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Does the Collatz Sequence Eventually Reach 1 for All Positive Integer Initial Values?

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Abstract: This study focuses on one of the most famous open problems in mathematics, namely the Collatz conjecture. The Collatz conjecture or $3x + 1$ Problem is perhaps today's most enigmatic unsolved mathematical problem. It is named after Lothar Collatz, who first proposed it in 1937. It may be stated as as follow: Take any positive integer n. If n is even then divide it by 2, else do "triple plus one" and get $3n + 1$. The conjecture is that this process will eventually reach the number 1, regardless of which positive integer is chosen initially. In this paper, we present a simple proof for the Collatz conjecture.

Keywords: Collatz conjecture; $3x + 1$ conjecture; base-2 numeral system

MSC: Primary 11Zxx; 11Bxx; Secondary 37P99

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1. Introduction

The Collatz conjecture is an unsolved conjecture in mathematics. It is named after Lothar Collatz, who first proposed it in 1937. The conjecture is also known as the $3n + 1$ conjecture, the Ulam conjecture (after Stanislaw Ulam), the Syracuse problem, as the hailstone sequence or hailstone numbers, or as Wondrous numbers per Gödel, Escher, Bach.

The Collatz problem is one of the most famous unsolved issues in mathematics. Paul Erdös was correct when he stated, "Mathematics is not ready for such problems". Lagarias [7] stated that the Collatz conjecture "is an extraordinarily difficult problem, completely out of reach of present day mathematics". Possibly, the interest to this problem is related to the fact that the question is very easy to state, but very hard to solve. In fact, it is even complicated to give partial answers. The problem addresses the following situation: Consider an iterative method over the set of positive integers defined in the following way. If n is even, then we consider the positive integer $\frac{n}{2}$ for the next step. On the other hand, if n is odd, then we consider the positive integer $3n + 1$ for the next step. The Collatz conjecture states that, independently of the chosen initial value for n, the number 1 is reached eventually.

In [7] discussion concerning the origin of the problem was given. In the last 50 years, the mathematical community has tried different approaches to the Collatz conjecture, but none of them is believed to provide a definitive path that would allow to solve the problem [8]. On one hand, there are theoretical arguments that allow proving statements which are similar to the conjecture, but a bit weaker [5,10]. On the other hand, there are numerical experiments that show that the conjecture holds for numbers which are smaller than a certain threshold N, or that the Collatz functions does not have non-trivial cycles with length less or equal to m. The values of m and N have been continuously improved, and, nowadays, we can ensure that the conjecture holds for $N < 5.78 \times 10^{18}$ [7] or that the length of a non-trivial cycle is, at least, 1.7×10^{7} [3]. These computational arguments have been feeding continuously the opinion that the Collatz conjecture is true, and that there is no non-trivial cycle.

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Besides, some authors have devoted efforts to rewrite the conjecture in other terms such as algebraic and boolean fractals [2]. Furthermore, some study has been developed concerning the representation and study of the Collatz conjecture in terms of graphs [1,4,6,9].

In this paper the authors propose a proof for the Collatz conjecture. For this purpose, we use the base-2 numeral system or binary numeral system representation. The reasons for using binary system are: a) The multiplication property of 11 (recall that for any positive integer $r = abcd$ we have $r \times 11 = (abcd) \times 11 = a(a+b)(b+c)(c+d)d$ and $r \times 11 = (abcd) \times 11 = efghd$, where $e = a+1$ if $a+b \geq 10$; f= ones digit of $(a+b)+1$ if $b+c \geq 10$; $g =$ ones digit of $(b+c)+1$ if $c+d \geq 10$; h= ones digit of $(c+d)$ if $c+d \geq 10$) and note that since $3 = (11)_2$ we can apply the above property in binary mode ; and b) In binary representation of any positive integer, the number of zeros appearing to the right determines that we can divide the number by which power of 2. For instance, if $P = (11000)_2$, then the number of zeros is 3, thus P is dividable by 2^3.

2. Preliminaries

In this section we first give some lemmas and then using these lemmas the main theorem is proved in the next section.

Lemma 2.1. If $n = (1010\cdots01)_2$, then the next step of sequence ends to 1.

Proof. Assume that n is multiplied by $3 = (11)_2$ and then the result is added to one. Hence, $3n+1 = (1\underbrace{000\cdots0}_{k\ times})_2 = 2^k$. Clearly, in this case after k times dividing by 2 we reach to 1. □

Corollary 1. If n is shown as $n = (\overline{a}11)_2, n = (\overline{a}001)_2$ or $n = (\overline{a}1101)_2$, where a shows a consecutive 0-digits and 1-digits, then in the next step $3n+1$ is dividable by $2^1, 2^2$ and 2^3, respectively.

Remark 1. In what follows we prove some lemmas which show that the generated numbers are ultimately converted to the given number in Lemma 2.1.

Lemma 2.2. Let $n = (\cdots 0\underbrace{11\cdots1}_{k-1\ times})_2$, that is, the first zero appears in k-th place. Then, the place of this zero is reduced to one unit at each iteration.

Proof. If $n = (\cdots 0\underbrace{11\cdots1}_{k-1\ times})_2$, then $3n+1 = (\cdots 0\underbrace{11\cdots1}_{k-2\ times}0)_2$. Thus, using corollary 1 in the next iteration $3n+1$ is dividable to 2^1 and as a result $n = (\cdots 0\underbrace{11\cdots1}_{k-1\ times})_2 \to \frac{3n+1}{2} = (\cdots 0\underbrace{11\cdots1}_{k-2\ times})_2 \to \cdots \to \frac{3n+1}{2} = (\cdots 01)_2$ □

Lemma 2.3. If $n = (\overline{a}01)_2$, then instead of computing $3n+1$ it is enough to compute $3a+1$.

Proof. In this case by corollary 1 it is clear that $3n+1$ is dividable by $4 = 2^2$. Suppose $n = (\cdots a_4a_301)_2$, so in the next step $3n+1 = (\cdots(a_4+a_3)(a_3+1)00)_2$. On the other hand

$$3a+1 = 3(\cdots a_4a_3)_2 + 1 = (11)_2(\cdots a_4a_3) + 1 = (\cdots(a_4+a_3)(a_3+1))_2 = 3n+1$$

□

3. Main Result

Now, we give a simple proof for Collatz conjecture.

Theorem The Collatz conjecture is correct.

Proof. Let $n = (a_k\cdots a_2a_1)_2$. If $a_1 = 0$, then we remove it (dividing by 2). Without loss of generality assume that $a_k = a_1 = 1$. If $a_2 = 1$,then using Lemma 2.2 instead of working on n, after some iterations, we consider $n' = (\cdots01)_2$ and finally by Lemma 2.3 we use $n'' = (\cdots101)_2$, instead of n', and Lemma 2.1 shows that the sequence with n'' is convergence to 1. □

In the following, we show that obtaining the sequence in Lemma 2.1 occurs much faster.

Lemma 3.1. Let $n = (\cdots 0\underbrace{11\cdots 1}_{k\ times}0\cdots)_2$, $k > 1$, then in the next step $3n+1 = (\cdots 0\ \underbrace{11\cdots 1}_{k-1\ times}\ 0\cdots)_2$ or $3n+1 = (\cdots 0\ \underbrace{11\cdots 1}_{k-2\ times}\ 01\cdots)_2$.

Proof. Let $n = (\cdots 0\underbrace{11\cdots 1}_{k\ times}0\cdots)_2$, then using the multiplication property of $3 = (11)_2$ we have $3n+1 = (*0\ \underbrace{1\cdots 1}_{k-1\ times}\ 0*)_2$ depends on the previous sums give us a 1 or $3n+1 = (*0\ \underbrace{1\cdots 1}_{k-2\ times}\ 01*)_2$ depends on we do not have 1, where $*$ stands for a consecutive 0-digits and 1-digits. Therefore in both cases, we have a sequence as $\underbrace{1\cdots 1}_{q\ times}$, where q is $k-1$ or $k-2$, and after at most $k-1$ times this sequence is converted to $(*010*)_2$. □

Indeed, the above lemma shows that at each iteration at least one of 1-digits is reduced and ultimately the given number approached to mentioned number in Lemma 2.1.

Lemma 3.2. Let $n = (\cdots 1\underbrace{0\cdots 0}_{k\ times}1)_2$. Then, after some iterations we have $3n+1 = (\cdots 101)_2$.

Proof. By corollary 1, in the next step 2^2 is a factor of $3n+1$ and $\frac{3n+1}{2^2} = (\cdots 1\ \underbrace{0\cdots 0}_{k-2\ times}\ 1)_2$. Hence, if k is an odd number and $k = 2l+1$, then after l iteration we reach to a number in which its two right digits are 01. If k is even the sequence leads to a number whose two right digits are 11. In this case by employing Lemma 3.1 we reach to a number with 01 in its right digits. □

In a similar way, we get $(*101*)_2$ for the case where there is a sequence of zeros between two ones.

Lemma 3.3. (The worst case) Let $n = (11\cdots 1)_2$ then after four iteration we reach to a sequence that its three left digits is 111 or 10101, periodically. That is, in this worst case, the Collatz's sequence goes to 5 and then to 1.

Proof. Since $n = (\underbrace{11\cdots 1}_{k\ times})_2$, thus $3n+1 = (1011\cdots)_2 \to 3n+1 = (100011\cdots)_2 \to 3n+1 = (110101\cdots)_2 \to 3n+1 = (10100001\cdots)_2$ and after this step three digits on the left have the following periodic behavior: $111 \to 10101 \to 111$ □

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