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Formal Calculation

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Keywords: Formal Calculation; nested sums; Gaussian coefficient; Stirling number; associated Stirling numbers; Eulerian number and polynomial; Wolstenholme theorem

Article

Formal Calculation

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Abstract: Formal Calculation uses an auxiliary form to calculate various nested sums and provides results in three forms. In addition to computation, it is also a powerful tool for analysis, allowing one to study various numbers in a unified way. This article contains many results of two types of Stirling numbers, associated Stirling numbers, and Eulerian numbers, making a great generalization of Euler polynomials, Wilson's theorem, and Wolstenholme's theorem, showing that they are just special cases. Formal Calculation provides a novel method for obtaining combinatorial identities and analyzing q-binomial.This article has obtained a large number of results in q-analogues, including inversion formulas for q-binomial coefficients. This article also introduces a theorem on symmetry.

Keywords: Formal Calculation; nested sums; Gaussian coefficient; Stirling number; associated Stirling numbers; Eulerian number and polynomial; Wolstenholme theorem

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1. Introduction

Formal Calculation is introduced in [1–3] , this article contains its summary and latest achievements.

Definition 1. *Recursive define* $\nabla^p, p \in \mathbb{Z}, \nabla^0 f(n) = f(n), \sum_{n=0}^{N-1} \nabla^1 f(n+1) = f(N), \sum_{n=0}^{N-1} f(n+1) = \nabla^{-1} f(N), \nabla^1 = \nabla$

Definition 2. *Recursive define SUM(N)=SUM(N,PS,PT).*$K_i, D_i \in$ *Ring with identity elements.*

$SUM(N, [K_1 : D_1], [T_1 = 1]) = \sum_{n=0}^{N-1} (K_1 + nD_1)$

$SUM(N, [K_1 : D_1, K_2 : D_2], [T_1, T_2 = T_1 + 2 - p]) = \sum_{n=0}^{N-1} (K_2 + nD_2) \nabla^p SUM(n+1, [K_1 : D_1], [T_1])$

If $f(N) = \sum A_i \binom{N_i}{M_i}$ *and* M_i *is not changed with* N, *then* $\nabla^p f(N) = \sum A_i \binom{N_i - p}{M_i - p}$
$[K_1 : D, K_2 : D ... K_M : D]$ is abbreviated as $[K_1, K_2 ... K_M] : D$,$[K_1, K_2 ... K_M] : 1$ is abbreviated as $[K_1, K_2 ... K_M]$.
By default,this paper use:
PS=$[K_1 : D_1, K_2 : D_2 ... K_M : D_M]$,PT=$[T_1, T_2 ... T_M]$,PS1=[PS,$K_{M+1} : D_{M+1}$],PT1=[PT,$T_{M+1}$]
This is actually nested summation. For example:

$SUM(N, PS, [1, 2, 3 ... M]) = \sum_{n=0}^{N-1} \prod_{i=1}^{M} (K_i + nD_i)$

$SUM(N, PS, [1, 3, 5 ... 2M - 1]) = \sum_{n_M=0}^{N-1} (K_M + n_M D_M) ... \sum_{n_2=0}^{n_3} (K_2 + n_2 D_2) \sum_{n_1=0}^{n_2} (K_1 + n_1 D_1)$

$SUM(N, PS, [1, 2, 4]) = \sum_{n_3=0}^{N-1} (K_3 + n_3 D_3) \sum_{n=0}^{n_3} (K_1 + nD_1)(K_2 + nD_2)$

$SUM(N, PS, [1, 3, 4]) = \sum_{n_3=0}^{N-1} (K_3 + n_3 D_3)(K_2 + n_3 D_2) \sum_{n=0}^{n_3} (K_1 + nD_1)$

The following use K to represent the set $[K_1, K_2 ... K_M]$, T to represent the set $[T_1, T_2 ... T_M]$.

Use the auxiliary form: $(K_1 + T_1)(K_2 + T_2)...(K_M + T_M) = \sum \prod_{i=1}^{M} X_i, X_i = T_i \ or \ K_i$

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Definition 3. *X(T)=Number of* $\{X_1, X_2...X_M\} \in T$

Definition 4. X_{T-1}*=Number of* $\{X_1, X_2...X_{i-1}\} \in T$,$X_{K-1}$*=Number of* $\{X_1, X_2...X_{i-1}\} \in K$
X_T*=Number of* $\{X_1, X_2...X_i\} \in T$*,and also define* X_K

Obviously:$X_{T-1} + X_{K-1} = i - 1$.

Theorem 1. *[1] SUM(N,PS,PT)=*

$$Form_1 \rightarrow \sum_{g=0}^{M} H_1(g)\binom{N+T_M-M}{N-1-g} = \sum_{g=0}^{M} H_1(g)\binom{N+T_M-M}{T_M-M+1+g}, B_i = \begin{cases}(T_i-X_{K-1})D_i, X_i=T_i\\ K_i+X_{T-1}D_i, X_i=K_i\end{cases}$$

$$Form_2 \rightarrow \sum_{g=0}^{M} H_2(g)\binom{N+T_M-M+g}{N-1} = \sum_{g=0}^{M} H_2(g)\binom{N+T_M-M+g}{T_M-M+1+g}, B_i = \begin{cases}(T_i-X_{K-1})D_i, X_i=T_i\\ K_i+(X_{K-1}-T_i)D_i, X_i=K_i\end{cases}$$

$$Form_3 \rightarrow \sum_{g=0}^{M} H_3(g)\binom{N+T_M-g}{N-1-g} = \sum_{g=0}^{M} H_3(g)\binom{N+T_M-g}{T_M+1}, B_i = \begin{cases}-K_i+(T_i-X_{T-1})D_i, X_i=T_i\\ K_i+X_{T-1}D_i, X_i=K_i\end{cases}$$

The factors of $\prod X_i$ *cannot be exchanged.* $H_i(g)$*,short for* $H_i(g, PS, PT)$*, is also defined above as* $\sum_{X(T)=g} \prod_{i=1}^{M} B_i$

The theorem is proved by induction.There have three forms because: $\sum_{n=0}^{N-1} n\binom{n+K}{M}$
$= (M+1)\binom{N+K}{M+2} + (M-K)\binom{N+K}{M+1} = (M+1)\binom{N+K+1}{M+2} - (1-K)\binom{N+K}{M+1} = (M-K)\binom{N+K+1}{M+2} +$
$(1+K)\binom{N+K}{M+2}$

Definition 5. $F_M^{K=\{K_1,K_2...K_M\}}$=$\sum K_{I_1}K_{I_2}...K_{I_M}, a < b, I_a < I_b$,$F_M^N$ *is short for* $F_M^{\{1,2...N\}}$,$F_0^K = 0$

Definition 6. $E_M^{K=\{K_1,K_2...K_M\}}$=$\sum K_{I_1}K_{I_2}...K_{I_M}, a < b, I_a \leq I_b$,$E_M^N$ *is short for* $E_M^{\{1,2...N\}}$,$E_0^K = 0$

Theorem 2. $\nabla SUM(N, PS, [1, 2...M]) = \prod_{i=1}^{M}(K_i + nD_i)$

Theorem 3. *In SUM(N,[...PS...],[...T+1,T+2...T+M...]),* K_i *can exchange orders.*

Theorem 4. $SUM(N, [L_1, L_2...L_q, PS], [L_1, L_2...L_q, PT]) = \prod_{i=1}^{q} L_i \times SUM(N, PS, PT)$*, so* T_1 *can great than 1,*$T_i \in \mathbb{N}$

Theorem 5. $SUM(N, [1, 1...1], [1, 2...M]) = SUM(N, [1, 1...1], [2, 3...M]) = 1^M + 2^M + ... + N^M$

Theorem 6. $SUM(N, [1, 1...1], [1, 3...2M-1]) = SUM(N, [1, 1...1], [3, 5...2M-1])$
$= \sum_{\lambda_1+...+\lambda_N=M, \lambda_i \geq 0} 1^{\lambda_1}2^{\lambda_2}...N^{\lambda_N} = E_M^N = S_2(N+M, N)$*.* S_2 *is Stirling numbers of the second kind.*

Theorem 7. $SUM(N, [1, 2...M], [1, 3...2M-1]) = SUM(N, [2, 3...M], [3, 5...2M-1])$
$= \sum_{1 \leq i_1 < i_2 < ... < i_M \leq N+M-1} i_1 i_2 ... i_M = F_M^{N+M-1} = S_1(N+M, N)$*.* S_1 *is unsigned Stirling numbers of the first kind.*

Example 1.1:
$Form = (1+T_1)(2+T_2)(3+T_3), \sum_{X(T)=1} \prod X_i = 1 \times 2 \times T_3 + 1 \times T_2 \times 3 + T_1 \times 2 \times 3$
$H_1(1) = 1 \times 2 \times (T_3 - X_{K-1}) + 1 \times (T_2 - X_{K-1}) \times (3 + X_{T-1}) + T_1 \times (2 + X_{T-1}) \times (3 + X_{T-1})$
$= 1 \times 2 \times (5-2) + 1 \times (3-1) \times (3+1) + 1 \times (2+1) \times (3+1) = 26$

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$$SUM(N,[1,2,3],[1,3,5]) = 1\times 3\times 5\binom{N+2}{6} + 35\binom{N+2}{5} + 26\binom{N+2}{4} + 1\times 2\times 3\binom{N+2}{3}$$

$$SUM(N,[2,3],[3,5]) = 3\times 5\binom{N+3}{6} + (2\times 4+3\times 4)\binom{N+3}{5} + 2\times 3\binom{N+3}{4}$$

It also can be calculated in the Ring with identity elements.K_i, D_i can be a matrix.

Theorem 8. *$\prod(K_i+nD_i)$ can be decomposed into three forms by 1 and ∇.*

2. Property

2.1. Relationships between $H(g)$

By definition:

1. $H_1(g,PS1,PT1) = H_1(g-1)(T_{M+1}-[M-(g-1)])D_{M+1} + H_1(g)(K_{M+1}+gD_{M+1})$
2. $H_2(g,PS1,PT1) = H_2(g-1)(T_{M+1}-[M-(g-1)])D_{M+1} + H_2(g)(K_{M+1}+[M-g-T_{M+1}]D_{M+1})$
3. $H_3(g,PS1,PT1) = H_3(g-1)(-K_{M+1}+(T_{M+1}-[g-1])D_{M+1}) + H_3(g)(K_{M+1}+gD_{M+1})$

Using these relationships and induction can prove:

Theorem 9. $H_1(g) = \sum_{k=g}^{M} H_2(k)\binom{k}{g} = \sum_{k=0}^{g} H_3(k)\binom{M-k}{M-g}$ [2]

Inversion→

Theorem 10. $H_2(g) = \sum_{k=g}^{M}(-1)^{k+g}H_1(k)\binom{k}{g}, H_3(g) = \sum_{k=0}^{g}(-1)^{k+g}H_1(k)\binom{M-k}{M-g}$

Calculation with 9 →

Theorem 11. $\sum_{g=0}^{M} H_1(g) = \sum_{g=0}^{M} H_2(g)2^g = \sum_{g=0}^{M} H_3(g)2^{M-g}$

Theorem 12. $\sum_{g=0}^{M} H_1(g)\binom{A}{B-g} = \sum_{g=0}^{M} H_2(g)\binom{A+g}{B} = \sum_{g=0}^{M} H_3(g)\binom{A+M-g}{B-g}, A,B\in\mathbb{N}$

This indicates $Form_1 = Form_2 = Form_3 \rightarrow \sum_{g=0}^{M} H_1(g)\binom{A}{g} = \sum_{g=0}^{M} H_2(g)\binom{A+g}{g} = \sum_{g=0}^{M} H_3(g)\binom{A+M-g}{M}$

Induction [2]→

Theorem 13. $\sum_{g=0}^{M} H_1(g)g\binom{A+1}{B-g} = \sum_{g=0}^{M} H_2(g)g\binom{A+g}{B-1} = \sum_{g=0}^{M}\{H_3(g)g\binom{A+M-g}{B-g} + M\times H_3(g)\binom{A+M-g}{B-1-g}\}$

2.2. Property of H(g)

$\prod X_i = (\prod X_i\in T)(\prod X_i\in K)$.In some cases, H(g) is easy to calculate.

Definition 7. $H(g,T) = H(g,T,PS,PT) = \prod_{X_i\in T} B_i, H(g,\sum T) = \sum\prod_{X_i\in T} B_i$.*Also define* $H(g,K), H(g,\sum K)$

If $D_i = 1$ and $T_i + 1 = T_{i+1}$,$H_1(g,T) = \prod_{i=1}^{g} T_i$. $H_1(g,T,PS,[1,2...M]) = g!.H_1(g,\sum K,[1,1...1],PT) = E_{M-g}^{g+1}$

Theorem 14. *If* $D_i = 1$,$H_1(g,\sum K) = F_{M-g}^K E_0^g + F_{M-g-1}^K E_1^g + ... + F_0^K E_{M-g}^g$

Theorem 15. *If* $D_i = 1$,$H_1(g,\sum T) = F_g^T E_0^{M-g} - F_{g-1}^T E_1^{M-g} + ... + (-1)^{M-g} F_0^T E_g^{M-g}$

Theorem 16. *If* $D_i = 1$ *and* $K_{i+1} - K_i = T_{i+1} - T_i = 1$,$H_1(g)$=$\binom{M}{g} T_1...T_g \times K_{g+1}...K_M$

Theorem 17. *If PS=PT,*$H_1(g) = \prod_{i=1}^{M} T_i \binom{M}{N}$, $H_2(M) = H_3(0) = \prod_{i=1}^{M} T_i$, $H_2(g < M) = H_3(g > 0) = 0$

Theorem 18. $H_1(g,[AD : D, PS],[A,PT]) = AD(H_1(g-1) + H_1(g)) \rightarrow H_1(g,[1,PS],[1,PT]) = H_1(g-1) + H_1(g)$

Definition 8. $E_p^q \odot ([T_1, T_2...T_M], C)$
$$= \sum_{\lambda_1+\lambda_2+...+\lambda_q=p,\lambda_i\geq 0} 1^{\lambda_1} 2^{\lambda_2} ... q^{\lambda_q} (T_1 + \lambda_1 C)(T_2 + \lambda_1 C + \lambda_2 C)...(T_{q-1} + \lambda_1 C + \lambda_2 C + ... + \lambda_{q-1} C)$$

[4] has proved:$\left\langle {M \atop g} \right\rangle = \left\langle {M \atop M-g-1} \right\rangle = E_{M-g-1}^{g+1} \odot ([1,1...1],1])$
$\left\langle {M \atop g} \right\rangle$ is Eulerian numbers.It is known that there exists Worpitzky identity:$N^M = \sum_{g=0}^{M-1} \left\langle {M \atop g} \right\rangle \binom{N+g}{M}$
$eg : \left\langle {5 \atop 2} \right\rangle = E_2^3 \odot ([1,1...],1) = \sum_{\lambda_1+\lambda_2+\lambda_3=2} 1^{\lambda_1} 2^{\lambda_2} 3^{\lambda_3} (1+\lambda_1)(1+\lambda_1+\lambda_2) = 66$

By simple calculation:
$H_1(g,[1,1...1],PT = [T_i = T_1 + (C+1)(i-1)]) = E_{M-g}^{g+1} \odot (PT, C)$
$H_3(g,[1,1...1],PT = [T_i = T_1 + (C+1)(i-1)]) = E_{M-g}^{g+1} \odot ([T_i = T_1 - 1 + C(i-1)], C+1)$

2.3. Shape of numbers

In this section,if not specifically mentioned,$T_1 = 1$, $T_{i+1} - T_i = 1$ *or* 2.
To calculate $\sum_{1\leq K_1<K_2<...<K_M\leq N} K_1 K_2 ... K_M$ (*),products needs to be divided into 2^{M-1} categories.
There are **M-1** intervals between factors.If the interval=1,define it as **Continuity**.If the interval>1,define it as **Discontinuity**. Continuities,Discontinuities and their Positions are defined as **Shape**.So there have 2^{M-1} Shapes.
From the definition of nested sum:

$$\sum_{1\leq K_1<K_2<K_3\leq N} K_1 K_2 K_3$$
$= SUM(N,[1,2,3],[1,2,3]) + SUM(N-1,[1,2,4],[1,2,4]) + SUM(N-1,[1,3,4],[1,3,4]) + SUM(N-2,[1,3,5],[1,3,5])$

Definition 9. *PB(PT)=Number of* $T_i + 1 < T_{i+1}$*=Number of discontinuities*

$(*) = \sum_{All\ of\ the\ Shapes\ with\ factors=M} SUM(N - PB(PT), PT, PT)$.From 17 we can obtain a simple formula:

Theorem 19. $SUM(N,PT,PT) = \prod_{i=1}^{M} T_i \binom{N+T_M-M}{T_M-M+1}$*,PT has no restrictions.*

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This generalizes the famous formula $\sum_{n=0}^{N-1}\binom{n}{M} = \binom{N}{M+1}$. It was discovered during the calculation of (*) which led to the birth of Formal Calculation.

Theorem 20. *Number of Products in SUM(N,PT,PT)* = $\binom{N+PB(PT)}{PB(PT)+1}$

Definition 10. $MIN_g(M) = \sum_{PB(PT)=g}\prod_{i=1}^{M} T_i = 1 \times \sum_{PB(PT)=g}\prod_{i=2}^{M} T_i$

This is the sum of the products of PTs with the same number of discontinuities.By definition:

Theorem 21. $MIN_g(M) = \sum \frac{(M+g)!}{i_1 i_2 ... i_g}, 2 \le i_1 < i_2 < ... < i_g \le M+g-1, i_{j+1} - i_j \ge 2$

Based on the concept of Shape rather than 21, it is easier to understand.
$eg : MIN_2(4) = (12357) + (12457) + (13457) + (12467) + (13467) + (13567)$.Here (...) is products.

From the definition of nested sum,there exists general classification principles:

Theorem 22. $SUM(N, [K_1 : D_1, K_2 : D_2 ... K_M : D_M], [T_1, T_2 ... T_M]) = SUM(N, PS, [T_1 ... T_i, T_{i+1} - 1 ... T_M - 1]) + SUM(N-1, [K_1 : D_1 ... K_i : D_i, K_{i+1} + D_{i+1} : D_{i+1} ... K_M + D_M : D_M], PT)$

$eg : SUM(N, [1,2,3], [1,3,5]) = SUM(N, [1,2,3], [1,2,4]) + SUM(N-1, [1,3,4], [1,3,5])$
$= SUM(N, [1,2,3], [1,2,3]) + SUM(N-1, [1,2,4], [1,2,4]) + SUM(N-1, [1,3,4], [1,3,4]) + SUM(N-2, [1,3,5], [1,3,5])$
$(*) = SUM(N, [1,2...M], [1,3...2M-1]) = \sum_{g=0}^{M-1} MIN_g(M)\binom{N}{g+1}$ *.It's exactly* 7.

2.4. H(g) and Associated Stirling Numbers

Associated Stirling Numbers of the first kind $S_{1,r}(n,k)$ is defined as the number of permutations of a set of n elements having exactly k cycles, all length >= r.

1. $S_{1,r}(n,k) = \frac{n!}{k!}\sum_{i_1+i_2+...+i_k=n, i_j \ge r}\frac{1}{i_1 i_2 ... i_k}$
2. $S_{1,r}(n+1,k) = nS_{1,r}(n,k) + (n)_{r-1}S_{1,r}(n-r+1,k-1), n \ge kr$
3. $\sum \frac{(n-1)!}{i_1 i_2 ... i_{k-1}}, r \le i_1 < i_2 < ... < i_{k-1} \le n-r, i_{j+1} - i_j \ge r$ [5]

Derived from 2 and definition of H(g) or 3 and 21:

Theorem 23. $MIN_g(M) = S_{1,2}(M+g+1, g+1)$

Table 1. Table of $MIN_g(M) = S_{1,2}(M+g+1, g+1)$.

	g=0	g=1	g=2	g=3	g=4	g=5	g=6
M=1	1						
M=2	2	3					
M=3	6	20	15				
M=4	24	130	210	105			
M=5	120	924	2380	2520	945		
M=6	720	7308	26432	44100	34650	10395	
M=7	5040	64224	303660	705320	866250	540540	135135

Associated Stirling Numbers of the second kind $S_{2,r}(n,k)$ is defined as the number of permutations of a set of n elements having exactly k blocks, all length >= r.

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1. $S_{2,r}(n,k) = \frac{n!}{k!} \sum_{i_1+i_2+...+i_k=n, i_j \geq r} \frac{1}{i_1!i_2!...i_k!}$
2. $S_{2,r}(n+1,k) = kS_{2,r}(n,k) + \binom{n}{r-1} S_{2,r}(n-r+1,k-1), n \geq kr$

Derived from 2:

Theorem 24. $H_2(g,[1,1...1],[3,5...2M-1]) = S_{2,2}(M+g+1,g+1)$

Table 2. Table of $H_2(g,[1,1...1],[3,5...2M-1]) = S_{2,2}(M+g+1,g+1)$.

	g=0	g=1	g=2	g=3	g=4	g=5	g=6
M=1	1						
M=2	1	3					
M=3	1	10	15				
M=4	1	25	105	105			
M=5	1	56	490	1260	945		
M=6	1	119	1918	9450	17325	10395	
M=7	1	246	6825	56980	190575	270270	135135

$$SUM(N,[2,3...M],[3,5...2M-1]) = S_{1,1}(N+M,N) = \frac{(N+M)!}{N!} \sum_{i_1+i_2+...+i_N=N+M, i_j \geq 1} \frac{1}{i_1 i_2 ... i_N}$$
$$= \sum_{g=0}^{M-1} S_{1,2}(M+g+1,g+1)\binom{N+M}{M+1+g} = \sum_{g=1}^{M} S_{1,2}(M+g,g)\binom{N+M}{M+g}$$
$$= \frac{(N+M)!}{N!} \sum_{g=1}^{M} [\sum_{i_1+...+i_N=N+M, i_j \geq 1, Number\ of\ i_j>1=g} \frac{1}{i_1 i_2 ... i_N}] = \sum_{g=1}^{M} f(*)$$
$$f(*) = \frac{(N+M)!}{N!}\binom{N}{N-g} \sum_{i_1+...+i_g=g+M, i_j \geq 1} \frac{1}{i_1 i_2 ... i_g} = \binom{N+M}{M+g} S_{1,2}(M+g,g)$$

Use the same way:

Theorem 25. $S_{1,r}(rN+M,N) = \sum_{g=1}^{M} \frac{1}{r^{N-g}} \frac{(rN-rg)!}{(N-g)!} \binom{rN+M}{rg+M} S_{1,r+1}(rg+M,g)$

Theorem 26. $S_{2,r}(rN+M,N) = \sum_{g=1}^{M} \frac{1}{(r!)^{N-g}} \frac{(rN-rg)!}{(N-g)!} \binom{rN+M}{rg+M} S_{2,r+1}(rg+M,g)$

2.5. Table of H(g)

Table 3. Table of H(g).

PS	PT	$H_1(g)$	$H_2(g)$	$H_3(g)$
[1,1...1]	[1,2...M]	$g!E^{g+1}_{M-g} = g!S_2(M+1,g+1)$	$(-1)^{M-g}g!S_2(M,g)$	$\left\langle {M \atop g} \right\rangle$
[1,1...1]	[2,3...M]	$(g+1)!S_2(M,g+1)$	$(-1)^{M-1-g}(g+1)!S_2(M,g+1)$	$\left\langle {M \atop g} \right\rangle$
[1,1...1]	[1,3...2M−1]	$E^{g+1}_{M-g} \odot (PT,1)$	$(-1)^{M-g}MIN_{g-1}(M)$	$E^{g+1}_{M-g} \odot ([0,1...],2)$
[1,1...1]	[3,5...2M−1]	$S_{2,2}(M+1+g,g+1)$	$(-1)^{M-1-g}MIN_g(M)$	$E^{g+1}_{M-1-g} \odot ([2,3...],2)$
[1,2...M]	[1,3...2M−1]	$MIN_{g-1}(M) + MIN_g(M)$	$1 \times (-1)^{M-g}E^g_{M-g} \odot ([3,5...],1)$	$1 \times E^{M-g}_g \odot ([2,3...],2)$
[2,3...M]	[3,5...2M−1]	$MIN_g(M)$	$(-1)^{M-1-g}S_{2,2}(M+1+g,g+1)$	$E^{M-g}_g \odot ([2,3...],2)$

3. Application

3.1. Number analysis

Theorem 27. 9, 11, 13 →

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1. $\sum_{g=0}^{M}\left\langle {M \atop g} \right\rangle = M!$, $\sum_{g=1}^{M}(-1)^{M-g}g!S_2(M,g) = 1$, $\sum_{g=1}^{M}(-1)^{M-g}g \times g!S_2(M,g) = 2^M - 1$
2. $g!S_2(M,g) = \sum_{k=g}^{M}(-1)^{M-k}k!S_2(M,k)\binom{K-1}{g-1} = \sum_{k=0}^{g-1}\left\langle {M \atop k} \right\rangle\binom{M-1-k}{M-g}, 1 \le g \le M$
3. $S_{1,2}(M+g,g) = \sum_{k=g}^{M}(-1)^{M-k}S_{2,2}(M+k,k)\binom{K-1}{g-1}, S_{2,2}(M+g,g) = \sum_{k=g}^{M}(-1)^{M-k}S_{1,2}(M+k,k)\binom{K-1}{g-1}$
4. $\sum_{g=1}^{M}g!S_2(M,g) = \sum_{g=1}^{M}(-1)^{M-g}g!S_2(M,g)2^{g-1} = \sum_{g=1}^{M}\left\langle {M \atop M-g} \right\rangle 2^{M-g}$
5. $\sum_{g=1}^{M}S_{2,2}(M+g,g) = \sum_{g=1}^{M}(-1)^{M-g}S_{1,2}(M+g,g)2^{g-1}$, $\sum_{g=1}^{M}S_{1,2}(M+g,g) = \sum_{g=1}^{M}(-1)^{M-g}S_{2,2}(M+g,g)2^{g-1}$
6. $\sum_{g=1}^{M}g!S_2(M,g)(g-1)\binom{A+1}{g} = \sum_{g=1}^{M}(-1)^{M-g}g!S_2(M,g)(g-1)\binom{A+g-1}{g}$
7. $\sum_{g=0}^{M-1}(-1)^{M-1-g}MIN_g(M) = \sum_{g=1}^{M}(-1)^{M-g}S_{1,2}(M+g,g) = 1$, $\sum_{g=1}^{M}(-1)^{M-g}S_{2,2}(M+g,g) = M!$

PS=PT=[1,2...M],$H_1(g) = H_1(M-g) = M!\binom{M}{g}$,14 $\rightarrow$

Theorem 28. $M!\binom{M}{g} = g!\sum_{i=0}^{M-g}S_1(M+1,g+1+i)S_2(g+i,g) = (M-g)!\sum_{i=0}^{g}S_1(M+1,M+1-i)S_2(M-i,M-g)$

$H_1(g,[1,1...1],[1,2...M]) = g!S_2(M+1,g+1), F_i^{\{1,1...1\}} = \binom{M}{i}$, 14 $\rightarrow$

Theorem 29. $S_2(M+1,g+1) = \sum_{i=0}^{M-g}S_2(M-i,g)\binom{M}{i}$

$H_1(g,[1,2...M],[1,2...M]) = H_1(g,[M,M-1...1],[1,2...M]) = M!\binom{M}{g}$ 15 $\rightarrow$

Theorem 30. $g!\binom{M}{g} = \sum_{i=0}^{g}S_1(M+1,M+1-g+i)S_2(M-g+i,M-g)(-1)^i$

$H_1(g,[K+i],[T+i])$,16 $\rightarrow \binom{M}{g}T_1...T_g \times K_{g+1}...K_M$,14$\rightarrow T_1...T_g(...)$,15$\rightarrow K_{g+1}...K_M(...)$

Theorem 31. $\prod_{i=g+1}^{M}(K+i)\binom{M}{g} = F_{M-g}^{\{K+i\}}E_0^g + ... + F_0^{\{K+i\}}E_{M-g}^g, \prod_{i=1}^{g}(T+i)\binom{M}{g} = F_g^{\{T+i\}}E_0^{M-g} + ... + (-1)^g F_0^{\{T+i\}}E_g^{M-g}$

3.2. Merge and Expand

Theorem 32. *SUM(N,PS,PT)=SUM(N,[1,1...1,PS],[1,1...1,PT]) expand* $\sum_{g=0}^{M}(...) = \sum_{g=0}^{M+(number\ of\ 1\ added)}(...)$

Any $\sum_{g=0}^{M}a_g\binom{X}{Y+g}$ can be converted to $\frac{a_M}{M!}\nabla^q SUM(N+P,[K_1,K_2...K_M],[1,2...M])$

10 provides the necessary and sufficient condition for $\sum_{g=0}^{M}H(g)\binom{X}{Y+g}$ to be merged into

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$\sum_{g=0}^{M-K}(...)\binom{X+K}{Y+K+g}$:

$$H_2(g)=\sum_{x=g}^{M}(-1)^x H(x)\binom{x}{g}=0, g<K \; or \; H_3(g)=\sum_{x=0}^{g}(-1)^x H(x)\binom{M-x}{M-g}=0, M-g<K$$

For example:

$SUM(N,[1,2...M],[1,2...M]), 19 \rightarrow \sum_{x=g}^{M}(-1)^x\binom{M}{x}\binom{x}{g}=0, 0\le g<M$

$\sum_{n=0}^{N-1}\binom{M+dn}{M}=\frac{1}{M!}SUM(N,[1,2...M]:d,[1,2...M])=\frac{1}{M!}\sum_{g=0}^{M}H_1(g)\binom{N}{1+g}$

$If\ M\ge kd, B_i(X_i=K_i)=i+(X_{K-1}-i)d \rightarrow H_2(g<k)=0\rightarrow \sum_{n=0}^{N-1}\binom{M+dn}{M}=\sum_{g=0}^{M-k}(...)\binom{N+k}{1+g+k}$

After a simple calculation,it can be written as:

Theorem 33. *Necessary and sufficient conditions for merging,* $0\le g<K\le M$:

1. $\sum_{n=0}^{M}H(n)\binom{X}{Y+n}\rightarrow\sum_{n=0}^{M-K}(...)\binom{X+K}{Y+K+n}:\sum_{x=0}^{M}(-1)^x H(x)\binom{P\pm x}{g}=0$
2. $\sum_{n=0}^{M}H(n)\binom{X+n}{Y+n}\rightarrow\sum_{n=0}^{M-K}(...)\binom{X+K+n}{Y+K+n}:\sum_{x=0}^{M}H(x)\binom{P\pm x}{g}=0$
3. $\sum_{n=0}^{M}H(n)\binom{X-n}{Y}\rightarrow\sum_{n=0}^{M-K}(...)\binom{X-K-n}{Y-K}:\sum_{x=0}^{M}H(x)\binom{P\pm x}{g}=0$

Theorem 34. $\sum_{g=0}^{M}\binom{M}{g}\binom{A+T+g}{A}\binom{X}{Y+g}=\sum_{g=0}^{A}\binom{A+T}{g+T}\binom{M+T+g}{M+T}\binom{X+M-A}{Y+M-A+g}, A\ge 0$

Proof. It can be proved by induction, but it is cumbersome.

SUM(N,[T+1,T+2...T+M],[T+A+1,T+A+2...T+A+M])

$=\sum_{g=0}^{M}\binom{M}{g}[T+A+g]_g[T+M]_{M-g}\binom{N+T+A}{T+A+1+g}$

$=\sum_{g=0}^{M}\binom{M}{g}\frac{(T+A+g)!}{(T+A)!}\frac{(T+M)!}{(T+g)!}\binom{N+T+A}{T+A+1+g}=\frac{A!(T+M)!}{(T+A)!}\sum_{g=0}^{M}\binom{M}{g}\binom{A+T+g}{A}\binom{N+T+A}{T+A+1+g}$

$=\frac{(T+M)!}{(T+A)!}SUM(N,[T+1,T+2...T+A],[T+M+1,T+M+2...T+M+A])$

$=\frac{(T+M)!}{(T+A)!}\sum_{g=0}^{A}\binom{A}{g}\frac{(T+M+g)!}{(T+M)!}\frac{(T+A)!}{(T+g)!}\binom{N+T+M}{T+M+1+g}$

$\sum_{g=0}^{M}\binom{M}{g}\binom{A+T+g}{A}\binom{N+T+A}{T+A+1+g}=\sum_{g=0}^{A}\frac{(T+M+g)!}{(T+M)!}\frac{(T+A)!}{(T+g)!}\frac{A!}{(A-g)!g!}\frac{1}{A!}\binom{N+T+M}{T+M+1+g}$ □

$$g:=M-g\rightarrow\sum_{g=0}^{M}\binom{M}{g}\binom{A+T+M-g}{A}\binom{X}{Y+g}=\sum_{g=0}^{A}\binom{A+T}{g}\binom{A+M+T-g}{M+T}\binom{X+M-A}{Y+M-A+g}, A\ge 0$$

When A>M, it is an expansion; When A<M, it is a merge.Combining with 33:

Theorem 35. $\sum_{g=0}^{M}(-1)^g\binom{M}{g}\binom{X_1\pm g}{A}\binom{X_2\pm g}{B}=0, M>A+B, A, B\ge 0$

$If\ f(n)=\sum_{i=0}^{B}a_i n^i, B<M\rightarrow\sum_{g=0}^{M}(-1)^g\binom{M}{g}f(g)=0\rightarrow\sum_{g=0}^{M}(-1)^g\binom{M}{g}\binom{X_1\pm g}{Y_1}\binom{X_2\pm g}{Y_2}...=0, \sum Y_i<M$

It can be proved by induction:

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Theorem 36. $\sum_{g=0}^{M}(-1)^g\binom{M}{g}\binom{T+g}{M+K}=\begin{cases}0, K<0\\(-1)^M\binom{T}{K}, K\geq 0\end{cases};\sum_{g=0}^{M}(-1)^g\binom{M}{g}\binom{T-g}{M+K}=\begin{cases}0, K<0\\\binom{T}{K}, K\geq 0\end{cases}\ T\pm g\in\mathbb{Z}$

This helps to understand the differential sequence.

3.3. Congruences

P is prime.K_i, D is any integer,D $\neq$0.

Theorem 37. $(P,D)=1, SUM(P,[K_1,K_2...K_M]:D,[1,2...M])\equiv\begin{cases}0 \pmod P, M<P-1\\-1 \pmod P, M=P-1\end{cases}$

Proof. If M=P-1,$SUM(P)\equiv H_1(P-1)\binom{P}{P}\equiv H_1(P-1)\equiv(P-1)!D^{P-1}\equiv -1 \pmod P$ □

If a product has a factor that is divisible by P then ignore it and change the factor to its minimum positive residue, then we can obtain many congruences. Wilson's Theorem is a special case.eg: $A,B,C\in\mathbb{N}$
$1^A2^B+2^A3^B+...+(P-2)^A(P-1)^B\equiv 1^A3^B+...+(P-3)^A(P-1)^B+(P-1)^A1^B\equiv\begin{cases}0 \pmod P, A+B<P-1\\-1 \pmod P, A+B=P-1\end{cases}$
$1^A2^B3^C+2^A3^B4^C+...+(P-3)^A(P-2)^B(P-1)^C\equiv\begin{cases}0 \pmod P, A+B+C<P-1\\-1 \pmod P, A+B+C=P-1\end{cases}$

Theorem 38. $\sum_{0<K_i,K_j<P,K_i\neq K_j}{K_1}^{\lambda_1}{K_2}^{\lambda_2}...{K_q}^{\lambda_q}\equiv\begin{cases}-1 \pmod P, \lambda_1+\lambda_2+...+\lambda_q=P-1\\0 \pmod P, \lambda_1+\lambda_2+...+\lambda_q<P-1\end{cases},\lambda_i\in\mathbb{N}$

Wolstenholme's Theorem is also a special case.P>3.

1. Wolstenholme's Theorem:$(P-1)!\sum_{n=1}^{P-1}\frac{1}{n}=\sum_{0<K_i,K_j<P,K_i\neq K_j}K_1K_2...K_{P-2}\equiv 0 \pmod{P^2}$
2. $\sum_{n=1}^{P-1}n^{P-2}\equiv 0 \pmod{P^2}$

They are two extremes.In fact,there have:

Theorem 39. $\sum_{0<K_i,K_j<P,K_i\neq K_j}{K_1}^{C_1}{K_2}^{C_2}...{K_q}^{C_q}\equiv 0 \pmod{P^2}, C_1+C_2+...+C_q=P-2, C_i>0$

Proof.
If $X \pmod{P^2}$ and $X+Y \pmod{P^2}$ then $Y \pmod{P^2}$.The Sum has symmetry.
For $\sum AB^{P-3}, A\neq B$,if $P-A\neq B$ then add AB^{P-3} with $(P-A)B^{P-3}$ to PB^{P-3},if $P-A=B$ then add AB^{P-3} with BB^{P-3} to PB^{P-3}.so $\sum AB^{P-3}+\sum B^{P-2}=xP\sum B^{P-3}, 0<x<P\rightarrow\sum AB^{P-3}\equiv 0 \pmod{P^2}$.

Similarly:
For $\sum ABC^{P-4}, A\neq B\neq C$,treat P-A=B,P-A=C,$P-A\neq B,C$ separately.
$\sum ABC^{P-4}+x\sum B^2C^{P-4}+y\sum BC^{P-3}\equiv 0 \bmod P^2, 0<x,y<P$
$\rightarrow\sum ABC^{P-4}+x\sum B^2C^{P-4}\equiv 0 \pmod{P^2}, 0<x<P$
$\sum ABC^{P-4}+\sum B^2C^{P-4}=\sum(\binom{P}{2}-B)BC^{P-4}+\sum B^2C^{P-4}=\binom{P}{2}\sum BC^{P-4}\equiv 0 \pmod{P^2}$
$\rightarrow\sum B^2C^{P-4}\equiv 0 \bmod P^2\rightarrow\sum ABC^{P-4}\equiv 0 \bmod P^2$
Prove the conclusion in a similar way... □

Theorem 40. $E_{P-2}^{P-1}=S_2(2P-3,P-1)\equiv 0 \pmod{P^2}; E_{P-2}^{P}=S_2(2P-2,P)\equiv 0 \pmod{P^2}$

$eg: S_2(7,4)=350, S_2(8,5)=1050\equiv 0 \pmod{25}, S_2(11,6)=179487, S_2(12,7)=627396\equiv 0 \pmod{49}$

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4. Combinatorial Identities

Definition 11. *R-FOLD SUM:*$\sum_{(r)}^{N} f(k) = \sum_{k_r=1}^{N} ... \sum_{k_2=1}^{k_3} \sum_{k_1=1}^{k_2} f(k_1) = \sum_{k_r=0}^{N-1} ... \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} f(k_1+1)$

By nested sum:

Theorem 41. $\sum_{(r)}^{N} \nabla^p SUM(k, PS, PT) = \nabla^{p-r} SUM(N, PS, PT)$

Theorem 42. $\sum_{k_r=1}^{N} ... \sum_{k_2=1}^{k_3} \sum_{k_1=1}^{k_2} \nabla SUM(k_r, PS, [1, 2...M]) = SUM(N, PS, PT = [T_i = i + r - 1])$

Proof. $PS1 = [1:0, 1:0...1:0, PS], PT1 = [1, 3...2(r-1)-1, 1+2(r-1), 2+2(r-1)...M+2(r-1)]$
$B_i = \begin{cases} (T_i - X_{T-1})D_i = 0, X_i \in T \\ K_i + X_{K-1}D_i = 1, X_i \in K \end{cases} \rightarrow H_1(g > M, PS1, PT1) = 0, H_1(g \leq M, PS1, PT1) = H_1(g, PS, PT)$
$SUM(N, PS1, PT1) = \sum_{g=0}^{M} H_1(g, PS, PT) \binom{N+r-1}{r+g} = \sum_{k_r=1}^{N} \nabla SUM(k_r, PS, [1, 2...M])... \sum_{k_2=1}^{k_3} \sum_{k_1=1}^{k_2} 1$ □

Theorem 43. $\sum_{k_x=1}^{N} ... \sum_{k_2=1}^{k_3} \sum_{k_1=1}^{k_2} \nabla SUM(k_r, PS, [1, 2...M]) = \nabla^{r-x} SUM(N, PS, [T_i = i + r - 1])$

$eg: (*) \sum_{k_r=1}^{N} ... \sum_{k_2=1}^{k_3} \sum_{k_1=1}^{k_2} \binom{k_i}{j} = \frac{1}{j!} \nabla^{i-r} SUM(N+1, [0, -1, -2... - (j-1)], [T_x = x + (i-1)])$
$H_1(g < j) = 0, H_1(j) = \frac{(j+i-1)!}{(i-1)!}, T_j - j = i - 1 \rightarrow$
$(*) = \frac{1}{j!} \nabla^{i-r} \frac{(j+i-1)!}{(i-1)!} \binom{(N+1)+(i-1)}{j+1+(i-1)} = \binom{j+i-1}{i-1} \binom{N+i-(i-r)}{j+i-(i-r)} = \binom{j+i-1}{i-1} \binom{N+r}{j+r}$ [6]

Using induction to prove:

Theorem 44. $\sum_{0 \leq n_1 \leq ... \leq n_M \leq N-1} (K + n_1 D_1 + ... + n_M D_M) = (D_1 + 2D_2 + ... + MD_M) \binom{N+M-1}{M+1} + K \binom{N+M-1}{M}$

$eg: \sum_{0 \leq n_1 \leq ... \leq n_p \leq n} (n_1 + ... + n_p) = \sum_{n_p=0}^{n} \sum_{n_{p-1}=0}^{n_p} ... \sum_{n_1=0}^{n_2} (n_1 + ... + n_p) = \binom{p+1}{2} \binom{n+p}{p+1} = \frac{pn}{2} \binom{n+p}{p}.$
[6]

2 , 3 , 4 can be used to derive combinatorial identities.

Theorem 45. $\binom{n+A}{A} \binom{n+M+B}{M} =$

1. $\sum_{g=0}^{M} \binom{A+g}{g} \binom{M+B}{M-g} \binom{n+A}{A+g}$
2. $\sum_{g=0}^{M} (-1)^{M-g} \binom{A+g}{g} \binom{A-B}{M-g} \binom{n+A+g}{A+g}$
3. $\sum_{g=0}^{M} \binom{A-B}{g} \binom{M+B}{M-g} \binom{n+A+M-g}{A+M}$

Proof.
$= \frac{1}{A!M!} \nabla SUM(N, [1, 2...A, B+1...B+M], [1, 2...A+M]) = \frac{1}{M!} \nabla SUM(N, [B+1...B+M], [A+1...A+M])$
$H_1(g) = \binom{M}{g} (A+1)...(A+g)(B+g+1)...(B+M) = \binom{M}{g} g! \binom{A+g}{g} (M-g)! \binom{B+M}{M-g}$
Using a similar method to obtain $H_2(g), H_3(g)$ □

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Theorem 46. $\binom{n+X}{A}\binom{n+Y}{M} = \sum_{x=0}^{A}\binom{M+x}{x}\binom{M+X-Y}{A-x}\binom{n+Y}{M+x}, 0 \le Y \le M$

Proof.
$= \frac{1}{A!M!}\nabla SUM(N,[X,X-1...X-A+1,Y,Y-1...Y-M+1],[1,2...A+M])$
$= \frac{1}{A!M!}\nabla SUM(N,[1,2...Y,0,-1,-2...-(M-Y)+1,X,X-1...X-A+1],[1,2...A+M])$
$= \frac{Y!}{A!M!}\nabla SUM(N,[0,-1,-2...-(M-Y)+1,X,X-1...X-A+1],[Y+1,Y+2...A+M])$
$If\ H_1(g) \ne 0\ then\ X_1,X_2...X_{M-Y} \in T \to H_1(g<M-Y)=0, Let\ C=A+M-Y$
$If\ H_1(g \ge M-Y) \ne 0, Number\ of\ X \in K = \binom{C-M+Y}{C-g} \to H_1(g) = \binom{C-M+Y}{C-g}[X+M-Y]_{C-g}[Y+1]^g$
x:=-(M-Y-g)→ $H_1(M-Y+x) = \frac{A!}{(A-x)!x!}\frac{(M+X-Y)!}{(M+X-Y-A+x)!}\frac{(M+x)!}{Y!}$ □

Theorem 47. $\prod_{i=1}^{M}(A+2i+n) = \binom{n+A}{A}^{-1}\sum_{g=0}^{M}(2(M-g)-1)!!\binom{2M-g}{g}[A+1]^g\binom{n+A+g}{A+g}, A \ge 0$

Proof. $PS=[A+2,A+4...A+2M], PT=[A+1,A+2...A+M], PT1=[1,3...2(M-g)-1]$
$H_2(g,\sum K) = SUM(g+1,PT1,PT1)) = (2(M-g)-1)!!\binom{2M-g}{g}, H_2(g,T) = [A+1]^g$
$\binom{n+A}{A}\prod_{i=1}^{M}(A+2i+n) = \frac{1}{A!}\nabla SUM(N,[1,2...A,PS],[1,2...A,PT]) = \nabla SUM(N,PS,PT)$ □

Theorem 48. *SUM(N,[A+1,A+3...A+2M-1],[1,3...2M-1])=* $\sum_{g=0}^{M}[A]^{M-g}(2g-1)!!\binom{M+g}{2g}\binom{N+M-1+g}{M+g}$

Proof. $H_2(g,\sum T) = SUM(M-g+1,[1,3...2g-1],[1,3...2g-1])) = (2g-1)!!\binom{M+g}{2g}, H_2(g,K) = [A]^{M-g}$ □

Theorem 49. *SUM(N,[A,A+1...A+M-1]:2,[1,3...2M-1])=*$\binom{M+N-1}{M}[A+M+N-2]_M$

Proof. $SUM(g+1,[A,A+1...A+M-1-g]:2,[1,3...2(M-g)-1])$
$= H_1(g,\sum K,[A,A+1...A+M-1],[1,2...M]) = \binom{M}{g}[A+M-1]_{M-g}$ □

$$SUM(N,[1,2...M]:2,[1,3...2M-1]) = M!\binom{N+M-1}{M}^2, 1+3+...+(2N-1) = N^2$$

49 and 45 → $\frac{1}{M!}H(g,[A+1,A+2...A+M]:2,[1,3...2M-1]) = H(g,[A+1,A+2...A+M],[M+1,M+2...2M])$

Theorem 50. $SUM(N,[A+2,A+4...A+2M]:3,[1,3...2M-1]) = \sum_{g=0}^{M}\binom{A+N-1+g}{g}\binom{N+M-1-g}{M-g}\frac{[M+N-g]^M}{2^{M-g}}$

Proof. $PS1=[A+2,A+4...A+2M], PT1=[A+1,A+2...A+M]$
$H_1(g,PS1,PT1) = [A+1]^g SUM(g+1,[A+2,A+4...A+2(M-g)]:3,[1,3...2(M-g)-1]))$
$= \sum_{k=g}^{M}H_2(g,PS1,PT1)\binom{k}{g}$, 47 →$= \sum_{k=g}^{M}(2(M-k)-1)!!\binom{2M-k}{k}[A+1]^k\binom{k}{g}$ □

5. Matrix of SUM(N)

Consider H(g) as variables,list SUM(N),SUM(N+1)...SUM(N+M),we can obtain a (M+1) ×(M+1) matrix.
Let $P = N + T_M - M, Q = N - 1$, corresponding to the three forms.define $A_{1,2,3}(P, Q, M) =$

$$\begin{pmatrix} \binom{P}{Q} & \cdots & \binom{P}{Q-M} \\ \vdots & \ddots & \vdots \\ \binom{P+M}{Q+M} & \cdots & \binom{P+M}{Q} \end{pmatrix}, \begin{pmatrix} \binom{P}{Q} & \cdots & \binom{P+M}{Q} \\ \vdots & \ddots & \vdots \\ \binom{P+M}{Q+M} & \cdots & \binom{P+2M}{Q+M} \end{pmatrix}, \begin{pmatrix} \binom{P+M}{Q} & \cdots & \binom{P}{Q-M} \\ \vdots & \ddots & \vdots \\ \binom{P+2M}{Q+M} & \cdots & \binom{P+M}{Q} \end{pmatrix}$$

Theorem 51. $\parallel A_1(P,Q,M) \parallel = \parallel A_2(P,Q,M) \parallel = \parallel A_3(P,Q,M) \parallel, \parallel A(P,Q,M) \parallel = \parallel A(P,P-Q,M) \parallel$ [2]

Theorem 52. $\parallel A(P,0,M) \parallel = 1, \parallel A(P,1,M) \parallel = \binom{P+M}{1+M}, \parallel A(P,Q>1,M) \parallel = \prod_{g=0}^{Q-1} \frac{\binom{P+M-g}{1+M}}{\binom{1+M-g}{1+M}}$ [2]

If SUM(N) or ∇ SUM(N) is easy to obtained,then H(g) can be calculated with the Cramer's law.Below,$T_M \geq M$

Theorem 53. $H_1(g) = \sum_{k=1}^{g+1} (-1)^{g+1+k} \binom{T_M-M+1+g}{T_M-M+k} SUM(k) = \sum_{k=1}^{g+1} (-1)^{g+1+k} \binom{T_M-M+g}{T_M-M+k-1} \nabla SUM(k)$

(1) $\nabla SUM(N,[1,1...1],[2,3...M]) = N^M \rightarrow S_2(M,g) = \frac{1}{g!} \sum_{k=0}^{g} (-1)^{g+k} \binom{g}{k} k^M = \frac{1}{g!} \sum_{k=0}^{g} (-1)^k \binom{g}{k} (g-k)^M$

Theorem 54. $z(k) = \sum_{i=1}^{k} (-1)^{i+k} \binom{k}{i} \nabla SUM(k), H_2(g) = \sum_{k=g+1}^{M+1} (-1)^{g+k-1} \binom{k-1}{g} z(k)$

$\nabla SUM(N,[1,1...1],[2,3...M]) = N^M \rightarrow z(k) = \sum_{i=1}^{k} (-1)^{i+k} \binom{k}{i} i^M = k! S_2(M,k) \rightarrow$ 27.2

Theorem 55. $H_3(g) = \sum_{k=1}^{g+1} (-1)^{g+1+k} \binom{2+T_M}{g+1-k} SUM(k) = \sum_{k=1}^{g+1} (-1)^{g+1+k} \binom{1+T_M}{g+1-k} \nabla SUM(k)$

(2) $\nabla SUM(N,[1,1...1],[2,3...M]) = N^M \rightarrow \left\langle {M \atop g} \right\rangle = \sum_{k=1}^{g+1} (-1)^{1+g+k} \binom{M+1}{g+1-k} k^M = \sum_{k=0}^{g} (-1)^k \binom{M+1}{k} (g+1-k)^M$
(1)(2) are already known formula.

6. Eulerian polynomials and Beyond

In this section,$q \neq 0, q \neq 1$.Inductive proof:

Theorem 56. $\sum_{n=0}^{N-1} q^n \binom{n+K}{M} = q^N \sum_{g=0}^{M} (-1)^g \frac{\binom{N+K-1-g}{M-g}}{(q-1)^{g+1}} + \frac{q^{M-K}}{(1-q)^{M+1}}$

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Definition 12. $A_q^M = \sum_{k=0}^{M} (1-q)^{M-k} q^k S_2(M,k)k!, A_q^0 = 1, A_q^1 = q$

Table 4. Table of A_q^M.

	M=0	M=1	M=2	M=3	M=4	M=5	M=6	OEIS
A_2^M	1	2	6	26	150	1082	9366	A000629
A_3^M	1	3	12	66	480	4368	47712	A123227
A_4^M	1	4	20	132	1140	12324	160020	A201355

$n^M = \nabla SUM(N,[1,1...1],[2,3...M]) = \sum_{g=0}^{M} S_2(M,g)g! \binom{n}{g} = \sum_{g=0}^{M} (-1)^{M-g} S_2(M,g)g! \binom{n+g}{g} = \sum_{g=0}^{M} \left\langle {M \atop g} \right\rangle \binom{n+g}{M}$

$\sum_{n=0}^{N-1} q^n n^M = \sum_{g=0}^{M} S_2(M,g)g! \sum_{n=0}^{N-1} q^n \binom{n}{g} = \sum_{g=0}^{M} S_2(M,g)g! \{q^N \sum_{k=0}^{g} (-1)^k \frac{\binom{N-1-k}{g-k}}{(q-1)^{k+1}} + \frac{q^g}{(1-q)^{g+1}}\}$

$= \frac{q^N}{(q-1)^{M+1}} \sum_{g=0}^{M} S_2(M,g)g! \sum_{k=0}^{M} (-1)^k \binom{N-1-k}{g-k} (q-1)^{M-k} + \frac{\sum_{g=0}^{M} S_2(M,g)g!(1-q)^{M-g}q^g}{(1-q)^{M+1}}$

$= \frac{q^N}{(q-1)^{M+1}} \sum_{k=0}^{M} (-1)^k (q-1)^{M-k} \sum_{g=0}^{M} S_2(M,g)g! \binom{N-1-k}{g-k} + \frac{A_q^M}{(1-q)^{M+1}}$

$= \frac{q^N}{(q-1)^{M+1}} \sum_{k=0}^{M} (-1)^k (q-1)^{M-k} \nabla^k (N-1)^M + \frac{A_q^M}{(1-q)^{M+1}} (*)$

Use the $From_2$ and $From_3$ of n^M,the first part of (*) keep same, we can obtain:

Theorem 57. $A_q^M = q \sum_{k=0}^{M} (q-1)^{M-k} S_2(M,k)k! = \sum_{g=0}^{M} \left\langle {M \atop g} \right\rangle q^{M-g} = \sum_{g=0}^{M} \left\langle {M \atop g} \right\rangle q^{1+g}$

$n^M = \nabla SUM(N,[1,1...1],[1,2...M]) = \sum_{g=0}^{M} S_2(M+1,g+1)g! \binom{n-1}{g} \rightarrow$

Theorem 58. $A_q^M = \sum_{k=0}^{M} (1-q)^{M-k} q^{k+1} S_2(M+1,k+1)k!, M > 0$

$S_2(M+1,k+1) = \frac{1}{(k+1)!} \sum_{j=0}^{k+1} (-1)^{j+k+1} \binom{k+1}{j} j^{M+1} = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{j+k} \binom{k}{j} (j+1)^M$

By definition of difference:

$\nabla^k (N-1)^M = \sum_{j=0}^{k} (-1)^j \binom{k}{j} (N-1-j)^M = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \sum_{g=0}^{M} \binom{M}{g} N^g (j+1)^{M-g} (-1)^{M-g}$

$= \sum_{g=0}^{M} (-1)^{M-g-k} \binom{M}{g} N^g \sum_{j=0}^{k} (-1)^{j+k} \binom{k}{j} (j+1)^{M-g} = \sum_{g=0}^{M} (-1)^{M-g-k} \binom{M}{g} N^g S_2(M-g+1,k+1)k! \rightarrow$

$(*) = \frac{q^N}{(q-1)^{M+1}} \sum_{k=0}^{M} (-1)^k (q-1)^{M-k} \{ \sum_{g=0}^{M} (-1)^{M-g-k} \binom{M}{g} N^g S_2(M-g+1,k+1)k! \} + \frac{A_q^M}{(1-q)^{M+1}}$

$= \frac{q^N}{(q-1)^{M+1}} \sum_{g=0}^{M} (-1)^{M-g} (q-1)^g \binom{M}{g} N^g \sum_{k=0}^{M} (q-1)^{M-k-g} S_2(M-g+1,k+1)k! + \frac{A_q^M}{(1-q)^{M+1}} (**)$

$N = 0, (*) = 0 \rightarrow \frac{q^N}{(q-1)^{M+1}} (...) + \frac{A_q^M}{(1-q)^{M+1}} = 0 \rightarrow$

Theorem 59. $A_q^M = \sum_{k=0}^{M} (q-1)^{M-k} S_2(M+1,k+1)k!$

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$$A_q^{M-g} = \sum_{k=0}^{M} (q-1)^{M-k-g} S_2(M-g+1,k+1)k!, (**) \rightarrow$$

Theorem 60. $\sum_{n=0}^{N-1} q^n n^M = \frac{q^N}{(q-1)^{M+1}} \sum_{g=0}^{M} (-1)^{M-g}(q-1)^g \binom{M}{g} A_q^{M-g} N^g + \frac{A_q^M}{(1-q)^{M+1}}$

The Eulerian polynomials: $A_M(t): \sum_{i=0}^{\infty} t^i i^M = \frac{tA_M(t)}{(1-t)^{M+1}}, A_M(t) = \sum_{g=0}^{M-1} \left\langle {M \atop g} \right\rangle t^g$

$|q| < 1, \lim_{N\to\infty} \sum_{n=0}^{N-1} q^n n^M = \frac{A_q^M}{(1-t)^{M+1}} \rightarrow A_t^M = tA_M(t)$. There have 5 expressions for $A_M(t)$.
Eulerian Numbers and Polynomials is just a special case,we can handler:

$$\sum_{n=0}^{N-1} q^n \nabla^p SUM(n+Y, PS, PT), X = T_M - M - p$$

$$= \begin{cases} \sum_{g=0}^{M} H_1(g) \sum_{n=0}^{N-1} q^n \binom{n+Y+X}{X+1+g} = \sum_{g=0}^{M} H_1(g)\{q^N \sum_{k=0}^{X+1+g} (-1)^k \frac{\binom{N+Y+X-1-k}{X+1+g-k}}{(q-1)^{k+1}} + \frac{q^{1+g-Y}}{(1-q)^{X+2+g}}\} \\ \sum_{g=0}^{M} H_2(g) \sum_{n=0}^{N-1} q^n \binom{n+Y+X+g}{X+1+g} = \sum_{g=0}^{M} H_2(g)\{q^N \sum_{k=0}^{X+1+g} (-1)^k \frac{\binom{N+Y+X+g-1-k}{X+1+g-k}}{(q-1)^{k+1}} + \frac{q^{1-Y}}{(1-q)^{X+2+g}}\} \\ \sum_{g=0}^{M} H_3(g) \sum_{n=0}^{N-1} q^n \binom{n+Y+X+M-g}{X+1+M} = \sum_{g=0}^{M} H_3(g)\{q^N \sum_{k=0}^{X+1+M} (-1)^k \frac{\binom{N+Y+X+M-g-1-k}{X+1+M-k}}{(q-1)^{k+1}} + \frac{q^{1+g-Y}}{(1-q)^{X+2+M}}\} \end{cases}$$

Theorem 61. $\sum_{n=0}^{M} H_1(g)(1-q)^{M-g} q^{g+1} = q \sum_{n=0}^{M} H_2(g)(1-q)^{M-g} = \sum_{n=0}^{M} H_3(g) q^{g+1}$

Here q can take any value,which is magical.$q = 0.5 \rightarrow$ 11

Definition 13. $A_q(PS, PT) =$ 61

Theorem 62. $X = T_M - p, \sum_{n=0}^{N-1} q^n \nabla^p SUM(n+Y) = \frac{q^N}{(q-1)^{X+2}} \sum_{k=0}^{M} (q-1)^{X-k}(-1)^k \nabla^{p+K-1} SUM(n+Y-2) + \frac{A_q(PS,PT)q^{-Y}}{(1-q)^{X+2}}$

Theorem 63. $|q| < 1, \sum_{n=0}^{\infty} q^n \nabla^p SUM(n+Y, PS, PT) = \frac{A_q(PS,PT)q^{-Y}}{(1-q)^{T_M+2-p}}$

We can handler 63 of SUM(N,[a,a...a]:d,[1,2...M]),SUM(N,[1,1...1,2,2...2...k,k...k],[1,2...kM]). Many results of [7,8] can be obtained by this.

7. Formal Calculation of q-Binomial

7.1. Concept

q-Binomial: $\left[{N \atop M}\right]_q = \frac{(q^N-1)(q^{N-1}-1)...(q^{N-M+1}-1)}{(q^M-1)(q^{M-1}-1)...(q^1-1)}, q \neq 0, 1$,abbreviated as G_M^N

1. $G_0^N = 1, G_{M<0,M>N}^N = 0, G_M^N = G_{N-M}^N$
2. $G_M^N = q^M G_M^{N-1} + G_{M-1}^{N-1} = G_M^{N-1} + q^{N-M} G_{M-1}^{N-1}$
3. $\sum_{n=0}^{N-1} q^n G_M^{n+K} = q^{M-K} G_{M+1}^{N+K}$
4. $G_K^M = \sum_{w \in \Omega(0^{M-K},1^K)} q^{inv(w)}$ [9].$w_1 \ldots w_M$ with M-K(zeros) and K(ones),inv(·) denotes the inversion statistic.

The Formal Calculation use $q^n(K_i + G_1^n D_i)$ instead of $K_i + q^n D_i$.

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Definition 14. *Recursive define* $\nabla_q^p, p \in \mathbb{N}, \nabla_q^0 f(n) = f(n), \sum_{n=0}^{N-1} q^n \nabla_q^1 f(n+1) = f(N), \sum_{n=0}^{N-1} q^n f(n+1)$ $= \nabla_q^{-1} f(N)$

Definition 15. *Recursive define* $SUM_q(N) = SUM_q(N, PS, PT)$.

$SUM_q(N, [K_1 : D_1], [T_1 = 1]) = \sum_{n=0}^{N-1} q^n (K_1 + G_1^n D_1)$

$SUM_q(N, [K_1 : D_1, K_2 : D_2], [T_1, T_2 = T_1 + 2 - p]) = \sum_{n=0}^{N-1} q^n (K_2 + G_1^n D_2) \nabla^p SUM_q(n+1, [K_1 : D_1], [T_1])$

Theorem 64. $\sum_{n=0}^{N-1} q^n G_1^n G_M^{n+K}, M > 0, M \geq K$

$$= q^{2(M-K)+1} G_1^{M+1} G_{M+2}^{N+K} + q^{M-K} G_1^{M-K} G_{M+1}^{N+K}$$
$$= q^{M-2K-1} G_1^{M+1} G_{M+2}^{N+K+1} + q^{M-K} (G_1^{M-K} - q^{-K-1} G_1^{M+1}) G_{M+1}^{N+K}$$
$$= (q^{2(M-K)+1} G_1^{M+1} - q^{2M-K+2} G_1^{M-K}) G_{M+2}^{N+K} + q^{M-K} G_1^{M-K} G_{M+2}^{N+K+1}$$

Use this to prove:

Theorem 65. *[2]* $SUM_q(N, PS, PT)$=

$$Form_1 \rightarrow \sum_{g=0}^{M} H_1^q(g) G_{N-1-g}^{N+T_M-M} = \sum_{g=0}^{M} H_1^q(g) G_{T_M-M+1+g}^{N+T_M-M}, B_i = \begin{cases} q^{1+(T_i-T_{i-1})X_{T-1}} G_1^{T_i-X_{K-1}} D_i, X_i = T_i \\ q^{(T_i-T_{i-1}-1)X_{T-1}} (K_i + G_1^{X_{T-1}} D_i), X_i = K_i \end{cases}$$

$$Form_2 \rightarrow \sum_{g=0}^{M} H_2^q(g) G_{N-1}^{N+T_M-M+g} = \sum_{g=0}^{M} H_2^q(g) G_{T_M-M+1+g}^{N+T_M-M+g}, B_i = \begin{cases} q^{-(T_i-X_{K-1})} G_1^{T_i-X_{K-1}} D_i, X_i = T_i \\ K_i - q^{-(T_i-X_{K-1})} G_1^{T_i-X_{K-1}} D_i, X_i = K_i \end{cases}$$

$$Form_3 \rightarrow \sum_{g=0}^{M} H_3^q(g) G_{N-1-g}^{N+T_M-g} = \sum_{g=0}^{M} H_3^q(g) G_{T_M+1}^{N+T_M-g}, B_i = \begin{cases} q^{1+(T_i-T_{i-1}-1)X_{T-1}} \{(q^{X_{T-1}} G_1^{T_i} - q^{T_i} G_1^{X_{T-1}}) D_i - K_i q^{T_i}\}, X_i = T_i \\ q^{(T_i-T_{i-1}-1)X_{T-1}} (K_i + G_1^{X_{T-1}} D_i), X_i = K_i \end{cases}$$

$$H^q(g) = \sum_{X(T)=g} \prod_{i=1}^{M} B_i, \lim_{q \to 1} SUM_q(N) = SUM(q), \lim_{q \to 1} H^q(g) = H(g)$$

7.2. Property

Theorem 66. $\nabla_q^1 SUM_q(N, PS, [1, 2...M]) = \prod_{i=1}^{M} (K_i + D_i G_1^n)$

Theorem 67. *In* $SUM_q(N, [...PS...], [...T+1, T+2...T+M...]), K_i$ *can exchange orders.*

$Form_2$ is simplest,$X = T_M - M - p, 3 \rightarrow$

Theorem 68. $\nabla_q^p SUM_q(N) = \sum_{g=0}^{M} H_1^q(g) G_{X+1+g}^{N+X} q^{-gp} = \sum_{g=0}^{M} H_2^q(g) G_{X+1+g}^{N+X+g} = \sum_{g=0}^{M} H_3^q(g) G_{X+M+1}^{N+X+M-g} q^{-gp}$

Definition 16. $n_{q-} = q^{-n} G_1^n, n_{q+} = q^n G_1^n, n_{q-}! = n_{q-}...2_{q-}1_{q-}, 0_{q-}! = 0, n_{q+}!$ *is similar.*

$Form_2 \rightarrow$

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Theorem 69. *$SUM_q(N,[L_{1_{q-}},L_{2_{q-}}...L_{Q_{q-}},PS],[L_1,L_2...L_Q,PT]) = \prod_{i=1}^{Q} L_{i_{q-}} SUM_q(N,PS,PT)$,$T_1$ can great than 1,$T_i \in \mathbb{N}$*

Theorem 70. $SUM_q(N,[T_{1_{q-}},T_{2_{q-}}...T_{M_{q-}}],[T_1,T_2...T_M]) = \prod_{i=1}^{M} T_{i_{q-}} G_{T_M+1}^{N+T_M}$

$$SUM_q(N,[1_{q-},2_{q-}...M_{q-}],[1,2...M]) \to \sum_{n=0}^{N-1} q^n G_1^{n+1} G_1^{n+2}...G_1^{n+M} = G_1^1 G_1^2...G_1^M G_{M+1}^{N+M}$$

By definition and 4

Theorem 71. *In $H^q(g)$, $\sum_{X_i \in K} \prod q^{X_T} = G_{M-g}^M = G_g^M$*

7.3. Application

Theorem 72. $\sum_{0 \le \lambda_1 \le \lambda_2 \le ... \le \lambda_M \le N} q^{\lambda_1+\lambda_2+...+\lambda_M} = G_M^{N+M} = G_N^{N+M}$ *[6]*

Proof. $SUM_q(N+1,[1,1...1] : 0,[1,3...2M-1]) \to H_2^q(g > 0) = 0, H_2^q(0) = 1 \to \sum_{\lambda_M=0}^{N} q^{\lambda_M} ... \sum_{\lambda_1=0}^{\lambda_2} q^{\lambda_1} = G_M^{N+M}$ □

Theorem 73. $\sum_{1 \le \lambda_1 < \lambda_2 < ... < \lambda_M \le N+M} q^{\lambda_1+\lambda_2+...+\lambda_M} = q^{\binom{M+1}{2}} G_M^{N+M}$

Proof. $\sum_{\lambda_M=0}^{N} q^{M+\lambda_M} ... \sum_{\lambda_2=0}^{\lambda_3} q^{2+\lambda_2} \sum_{\lambda_1=0}^{\lambda_2} q^{1+\lambda_1} = q^{1+2+...+M} \sum_{\lambda_M=0}^{N} q^{\lambda_M} ... \sum_{\lambda_2=0}^{\lambda_3} q^{\lambda_2} \sum_{\lambda_1=0}^{\lambda_2} q^{\lambda_1}$ □

Theorem 74. $\sum_{A \le \lambda_1 < \lambda_2 < ... < \lambda_M \le B} q^{\lambda_1+\lambda_2+...+\lambda_M} = q^{\binom{M+1}{2}+(A-1)M} G_M^{B-A+1}, A, B \in \mathbb{Z}$

By simply following the definition of product, we can obtain:

Theorem 75. $\prod_{i=1}^{M} (1+q^{A+i}z) = \sum_{g=0}^{M} q^{\binom{g+1}{2}+Ag} G_g^M z^g$

A=-1 or 1,it's **q-Binomial Theorem**: $\prod_{i=1}^{M} (1+q^{i-1}z) = \sum_{g=0}^{M} q^{\binom{g}{2}} G_g^M z^g$, $\prod_{i=1}^{M} (1+q^i z) = \sum_{g=0}^{M} q^{\binom{g+1}{2}} G_g^M z^g$

If $T_i + 1 = T_{i+1}, D_i = 1, K_i = K_{i_{q-}}$:

B_i *of* $H_{1,2,3}^q(g) = \begin{cases} X_i = T \\ X_i = K_{i_{q-}} \end{cases} = \begin{cases} q^{X_T} G^{T_i - X_{K-1}} \\ q^{-K_i} G^{K_i + X_{T-1}} \end{cases}, = \begin{cases} q^{-(T_i - X_{K-1})} G^{T_i - X_{K-1}} \\ q^{-K_i} G^{K_i - T_i + X_{K-1}} \end{cases}, = \begin{cases} q^{1+X_T} G^{T_i - K_i - X_{T-1}} \\ q^{-K_i} G^{K_i + X_{T-1}} \end{cases}$

It's similar to 1.Replace each B_i to $G_1^{B_i}$ in $H(g,[K_1,K_2...K_M],PT)$ and multiply by $q^?$ to obtain $H^q(g)$. If there is another $K_i + 1 = K_{i+1}$,74 can be used to obtain general formulas.

Theorem 76. *Expansion of 2:*$G_{M+1}^{N+M} = \sum_{g=0}^{M} q^{(g+1)g} G_g^M G_{1+g}^N, G_{M+1+P}^{N+M+P} = \sum_{g=0}^{M} q^{(g+1+P)g} G_g^M G_{P+1+g}^{N+P}$

Proof.

$SUM_q(N,[1_{q-},2_{q-}...M_{q-}],[1,2...M]) = M_{q-}!G_{M+1}^{N+M} = \sum_{g=0}^{M} H_1^q(g) G_{1+g}^N =$

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$SUM_q(N, [M_{q-}, ..., 1_{q-}], PT)$

$$H_1^q(g,T) = g_{q+}!, H_1^q(g,\sum K) = G_1^M G_1^{M-1}...G_1^{g+1} q^{-(M+1)(M-g)} \sum_{1\le\lambda_1<...<\lambda_{M-g}\le M} q^{\lambda_1+...+\lambda_{M-g}}$$

$$\frac{H_1^q(g)}{M_{q-}!} = \frac{q^{\binom{g+1}{2}} q^{-(M+1)(M-g)} \sum_{1\le\lambda_1<...<\lambda_{M-g}\le M} q^{\lambda_1+...+\lambda_{M-g}}}{q^{-(1+2+...+M)}} = \frac{q^{\binom{g+1}{2}} q^{-(M+1)(M-g)} q^{\binom{M-g+1}{2}} G_{M-g}^M}{q^{-\binom{M+1}{2}}} = q^{(g+1)g} G_g^M \quad \square$$

Theorem 77. $G_M^N = \sum_{g=0}^{M} (-1)^g q^{\frac{g^2}{2}} G_g^M \begin{bmatrix} N+M-g \\ 2M \end{bmatrix}_{q^{\frac{1}{2}}}$

Proof.
$SUM_q(N+1, [q^1 : (q-1)q^1, q^2 : (q-1)q^2...q^M : (q-1)q^M], [1,3...2M-1])$
$= \sum_{\lambda_M=0}^{N} q^{2\lambda_M+M} ... \sum_{\lambda_2=0}^{\lambda_3} q^{2\lambda_2+2} \sum_{\lambda_1=0}^{\lambda_2} q^{2\lambda_1+1} = q^{\binom{M+1}{2}} \sum_{0\le\lambda_1\le...\le\lambda_M\le N} q^{2(\lambda_1+...+\lambda_M)} = q^{\binom{M+1}{2}} \begin{bmatrix} N+M \\ M \end{bmatrix}_{q^2}$

$$B_i \; of \; H_3^q(g) = \begin{cases} q^{1+X_{T-1}}\{(q^{X_{T-1}} G_1^{T_i} - q^{T_i} G_1^{X_{T-1}}) D_i - K_i q^{T_i}\} = -q^{i+1+2X_{T-1}} = -q^{i-1+2X_T}, X_i = T_i \\ q^{X_{T-1}}(K_i + G_1^{X_{T-1}} D_i) = q^{X_{T-1}}(q^i + G_1^{X_{T-1}} q^i (q-1)) = q^{2X_{T-1}+i} = q^{i+2X_T}, X_i = K_i \end{cases}$$

Factor q^i is present in all of them here, so $q^{(1+2+...+M)}$ can be extracted.
$H_3^q(g) = (-1)^g q^{(1+2+...+M)-g+2(1+2+...+g)} \sum_{X_i\in K} \prod q^{2X_T} = (-1)^g q^{\binom{M+1}{2}+g^2} \begin{bmatrix} M \\ g \end{bmatrix}_{q^2}$

$$q^{\binom{M+1}{2}} \begin{bmatrix} N+M \\ M \end{bmatrix}_{q^2} = \sum_{g=0}^{M} H_3^q(g) G_{2M}^{N+2M-g} = q^{\binom{M+1}{2}} \sum_{g=0}^{M} (-1)^g q^{g^2} \begin{bmatrix} M \\ g \end{bmatrix}_{q^2} G_{2M}^{N+2M-g}$$
$\square$

Definition 17. ${}_qE_M^N = \sum_{1\le\lambda_1\le\lambda_2\le..\le\lambda_M\le N} G_1^{\lambda_1} G_1^{\lambda_2} ... G_1^{\lambda_M} = S_2^q(N+M, N)$

Theorem 78. $(G_1^N)^M = (1+q+...+q^{N-1})^M = \sum_{g=1}^{M} g_{q+}! S_2^q(M,g) G_g^N q^{-g}$

Proof.
$\nabla_q^1 SUM(N, [1_{q-}, 1_{q-}...1_{q-}], [1,2...M]) = \prod(\frac{1}{q} + \frac{q^n-1}{q-1}) = q^{-M}(G_1^{n+1})^M = q^{-M}(G_1^N)^M$
$= q^{-1}\nabla_q^1 SUM(N, [1_{q-}, 1_{q-}...1_{q-}], [2,3...M]) = q^{-1} \sum_{g=0}^{M-1} H_1^q(g) G_{1+g}^N q^{-g}$
$$B_i = \begin{cases} q^{1+X_{T-1}} G_1^{(i+1)-X_{K-1}} = q^{X_T} G_1^{1+X_T} = q^{-1} q^{1+X_T} G_1^{1+X_T}, X_i \in T \\ q^{-1} + G_1^{X_{T-1}} = q^{-1} G_1^{1+X_{T-1}} = q^{-1} G_1^{1+X_T}, X_i \in K \end{cases}$$
$H_1^q(g,T) = q^{-(g+1)}(g+1)_{q+}!, H_1^q(g,\sum K) = q^{-(M-1-g)} {}_qE_{M-1-g}^{g+1}$
$(G_1^N)^M = q^{M-1} \sum_{g=0}^{M-1} q^{-(g+1)}(g+1)_{q+} q^{-(M-1-g)} S_2^q(M, g+1)! G_{1+g}^N q^{-g} \quad \square$

Use $PS = [K_i + 1 : D_i(q-1)], PT = [1,2...M] \rightarrow$

Theorem 79. $\sum_{n=0}^{N-1} q^n \prod_{i=1}^{M} (K_i + D_i q^n) = \sum_{g=0}^{M} H_1^q(g) G_{1+g}^N, B_i = \begin{cases} q^{X_T}(q^{X_T}-1) D_i, X_i \in T \\ K_i + q^{X_T} D_i, X_i \in K \end{cases}$

$$K_i = 1, D_i = q^{-i}z \rightarrow \nabla SUM(M+1) = \prod_{i=1}^{M} (1 + q^{M-i}z) = \sum_{g=0}^{M} H_1^q(g) G_g^M q^{-g} (*)$$
$$B_i = \begin{cases} q^{X_T}(q^{X_T}-1) q^{-i} z, X_i = T_i \\ 1 + q^{X_T} q^{-i} z = 1 + q^{-1-X_{K-1}} z = 1 + q^{-X_K} z, X_i = K_i \end{cases}$$
$H_1^q(g) = q^{1+2+...+g} z^g (q^1-1)...(q^g-1)[\sum_{-M\le\lambda_1<...<\lambda_g\le-1} q^{\lambda_1+...+\lambda_g}](1+q^{-1}z)...(1+q^{-(M-g)}z)$

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$(*) = \sum_{g=0}^{M} q^{\binom{g+1}{2}} z^g (q^1-1)...(q^g-1) q^{\binom{g+1}{2}-g(M+1)} G_g^M (1+q^{-1}z)...(1+q^{-(M-g)}z) G_g^M q^{-g}$
$= \sum_{g=0}^{M} q^{g^2-g(M+1)-\binom{M-g+1}{2}} z^g (q^M-1)...(q^{M-g+1}-1)(q^1+z)...(q^{(M-g)}+z) G_g^M$
Combining q-Binomial Theorem →

Theorem 80. $\sum_{g=0}^{M} q^{\binom{g}{2}} z^g G_g^M (q^M-1)...(q^{M-g+1}-1)(q^{M-g}+z)...(q^1+z) = q^{\binom{M+1}{2}} \sum_{g=0}^{M} q^{\binom{g}{2}} z^g G_g^M$

Use 79 and induction →

Theorem 81. $\sum_{n=0}^{N-1} \prod_{i=1}^{M} (K_i + D_i q^n) = \sum_{g=1}^{M} f(g) G_g^N + N \prod K_i, f(g) = \sum \prod B_i =$
$\begin{cases} 1, X_i \in T, X_{T-1} = 0 \\ q^{X_{T-1}}(q^{X_{T-1}}-1) D_i, X_i \in T, X_{T-1} > 0 \\ K_i, X_i \in K, X_{T-1} = 0 \\ K_i + q^{X_{T-1}-1} D_i, X_i \in K, X_{T-1} > 0 \end{cases}$

Theorem 82. $q^{Mn} = \sum_{g=0}^{M} G_g^M \prod_{i=0}^{g-1} (q^n - q^i) = q^{-M} \sum_{g=0}^{M} q^g \begin{bmatrix} M \\ g \end{bmatrix}_{q^{-1}} \prod_{i=1}^{g} (q^n - q^{-i}) =$
$\sum_{g=0}^{M} G_g^M (-1)^g q^{\binom{g}{2}} G_M^{n+M-g}$

Proof.
$q^{Mn} = \nabla_q^1 SUM(N, [1,1...1] : q-1, [1,2...M])$
$= \sum_{g=0}^{M} H_1^q(g) G_g^n q^{-g} = \sum_{g=0}^{M} H_2^q(g) G_g^{n+g} = \sum_{g=0}^{M} H_3^q(g) G_M^{n+M-g} q^{-g}$
$B_i = \begin{cases} q^{1+X_{T-1}} G_1^{i-X_{K-1}}(q-1) = q^{X_T}(q^{X_T}-1), X_i \in T \\ 1 + G_1^{X_{T-1}}(q-1) = q^{X_{T-1}} = q^{X_T}, X_i \in K \end{cases}, H_1^q(g,T) = (q-1)^g g_{q+}!, H_1^q(g, \sum K) = G_g^M$
$B_i = \begin{cases} q^{-(1+X_{T-1})}(q^{1+X_{T-1}}-1) = q^{-X_T}(q^{X_T}-1), X \in T \\ q^{-(1+X_{T-1})} = q^{-1} q^{-X_T}, X \in K \end{cases}, H_2^q(g,T) = (q-1)^g g_{q-}!, H_2^q(g, \sum K) =$
$q^{-(M-g)} \begin{bmatrix} M \\ g \end{bmatrix}_{q^{-1}}$
$B_i = \begin{cases} -q^{1+X_{T-1}} = -q^{X_T}, X \in T \\ q^{X_{T-1}} = q^{X_T}, X \in K \end{cases}, H_3^q(g,T) = (-1)^g q^{\binom{g+1}{2}}, H_3^q(g, \sum K) = G_g^M$
□

Theorem 83. $\sum_{k_r=1}^{N} ... \sum_{k_2=1}^{k_3} \sum_{k_1=1}^{k_2} q^{k_1+k_2+...+k_r} \nabla_q^p SUM_q(k_1, PS, PT) = \nabla_q^{p-r} SUM_q(N, PS, PT)$

Theorem 84. $\sum_{k_x=1}^{N} ... \sum_{k_2=1}^{k_3} \sum_{k_1=1}^{k_2} q^{k_1+k_2+...+k_x} \nabla_q^1 SUM_q(k_r, PS, [1,2...M]) = \nabla_q^{r-x} SUM_q(N, PS, [T_i = i + (r-1)])$

Theorem 85. $\sum_{0 \le n_1 \le ... \le n_M \le N-1} q^{n_1+n_2+...+n_M} (K + G_1^{n_1} D_1 + ... + G_1^{n_M} D_M)$
$= \{q^1 D_M + q^2(D_M + D_{M+1}) + ... + q^M(D_M + D_{M+1} + ... + D_1)\} G_{M+1}^{N+M-1} + K G_M^{N+M-1}$

Theorem 86. $(K+qD)^M = (q-1) D \sum_{g=0}^{M-1} (K+D)^g (K+qD)^{M-1-g} + (K+D)^M$

Proof. $PS = [K+D, K+D...K+D] : (q-1)D, PT = [1,2...M]$
$B_i = \begin{cases} q^{X_T}(q^{X_T}-1)D, X \in T \\ K+Dq^{X_T}, X \in K \end{cases}, H_1^q(1) = q^1(q^1-1)D \sum_{a+b=M-1,a,b\geq 0} (K+D)^a(K+qD)^b, H_1^q(0) = (K+D)^M$
$SUM_q(2) - SUM_q(1) = H_1^q(1) + H_1^q(0)G_1^2 - H_1^q(0) = H_1^q(1) + qH_1^q(0) = q(K+qD)^M$ □

7.4. Relationships between $H^q(g)$

Theorem 87. $PT = [1,2...M], H_1^q(g) = q^{g(g+1)} \sum_{k=g}^{M} H_2^q(k)G_g^k$[3]

Theorem 88. $PT = [1,2...M], H_1^q(g) = \sum_{k=0}^{g} H_3^q(k)G_{M-g}^{M-k}q^{(g+1)(g-k)}$

Proof. Direct verification when M=1, assuming M holds.
$(*)H_1^q(PS1, PT1, g) = q^g G_1^g H_1^q(g-1) + (K_{M+1} + G_1^g D_{M+1})H_1^q(g)$
$= q^g G_1^g H_1^q(g-1) \sum_{x=0}^{M} H_3^q(x)G_{M-g+1}^{M-x}q^{g(g-1-x)} + (K_{M+1} + G_1^g D_{M+1}) \sum_{x=0}^{M} H_3^q(x)G_{M-g}^{M-x}q^{(g+1)(g-x)}$
$(**)H_1^q(PS1, PT1, g) = \sum_{x=0}^{M+1} H_3^q(PS1, PT1, x)G_{M+1-g}^{M+1-x}q^{(g+1)(g-x)}$
$= \sum_{x=0}^{M} H_3^q(x)\{\frac{q^{M+2}-q^{X+1}}{q-1}D_{M+1} - K_{M+1}q^{M+2}\}G_{M+1-g}^{M-x}q^{(g+1)(g-x-1)}$
$+ \sum_{x=0}^{M} H_3^q(x)(K_{M+1} + G_1^X D_{M+1})G_{M+1-g}^{M+1-x}q^{(g+1)(g-x)}$
Items containing K_{M+1}:
$K_{M+1}G_{M+1-g}^{M+1-x}q^{(g+1)(g-x)} - q^{M+2}K_{M+1}G_{M+1-g}^{M-x}q^{(g+1)(g-x-1)} = K_{M+1}G_{M-g}^{M-x}q^{(g+1)(g-x)}$

Items does not contain K_{M+1} :
$In\ (*) = q^g G_1^g D_{M+1}G_{M-g+1}^{M-x}q^{g(g-1-x)} + G_1^g D_{M+1}G_{M-g}^{M-x}q^{(g+1)(g-x)}$
$Divide\ by\ D_{M+1}q^g q^{g(g-1-x)}(q-1)^{-1} = (q^g-1)G_{M+1-g}^{M-x} + (q^g-1)G_{M-g}^{M-x}q^{g-x} = (q^g-1)G_{M+1-g}^{M+1-x}$
$In\ (**) = \frac{q^{M+2}-q^{x+1}}{q-1}D_{M+1}G_{M+1-g}^{M-x}q^{(g+1)(g-x-1)} + G_1^x D_{M+1}G_{M+1-g}^{M+1-x}q^{(g+1)(g-x)}$
$Divide\ by\ D_{M+1}q^g q^{g(g-1-x)}(q-1)^{-1}$
$= (q^{M+1-x}-1)G_{M+1-g}^{M-x} + (q^g - q^{g-x})G_{M+1-g}^{M+1-x} = (q^g-1)G_{M+1-g}^{M+1-x}$ □

Theorem 89. $(q^M - 1)...(q^{M-K+1} - 1) = \sum_{g=0}^{K} (-1)^g q^{\binom{K-g+1}{2}} G_g^M G_{M-K}^{M-g} = \sum_{g=0}^{K} (-1)^g q^{\binom{K-g+1}{2}+(K-g)(M-K)} G_g^K$

Proof. 88 and 82 can obtain the first equation, the second equation is derived from 75. □

Similarly, using induction to prove:

Theorem 90. $PT = [1,2...M], H_2^q(g) = \sum_{k=g}^{M} (-1)^{k+g} G_g^k q^{-k(k+1)+\binom{k-g}{2}} H_1^q(k)$

Theorem 91. $PT = [1,2...M], H_3^q(g) = \sum_{k=0}^{g} (-1)^{k+g} G_{M-g}^{M-k} q^{(g+1)(g-k)-\binom{g-k}{2}} H_1^q(k)$

Theorem 92. *PT = [1,2...M] and* $D_i = 1, H_1^q(g, \sum K) = F_{M-g}^K \times {}_qE_0^g + F_{M-g-1}^K \times {}_qE_1^g + ... + F_0^K \times {}_qE_{M-g}^g$

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This indicates any $\sum_{g=0}^{M} a_g G_{Y+g}^{X}$ can be converted to $\frac{a_M}{M_{q+}!}\nabla^A SUM_q(N+B,[K_1,K_2...K_M],[1,2...M])$
Use 90 , 91 and 68 ,we can obtain the inversion formulas.

Theorem 93. *If* $\sum_{g=0}^{M} a_g G_{Y+g}^{X} = \sum_{g=0}^{M} b_g G_{Y+g}^{X+g} = \sum_{g=0}^{M} c_g G_{Y+M}^{X+M-g}$

1. $a_g q^{(1-Y)g} = q^{g(g+1)} \sum_{k=g}^{M} b_k G_g^k = \sum_{k=0}^{g} c_k G_{M-g}^{M-k} q^{(g+1)(g-k)} q^{(1-Y)k}$
2. $b_g = \sum_{k=g}^{M} (-1)^{k+g} G_g^k q^{-k(k+1)+\binom{k-g}{2}} a_k q^{(1-Y)k}$
3. $c_g q^{(1-Y)g} = \sum_{k=0}^{g} (-1)^{k+g} G_{M-g}^{M-k} q^{(g+1)(g-k)-\binom{g-k}{2}} a_k q^{(1-Y)k}$

7.5. Merge and Expand

Theorem 94. *Necessary and sufficient conditions for merging,*$0 < K \le M$:

1. $\sum_{n=0}^{M} H(n) G_{Y+n}^{X} \to \sum_{n=0}^{M-K} (...) G_{Y+K+n}^{X+K} : \sum_{x=g}^{M} (-1)^x G_g^x q^{-x(x+1)+\binom{x-g}{2}} H(x) q^{(1-Y)x} = 0, 0 \le g < K$
2. $\sum_{n=0}^{M} H(n) G_{Y+n}^{X} \to \sum_{n=0}^{M-K} (...) G_{Y+K+n}^{X+K} : \sum_{x=0}^{g} (-1)^x G_{M-g}^{M-x} q^{(g+1)(g-x)-\binom{g-x}{2}} H(x) q^{(1-Y)x} = 0, 0 \le M-g < K$
3. $\sum_{n=0}^{M} H(n) G_{Y+n}^{X+n} \to \sum_{n=0}^{M-K} (...) G_{Y+K+n}^{X+K+n} : \sum_{x=g}^{M} H(x) G_g^x = 0, 0 \le g < K$
4. $\sum_{n=0}^{M} H(n) G_{M+Y}^{X-n} \to \sum_{n=0}^{M-K} (...) G_{M+Y-K}^{X-K-n} : \sum_{x=0}^{g} H(x) G_{M-g}^{M-x} q^{(g+1)(g-x)} q^{(1-Y)x} = 0, 0 \le M-g < K$

Y=1,76 $\to$:

Theorem 95. $\sum_{x=0}^{M} (-1)^x G_g^x G_x^M q^{\binom{x-g}{2}} = 0, 0 \le g < M, \sum_{x=0}^{M} (-1)^x G_x^M q^{\binom{x}{2}} = 0$

Theorem 96. *P=A+T+1-Y,M > A ≥ 0,*
$\sum_{g=0}^{M} G_g^M G_A^{A+T+g} G_{Y+g}^{X} q^{g(g+1+T-P)} = \sum_{x=0}^{A} G_{x+T}^{A+T} G_{M+T}^{M+T+x} G_{Y+M-A+x}^{X+M-A} q^{x(x+1+T-P)}$

Proof.

$SUM_q(N,[(T+1)_{q-},(T+2)_{q-}...(T+M)_{q-}],[T+A+1...T+A+M]), B_i \quad of \quad H_1^q(g) = \begin{cases} q^{X_T} G_1^{T+A+i-X_T}, X_i=T_i \\ q^{-(T+i)} G_1^{T+i+X_T}, X_i=K_i \end{cases}$

$= \sum_{g=0}^{M} G_{A+T+1+g}^{X} q^{\frac{g(1+g)}{2}} G_1^{T+A+1}...G_1^{T+A+g} \times G_1^{T+M}...G_1^{T+g+1} q^{-(M-g)(T+M+1)} \sum_{1\le\lambda_1<...<\lambda_{M-g}\le M} q^{\sum\lambda_i}$

$= \frac{(q^A-1)...(q-1)}{(q-1)^M} \sum_{g=0}^{M} G_{A+T+1+g}^{X} q^{\frac{g(1+g)}{2}-(M-g)(T+M+1)+\frac{(M-g+1)(M-g)}{2}} \times (q^{T+M}-1)...(q^{T+A+1}-1) G_A^{A+T+g} G_g^M$

$= q^{-(T+A+1)} G_1^{T+A+1} ... q^{-(T+M)} G_1^{T+M} SUM_q(N,[(T+1)_{q-}....(T+A)_{q-}],[T+M+1...T+M+A])$

$= q^{\frac{-(M-A)(T+A+1+T+M)}{2}} G_1^{T+A+1}...G_1^{T+M} \sum_{x=0}^{A} G_{M+T+1+x}^{X+M-A} q^{\frac{x(1+x)}{2}-(A-x)(T+A+1)+\frac{(A-x+1)(A-x)}{2}}$
$\times G_1^{T+M+1}..G_1^{T+M+x} \times G_1^{T+A}...G_1^{T+x+1} G_x^A$

$\to \sum_{g=0}^{M} G_{A+T+1+g}^{X} q^{\frac{g(1+g)-(M-g)(g+M+1+2T)}{2}} G_A^{A+T+g} G_g^M$

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$$= q^{\frac{-(M-A)(T+A+1+T+M)}{2}} \sum_{x=0}^{A} G_{M+T+1+x}^{X+M-A} q^{\frac{x(1+x)-(A-x)(x+A+1+2T)}{2}} G_{M+T}^{M+T+x} G_{x+T}^{A+T}$$

$$\rightarrow \sum_{g=0}^{M} G_g^M G_A^{A+T+g} G_{A+T+1+g}^{X} q^{g(g+1+T)} = \sum_{x=0}^{A} G_{x+T}^{A+T} G_{M+T}^{M+T+x} G_{M+T+1+x}^{X+M-A} q^{x(x+1+T)} \quad \square$$

The difference between this and 34 is that M>A is required.

$$A = 0 \rightarrow \sum_{g=0}^{M} G_g^M G_{T+1+g}^{N} q^{g(g+1+T)} = \sum_{g=0}^{M} G_g^M G_{T+1+M-g}^{N} q^{(M-g)(M-g+1+T)} = G_{T+1+M}^{N+M}$$

$K = T + 1 + M \rightarrow \sum_{g=0}^{K} G_g^M G_{K-g}^{N} q^{(M-g)(K-g)} = G_K^{N+M}$. It's the **q-Vandermorde Theorem**.

$$0 \leq B + A < M, 94 \rightarrow \sum_{g=0}^{M} G_g^M G_A^{A+T+g} G_B^g q^{-gA+\binom{g-B}{2}} (-1)^g = 0 \rightarrow$$

$$\sum_{g=0}^{M} G_g^M G_A^{X_1+g} G_B^g q^{\binom{g}{2}-g(A+B)} (-1)^g = 0$$

A and B have symmetry,2→.

Theorem 97. $\sum_{g=0}^{M} G_g^M G_A^{X_1+g} G_B^{X_2+g} q^{\binom{g}{2}-g(A+B)} (-1)^g = 0, 0 \leq B + A < M$

Theorem 98. $\sum_{g=0}^{M} (-1)^g G_g^M G_{M+K}^{X+g} q^{\binom{M+1-g}{2}+(M-g)K} = (-1)^M G_K^X, X + g \geq 0, K \geq 0$

Proof.

$SUM_q(N, [(T+1)_{q-}, (T+2)_{q-} ... (T+M)_{q-}], [T+K+M+1, T+K+M+2 ... T+K+2M])$

$H_1^q(g) = q^{\frac{g(1+g)}{2}-(M-g)(T+M+1)+\binom{M-g+1}{2}} \times G_1^{T+K+M+1} ... G_1^{T+K+M+g} \times G_1^{T+M} ... G_1^{T+1+g} \times G_1^{T+M+1} G_g^M$

$H_2^q(0) = \sum_{g=0}^{M} (-1)^g H_1^q(g) q^{-g(g+1)+\frac{g(g-1)}{2}-(T+K+M)g} = (-1)^M q^{-M(T+1+T+M)} G_1^{K+M} G_1^{K+M-1} ... G_1^{K+1}$

$\rightarrow (-1)^M G_1^{K+M} ... G_1^{K+1} = \sum_{g=0}^{M} (-1)^g q^{\binom{M-g+1}{2}+(M-g)K} \times G_1^{T+K+M+g} ... G_1^{T+K+M+1} \times G_1^{T+M} ... G_1^{T+1+g} \times G_g^M$

$$\rightarrow (-1)^M G_K^{T+K+M} = \sum_{g=0}^{M} (-1)^g q^{\binom{M-g+1}{2}+(M-g)K} \times G_{M+K}^{T+K+M+g} G_g^M \quad \square$$

7.6. Matrix of $SUM_q(N)$

Let $P = N + T_M - M, Q = N - 1$, corresponding to the three forms,define $A_{1,2,3}^q(P, Q, M) =$

$$\begin{pmatrix} G_Q^P & \cdots & G_{Q-M}^P \\ \vdots & \ddots & \vdots \\ G_{Q+M}^{P+M} & \cdots & G_Q^{P+M} \end{pmatrix}, \begin{pmatrix} G_Q^P & \cdots & G_Q^{P+M} \\ \vdots & \ddots & \vdots \\ G_{Q+M}^{P+M} & \cdots & G_{Q+M}^{P+2M} \end{pmatrix}, \begin{pmatrix} G_Q^{P+M} & \cdots & G_{Q-M}^P \\ \vdots & \ddots & \vdots \\ G_{Q+M}^{P+2M} & \cdots & G_Q^{P+M} \end{pmatrix}$$

Theorem 99. $\| A_2^q(P, Q, M) \| = \| A_2^q(P, P-Q, M) \| = \frac{G_Q^P}{G_0^P} \frac{G_{Q+1}^{P+2}}{G_1^{P+2}} ... \frac{G_{Q+M}^{P+2M}}{G_M^{P+2M}} q^{(P+1)+2(P+2)+...+M(P+M)}$

$\| A_2^q(P, 0, M) \| = q^{(P+1)+2(P+2)+...+M(P+M)}, \| A_2^q(P, 1, M) \| = G_{M+1}^{P+M} q^{(P+1)+2(P+2)+...+M(P+M)}$

Theorem 100. $\| A_{1,3}^q(P, Q, M) \| = \frac{G_Q^P}{G_0^P} \frac{G_{Q+1}^{P+2}}{G_1^{P+2}} ... \frac{G_{Q+M}^{P+2M}}{G_M^{P+2M}} q^{Q\binom{M+1}{2}}$

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8. Multi-parameter Formal Calculation

2-parameters Formal Calculation calculate nested sum of $K_i + \binom{n}{1} D_{1,i} + \binom{n}{2} D_{2,i}$.

$SUM(N, [K_1], [T_{1,1} = 1 : D_{1,1}], [T_{2,1} = 1 : D_{2,1}]) = \sum\limits_{n=0}^{N-1} (K_1 + D_{1,1}n + D_{2,1} \binom{n}{2})$

$SUM(N, [K_1, K_2], [T_{1,1} : D_{1,1}, T_{1,2} = T_{1,1} + 2 - p : D_{1,2}], [T_{2,1} = T_{1,1} : D_{2,1}, T_{2,2} = T_{1,2} : D_{2,2}])$

$= \sum\limits_{n=0}^{N-1} (K_2 + D_{1,2}n + D_{2,2} \binom{n}{2}) \nabla^p SUM(N, [K_1], [T_{1,1} : D_{1,1}], [T_{2,1} : D_{2,1}])$

Recursive define $SUM(N, PS, PT_1, PT_2)$. There is always $T_i = T_{1,i} = T_{2,i}$

Use the Form: $(K_1 + T_{1,1} + T_{2,1})(K_2 + T_{1,2} + T_{2,2})...(K_M + T_{1,M} + T_{2,M})$

Use T_1 to represent the set $\{T_{1,1}, T_{1,2}...T_{1,M}\}$,$T_2$ to represent the set $\{T_{2,1}, T_{2,2}...T_{2,M}\}$.

Definition 18. *X(PT_1)=Number of $\{X_1...X_M\} \in T_1$, X(PT_2)=Number of $\{X_1...X_M\} \in T_2$,X(PT)=X(PT_1)+2X(PT_2)*

Definition 19. X_{PT1}*=Number of* $\{X_1...X_i\} \in T_1$,X_{PT2}*=Number of* $\{X_1...X_i\} \in T_2$,$X_{PT} = X_{PT1} + 2X_{PT2}$

Theorem 101. $SUM(N, PS, PT_1, PT_2)$

$$= \sum_{g=0}^{2M} H_1(g) \binom{N+T_M-M}{T_M-M+1+g}, B_i = \begin{cases} K_i + X_{PT}D_{1,i} + \binom{X_{PT}}{2} D_{2,i}, X_i = K_i \\ (T_i - i + X_{PT})D_{1,i} + (T_i - i + X_{PT})(X_{PT} - 1)D_{2,i}, X_i = T_{1,i} \\ \binom{T_i-i+X_{PT}}{2} D_{2,i}, X_i = T_{2,i} \end{cases}$$

$$= \sum_{g=0}^{2M} H_2(g) \binom{N+T_M-M+g}{T_M-M+1+g}, B_i = \begin{cases} K_i - (T_i - i + X_{PT} + 1)D_{1,i} + \binom{T_i-i+X_{PT}+2}{2} D_{2,i}, X_i = K_i \\ (T_i - i + X_{PT})D_{1,i} - (T_i - i + X_{PT})(T_i - i + X_{PT} + 1)D_{2,i}, X_i = T_{1,i} \\ \binom{T_i-i+X_{PT}}{2} D_{2,i}, X_i = T_{2,i} \end{cases}$$

$$= \sum_{g=0}^{2M} H_3(g) \binom{N+T_M+M-g}{T_M+M+1}, B_i = \begin{cases} K_i + X_{PT-1}D_{1,i} + \binom{X_{PT-1}}{2} D_{2,i}, X_i = K_i \\ -2K_i + (T_i + i - 1 - 2X_{PT-1})D_{1,i} + (T_i + i - X_{PT-1})D_{2,i}, X_i = T_{1,i} \\ K_i - (T_i + i - 1 - X_{PT-1})D_{1,i} + \binom{T_i+i-X_{PT-1}}{2} D_{2,i}, X_i = T_{2,i} \end{cases}$$

According to this way, it can be extended to multi-parameter $SUM(N)$ and $SUM_q(N)$. This formula is complex and has not been studied in terms of analysis yet.

9. A theorem of symmetry

In this section,$T_i \geq i$.

17$\rightarrow H_1(g, PT, PT) = \prod\limits_{i=1}^{M} T_i \binom{M}{g} = \prod\limits_{i=1}^{M} T_i \binom{M}{g, M-g}$.Promoted it:Set T come from p Source:$S_1, S_2...S_p$.

Definition 20. $Diff(S_x, S_x) = 0, Diff(S_x, S_y) = -Diff(S_y, S_x) = 1, x > y$

Definition 21. $Diff(T_i, T_j) = Diff(S_x, S_y), T_i \in S_x, T_j \in S_y$

Definition 22. $W(g_1, g_2...g_p, [T_1, T_2...T_M]) = \sum\limits_{g_1+g_2+...+g_p=M, g_i=|S_i|} \prod\limits_{i=1}^{M} (T_i + \sum\limits_{j<i} Diff(T_j, T_i))$
In set T, g_1 *come from* S_1,g_2 *comes from* S_2...g_M *comes from* S_M.

Theorem 102. $W(g_1, g_2...g_p, [T_1, T_2...T_M]) = \prod\limits_{i=1}^{M} T_i \binom{M}{g_1, g_2, ..., g_M}$

Definition 23. $W_q(g_1, g_2...g_p, [T_1, T_2...T_M]) = \sum\limits_{g_i+g_2+...+g_p=M, g_i=|S_i|} \prod\limits_{i=1}^{M} G_1^{T_i + \sum\limits_{j<i} Diff(T_j, T_i)} q^{\sum\limits_{j<i, Diff(T_j, T_i)=-1} 1}$

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Definition 24. $G^M_{g_1,g_2\ldots g_p} = \frac{(q^M-1)(q^{M-1}-1)\ldots(q^1-1)}{\prod_{i=0}^{p}(q^{g_i}-1)(q^{g_i-1}-1)\ldots(q^i-1)}, g_1+g_2+\ldots+g_p = M$

Theorem 103. $W_q(g_1,g_2\ldots g_p,[T_1,T_2\ldots T_M]) = (\prod_{i=1}^{M} G_1^{T_i})G^M_{g_1,g_2\ldots g_p}, T_i \geq i$

Proof.
$W(1,1,[T_1,T_2]) = G_1^{T_1}G_1^{T_2+1} + G_1^{T_1}G_1^{T_2-1}q = G_1^{T_1}G_1^{T_2}G_1^2$.Holds
Suppose $W_q(g_1,g_2,PT)$ holds,$W_q(g_1,g_2+1,[PT,T_{M+1}]) = T_{M+1} \in Source_1 + T_{M+1} \in Source_2$
$= W_q(g_1,g_2,PT)G_1^{T_{M+1}+g_1} + W_q(g_1-1,g_2+1,PT)G_1^{T_{M+1}-(g_2+1)}q^{g_2+1}$
$= (\prod_{i=1}^{M} G_1^{T_i})G^M_{g_1,g_2}G_1^{T_{M+1}+g_1} + (\prod_{i=1}^{M} G_1^{T_i})G^M_{g_1-1,g_2+1}G_1^{T_{M+1}-(g_2+1)}q^{g_2+1}$
Just need to prove:$G^M_{g_1}G_1^{T_{M+1}+g_1} + G^M_{g_1-1}G_1^{T_{M+1}-(M-g_1+1)}q^{M-g_1+1} = G_1^{T_{M+1}}G^{M+1}_{g_1}$
$(Right\ side) \times \frac{(q^{M-g_1+1}-1)}{G^M_{g_1}} = (q^{T_{M+1}-1}+\ldots+q+1)(q^{M+1}-1) = (1)$
$(Left\ side) \times \frac{(q^{M-g_1+1}-1)}{G^M_{g_1}} = (q^{M-g_1+1}-1)G_1^{T_{M+1}+g_1} + (q^{g_1}-1)G_1^{T_{M+1}-(M-g_1+1)}q^{M-g_1+1}$
$= (q^{M-g_1+1}-1)(q^{T_{M+1}+g_1-1}+\ldots+q+1) + (q^{g_1}-1)(q^{T_{M+1}-1}+\ldots+q^{M-g_1+2}+q^{M-g_1+1}) = (2)$
(1)-(2)=0→It's holds when p=2.

$$W_q(g_1,g_2+g_3,[T_1,T_2\ldots T_M]) = (\prod_{i=1}^{M} G_1^{T_i})G^{g_1+g_2+g_3}_{g_1,g_2+g_3}$$

Every product has g_2+g_3 factors come from $Source_2$,divide them to $g_2 \times Source_2 + g_3 \times Source_3$
g_1-factors are invariant,(g_2+g_3)-factors are variant.
$\sum\prod(variant\ factors) = W_q(g_2,g_3,[X_1,X_2\ldots X_{g_2+g_3}]) = \prod_{i=1}^{g_2+g_3} G_1^{X_i}G^{g_2+g_3}_{g_2,g_3}$
$W_q(g_1,g_2,g_3,[T_1,T_2\ldots T_M]) = (\prod_{i=1}^{M} G_1^{T_i})G^{g_1+g_2+g_3}_{g_1,g_2+g_3}G^{g_2+g_3}_{g_2,g_3} = (\prod_{i=1}^{M} G_1^{T_i})G^{g_1+g_2+g_3}_{g_1,g_2,g_3}$ □

$eg : W_q(1,2,[1,2,3]) = S_1S_2S_2 + S_2S_1S_2 + S_2S_2S_1 = G_1^1G_1^3G_1^4 + G_1^1G_1^1qG_1^4 + G_1^1G_1^2G_1^1q^2 = G_1^1G_1^2G_1^3G^3_{1,2}$
$eg : When\ W_q(1,2,[1,2,3])\ changes\ to\ W_q(1,1,1,[1,2,3])$
$= S_1S_2S_3 + S_1S_3S_2 + S_2S_1S_3 + S_3S_1S_2 + S_2S_3S_1 + S_3S_2S_1$
$= G_1^1\{G_1^3G_1^5 + G_1^3G_1^3q\} + G_1^1q\{G_1^1G_1^5 + G_1^1G_1^3q\} + G_1^1q^2\{G_1^1G_1^3 + G_1^1G_1^1q\}$
$= G_1^1\{G_1^3G_1^4G^2_{1,1}\} + G_1^1q\{G_1^1G_1^4G^2_{1,1}\} + G_1^1q^2\{G_1^1G_1^2G^2_{1,1}\} = G_1^1G_1^2G_1^3G^3_{1,2} \times G^2_{1,1} = G_1^1G_1^2G_1^3G^3_{1,1,1}$

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