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Fractional Differential Boundary Value Equation utilizing the Convex Interpolation for Symmetry of Variables

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Article

Fractional Differential Boundary Value Equation Utilizing the Convex Interpolation for Symmetry of Variables

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Abstract: The main objective of the study is to introduce a novel form of interpolative convex contraction and develop some new conclusions for it utilising the progressive method of interpolative convex contractions. With orbitally S-complete and Suzuki type contractions in F-metric spaces, the paper draws certain results. The purpose of my research is to evaluate the effectiveness of the fixed point approach in solving fractional differential equations with boundary condition.

Keywords: interpolative convex contraction; Suzuki convex contraction; fixed point; fractional differential equation

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1. Introduction

Reich [16,26] has used Fixed Point Theory(FPT) to develop unique global tactics. The idea of the interpolative class of contractions was first introduced by Erdal et al. [18], who also implemented a few fixed point results in a partial metric space. In order to determine the Hardy-Rogers findings, Erdal revised Kannan's interpolative contraction in [17] and employed an interpolative method.

Erdal updated Kannan's interpolative contraction in [17] and used an interpolative approach to determine the Hardy-Rogers findings in [19]. Additionally, he created a novel interpolative contraction technique in [4].

Aydi et al. [1,2], originally introduced interpolative and ω-interpolative Reich-Rus type contractions and also showed some relevant fixed point findings for these mappings.

Aftab [7–9], recently expanded on this idea of Erdal and published a few findings pertaining to these kinds of novel contractions. Nazam et al. [23–25], introduced (Ψ, Φ)-orthogonal interpolative contractions very recently and make a few observations on the literature.

Jleli and Samet [14] added brand new form of $\mathcal{F}$-metric space(FMS).

Definition 1.1. *[14]. Let $\mathcal{F}$ represent the group of functions $g : (0, +\infty) \to \mathbb{R}$ assuring the requirements:*

$(\mathcal{F}_1)$ g is increasing, meaning that for every $u > v > 0, \Longrightarrow g(u) \leq g(v)$;
$(\mathcal{F}_2)$ each sequence $\{v_n\} \subset (0, +\infty)$, s.t

$$\lim_{n\to+\infty} v_n = 0 \iff \lim_{n\to+\infty} g(v_n) = -\infty.$$

The following is a thorough definition of a FMS:

Definition 1.2. *[14]. Let Đ: $\chi \times \chi \to [0, +\infty)$ be a mapping and let χ be a nonempty set. Suppose that there exists $(g, \mu) \in \mathcal{F} \times [0, +\infty)$ s.t*

(Đ$_1$) $(\jmath, \ell) \in \chi \times \chi$, Đ$(\jmath, \ell) = 0 \iff \jmath = \ell$.
(Đ$_2$) Đ$(\jmath, \ell) =$Đ$(\ell, \jmath)$ for all $(\jmath, \ell) \in \chi \times \chi$.

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(Đ$_3$) For every $(\jmath,\ell) \in \chi \times \chi$, every $2 \leq N \in \mathbb{N}$, each $(u_i)_{i\in\mathbb{N}} \subset \chi$ with $(u_1, u_N) = (\jmath, \ell)$, we have

$$Đ(u_1, u_N) > 0 \implies g(Đ(u_1, u_N)) \leq g(\sum_{i=1}^{N-1} d(u_i, u_{i+1})) + \mu.$$

The pair (χ,Đ) is then referred to as an FMS.

Example 1. *[14]. Let an FMS is a set of $\mathbb{N}$ if we define Đ by,*

$$Đ(\jmath,\ell) = \begin{cases} (\jmath-\ell)^2, & \text{if } (\jmath,\ell) \in [0,3]\times[0,3] \\ |\jmath-\ell|, & \text{if } (\jmath,\ell) \notin [0,3]\times[0,3], \end{cases}$$

for all $(\jmath,\ell) \in \chi \times \chi$ with $f(v) = \ln(v)$ and $\mu = \ln(3)$, and not a FM. You should be aware that any measure on χ is an FM.

Definition 1.3. *[14]. Assume that (χ,Đ) is an FMS. Let $\{\jmath_n\}$ represent a sequence in χ.*
(i) If $\lim_{n,m\to\infty}Đ(\jmath_n, \jmath_m) = 0$, we declare that $\{\jmath_n\}$ is $\mathcal{F}$-Cauchy,
(ii) We say that (χ,Đ) is $\mathcal{F}$-complete(FC) if every $\mathcal{F}$-Cauchy sequence in χ is $\mathcal{F}$-convergent to a specific element in χ.

Introducing the Banach Contraction Principle was Jleli and Samet:

Theorem 1.1. *[14]. Let $h : \chi \to \chi$ be a predetermined mapping and (χ,Đ) said to an FMS. Assume that the subsequent criteria are met:*
(i) It is FC for (χ,Đ).
(ii) $k \in (0,1)$ occurs in such a way that

$$Đ(h(\jmath), h(\ell)) \leq kĐ(\jmath,\ell), \ (\jmath,\ell) \in \chi \times \chi.$$

Then g has a distinct fixed point $\jmath^* \in \chi$. In addition, $\jmath_0 \in \chi$, the sequence $\{\jmath_n\} \subset \chi$ defined by $\jmath_{n+1} = h(\jmath_n)$, $n \in \mathbb{N}$, is $\mathcal{F}$-convergent for any to $\jmath^*$ as well.

Delivered a class of α-admissible mapping and installed the Banach contraction Principle using α-admissible mapping, according to Samet et al. (2012). The following is the basic defintion of α-admissible mapping:

Definition 1.4. *[28]. Let $S : \chi \to \chi$ and $\alpha : \chi \times \chi \to [0, +\infty)$. We say that S is an α-admissible if $\jmath, \ell \in \chi$, $\alpha(\jmath,\ell) \geq 1$ implies that $\alpha(S\jmath, S\ell) \geq 1$.*

The idea of α-admissible mapping was then modified by Salimi et al. [27] as follows.

Definition 1.5. *[27]. Let $S : \chi \to \chi$ and two functions $\alpha, \eta : \chi \times \chi \to \mathbb{R}^+$. We claim that S is an α-admissible mapping regarding to η then $\jmath, \ell \in \chi$, $\alpha(\jmath,\ell) \geq \eta(\jmath,\ell)$ implies that $\alpha(S\jmath, S\ell) \geq \eta(S\jmath, S\ell)$.*

Definition 1.6. *[12] Let (χ, d) be a metric space. Let's say that there are two functions $\alpha, \eta : \chi \times \chi \to [0, +\infty)$ and $S : \chi \to \chi$. A map S is considered as α-η-continuous map on (χ, d) whenever given $\jmath \in \chi$, and a sequence $\{\jmath_n\}$ is as follows*

$$\lim_{n\to\infty} \jmath_n = \jmath.\ \alpha(\jmath_n, \jmath_{n+1}) \geq \eta(\jmath_n, \jmath_{n+1}) \text{ for all } n \in \mathbb{N} \implies S\jmath_n \to S\jmath.$$

For more details see [20,21].

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If $\lim_{n\to\infty} S^n \jmath = v$ implies that $\lim_{n\to\infty} SS^n \jmath = Sv$, then a mapping $S : \chi \to \chi$ is pronounce to be orbitally continuous at v. If S is orbitally continuous for all v, then the mapping S is orbitally continuous on v.

Remark 1.1. *Observe that some papers are not in corrected form without this property see [5,8,9].*

Observation 1 [24]. The inequality applies to all, $\text{þ}, q \geq 2$ and $r \geq 1$,

$$(\text{þ} + q)^r \leq (\text{þ}q)^r.$$

2. Interpolative Convex Reich-type Contraction

In this section, we offer a novel interpolative convex contraction and establish some new discoveries for interpolative convex Reich-type α-η-contraction in the context of F-complete FMS.

Definition 2.1. *Let $(\chi, Đ)$ is be an FMS. Let there are two functions $\alpha, \eta : \chi \times \chi \to [0, +\infty)$ and $S : \chi \to \chi$. If there are constants $\lambda \in [0,1)$ and $\alpha, \beta, \gamma \in (0,1)$ such that whenever $\alpha(\jmath, \ell) \geq \eta(\jmath, \ell)$, we say that S is an interpolative convex Reich-type α-η-contraction.*

$$Đ(S\jmath, S\ell)^{þ} \leq \lambda \left[Đ(\jmath, \ell)^{þ\beta + q\alpha} \cdot Đ(\ell, S\ell)^{þ\gamma - q\gamma} \cdot Đ(\jmath, S\jmath)^{þ(1-\beta-\gamma)+q(\gamma-\alpha)} \right], \tag{1}$$

for all $\jmath, \ell \in \chi \backslash Fix(S)$, where $þ, q \in [1, \infty)$.

Example 2. *Let $\chi = \{0, 1, 2, 3\}$ be endowed with FMS given by*

$$Đ(\jmath, \ell) = \begin{cases} (\jmath - \ell)^2, \text{ if } (\jmath, \ell) \in \chi \times \chi \\ |\jmath - \ell|, \text{ if } (\jmath, \ell) \notin \chi \times \chi, \end{cases}$$

with $f(v) = \ln(v)$ and $\mu = \ln(3)$. Define $S : \chi \to \chi$ by

$$S0 = 0, S1 = 1, S2 = S3 = 0.$$

and $\alpha, \eta : \chi \times \chi \to [0, +\infty)$ by

$$\alpha(\jmath, \ell) = \begin{cases} 1, \text{ if } \jmath, \ell \in \chi \\ 0, \text{ otherwise} \end{cases} \quad \text{and } \eta(\jmath, \ell) = \begin{cases} \frac{1}{2}, \text{ if } \jmath, \ell \in \chi \\ 0, \text{ otherwise} \end{cases}.$$

If $\jmath, \ell \in \chi$. Clearly $\alpha(\jmath, \ell) \geq \eta(\jmath, \ell)$ then

$$\begin{aligned} 0 &= Đ(S2, S3)^{þ} \leq \lambda \left[Đ(2,3)^{þ\beta+q\alpha} \cdot Đ(2, S2)^{þ(1-\beta-\gamma)+q(\gamma-\alpha)} \cdot Đ(3, S3)^{þ\gamma - q\gamma} \right] \\ &= \lambda \left[(1)^{þ\beta+q\alpha} \cdot Đ(2,0)^{þ(1-\beta-\gamma)+q(\gamma-\alpha)} \cdot Đ(3,0)^{þ\gamma-q\gamma} \right] \\ &= \lambda \left[(4)^{þ(1-\beta-\gamma)+q(\gamma-\alpha)} \cdot \left(\frac{9}{4}\right)^{þ\gamma-q\gamma} \right] \end{aligned}$$

By taking any value of constants $\lambda \in [0,1)$, $\alpha, \beta, \gamma \in (0,1)$ and $þ, q \in [1, \infty)$. Clearly, (1) holds for all $\jmath, \ell \in \chi \backslash Fix(S)$. Keep in mind that S has two fixed points, 0 and 1. see for more information and examples [18].

Now public and substantiate the key theorem.

Theorem 2.1. *Let $(\chi, Đ)$ be an F-Complete FMS and S is an interpolative convex Reich type α-η-contraction assuring the accompanying conditions:*

(i) S is an α-admissible in term of η;
(ii) there is a $\jmath_0 \in \chi$ such that $\alpha(\jmath_0, S\jmath_0) \geq \eta(\jmath_0, S\jmath_0)$;
(iii) S is α-η-continuously.
Consequently, S attains a fixed point in χ.

Proof. Let $\jmath_0$ in χ such that $\alpha(\jmath_0, S\jmath_0) \geq \eta(\jmath_0, S\jmath_0)$. For $\jmath_0 \in \chi$, we construct a sequence $\{\jmath_n\}_{n=1}^{\infty}$ such that $\jmath_1 = S\jmath_0$, $\jmath_2 = S\jmath_1 = S^2\jmath_0$. Continue this approach until for every $n \in \mathbb{N}, \jmath_{n+1} = S\jmath_n = S^{n+1}\jmath_0$. . Because of (i), S is an α-admissible in term of η after that $\alpha(\jmath_0, \jmath_1) = \alpha(\jmath_0, S\jmath_0) \geq \eta(\jmath_0, S\jmath_0) = \eta(\jmath_0, \jmath_1)$. By carrying out this procedure further, we have

$$\alpha(\jmath_{n-1}, \jmath_n) \geq \eta(\jmath_{n-1}, S\jmath_{n-1}) = \eta(\jmath_{n-1}, \jmath_n), \text{ for all } n \in \mathbb{N}. \tag{2}$$

On condition that $\jmath_{n+1} = \jmath_n$ a few $n \in \mathbb{N}$, afterwards $\jmath_n = \jmath^*$, is a fixed point of S. Thus, we presume $\jmath_n \neq \jmath_{n+1}$ with

$$Đ(S\jmath_{n-1}, S\jmath_n) = Đ(\jmath_n, S\jmath_n) > 0, \text{ for all } n \in \mathbb{N}.$$

Since S is an interpolative convex Reich-type α-η-contraction, for any $n \in \mathbb{N}$, give us

$$\begin{aligned} Đ(\jmath_n, \jmath_{n+1})^{þ} &= Đ(S\jmath_{n-1}, S\jmath_n)^{þ} \\ &\leq \lambda \left[Đ(\jmath_{n-1}, \jmath_n)^{þ\beta+q\alpha} \cdot Đ(\jmath_{n-1}, S\jmath_{n-1})^{þ(1-\beta-\gamma)+q(\gamma-\alpha)} \cdot Đ(\jmath_n, S\jmath_n)^{þ\gamma-q\gamma}\right], \\ &= \lambda \left[Đ(\jmath_{n-1}, \jmath_n)^{þ\beta+q\alpha} \cdot Đ(\jmath_{n-1}, \jmath_n)^{þ(1-\beta-\gamma)+q(\gamma-\alpha)} \cdot Đ(\jmath_n, \jmath_{n+1})^{þ\gamma-q\gamma}\right], \\ &= \lambda \left[Đ(\jmath_{n-1}, \jmath_n)^{þ(1-\gamma)+q\gamma} \cdot Đ(\jmath_n, \jmath_{n+1})^{þ\gamma-q\gamma}\right], \end{aligned}$$

we acquire

$$Đ(\jmath_n, \jmath_{n+1})^{þ(1-\gamma)+q\gamma} \leq \lambda Đ(\jmath_{n-1}, \jmath_n)^{þ(1-\gamma)+q\gamma}. \tag{3}$$

Afterward decide that $\{Đ(\jmath_{n-1}, \jmath_n)\}$ is decreasing terms. As a result there is a positive term ϱ s.t. $\lim_{n\to\infty} Đ(\jmath_{n-1}, \jmath_n) = \varrho$. Take note $\varrho \geq 0$, if we extrapolate use (2.3), gives

$$Đ(\jmath_n, \jmath_{n+1}) \leq \lambda Đ(\jmath_{n-1}, \jmath_n) \leq \lambda^n Đ(\jmath_0, \jmath_1).$$

Which provide

$$m > n, \sum_{i=n}^{m-1} Đ(\jmath_i, \jmath_{i+1}) \leq \frac{\lambda^n}{1-\lambda} Đ(\jmath_0, \jmath_1).$$

Subsequently

$$\lim_{n\to+\infty} \frac{\lambda^n}{1-\lambda} Đ(\jmath_0, \jmath_1) = 0.$$

There are some $N \in \mathbb{N}$ therefore

$$N \leq n, \Rightarrow \; 0 < \frac{\lambda^n}{1-\lambda} Đ(\jmath_0, \jmath_1) < \delta.$$

Let $\epsilon > 0$ be fixed and $(g, \mu) \in \mathcal{F} \times \mathbb{R}^+$ be satisfied the $(Đ_3)$. Next to $(\mathcal{F}_2)$, there exist $\delta > 0$ such that

$$0 < v < \delta \text{ suggests } g(v) < g(\epsilon) - a. \tag{4}$$

Due to (4) and $(\mathcal{F}_1)$, we obtained

$$g\left[\sum_{i=n}^{m-1} Đ(\jmath_i, \jmath_{i+1})\right] \leq g\left[\frac{\lambda^n}{1-\lambda} Đ(\jmath_0, \jmath_1)\right] < g(\epsilon) - \mu, \tag{5}$$

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where $Đ(\jmath_n, \jmath_m) > 0$ and $m, n \in \mathbb{N}$ in like a way that $m > n \geq N$. Consequently, combining (5) and $(Đ_3)$ gives that

$$g\left[Đ(\jmath_m, \jmath_n)\right] \leq g\left[\sum_{i=n}^{m-1} Đ(\jmath_i, \jmath_{i+1})\right] + \mu < g(\epsilon),$$

which suggest that $(\mathcal{F}_1)$, we obtain

$$Đ(\jmath_m, \jmath_n) < \epsilon, m > n \geq N.$$

As a result $\{\jmath_n\}$ is an $\mathcal{F}$-Cauchy sequence. There exists $\jmath^* \in \chi$ such that $\jmath_n$ is $\mathcal{F}$-convergent to $\jmath^*$, because $(\chi, Đ)$ is a F-Complete MS that is,

$$\lim_{n\to\infty} Đ(\jmath_n, \jmath^*) = 0. \tag{6}$$

S is α-η-continuous and has the properties $\alpha(\jmath_{n-1}, \jmath_n) \geq \eta(\jmath_{n-1}, \jmath_n)$, as every $n \in \mathbb{N}$ afterwards apply limit approaches to infinity $\jmath_{n+1} = S\jmath_n \to S\jmath^*$ a certian $\jmath^* = S\jmath^*$. We will now demonstrate that $\jmath^*$ is a fixed point of S. We use contradiction to argue by assuming that $Đ(S\jmath^*, \jmath^*) > 0$. (Đ3), gives us

$$g(Đ(S\jmath^*, \jmath^*)) \leq g\left(Đ(S\jmath^*, S\jmath_n)^{\text{þ}} + Đ(S\jmath_n, \jmath^*)\right) + \mu,\ n \in \mathbb{N}.$$

By using $(\mathcal{F}_1)$ and the contractive condition gives

$$g(Đ(S\jmath^*, \jmath^*)) \leq g\left(\lambda Đ(\jmath^*, \jmath_n)^{\text{þ}\beta+q\alpha} \cdot Đ(S\jmath^*, \jmath^*)^{\text{þ}(1-\beta-\gamma)+q(\gamma-\alpha)} \cdot Đ(\jmath_n, \jmath_{n+1})^{\text{þ}\gamma-q\gamma} + Đ(\jmath_{n+1}, \jmath^*)\right) + \mu,$$

for every $n \in \mathbb{N}$. In otherway, by using $(\mathcal{F}_2)$ together with (6), possess

$$\lim_{n\to\infty} g\left(\lambda Đ(\jmath^*, \jmath_n)^{\text{þ}\beta+q\alpha} + Đ(\jmath_{n+1}, \jmath^*)\right) + \mu = -\infty,$$

that result in a contradiction. In light of the fact that $Đ(S\jmath^*, \jmath^*) = 0$. Finally, $\jmath^*$ is a fixed point of S. □

Theorem 2.2. *The hypothesis of Theorem 2.1 holds true with assertion (i) to (ii).*

(iii) if any sequence $\{\jmath_n\}$ in χ such that $\alpha(\jmath_n, \jmath_{n+1}) \geq \eta(\jmath_n, \jmath_{n+1})$ together $\lim_{n\to\infty} \jmath_n = \jmath^*$ at that occasion $\alpha(\jmath_n, \jmath^*) \geq \eta(\jmath_n, \jmath^*)$ hold for every $n \in \mathbb{N}$.

Consequently, S posses a fixed point in χ.

Proof. In a manner similar to the proof of Theorem 2.3. We obtain $\alpha(\jmath_n, \jmath^*) \geq \eta(\jmath_n, \jmath^*)$ for every $n \in \mathbb{N}$. (D3) gives us

$$g\left(Đ(S\jmath^*, \jmath^*)\right) \leq g\left(Đ(S\jmath^*, S\jmath_n) + Đ(\jmath_n, \jmath^*)\right) + \mu.$$

(1) and $(\mathcal{F}_1)$, give us

$$\begin{aligned} g\left(Đ(S\jmath^*, \jmath^*)\right) &\leq g\left(\left(Đ(S\jmath^*, S\jmath_n)^{\text{þ}}\right) + Đ(S\jmath_n, \jmath^*)\right) + \mu \\ &\leq g\left(\lambda\left[Đ(\jmath^*, \jmath_n)^{\text{þ}\beta+q\alpha} \cdot Đ(\jmath^*, S\jmath^*)^{\text{þ}(1-\beta-\gamma)+q(\gamma-\alpha)} \cdot Đ(\jmath_n, \jmath_{n+1})^{\text{þ}\gamma-q\gamma}\right] + Đ(\jmath_n, \jmath^*)\right) + \mu. \end{aligned}$$

Using (6) information

$$\lim_{n\to\infty} Đ(\jmath_n, \jmath^*) = 0 = \lim_{n\to\infty} Đ(\jmath_{n+1}, \jmath^*),$$

we obtain

$$g\left(Đ(\jmath^*, S\jmath^*)\right) \leq g\left(Đ(\jmath^*, S\jmath^*)\right) + \mu$$

Using $(\mathcal{F}_2)$,gives that

$$\lim_{n\to\infty} g\left(Đ(\jmath^*, S\jmath^*)\right) + \mu = -\infty,$$

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that result in a contradiction. In light of the fact that $Ð(\jmath^*, S\jmath^*) = 0$ it is an established point $\jmath^*$ possess fixed point of S. □

Example 3. *Assume* $\chi = \mathbb{R}$ *to FMS* $Ð$: $\chi \times \chi \to \mathbb{R}^+$ *by*

$$Ð(\jmath, \ell) = \begin{cases} (\jmath - \ell)^2, \text{ if } (\jmath, \ell) \in \mathbb{N} \times \mathbb{N} \\ |\jmath - \ell|, \text{ if } (\jmath, \ell) \notin \mathbb{N} \times \mathbb{N}, \end{cases}$$

with $\mu = \ln(100)$ *and* $f(v) = \ln(v)$. *Define* $S : \chi \to \chi$ *by*

$$S\jmath = \begin{cases} 1 - \frac{\jmath}{2}, \text{ if } \jmath \in \mathbb{N} \\ 0, \quad \text{if } \jmath \notin \mathbb{N} \end{cases}$$

and $\alpha, \eta : \chi \times \chi \to [0, +\infty)$ *by*

$$\alpha(\jmath, \ell) = \begin{cases} 2, \text{ if } \jmath, \ell \in [0, \infty) \\ 0, \text{ otherwise} \end{cases} \text{ and } \eta(\jmath, \ell) = \begin{cases} 1, \text{ if } \jmath, \ell \in [0, \infty) \\ 0, \text{ otherwise} \end{cases}.$$

Case 1: If $\jmath = \ell$. Evidently $Ð(\jmath, \ell) = 0$.
As a result, Theorem 2.1's requirements are all met.
Case 2: If $\jmath, \ell$ are in $\mathbb{N}$, but $S\jmath \notin \mathbb{N}, S\ell \notin \mathbb{N}$, then

$$Ð(S\jmath, S\ell)^{þ} = Ð(1 - \frac{\jmath}{2}, 1 - \frac{\ell}{2}) = \left[\frac{1}{2}|\jmath - \ell|\right]^{þ}.$$

It is evident that S is an α-admissible in term of η for whenever $\alpha(\jmath, \ell) \geq \eta(\jmath, \ell)$, implies

$$Ð(S\jmath, S\ell)^{þ} = \left[\frac{1}{2}|\jmath - \ell|\right]^{þ} \leq \lambda \left[(\jmath - \ell)^{2þ\beta + 2q\alpha} \cdot \left|\frac{3}{2}\jmath - 1\right|^{þ(1-\beta-\gamma)+q(\gamma-\alpha)} \cdot \left|\frac{3}{2}\ell - 1\right|^{þ\gamma - q\gamma}\right],$$

by taking constants $\lambda \in [0, 1)$, $þ, q \in [1, \infty)$ and $\alpha, \beta, \gamma \in (0, 1)$, for all $\jmath, \ell \in \mathbb{N} \backslash Fix(S)$.
Although (i) either $\jmath$ nor ℓ are in $\mathbb{N}$, gives

$$Ð(S\jmath, S\ell)^{þ} = 0.$$

whenever $\alpha(\jmath, \ell) \geq \eta(\jmath, \ell)$,It is evidently S is an α-admissible mapping regard to η, such that

$$Ð(S\jmath, S\ell)^{þ} = 0 \leq \lambda \left[|\jmath - \ell|^{þ\beta + q\alpha} \cdot |\jmath|^{þ(1-\beta-\gamma)+q(\gamma-\alpha)} \cdot |\ell|^{þ\gamma - q\gamma}\right],$$

where $\lambda \in [0, 1)$, $þ, q \in [1, \infty)$ and $\alpha, \beta, \gamma \in (0, 1)$, for all $\jmath, \ell \in \mathbb{N} \backslash Fix(S)$.
(ii). One belong to $\mathbb{N}$ other outside of $\mathbb{N}$

$$Ð(S\jmath, S\ell)^{þ} = Ð(1 - \frac{\jmath}{2}, 0)^{þ} = \left|1 - \frac{\jmath}{2}\right|^{þ}.$$

It is evident that S is an α-admissible mapping w.r.t η for whenever $\alpha(\jmath, \ell) \geq \eta(\jmath, \ell)$, such that

$$Ð(S\jmath, S\ell)^{þ} = \left|1 - \frac{\jmath}{2}\right|^{þ} \leq \lambda \left[|\jmath - \ell|^{þ\beta + q\alpha} \cdot \left|\frac{3}{2}\jmath - 1\right|^{þ(1-\beta-\gamma)+q(\gamma-\alpha)} \cdot |\ell|^{þ\gamma - q\gamma}\right],$$

by taking constants $\lambda \in [0, 1)$, $þ, q \in [1, \infty)$ and $\alpha, \beta, \gamma \in (0, 1)$, for all $\jmath, \ell \in \mathbb{N} \backslash Fix(S)$.
As a result, our Theorem 2.1's requirements are all met. Thus S is convex interpolative Reich-type α-η-contraction as a result.

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Definition 2.2. *Assume that* $(\chi,Đ)$ *is an FMS and* $\alpha, \eta : \chi \times \chi \to [0, +\infty)$ *be two functions. The FMS in* χ *is considered to be* α*-*η*-complete iff each* $\mathcal{F}$*-Cauchy sequence* $\{\jmath_n\}$ *must contain*

$$\alpha(\jmath_n, \jmath_{n+1}) \geq \eta(\jmath_n, \jmath_{n+1}) \text{ as each } n \in \mathbb{N}.$$

in χ, $\mathcal{F}$ *-converges.*

Remark 2.1. *The Theorem 2.1 and 2.2 also apply to* α*-*η*-complete FMS instead of FCFMS (see for more information [10]).*

3. Convex Interpolative Kannan-type α-η-contraction

In this stage, we develop several fixed point theorems inside the positioning of F-Complete FMS and provide new convex interpolative Kannan-type contractions. The following is an explanation of interpolative convex Kannan type α-η-contraction:

Definition 3.1. *Let* $(\chi,Đ)$ *is an FMS. Let there are two functions* $\alpha, \eta : \chi \times \chi \to [0, +\infty)$ *and* $S : \chi \to \chi$*. If there are constants* $\lambda \in [0, 1)$ *and* $\alpha, \beta \in (0, 1)$ *such that whenever* $\alpha(\jmath, \ell) \geq \eta(\jmath, \ell)$*, we say that S is an convex interpolative convex Kannan-type* α*-*η*-contraction.*

$$\left[Đ(S\jmath, S\ell)^{þ+q}\right] \leq \lambda \left[Đ(\jmath, S\jmath)\right]^{þ(1-\beta)+q\alpha} \cdot \left[Đ(\ell, S\ell)\right]^{þ\beta+q(1-\alpha)}, \tag{7}$$

where $þ, q \in [1, \infty)$ *for all* $\jmath, \ell \in \chi$ *with* $\jmath \neq S\jmath$*.*

Now present and prove our second important theorem.

Theorem 3.1. *The hypotheses of Theorem 2.1 hold true with assertion (i) to (iii). Let S is a convex interpolative Kannan-type* α*-*η*-contraction.*

Consequently, S possess a fixed point in χ.

Proof. The same procedures are used in the proof of Theorem 3.1.

Since S is convex interpolative Kannan-type α-η-contraction, give us

$$\begin{aligned} Đ(S\jmath_{n-1}, S\jmath_n)^{þ+q} &\leq \lambda \left[Đ(\jmath_{n-1}, S\jmath_{n-1})\right]^{þ(1-\beta)+q(1-\alpha)} \cdot \left[Đ(\jmath_n, S\jmath_n)\right]^{þ\beta+q\alpha} \\ &= \lambda \left[Đ(\jmath_{n-1}, \jmath_n)\right]^{þ(1-\beta)+q(1-\alpha)} \cdot \left[Đ(\jmath_n, \jmath_{n+1})\right]^{þ\beta+q\alpha}. \end{aligned} \tag{8}$$

It is implied from equation (8) that

$$\left[Đ(\jmath_n, \jmath_{n+1})\right]^{þ(1-\beta)+q(1-\alpha)} \leq \lambda \left[Đ(\jmath_{n-1}, \jmath_n)\right]^{þ(1-\beta)+q(1-\alpha)}. \tag{9}$$

The rest of the proof follows the structure of Theorem 2.1 and proceeds in a similar manner. □

Theorem 3.2. *Theorem 3.1 from (i) to (ii) and Theorem 2.2 solely (iii) are satisfying hypotheses.*

S thus has a fixed point in χ.

Proof. Carried out in a manner similar to that of Theorem 2.2. (7) and (F1) give us

$$g\left(Đ(S\jmath^*, \jmath^*)^{þ+q}\right) \leq g\left(\lambda\left(Đ(\jmath^*, S\jmath^*)^{þ(1-\beta)+q(1-\alpha)} \cdot Đ(\jmath^*, \jmath_n)^{þ\beta+q\alpha}\right) + Đ(S\jmath_n, \jmath^*)\right) + \mu$$

Utilizing (6) and the information

$$\lim_{n\to\infty} Đ(\jmath_n, \jmath^*) = 0 = \lim_{n\to\infty} Đ(\jmath_{n+1}, \jmath^*).$$

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We achieve

$$g(Đ(\jmath^*, S\jmath^*)) \leq g(Đ(\jmath^*, S\jmath^*)) + \mu,$$

This is incongruous. $Đ(\jmath^*, S\jmath^*) = 0$ as a result, meaning that is a fixed point of S. □

The following corollaries derive from the Theorems 2.1, 2.2, 3.1, and 3.2 if $\eta(\jmath, \ell) = 1$.

Corollary 3.1. *Let $(\chi, Đ)$ be an F-Complete FMS and S be an interpolative convex Reich type α-η-contraction assuring the accompanying assertions:*

(i) S is an α-admissible;
(ii) there is a $\jmath_0 \in \chi$ such that $\alpha(\jmath_0, S\jmath_0) \geq 1$;
(iii) S is continuous.
Consequently, S has a fixed point in χ.

Corollary 3.2. *Assertions (i) through (ii) in Corollary 3.1 are true.*

(iii) if any sequence $\{\jmath_n\}$ in χ such that $\alpha(\jmath_n, \jmath_{n+1}) \geq 1$ together $\lim_{n\to\infty} \jmath_n = \jmath^*$ at that occasion $\alpha(\jmath_n, \jmath^*) \geq 1$ satisfy for every $n \in \mathbb{N}$.
S therefore has a fixed point.

Corollary 3.3. *Assertions (i) through (iii) in Corollary 3.1 are true. Assuming S is a convex interpolative Kannan type contraction, the following claims can be made:*

S thus has a fixed point in χ.

Corollary 3.4. *Assertions (i) through (iii) in Corollary 3.1 are true. Assuming S is a convex interpolative Kannan type contraction, the following claims can be made:*

(iii) if any sequence $\{\jmath_n\}$ in χ such that $\alpha(\jmath_n, \jmath_{n+1}) \geq 1$ together $\lim_{n\to\infty} \jmath_n = \jmath^*$ at that occasion $\alpha(\jmath_n, \jmath^*) \geq 1$ satisfy for every $n \in \mathbb{N}$.
S therefore has a fixed point in χ.

4. Findings

Our findings lead to some conclusions on Suzuki contractions, orbitally S-complete, and continuous maps in FMS.

Theorem 4.1. *Let S be a continuous self-map on χ and $(\chi, Đ)$ be a F-Complete FMS. Assume $r \in [0,1)$ and $\alpha, \beta, \gamma \in (0,1)$ are present and in such a way that*

$$Đ(\jmath, S\jmath) \leq Đ(\jmath, \ell) \implies Đ(S\jmath, S\ell)^{\flat} \leq r\,[Đ(\jmath, \ell)]^{\flat\beta + q\alpha} \cdot [Đ(\jmath, S\jmath)]^{\flat(1-\beta-\gamma)+q(\gamma-\alpha)} \cdot [Đ(\ell, S\ell)]^{\flat\gamma - q\gamma},$$

where $\flat, q \in [1, \infty)$, for every $\jmath, \ell \in \chi$.

Consequently, S has a fixed point in χ.

Proof. Set two functions $\alpha, \eta : \chi \times \chi \to [0, +\infty)$ by

$$\alpha(\jmath, \ell) = Đ(\jmath, \ell) \text{ and } \eta(\jmath, \ell) = Đ(\jmath, \ell), \text{ for all } \jmath, \ell \in \chi,$$

and $\beta, \gamma \in (0,1)$, and $r \in [0,1)$. It is clear that

$$\eta(\jmath, \ell) \leq \alpha(\jmath, \ell), \text{ for all } \jmath, \ell \in \chi,$$

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that is, our Theorem 2.1's criteria (i) through (iii) are satisfied. Let

$$\eta(\jmath, S\jmath) \leq \alpha(\jmath, \ell) \text{ then } Ð(\jmath, S\jmath) \leq Ð(\jmath, \ell),$$

it suggests a contractive condition

$$Ð(S\jmath, S\ell)^{þ} \leq r\,[Ð(\jmath, \ell)]^{þ\beta+q\alpha}\,.\,[Ð(\jmath, S\jmath)]^{þ(1-\beta-\gamma)+q(\gamma-\alpha)}\,.\,[Ð(\ell, S\ell)]^{þ\gamma-q\gamma}\,.$$

As a result, Theorem 3's criteria are all satisfied. Hence S attain a fixed point in $\jmath$. □

Theorem 4.2. *Suppose a continuous map S and $(\chi,Ð)$ be a F-Complete FMS. Assume $r \in [0,1)$ and $\alpha, \beta \in (0,1)$ are present and in such a way that*

$$Ð(\jmath, S\jmath) \leq Ð(\jmath, \ell) \implies Ð(S\jmath, S\ell)^{þ+q} \leq r\,[Ð(\jmath, S\jmath)]^{þ(1-\beta)+q(1-\alpha)}\,.\,[Ð(\ell, S\ell)]^{þ\beta+q\alpha}$$

where $þ, q \in [1,\infty)$, for all $\jmath, \ell \in \chi$.

Therefore S attain a fixed point.

Corollary 4.1. *Suppose a continuous map S and $(\chi,Ð)$ be a F-Complete FMS. Assume $r \in [0,1)$ in such a way that*

$$Ð(\jmath, S\jmath) \leq Ð(\jmath, \ell) \implies Ð(S\jmath, S\ell) \leq rÐ(\jmath, \ell),$$

for all $\jmath, \ell \in \chi$. Then S possess a fixed point.

Theorem 4.3. *Suppose S a self-map and $(\chi,Ð)$ be a FMS in χ. Surmise the given claims are true:*

(i) $(\chi,Ð)$ is an orbitally S-complete FMS;
(ii) $r \in [0,1)$ and $\alpha, \beta, \gamma \in (0,1)$ exist such that

$$Ð(S\jmath, S\ell)^{þ} \leq r\,[Ð(\jmath, \ell)]^{þ\beta+q\alpha}\,.\,[Ð(\jmath, S\jmath)]^{þ(1-\beta-\gamma)+q(\gamma-\alpha)}\,.\,[Ð(\ell, S\ell)]^{þ\gamma-q\gamma}$$

where $þ, q \in [1,\infty)$ for all $\jmath, \ell \in O(\omega)$ for some $\omega \in \chi$, where $O(\omega)$ is an orbit of ω;
where $O(\omega)$ is an orbit of ω, and $þ, q$ are in $[1,\infty)$ for every $\jmath, \ell \in O(\omega)$ and for some $\omega \in \chi$;
(iii) if $\{\jmath_n\}$ is a sequence where $\{\jmath_n\} \subseteq O(\omega)$ along $\lim_{n\to\infty} \jmath_n = \jmath^*$ then $\jmath^* \in O(\omega)$.

As a result, S has a fixed point.

Proof. Set $\alpha, \eta : \chi \times \chi \to [0,+\infty)$, by $\alpha(\jmath, \ell) = 3$ on $O(\omega) \times O(\omega)$ and $\alpha(\jmath, \ell) = 0$ otherwise and $\eta(\jmath, \ell) = 1$ for all $\jmath, \ell \in \jmath$(see Remark 6 [10]). Then $(\chi,Ð)$ is an α-η-complete $\mathcal{F}$-metric and S is an α-admissible regard to η. Let $\alpha(\jmath, \ell) \geq \eta(\jmath, \ell)$, later $\jmath, \ell \in O(\omega)$, afterwards from (ii) give us

$$Ð(S\jmath, S\ell)^{þ} \leq r\,[Ð(\jmath, \ell)]^{þ\beta+q\alpha}\,.\,[Ð(\jmath, S\jmath)]^{þ(1-\beta-\gamma)+q(\gamma-\alpha)}\,.\,[Ð(\ell, S\ell)]^{þ\gamma-q\gamma}$$

That is S is Interpolative convex α-η-contraction of the Reich-type. Let a sequence $\{\jmath_n\}$ applies it read $\alpha(\jmath_n, \jmath_{n+1}) \geq \eta(\jmath_n, \jmath_{n+1})$ and $\lim_{n\to\infty} \jmath_n = \jmath^*$. Therefore $\{\jmath_n\} \subseteq O(\omega)$. The expression it taken from (iii) $\jmath^* \in O(\omega)$, $\alpha(\jmath_n, \jmath^*) \geq \eta(\jmath_n, \jmath^*)$. As a result, Theorem 2.2's criteria are all fulfilled. S therefore has a fixed point. □

Theorem 4.4. *Similar to Theorem 4.3's hypotheses, satisfies*

$$Ð(S\jmath, S\ell)^{þ+q} \leq r\,[Ð(\jmath, S\jmath)]^{þ(1-\beta)+q(1-\alpha)}\,.\,[Ð(\ell, S\ell)]^{þ\beta+q\alpha}$$

Therefore S attain a fixed point.

Theorem 4.5. *Let S be a self map and $(\chi,Đ)$ a FCFMS. Suppose the subsequent claims are true:*

(i) for all $\jmath,\ell \in O(\omega)$, there exists $r \in [0,1)$ and $\alpha,\beta,\gamma \in (0,1)$, $þ,q \in [1,\infty)$ such that

$$Đ(S\jmath,S\ell)^{þ} \leq r\,[Đ(\jmath,\ell)]^{þ\beta+q\alpha}\,.\,[Đ(\jmath,S\jmath)]^{þ(1-\beta-\gamma)+q(\gamma-\alpha)}\,.\,[Đ(\ell,S\ell)]^{þ\gamma-q\gamma},$$

for some $\omega \in \chi$;

(ii) S is orbitally continuous.

Afterwards S possess a fixed point.

Proof. Define $\alpha,\eta : \chi\times\chi \to [0,+\infty)$, by $\alpha(\jmath,\ell) = 3$ on $O(\omega)\times O(\omega)$ and $\alpha(\jmath,\ell) = 0$ otherwise and $\eta(\jmath,\ell) = 1$(see Remark 1.1 [11]), we know S is α-η-continuous map. Assume $\alpha(\jmath,\ell) \geq \eta(\jmath,\ell)$, afterwards $\jmath,\ell \in O(\omega)$. Therefore $S\jmath, S\ell \in O(\omega)$ that is $\alpha(S\jmath,S\ell) \geq \eta(S\jmath,S\ell)$. In light of, S is therefore a mapping that is α-admissible. We have from (i)

$$Đ(S\jmath,S\ell)^{þ} \leq r\,[Đ(\jmath,\ell)]^{þ\beta+q\alpha}\,.\,[Đ(\jmath,S\jmath)]^{þ(1-\beta-\gamma)+q(\gamma-\alpha)}\,.\,[Đ(\ell,S\ell)]^{þ\gamma-q\gamma}.$$

That is to say, S is a Reich-type interpolative convex α-η-contraction. As a result, Theorem 2.1's entire premise is true. S therefore attains a fixed point. □

Theorem 4.6. *Theorem 4.5 (i) through (ii)hypotheses's are true then*

$$Đ(S\jmath,S\ell)^{þ+q} \leq r\,[Đ(\jmath,S\jmath)]^{þ(1-\beta)+q(1-\alpha)}\,.\,[Đ(\ell,S\ell)]^{þ\beta+q\alpha}$$

Consequently S attains a fixed point.

Corollary 4.2. *Let $(\chi,Đ)$ be an F-Complete FMS and S be a self map. Assuming the assertions are true:*

(i) there exist $r \in [0,1)$ such that for every $\jmath,\ell \in O(\omega)$,

$$Đ(S\jmath,S\ell) \leq r\,(Đ(\jmath,\ell))$$

for some $\omega \in \chi$;

(ii) S is orbitally continuous.

As a result, S possess a fixed point.

5. Application

Recent research has shown that the local and nonlocal fractional differential equations are useful tools for simulating a wide range of phenomena in a variety of scientific and architectural domains. Numerous fields, including viscoelasticity etc., make use of the fractional order differential equations. For more information, see [3,6,22]. Fractional generlized derivative in sense of Riemann involving a boundary condition, we want to demonstrate the existence and uniqueness of a bounded solution.

The left Riemann Liouville fraction of a Lebesgue integrable function g regards to an increasing function h is provided by [15].

$${}_aI_h^{\alpha}\, g(v) = \frac{1}{\Gamma(\alpha)}\int_a^v (h(v)-h(\omega))^{\alpha-1}\, f(\omega)h'(\omega)d\omega, \text{ where } \alpha > 0. \tag{10}$$

With regard to the identical rising function h, the related left Riemann Liouville fractional derivative of g is given by [15]

$${}_aĐ_h^{\alpha} g(v) = \left(\frac{1}{h'(v)}\frac{d}{dv}\right)^n I^{(n-\alpha)} g(v)$$

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$$= \left(\frac{1}{h'(v)}\frac{d}{dv}\right)^n \frac{1}{\Gamma(\alpha)} \int_a^v (h(v)-h(\omega))^{n-\alpha-1}\, g(\omega)h'(\omega)d\omega, \tag{11}$$

where $[\alpha]$ is the largest integer, $\alpha \geq 0$ and $n = [\alpha]+1$. The fractional integral and fractional derivative are combined in the following theorem.

Theorem 5.1. *[13] Let $\alpha > 0$, $n = -[-\alpha], g \in L[c,d]$ and ${}_aI_h^\alpha g \in AC_h^n[c,d]$. Then*

$${}_aI_{ha}^\alpha Đ_h^\alpha g(v) = g(v) - \sum_{k=1}^{n} c_k\,(h(v)-h(a))^{\alpha-k}.$$

We are thinking about the ensuing boundary value problem

$${}_cĐ_h^\alpha \ell(v) + g(v,\ell(v)) = 0, \text{ with } \ell(c) = \ell(d) = 0, \text{ where } 1 < \alpha \leq 2. \tag{12}$$

Lemma 5.1. *Let $\alpha > 0$, $n = -[-\alpha], g \in L[c,d]$ and ${}_cI_h^\alpha g \in AC_h^n[c,d]$ exist. If and only if, ℓ is a solution to the boundary value problem (12),*

$$\ell(v) = \int_c^d \aleph(\omega,v)\, g(\omega,\ell(\omega))h'(\omega)d\omega,$$

where the Greens' function

$$\aleph(\omega,v) = \frac{1}{\Gamma(\alpha)} \begin{cases} \left(\frac{(h(d)-h(\omega))(h(v)-h(c))}{(h(d)-h(c))}\right)^{\alpha-1} - (h(v)-h(\omega))^{\alpha-1}, & c < \omega \leq v \\ \left(\frac{(h(d)-h(\omega))(h(v)-h(c))}{(h(d)-h(c))}\right)^{\alpha-1}, & v \leq \omega < d \end{cases}$$

satisfies the following:

- $\aleph(\omega,v) \geq 0.$
- $\max_{c\leq\omega,v\leq d} \aleph(\omega,v) = \frac{1}{\Gamma(\alpha)}\left(\frac{h(d)-h(c)}{4}\right)^{\alpha-1}.$

Proof. Apply the integral (10) to (12) we get

$${}_cI_{hc}^\alpha Đ_h^\alpha \ell(v) = -\,{}_cI_h^\alpha\, g(v,\ell(v)) = -\frac{1}{\Gamma(\alpha)} \int_c^v (h(v)-h(\omega))^{\alpha-1}\, g(\omega)h'(\omega)d\omega,$$

using Theorem 5.1 we obtain

$$\ell(v) = c_1\,(h(v)-h(c))^{\alpha-1} + c_2\,(h(v)-h(c))^{\alpha-2} - \frac{1}{\Gamma(\alpha)} \int_c^v (h(v)-h(\omega))^{\alpha-1}\, g(\omega)h'(\omega)d\omega,$$

$$\ell(c) = 0, \text{ gives } c_2 = 0.$$

$$\begin{aligned} \ell(d) &= 0, \text{ gives} \\ c_1 &= \frac{(h(d)-h(c))^{1-\alpha}}{\Gamma(\alpha)} \int_c^d (h(d)-h(\omega))^{\alpha-1}\, g(\omega,\ell(\omega))h'(\omega)d\omega. \end{aligned}$$

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Therefore

$$\begin{aligned}\ell(v) &= \frac{1}{\Gamma(\alpha)}\int_c^d \frac{(h(d)-h(\omega))\,(h(v)-h(c))^{\alpha-1}}{(h(d)-h(c))}\,g(\omega,\ell(\omega))h'(\omega)d\omega \\ &-\frac{1}{\Gamma(\alpha)}\int_c^v (h(v)-h(\omega))^{\alpha-1}\,g(\omega,\ell(\omega))h'(\omega)d\omega.\end{aligned}$$

Hence

$$\ell(v)=\int_c^d \aleph(\omega,v)\,g(\omega,\ell(\omega))h'(\omega)d\omega,\ \text{ where}$$

$$\aleph(\omega,v)=\frac{1}{\Gamma(\alpha)}\begin{cases}\left(\frac{(h(d)-h(\omega))(h(v)-h(c))}{(h(d)-h(c))}\right)^{\alpha-1}-(h(v)-h(\omega))^{\alpha-1}, & c<\omega\le v\\ \left(\frac{(h(d)-h(\omega))(h(v)-h(c))}{(h(d)-h(c))}\right)^{\alpha-1}, & v\le\omega<d\end{cases}.$$

There is no doubt that $\aleph(\omega,v)\geq 0$, for $\omega\geq v$.

For example, $c\leq\omega<v$, one can demonstrateIt is clear that $\aleph(\omega,v)\geq 0$, when $\omega\geq v$.

$$\aleph(\omega,v)=\left(\frac{(h(v)-h(c))}{(h(d)-h(c))}\right)^{\alpha-1}\left[(h(d)-h(c))^{\alpha-1}-\left(h(d)-\left(h(c)+\frac{(h(\omega)-h(c))\,(h(d)-h(c))}{(h(v)-h(c))}\right)^{\alpha-1}\right)\right].$$

Since

$$h(c)+\frac{(h(\omega)-h(c))\,(h(d)-h(c))}{(h(v)-h(c))}\geq h(\omega),$$

It follows that $\aleph(\omega,v)\geq 0$, where $\omega\leq v$ and for $v\leq\omega$,

$$\frac{\partial\aleph}{\partial v}=\frac{1}{\Gamma(\alpha)}\left(\frac{h(d)-h(\omega)}{(h(d)-h(c))}\right)^{\alpha-1}\cdot(\alpha-1)\,(h(v)-h(c))^{\alpha-2}h'(v)\geq 0,$$

thus $\aleph(\omega,v)$ is rising in proportion to v.

Now $\omega\leq v$

$$\frac{\partial\aleph}{\partial v}=\frac{h'(v)\,(\alpha-1)}{\Gamma(\alpha)}\cdot\left[-(h(v)-h(\omega))^{\alpha-2}+\left(\frac{h(d)-h(\omega)}{h(d)-h(c)}\right)^{\alpha-1}(h(v)-h(c))^{\alpha-2}\right]$$

$$=\frac{h'(v)}{\Gamma(\alpha-1)}\cdot\left(\frac{h(v)-h(c)}{h(d)-h(c)}\right)^{\alpha-2}\left[\left(\frac{h(d)-h(\omega)}{h(d)-h(c)}\right)^{\alpha-1}-\left(\frac{(h(d)-h(c))\,(h(v)-h(\omega))}{h(d)-h(c)}\right)^{\alpha-2}\right]$$

$$\begin{aligned}&\leq \frac{h'(v)}{\Gamma(\alpha-1)}\cdot\left(\frac{h(v)-h(c)}{h(d)-h(c)}\right)^{\alpha-2}[(h(d)-h(c))^{\alpha-2}\\ &-\left(h(d)-\left(h(c)+\frac{h(d)-h(c)}{h(v)-h(c)}(h(\omega)-h(c))\right)^{\alpha-2}\right)]\\ &< 0.\end{aligned}$$

Thus $\aleph(\omega,v)$ is decreasing when $\omega\leq v$. Therefore, at $\omega=v$, $\aleph(\omega,v)$ reaches its maximum.

$$\aleph(\omega,\omega)=\frac{1}{\Gamma(\alpha)}\frac{(h(d)-h(\omega))^{\alpha-1}\,(h(\omega)-h(c))^{\alpha-1}}{(h(d)-h(c))^{\alpha-1}}=\hat{G}(\omega)$$

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$$\begin{aligned}\hat{G}'(\omega) &= -\frac{1}{\Gamma(\alpha)}(\alpha-1)\frac{(h(d)-h(\omega))^{\alpha-2}}{(h(d)-h(c))^{\alpha-1}}h'(\omega)\cdot(h(\omega)-h(c))^{\alpha-1} \\ &\quad+\frac{1}{\Gamma(\alpha)}\frac{(h(d)-h(\omega))^{\alpha-1}(\alpha-1)(h(\omega)-h(c))^{\alpha-2}h'(\omega)}{(h(d)-h(c))^{\alpha-1}} \\ &= 0\end{aligned}$$

yields

$$h(\omega)=\frac{h(c)+h(d)}{2},$$

or the critical point

$$\omega^* = h^{-1}\left(\frac{h(c)+h(d)}{2}\right).$$

Therefore, $h(\omega, v)$ maximum's value is

$$\check{\aleph}(\omega^*)=\frac{1}{\Gamma(\alpha)}\left(\frac{h(d)-h(c)}{4}\right)^{\alpha-1},$$

$$|\aleph(\omega,v)|\leq\frac{1}{\Gamma(\alpha)}\left(\frac{h(d)-h(c)}{4}\right)^{\alpha-1}.$$

The Riemann Stieltjes integrable function of w w.r.t ω and g is denoted as follows: A continuous function $g:[0,1]\times\mathbb{R}\to\mathbb{R}$. Assume that $\hat{C}_i$ is the linear space of all continuous functions defined on $I=[0,1]$, and that

$$Đ(w,v)=\|w-v\|_\infty^2=\max_{v\in I}|w(v)-v(v)|^2 \text{ for every } w,v\in\hat{C}_i.$$

So $(\hat{C}_i, D)$ is a metric space that is F-Complete.

We take into account the following situations:

(a) there exist $r\in[0,1)$, $\zeta:\mathbb{R}^2\to\mathbb{R}$ is a function for each $c,d\in\mathbb{R}$ with $\zeta(c,d)\geq\xi(c,d)$, such that

$$|g(\omega,w(\omega))d\omega-g(\omega,v(\omega))d\omega|^{þ}\leq|w(\omega)-v(\omega)|^{2(þ\gamma+q\beta)}\cdot|w(\omega)-Sw(\omega)|^{2(þ(1-\gamma-\hat{w})+q(\hat{w}-\beta))}\cdot|v(\omega)-vv(\omega)|^{2(þ\hat{w}-q$$

where $þ,q\geq[1,\infty)$, $\beta,\gamma,\hat{w}\in(0,1)$, ;

(b) for every $w_1\in\hat{C}_i$ there exists such that

$$\zeta\left(w_1(v),\int_c^d\aleph(v,\omega)g(\omega,w_1(\omega))h'(\omega)d\omega\right)\geq\xi\left(w_1(v),\int_c^d\aleph(v,\omega)g(\omega,w_1(\omega))h'(\omega)d\omega\right),$$

satisfies for each $v\in I$.

(c) There exists a $w_1,v_1\in\hat{C}_i$ for each $w,v\in\hat{C}_i$, such that

$$\begin{aligned}\zeta(w(v),v(v)) &\geq \xi(w(v),v(v)) \text{ implies } \zeta\left(\int_c^d\aleph(v,\omega)g(\omega,w_1(\omega))h'(\omega)d\omega,\int_c^d\aleph(v,\omega)g(\omega,v_1(\omega))h'(\omega)d\omega\right) \\ &\geq \xi\left(\int_c^d\aleph(v,\omega)g(\omega,w_1(\omega))h'(\omega)d\omega,\int_c^d\aleph(v,\omega)g(\omega,v_1(\omega))h'(\omega)d\omega\right),\end{aligned}$$

holds for all values of $v\in I$.

(d) for any group of points w in a sequence $\{w_n\}$ of points in $\hat{C}_i$ will have

$$\zeta(w_n,w_{n+1})\geq\xi(w_n,w_{n+1}),\ \lim_{n\to\infty}\inf\zeta(w_n,w)\geq\lim_{n\to\infty}\inf\xi(w_n,w).$$

Theorem 5.2. *Assume that the conditions (a) through (d) are met. So (12) has at least one $w\in\hat{C}_i$ solution.*

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Proof. We know that $w \in \hat{C}_i$ is a solution of the fractional order integral equation, if and only $w \in \hat{C}_i$ is a solution of (5.3),

$$w(v) = \lambda \int_a^b \aleph(v,s) g(s, w(s)) h'(s) ds \text{ for all } v \in I,$$

where $0 \leq \lambda < 1$. Define a map $S : \hat{C}_i \to \hat{C}_i$ by

$$Sw(v) = \lambda \int_a^b \aleph(v,s) g(s, w(s)) h'(s) ds \text{ for all } v \in I.$$

Then, solving problem (12) is identical to discovering $w^* \in \hat{C}_i$, a fixed point of S. Let $w, v \in \hat{C}_i$, be such that for all $v \in I$, $\zeta(w(v), v(v)) \geq 0$. Using (a), we obtain

$$\begin{aligned} |Sw(v) - Sv(v)|^p &= \left| \lambda \int_a^b \aleph(v,s) \left[g(s, w(s)) - g(s, v(s)) \right] h'(s) ds \right|^p \\ &\leq |\lambda| \int_a^b |\aleph(v,s)| \left| g(s, w(s)) - g(s, v(s)) h'(s) ds \right|^p \end{aligned}$$

$$\leq |\lambda| \int_a^b |\aleph(v,s)| \, h'(s) r ds \, |w(s) - v(s)|^{2(p\gamma + q\beta)} \cdot |w(s) - Sw(s)|^{2(p(1-\gamma-\hat{w}) + q(\hat{w}-\beta))} \cdot |v(s) - vv(s)|^{2(p\hat{w} - q\hat{w})}$$

$$\begin{aligned} &\leq \frac{1}{\Gamma(\alpha)} \left(\frac{h(b) - h(a)}{4} \right)^{\alpha - 1} (h(b) - h(a)) \, \| w(s) - v(s) \|_\infty^{2(p\gamma + q\beta)} \\ &\quad \cdot \| w(s) - Sw(s) \|_\infty^{2(p(1-\gamma-\hat{w}) + q(\hat{w}-\beta))} \cdot \| v(s) - Sv(s) \|_\infty^{2(p\hat{w} - q\hat{w})} \end{aligned}$$

$$\leq r \, \| w(s) - v(s) \|_\infty^{2(p\gamma + q\beta)} \cdot \| w(s) - Sw(s) \|_\infty^{2(p(1-\gamma-\hat{w}) + q(\hat{w}-\beta))} \cdot \| v(s) - Sv(s) \|_\infty^{2(p\hat{w} - q\hat{w})}.$$

Thus

$$D(Sw, Sv)^p < \left\{ |w(s) - v(s)|^{2(p\gamma + q\beta)} \cdot |w(s) - Sw(s)|^{2(p(1-\gamma-\hat{w}) + q(\hat{w}-\beta))} \cdot |v(s) - Sv(s)|^{2(p\hat{w} - q\hat{w})} \right\}$$

holds for each $w, v \in \hat{C}_i$ such that $\zeta(w(v), v(v)) \geq \xi(w(v), v(v))$ for each $v \in I$.

We define $\alpha : \hat{C}_i \times \hat{C}_i \to [0, \infty)$ by

$$\alpha(w, v) = \left\{ \begin{array}{ll} 2, & \text{if } \zeta(w(v), v(v)) \geq 0, \ v \in I, \\ 0, & \text{otherwise} \end{array} \right\} \text{ and } \eta(w, v) = \left\{ \begin{array}{ll} \frac{1}{3}, & \text{if } \xi(w(v), v(v)) \geq 0, \ v \in I, \\ 0, & \text{otherwise} \end{array} \right\}$$

Then, for all $w, v \in \hat{C}_i$, $\alpha(w, v) \geq \eta(w, v)$, we have

$$D(Sw, Sv)^p \leq r \left\{ |w(s) - v(s)|^{2(p\gamma + q\beta)} \cdot |w(s) - Sw(s)|^{2(p(1-\gamma-\hat{w}) + q(\hat{w}-\beta))} \cdot |v(s) - Sv(s)|^{2(p\hat{w} - q\hat{w})} \right\}.$$

Obviously, $\alpha(w, v) \geq \eta(w, v)$ for every $w, v \in \hat{C}_i$. If $\alpha(w, v) \geq \eta(w, v)$ for each $w, v \in \hat{C}_i$, then $\zeta(w(v), v(v)) \geq \xi(w(v), v(v))$.

From (c), we have $\zeta(Sw(v), Sv(v)) \geq \xi(Sw(v), Sv(v))$ and so $\alpha(Sw, Sv) \geq \eta(Sw, Sv)$.

Thus, S is α-admissible map concerning η.

From (b) there subsist $w_1 \in \hat{C}_i$ parallel to $\alpha(w_1, Sw_1) = \eta(w_1, Sw_1)$.

By (d), we know that any group of points in a sequence $\{w_n\}$ of points in $\hat{C}_i$ with w will have

$$\alpha(w_n, w_{n+1}) = \eta(w_n, w_{n+1}),$$

and

$$\lim_{n\to\infty} \inf \alpha(w_n, w) = \lim_{n\to\infty} \inf \eta(w_n, w).$$

By using Theorem 2.1, it can be shown that S attains a fixed point in $\hat{C}_i$. Finally w^* is a solution to the equation $Sw^* = w^*$ in $\hat{C}_i$ (12). □

Conclusion:

In the context of FMS, this study focuses on a novel notion of convex interpolative contraction of the Reich and Kannan type that is more inclusive than standard metric. Results for the Suzuki type fixed point are driven in the FMS. In order to demonstrate our theorems and as an application, we find a solution to the fractional differential equation problem. These new studies and uses would increase the effectiveness of the new arrangement.

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