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Article

CALCULATION OF THE ELECTROSTATIC FIELD OF A CIRCULAR CYLINDER WITH A SLOT BY THE WIENER-HOPF METHOD

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Abstract: The article considers an internal boundary value problem of the distribution of an electrostatic field in a lens formed by two identical semi-infinite circular cylinders coaxially located inside an infinite external cylinder. The problem is reduced to solving a system of singular Wiener-Hopf integral equations, which is further solved by the Wiener-Hopf method using factorized Bessel functions. Solutions to the problem for each region inside the infinite outer cylinder are presented as exponentially converging series in terms of eigenfunctions and eigenvalues.

Keywords: time-of-flight mass spectrometer; electron microscope; electrostatic lens; electrostatic mirror; relativistic effect; system of singular integral equations; factorized functions; eigenfunctions; eigenvalues.

1. Introduction

Electrostatic mirrors have become indispensable structural elements of modern scientific and technological instrumentation, which determine the quality of focusing of such instruments as time-of-flight mass spectrometers and electron microscopes. In this regard, electrostatic mirrors of rotational symmetry are of particular interest, since they perform stigmatic focusing of electron beams, i.e. create the correct electron-optical image of the object. However, the most studied and highly-demanded in practical implementation are the designs of mirrors constructed as sets of coaxial circular cylinders. The advantage of cylindrical electrodes is the possibility of shielding the beam from external electric fields. In works [1,2] devoted to the study of the focusing properties of electrostatic mirrors with cylindrical electrodes, the calculation was performed under the assumption that the width of the gap between the electrodes is infinitely small. However, practical application of such mirrors in high-voltage electron microscopy [3,4] imposes high requirements on the width of the gap between the electrodes in terms of ensuring electrostatic strength at high field intensity. The aim of the work is to calculate the electrostatic field in a lens formed by two identical semi-infinite coaxially-located circular cylinders, separated by gaps (slits) of finite width and located inside an infinite outer cylinder. Such an electrode design makes it possible to simultaneously provide electrostatic strength at high field intensities and screening of the electron beam from external electric fields at large gap widths between the internal electrodes.

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In this work, for the first time, the Wiener-Hopf (WH) method is used to solve the boundary value problem of an electrostatic lens taking into account the slit width in it. The boundary value problem, as a rule, is reduced to solving pairwise integral equations with kernels of Bessel functions, which were studied by L.A. Weinstein [5], Titchmarsh [6], Noble [7], Erdelyi and Sneddon [8] and others.

A comprehensive review of the historical development of pairwise integral equations is given by Eswaran [9] and Sneddon [10], where they are reduced to a system of algebraic equations, or to a Fredholm type equation. The methodology for solving paired integral equations is considered in detail in the works of N.N. Lebedev [11], V.A. Fock, P.L. Kapitsa and L.A. Weinstein[12].

In these works pairwise integral equations describing the problem of a conducting hollow cylinder of finite length were reduced to the Fredholm integral equation of the second kind [13], or solved by the variational method when the length of the cylinder is large enough compared to its diameter [14,15]. However, the proposed methods are very cumbersome and require a large amount of computational time.

It is known that the WH method [5,7,9,16–20], like the Riemann method, is a rigorous method for solving pairwise singular integral equations (SIEs) for semi-infinite structures whose solutions automatically satisfy the additional Meixner condition or the so-called condition on sharp edge, which determines the uniqueness of the solution to the problem, as well as the behavior of the field at small distances from the sharp edge. Note that, as a rule, this condition is not mentioned in approximate methods.

It should be noted that N.N. Lebedev [11] used the WH method to solve the boundary value problem of the electrostatic field of an electron lens consisting of a semi-infinite circular cylinder coaxially located inside an infinite cylinder, which is a key problem for solving a number of other problems. However, a well-known powerful WH method has not been used since then, even for an electrode system with two semi-infinite cylinders.

2. Statement of the problem

Let us consider a lens consisting of two thin semi-infinite cylinders of radius a with given potentials V_1 and V_2, coaxially located towards each other inside a shielding infinite cylinder of radius b, which is at zero potential (Fig. 1).

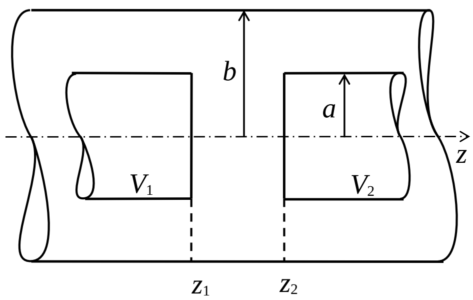

Figure 1. A cylinder with a slit

The desired potential $\varphi(r,z)$ satisfies the Laplace equation

$$\triangle\varphi(r,z) = 0$$

and boundary conditions

$$\varphi(b,z) = 0, \quad \varphi(a,z<a) = V_1, \quad \varphi(a,z>a) = V_2, \tag{1}$$

$$\varphi(a-0,z) - \varphi(a+0,z) = 0, \tag{2}$$

$$\left(\tfrac{\partial}{\partial r}\varphi(r-0,z) - \tfrac{\partial}{\partial r}\varphi(r+0,z)\right)\Big|_{r=a} = 0 \quad \text{at} \quad z_1 < z < z_2. \tag{3}$$

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Let us introduce the notation

$$
\begin{gathered}
L(r,w) = \frac{\pi}{2\ln\frac{a}{b}\mathrm{J}_0(vb)}\begin{cases}\mathrm{J}_0(vr)(a,b), & \text{at } 0 \le r \le a;\\ \mathrm{J}_0(va)(r,b), & \text{at } a \le r \le b,\end{cases}\\
(r,b) = \mathrm{N}_0(vr)\mathrm{J}_0(vb) - \mathrm{N}_0(vb)\mathrm{J}_0(vr),\\
v = \sqrt{k^2 - w^2}, \quad \mathrm{Im}v > 0,
\end{gathered} \tag{4}
$$

where $\mathrm{J}_0(vr)$, $\mathrm{N}_0(vr)$ – are zero-order Bessel and Neumann functions, (r,b) is a combination of Bessel functions, and search for a solution in the form

$$
\varphi(r,z) = \frac{1}{2\pi i}\int\limits_{-\infty}^{\infty} e^{iwz} L(r,w)\frac{F(w)}{L(a,w)}dw \tag{5}
$$

with respect to the desired function $F(w)$.

For electrostatic problems, k should be assumed to have a vanishingly small positive imaginary part, and we transfer to the limit $|k| \to 0$ only in finite expressions.

The cuts of the function L in (4) are located in the plane of the complex variable w on the curves $\mathrm{Im}v = 0$ (see Fig. 2).

Due to the properties of the Bessel functions and boundary conditions (1) and (3), we obtain a system of singular integral equations (SIE)

$$
\frac{1}{2\pi i}\int\limits_{-\infty}^{\infty} e^{iwz}F(w)dw = V_1, \quad z \le z_1, \tag{6}
$$

$$
\frac{1}{2\pi i}\int\limits_{-\infty}^{\infty} e^{iwz}F(w)dw = V_2, \quad z \ge z_2, \tag{7}
$$

$$
\int\limits_{-\infty}^{\infty} e^{iwz}L^{-1}(a,w)F(w)dw = 0, \quad z_1 < z < z_2. \tag{8}
$$

The last equation (8) can be obtained from the boundary condition (3) due to the property of the Bessel functions

$$
\frac{\partial}{\partial r}\left(\frac{\mathrm{J}_0(vr)(a,b)}{\mathrm{J}_0(vb)} - \frac{\mathrm{J}_0(va)(r,b)}{\mathrm{J}_0(vb)}\right)\Bigg|_{r=a} \equiv -\frac{2}{\pi a},
$$

.

3. Solution of a system of integral equations

The solution of SIE (6 – 8) is constructed by the Wiener-Hopf method in the form [21]

$$
\begin{aligned}
F(w) = L_-(a,w)\Big(A_+^1(w) + B_+^1(w)\Big)e^{-iwz_1} + \\
+L_+(a,w)\Big(A_-^2(w) + B_-^2(w)\Big)e^{-iwz_2}.
\end{aligned} \tag{9}
$$

Here, the functions with subscripts '+' correspond to holomorphic functions in the upper half-plane (UHP) $\mathrm{Im}w \ge 0$ and do not have zeros and poles there, while those with indices "–" satisfy the same conditions in the lower half-plane (LHP) $\mathrm{Im}w \le 0$, the functions L_+ and L_- are factorized functions, $L = L_- \cdot L_+$ [5,7].

It should be noted that for the SIE solution to be unique, the desired function must satisfy the Meixner condition [22,23] on the edge or the so-called sharp edge condition ($E_z, E_r \sim \rho^{-1/2}$, $E_\phi \sim \sqrt{\rho}$), which is equivalent to the behavior of the function $F(|w|) \sim |w|^{-3/2}$ at infinity $|w| \to \infty$.

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Note that L_+ has zeros at the points $-w_n^a$, $-w_n^c$ and poles $-w_n^b$ $(n = 1, 2, \dots)$ in the LHP of the complex variable w. The function L_- has the same zeros and poles only in the UHP, due to the property of factorized functions

$$L_+(a, -w) = L_-(a, w). \tag{10}$$

The roots of the functions $\mathrm{J}_0(va)$, $\mathrm{J}_0(vb)$, and (a, b) with respect to the variable v in (4) are $v_n^a = \gamma_n/a$; $v_n^b = \gamma_n/b$ and $v_n^c = \delta_n/(b-a)$ $(n = 1, 2, \dots)$, where γ_n and δ_n denote the corresponding roots [24] of the equations

$$\mathrm{J}_0(\gamma) = 0,$$
$$\mathrm{N}_0\left(\tfrac{a\delta}{b-a}\right)\mathrm{J}_0\left(\tfrac{b\delta}{b-a}\right) - \mathrm{N}_0\left(\tfrac{b\delta}{b-a}\right)\mathrm{J}_0\left(\tfrac{a\delta}{b-a}\right) = 0.$$

Relation (6) will be satisfied due to the function A_+^1 if the function $F(w)$ is holomorphic everywhere in LHP ($\mathrm{Im}w \leq 0$) except for a single simple pole at the point $w = -k$ and uniformly tends to zero as $|w| \to \infty$. Therefore, the remaining poles contained in the LHP at the points $w = -w_n^b (n = 1, 2, \dots)$ of the function L_+ must be compensated using the function $B_+^1(w)$.

To find a solution to equation (7), we require the same conditions for the functions $A_-^2(w)$ and $B_-^2(w)$ in the UHP ($\mathrm{Im}w \geq 0$).

Thus, using the theory of residues to calculate the integrals (6) and (7), as well as compensating all singular points inside the integration contour (IC), except for the poles $\pm k$, we obtain the desired functions in (9):

$$A_+^1(w) = -\frac{V_1}{w+k}, \tag{11}$$

$$A_-^2(w) = \frac{V_2}{w-k} \quad (0 < \mathrm{Im}(k),\ |k| \to 0), \tag{12}$$

$$\begin{cases} B_+^1(w) = -\displaystyle\sum_{n=1}^{\infty} \frac{e^{iw_n^b(z_2-z_1)}}{(w+w_n^b)} \frac{L_+^*(a,-w_n^b)}{L_-(a,-w_n^b)} \left(A_-^2(-w_n^b) + B_-^2(-w_n^b)\right), \\ B_-^2(w) = -\displaystyle\sum_{n=1}^{\infty} \frac{e^{iw_n^b(z_2-z_1)}}{(w-w_n^b)} \frac{L_-^*(a,w_n^b)}{L_+(a,w_n^b)} \left(A_+^1(w_n^b) + B_+^1(w_n^b)\right), \end{cases} \tag{13}$$

where

$$L_+^*(a,-w_n^b) = \lim_{w\to -w_n^b}(w+w_n^b)L_+(a,w) =$$
$$= -\lim_{w\to w_n^b}(w-w_n^b)L_-(a,w) = -L_-^*(a,w_n^b).$$

The validity of the obtained solution of SIE (6) – (8) can be checked directly by substituting it into the equation and closing the IC in the LHP or UHP w, according to the Jordan lemma, then calculating the residues at all poles of the integrand inside this IC.

It should be noted that the resulting solution (9) automatically satisfies (8), since the integrand turns out to be holomorphic inside the corresponding IC.

3.1. Solution of a system of functional equations

The exact solution of the system of functional equations (13) can be represented in the form of rapidly convergent infinite series

$$
\begin{aligned}
B_+^1(w) = -V_2\Bigg(\sum_{n_1}^{\infty}\frac{g_{n_1}}{(w+w_{n_1}^b)l^{(1)}} + \sum_{n_1,n_2,n_3}^{\infty}\frac{g_{n_1}g_{n_2}g_{n_3}}{(w+w_{n_3}^b)l^{(3)}} + \dots \\
+ \sum_{n_1,\dots,n_{2i-1}}^{\infty}\frac{\prod_{k=1}^{2i-1} g_k}{(w+w_{2i-1}^b)l^{(2i-1)}}\Bigg) - V_1\Bigg(\sum_{n_1,n_2}^{\infty}\frac{g_{n_1}g_{n_2}}{(w+w_{n_2}^b)l^{(2)}} + \\
\sum_{n_1,n_2,n_3,n_4}^{\infty}\frac{g_{n_1}g_{n_2}g_{n_3}g_{n_4}}{(w+w_{n_4}^b)l^{(4)}} + \dots \sum_{n_1,\dots,n_{2i}}^{\infty}\frac{\prod_{k=1}^{2i} g_k}{(w+w_{2i}^b)l^{(2i)}}\Bigg),
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
B_-^2(w) = V_1\Bigg(\sum_{n_1}^{\infty}\frac{g_{n_1}}{(w-w_{n_1}^b)l^{(1)}} + \sum_{n_1,n_2,n_3}^{\infty}\frac{g_{n_1}g_{n_2}g_{n_3}}{(w-w_{n_3}^b)l^{(3)}} + \dots \\
+ \sum_{n_1,\dots,n_{2i-1}}^{\infty}\frac{\prod_{k=1}^{2i-1} g_k}{(w+w_{2i-1}^b)l^{(2i-1)}}\Bigg) + V_2\Bigg(\sum_{n_1,n_2}^{\infty}\frac{g_{n_1}g_{n_2}}{(w-w_{n_2}^b)l^{(2)}} + \\
+ \sum_{n_1,n_2,n_3,n_4}^{\infty}\frac{g_{n_1}g_{n_2}g_{n_3}g_{n_4}}{(w-w_{n_4}^b)l^{(4)}} + \dots \sum_{n_1,\dots,n_{2i}}^{\infty}\frac{\prod_{k=1}^{2i} g_k}{(w-w_{2i}^b)l^{(2i)}}\Bigg),
\end{aligned}
\tag{15}
$$

where the following notations are used

$$
\begin{aligned}
g_n = \frac{L_-^*(a,w_n^b)}{L_+(a,w_n^b)}e^{iw_n^b(z_2-z_1)}, \\
l^{(k)} = \underbrace{w_{n_1}^b(w_{n_1}^b+w_{n_2}^b)\cdots(w_{n_{k-1}}^b+w_{n_k}^b)}_{k}, \\
n_k = 1,2,\dots,\ k=1,2,\dots.
\end{aligned}
\tag{16}
$$

Factorized function $L_-^*(a,w_n^b)$ is calculated by formula (23). Indeed, system (13) can be easily divided into separate recursive equations

$$
\begin{aligned}
B_+^1(w) = -V_2\sum_{n_1=1}^{\infty}\frac{g_{n_1}}{(w+w_{n_1}^b)w_{n_1}^b} - V_1\sum_{n_1,n_2=1}^{\infty}\frac{g_{n_1}g_{n_2}}{(w+w_{n_1}^b)(w_{n_1}^b+w_{n_2}^b)w_{n_2}^b} + \\
+ \sum_{n_1,n_2=1}^{\infty}\frac{g_{n_1}g_{n_2}}{(w+w_{n_1}^b)(w_{n_1}^b+w_{n_2}^b)}B_+^1(w_{n_2}^b), \\
B_-^2(w) = V_1\sum_{n_1=1}^{\infty}\frac{g_{n_1}}{(w-w_{n_1}^b)w_{n_1}^b} + V_2\sum_{n_1,n_2=1}^{\infty}\frac{g_{n_1}g_{n_2}}{(w-w_{n_1}^b)(w_{n_1}^b+w_{n_2}^b)w_{n_2}^b} + \\
+ \sum_{n_1,n_2=1}^{\infty}\frac{g_{n_1}g_{n_2}}{(w-w_{n_1}^b)(w_{n_1}^b+w_{n_2}^b)}B_-^2(-w_{n_2}^b),
\end{aligned}
\tag{17}
$$

from which we directly obtain solutions in (17), cyclically using the equation itself in its right side.

4. Potential distribution in the lens

Substituting the resulting solution F into (5) and calculating the integral along the real axis w using the residue theory, we determine the potential φ.

Let us consider each case for different cylinder regions separately.

1. $z \le z_1$, $0 \le r \le a$.

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In this case, the IC should be closed in the LHP w. After making calculations and passing to the limit $k \to 0$, we obtain the potential distribution inside the semi-infinite cylinder

$$\varphi(r,z) = V_1 - \sum_{n=1}^{N} \frac{J_0(v_n^a r)}{J_0^*(-w_n^a a)} \{L_-(a,w)(A_+^1(w) + B_+^1(w))e^{iw(z-z_1)}\}_{w=-w_n^a} = V_1 - \sum_{n=1}^{N} e^{\frac{\gamma_n}{a}(z-z_1)} \frac{J_0(\gamma_n \frac{r}{a})}{\gamma_n J_1(\gamma_n)} L_+(a, i\tfrac{\gamma_n}{a})\Big(V_1 + i\frac{\gamma_n}{a} B_+^1(-i\tfrac{\gamma_n}{a})\Big). \quad (18)$$

Here we took into account that

$$J_0^*(-w_n^a a) = \lim_{w \to -w_n^b} (w + w_n^b)^{-1} J_0(va) = -iJ_1(v_n^b b) = -iJ_1(\gamma_n) \quad (19)$$

and the property of factorized functions $L_\pm$ in (10).

2. $z \le z_1$, $a \le r \le b$.

Integration along the real axis w in this case must also be closed, according to the Jordan lemma, in the LHP, then the integral can be easily transformed into a series of residues

$$\varphi(r,z) = V_1 \frac{\ln \frac{b}{r}}{\ln \frac{b}{a}} - \sum_{n=1}^{N} \left\{ \frac{(r,b)}{(a,b)^*} L_-(a,w)(A_+^1(w) + B_+^1(w))e^{iw(z-z_1)} \right\}_{-w_n^c} = V_1 \frac{\ln \frac{b}{r}}{\ln \frac{b}{a}} - \frac{\pi}{2} \sum_{n=1}^{N} \frac{J_0(\frac{\delta_n a}{b-a}) J_0(\frac{\delta_n b}{b-a})}{J_0^2(\frac{\delta_n b}{b-a}) - J_0^2(\frac{\delta_n a}{b-a})} (N_0(\tfrac{\delta_n r}{b-a}) J_0(\tfrac{\delta_n b}{b-a}) - N_0(\tfrac{\delta_n}{b-a} b) J_0(\tfrac{\delta_n r}{b-a})) L_+(a, i\tfrac{\delta_n}{b-a})\Big(V_1 + \frac{i\delta_n}{b-a} B_+^1(-i\tfrac{\delta_n}{b-a})\Big) e^{\frac{\delta_n}{b-a}(z-z_1)}. \quad (20)$$

Note that there is a transformation

$$(a,b)^*_{-w_n^c} = \lim_{w \to -w_n^c} (w + w_n^c)^{-1} (a,b) = -i\frac{2}{\pi v_n^c} \left(\frac{(a',b)}{(a',a)} - \frac{(a,b')}{(b,b')} \right)_{v_n^c} = -i\frac{2(b-a)}{\pi \delta_n} \left(\frac{J_0(\frac{\delta_n b}{b-a})}{J_0(\frac{\delta_n a}{b-a})} - \frac{J_0(\frac{\delta_n a}{b-a})}{J_0(\frac{\delta_n b}{b-a})} \right), \quad (21)$$

where in the derivation the Wronskian $(z', z) = 2/\pi z$ [24].

3. $z_1 \le z \le z_2$, $0 \le r \le b$.

Closing the IC in (5) for terms with an exponential factor $e^{iw(z-z_1)}$ according to the Jordan lemma in the UHP, and with a factor $e^{iw(z-z_2)}$ in the LHP w, taking into account all the contributions of the poles inside the IC, we similarly find the potential distribution in the slit region

$$\varphi(r,z) = \sum_{n=1}^{N} \frac{J_0(\gamma_n \frac{r}{b})}{J_0(\gamma_n \frac{a}{b})} L_-^*(a, w_n^b)\Big((-V_1/w_n^b + B_+^1(w_n^b))e^{iw_n^b(z-z_1)} + (V_2/w_n^b + B_-^2(-w_n^b))e^{-iw^b(z-z_2)}\Big),$$

or it can be written as

$$\varphi(r,z) = \frac{1}{\ln \frac{b}{a}} \sum_{n=1}^{N} \frac{J_0(\gamma_n \frac{r}{b}) J_0(\gamma_n \frac{a}{b})}{\gamma_n^2 J_1^2(\gamma_n)} L_+^{-1}(a, i\tfrac{\gamma_n}{b})\Big(e^{-\frac{\gamma_n}{b}(z-z_1)}(V_1 - \tfrac{i\gamma_n}{b} B_+^1(i\tfrac{\gamma_n}{b})) + e^{\frac{\gamma_n}{b}(z-z_2)}(V_2 - \tfrac{i\gamma_n}{b} B_-^2(-i\tfrac{\gamma_n}{b}))\Big), \quad (22)$$

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using the properties of the Bessel functions

$$\frac{(a,b)_{v_n^b}}{\mathrm{J}_0(\gamma_n \frac{a}{b})} = \frac{(b',b)_{v_n^b}}{\mathrm{J}_1(\gamma_n)} = -\frac{2}{\pi\gamma_n \mathrm{J}_1(\gamma_n)},$$

in the expression for

$$L_-^*(a,w_n^b) = \lim_{w\to w_n^b}(w-w_n^b)L_-(a,w) = \frac{i}{b\ln\frac{a}{b}}\frac{\mathrm{J}_0^2(\gamma_n\frac{a}{b})}{\gamma_n \mathrm{J}_1^2(\gamma_n)L_+(a,w_n^b)}. \quad (23)$$

4. $z_2 \le z, 0 \le r \le a$.
Similarly, closing the IC in the UHP of the complex variable w, we get

$$\varphi(r,z) = V_2 - \sum_{n=1}^{N} e^{-\frac{\gamma_n}{a}(z-z_2)}\frac{\mathrm{J}_0(\gamma_n\frac{r}{a})}{\gamma_n \mathrm{J}_1(\gamma_n)}L_+(a,i\tfrac{\gamma_n}{a}) \left(V_2 + i\frac{\gamma_n}{a}B_-^2(i\tfrac{\gamma_n}{a})\right). \quad (24)$$

5. $z_2 \le z, a \le r \le b$.
Deforming the IC upwards, we also obtain

$$\varphi(r,z) = V_2\frac{\ln\frac{b}{r}}{\ln\frac{b}{a}} - \frac{\pi}{2}\sum_{n=1}^{N}\frac{\mathrm{J}_0(\frac{\delta_n a}{b-a})\mathrm{J}_0(\frac{\delta_n b}{b-a})}{\mathrm{J}_0^2(\frac{\delta_n b}{b-a}) - \mathrm{J}_0^2(\frac{\delta_n a}{b-a})}L_+(a,i\tfrac{\delta_n}{b-a})(r,b)\big|_{v_n^c} \left(V_2 + \frac{i\delta_n}{b-a}B_-^2(i\tfrac{\delta_n}{b-a})\right)e^{-\frac{\delta_n}{b-a}(z-z_2)}. \quad (25)$$

5. Discussion

Thus, the exact solution (9) of the boundary value problem for the potential φ in (5) is found, where the auxiliary functions B_+^1 and B_-^2 are represented as rapidly convergent infinite series, as well as the factorized Bessel functions (see Appendix A).

As expected, when passing to the limit $z_1 \to -\infty$, when the end of the first semi-infinite cylinder is shifted by a considerable distance to the left, the expressions for the potential (22) – (25) coincide with the final results of N.N. Lebedev [11].

The numerical implementation of the factorized Bessel functions can be performed optimally using formula (A5) with a given accuracy, which is expressed through the functions P and Q [11] (Appendix A).

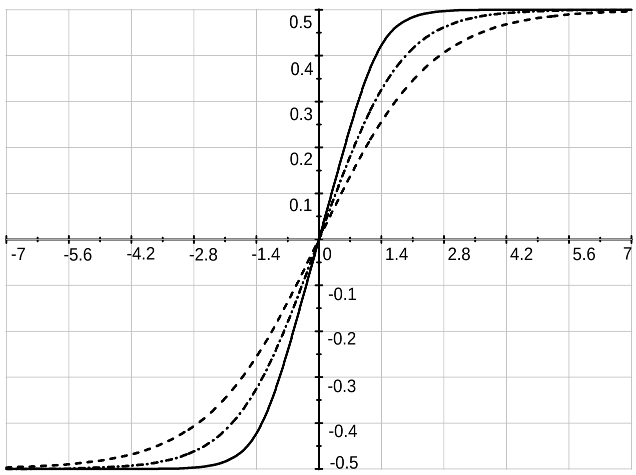

Figure 2. Potential distribution along the lens axis
$b = 6 : a = 1$ is a solid line, $a = 2$ is a dash-dotted line, $a = 3$ is a dashed line

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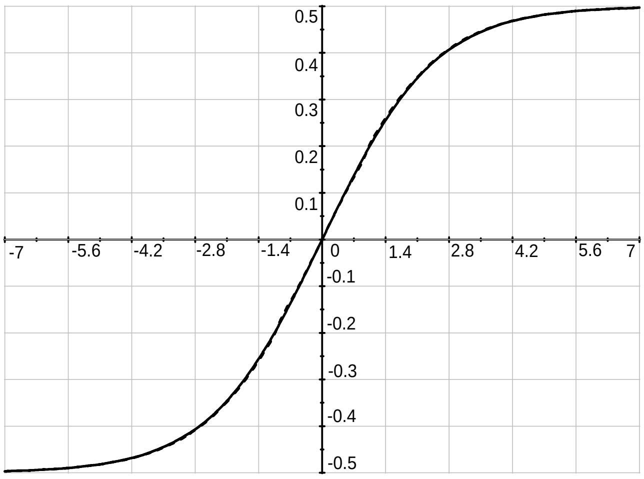

Figure 3. Influence of the outer cylinder radius on the potential distribution
$a = 3$: $b = 6$ is a solid line, $b = 12$ is a dashed line

It should be noted that the potential distribution is ultimately expressed by a real function φ expressed in the form of exponentially convergent series in (18) – (25).

Let us consider the distribution of the potential along the lens axis calculated by formulas (18), (22) and (24).

As can be seen from Fig. 2, as the radius of the semi-infinite cylinder increases, the steepness of the curve decreases.

It should be noted, as calculations show (see Fig. 3), that the radius of an infinite cylinder b has an insignificant effect on the potential distribution along the z axis.

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Abbreviations

The following abbreviations are used in this manuscript:

WH	Wiener-Hopf
SIE	Singular integral equation
UHP	Upper half-plane
LHP	Lower half-plane
IC	Integration contour

Appendix A. Factorization of Bessel functions

Using a standard decomposition of integer functions into factorized functions, we can represent the Bessel functions and their combinations as

$$L(a, w) = L(a, w)_+ L(a, w)_-,$$

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where [5,7]

$$L(a,w)_+ = \frac{\prod_{n=1}^{\infty}\left(1+\frac{w}{w_n^a}\right)e^{-\frac{w}{w_n^a}}\prod_{n=1}^{\infty}\left(1+\frac{w}{w_n^c}\right)e^{-\frac{w}{w_n^c}}}{\prod_{n=1}^{\infty}\left(1+\frac{w}{w_n^b}\right)e^{-\frac{w}{w_n^b}}}e^{-iw\left(T/\pi+(b-a)S\right)},$$

$$T = a\ln a + (b-a)\ln(b-a) - b\ln b,\ S = \sum_{n=1}^{\infty}\left(\frac{1}{\delta}_n - \frac{1}{\gamma}_n\right). \tag{A1}$$

As the function L_+ in its poles and zeros, which are imaginary, takes real values, it is convenient to express it through the gamma function

$$L(a,\alpha)_+ = \frac{e^{\alpha\frac{T}{\pi}}\,\Gamma(\frac{3}{4}+\alpha\frac{b}{\pi})}{\Gamma(\frac{3}{4}+\alpha\frac{a}{\pi})\Gamma(1+\alpha\frac{b-a}{\pi})} \prod_{n=1}^{\infty}\left(\frac{1+\alpha\frac{a}{\gamma_n}}{1+\alpha\frac{a}{\gamma_n'}}\right)\left(\frac{1+\alpha\frac{b-a}{\delta_n}}{1+\alpha\frac{b-a}{\pi n}}\right)\left(\frac{1+\alpha\frac{b}{\gamma_n}}{1+\alpha\frac{b}{\gamma_n'}}\right)^{-1}. \tag{A2}$$

Here, as can be seen, the fast convergence in infinite products occurs due to the asymptotics of the roots of the Bessel functions $\gamma_n' = \pi(n-\frac{1}{4})$ and $\delta_n' = \pi n$ $(n = 1, 2, \ldots)$. For convenience, denoting the infinite products in (A2) as

$$P(x) = \prod_{n=1}^{\infty}\frac{1+\frac{x}{\gamma_n}}{1+\frac{x}{\gamma_n'}} = \exp\sum_{n=1}^{\infty}\left(\ln(1+\tfrac{x}{\gamma_n}) - \ln(1+\tfrac{x}{\gamma_n'})\right), \tag{A3}$$

$$Q(x) = \prod_{n=1}^{\infty}\frac{1+\frac{x}{\delta_n}}{1+\frac{x}{\pi n}} = \exp\sum_{n=1}^{\infty}\left(\ln(1+\tfrac{x}{\delta_n}) - \ln(1+\tfrac{x}{\pi n})\right), \tag{A4}$$

we finally obtain the optimal formula for the numerical calculation with sufficient accuracy:

$$L(a,\alpha)_+ = \frac{e^{\alpha\frac{T}{\pi}}\,\Gamma(\frac{3}{4}+\alpha\frac{b}{\pi})}{\Gamma(\frac{3}{4}+\alpha\frac{a}{\pi})\Gamma(1+\alpha\frac{b-a}{\pi})}\frac{P(a\alpha)Q((b-a)\alpha)}{P(b\alpha)}$$

$$(L(a,+w)_\pm = L(a,-w)_\mp). \tag{A5}$$

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