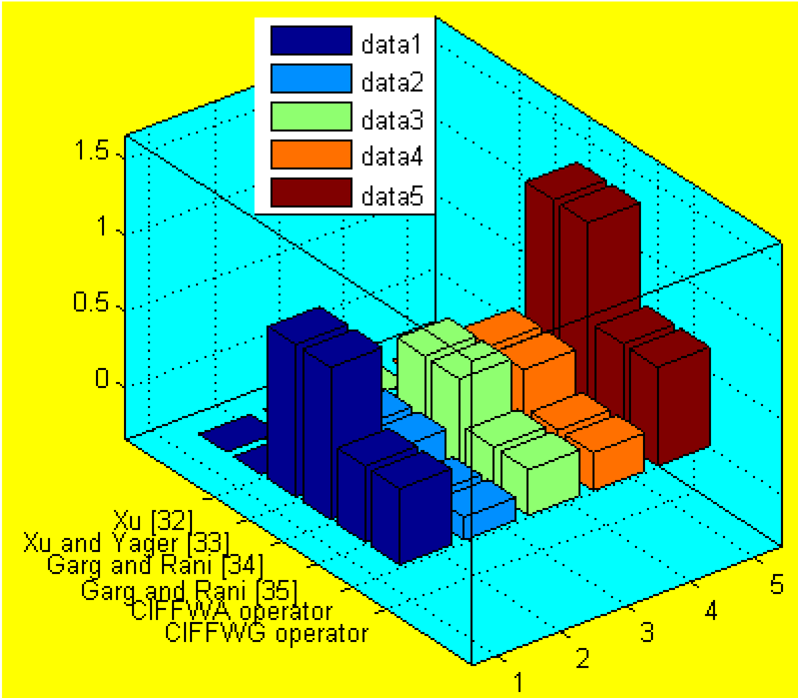


**Figure 7.** Geometrical shape of information in Table 7.

From the information in Table 7, we clear that the operators defined in Ref. [34, 35] are easily evaluated the information given in Table 2, but the operators defined based on IFI in Ref. [32, 33] are failed to evaluate the theory given in Table 2. AOs under the consideration of IFI were pioneered by Xu [32] and geometric AOs under the availability of IFI were evaluated by Xu and Yager [33] have a lot of limitations. But we try to show a little bit of supremacy of the existing information. For this, we use the information in Table 2 (without imaginary parts), and the comparative analysis of the diagnosed and existing operators is presented in Table 8.

**Table 8.** Represented the ranking values, using the information in Table 2 "without imaginary parts".

| Methods | Score values | Ranking values |
|---|---|---|
| **Xu [32]** | 0.6006,0.1003,0.2003,0.3003,0.7007 | $I_5 \geq I_1 \geq I_4 \geq I_3 \geq I_2$ |
| **Xu and Yager [33]** | 0.5999,0.0998,0.1998,0.2499,0.6999 | $I_5 \geq I_1 \geq I_4 \geq I_3 \geq I_2$ |
| **Garg and Rani [34]** | 1.001,0.3006,0.6009,0.5006,1.3013 | $I_5 \geq I_1 \geq I_3 \geq I_4 \geq I_2$ |
| **Garg and Rani [35]** | 0.9997,0.2996,0.5997,0.4997,1.2998 | $I_5 \geq I_1 \geq I_3 \geq I_4 \geq I_2$ |
| **CIFFWA Operator** | 0.60058,0.1003,0.2003,0.3003,0.7007 | $I_5 \geq I_1 \geq I_4 \geq I_3 \geq I_2$ |
| **CIFFWG Operator** | 0.5999,0.0998,0.1998,0.2999,0.6999 | $I_5 \geq I_1 \geq I_4 \geq I_3 \geq I_2$ |

Mean that, the word osteoporosis means "porous bone" and the condition is distinguished by gradual bone weakening. Osteoporosis is a result of discrepancies between new bone growth and old bone resorption. More than fifty million people over the age of fifty have osteoporosis issues due to low bone mass, disease contribution, and not exercising continuously. Certain most important risk factors and causes of osteoporosis contain age, gender, hormones, visits to inexperienced doctors, and certain medical conditions. From the information in Table 8, we clear that the operators defined in Ref. [34, 35] are easily evaluated the information given in Table 2 (without imaginary parts), and the operators defined based on IFI in Ref. [32, 33] are also evaluated the theory given in Table 2 (without imaginary parts). AOs under the consideration of IFI were pioneered by Xu [32] and geometric AOs under the availability of IFI were evaluated by Xu and Yager [33] have resolved the above information under the availability of their limitations. It mean that the diagnosed operators are beneficial and more dominant as compared to existing operators [32-35].

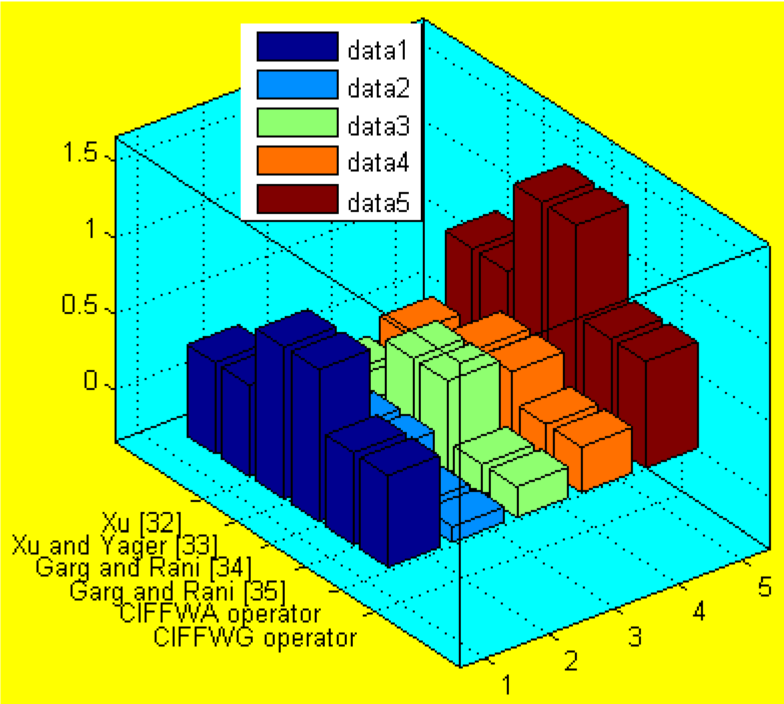


**Figure 8.** Geometrical shape of information in Table 8.

## 6. Conclusion

In this manuscript, we computed the following ideologies:

1. We diagnosed certain frank operational laws under the availability of CIF information.
2. We pioneered the theory of CIFFWA, CIFFOWA, CIFFHA, CIFFWG, CIFFOWG, and CIFFHG operators, and described their beneficial and valuable results with certain useful properties.
3. We evaluated the most dangerous type of osteoporosis based on their symptoms, causes, and risk factors using diagnosed approaches.
4. We enhanced the worth of the evaluated operators, compared the diagnosed result with certain prevailing results, and illustrated their geometrical representation with the help of MATLAB

In the future, we review the information of complex Pythagorean fuzzy N-soft sets [36], complex Fermatean fuzzy N-soft sets [37], complex spherical fuzzy N-soft sets [38], complex spherical fuzzy sets [39], complex T-spherical fuzzy relation [40], linear Diophantine fuzzy sets [41], spherical linear Diophantine fuzzy sets [42], linear Diophantine fuzzy soft rough sets [43] and try to improve the limitations of these theories or we try to utilize different types of operators, measures and methods.

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