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Some Topological Indices of Order Divisor Graphs of Cyclic Groups

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Abstract: We characterize the form of the order divisor graph of cyclic groups based on the group order. Furthermore, we describe some topological indices of the order divisor graph of cyclic groups, including the Wiener index, Harary index, first Zagreb index, and second Zagreb index.

Keywords: order divisor graphs; cyclic groups; Wiener index; Harary index; Zagreb index

1. Introduction

In 1878, Arthur Cayley represented groups in a graph called a Cayley graph [2]. The representation of groups in graphs has been growing and is now often referred to as algebraic graphs. In [11] the authors introduced prime graphs of finite groups. Do to the commutative concept, is also defined a non-commuting graph of non-abelian groups is defined [1]. In 2016, a non-coprime graph of finite groups was introduced which using the concept of elements order of the group [7]. Another algebraic graph that is using the order of the group elements is the order divisor graph of finite groups [9]. An order divisor graph of finite groups G, denoted by $OD(G)$, is a graph with vertex set G and two distinct vertices a and b of different orders are adjacent if and only if $|a|$ divides $|b|$ or $|b|$ divides $|a|$ where $|a|$ is the order of a, i.e. $|a|$ is a divisor of $|b|$ or $|b|$ is a divisor of $|a|$. More over in [10] it is explained signed total domination number of order divisor graphs of some finite groups. In [8], the generalization of order divisor graphs is given.

There are many studies on graph theory, one of which is related to topological indices (e.g., see [3,4,6,12]). Some well-known topological indices are the Wiener index, the Harary index, the first Zagreb index and the second Zagreb index. The Wiener index of graph Γ, denoted by $W(\Gamma)$, is defined as

$$W(\Gamma) = \sum_{u,v \in V(\Gamma)} d(u,v),$$

where $d(u,v)$ is the distance between u and v [4]. Analogous to the Wiener index, the Harary index of a graph Γ is defined as

$$H(\Gamma) = \sum_{u,v \in V(\Gamma)} \frac{1}{d(u,v)}$$

[3]. First Zagreb index of a simple connected graph Γ, denoted by $M_1(\Gamma)$, is defined as

$$M_1(\Gamma) = \sum_{v \in V(\Gamma)} (deg(v))^2,$$

where $deg(v)$ is the degree of v, i.e. the number of edges that incident to v [6]. The second Zagreb index of a graph Γ is defined as the sum of all the multiplications between the degrees of the two adjacent vertices, i.e.

$$M_2(\Gamma) = \sum_{uv \in E(\Gamma)} deg(u)deg(v)$$

[6].

So far, there is no investigation related to the Wiener index, Harary index, first Zagreb index, and second Zagreb index of the order divisor graph. Therefore, we are interested in examining these

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indices, particularly in the order divisor graphs of cyclic groups. In this paper, all concepts and notations related to groups refer to [5]. All groups in this paper are finite and the identity element is denoted by e.

2. Main Results

In this section, we will discuss several topological indices of an order divisor graph of a cyclic group of order n. The discussion is divided into several cases of n, i.e. for prime n, $n = p^k$, $n = p_1p_2$, and $n = p_1^{k_1}p_2^{k_2}\dots p_m^{k_m}$ where $p, p_1, p_2, \dots, p_m$ are prime.

2.1. For prime n.

Theorem 2.1. [9] Order divisor graph $OD(\mathbb{Z}_n)$ is a star graph if and only if n is a prime.

corollary 2.2. Let G be a cyclic group of order n. Order divisor graph $OD(G)$ is a star graph if and only if n is a prime.

The Wiener index, the Harary index, and the first and second Zagreb indices of the order divisor graph of cyclic groups of order prime n are given in the following theorems.

Theorem 2.3. Let G be a cyclic group of order n. If n is a prime, then the Wiener index of $OD(G)$ is $W(OD(G)) = (n-1)^2$.

Proof. Let G be a cyclic group of order prime number n. Vertex e is adjacent to all other vertices, so for any two vertices that are not adjacent $u, v \in V(OD(G)) \setminus \{e\}$ hold $d(u,v) = 2$. By Corollary 2.2, $OD(G)$ is a star graph, then we have

$$\begin{aligned}
W(OD(G)) &= \sum_{u,v \in V(OD(G))} d(u,v) \\
&= \sum_{v \in V(OD(G)) \setminus \{e\}} d(e,v) + \sum_{u,v \in V(OD(G)) \setminus \{e\}} d(u,v) \\
&= (n-1).1 + \binom{n-1}{2}.2 \\
&= (n-1) + (n-1)(n-2) \\
&= (n-1)^2.
\end{aligned}$$

□

Theorem 2.4. Let G be a cyclic group of order n. If n is a prime, then the Harary index of $OD(G)$ is $H(OD(G)) = (n-1)\left(\frac{n+2}{4}\right)$.

Proof. Let G be a cyclic group of order prime number n. Vertex e is adjacent to all other vertices, so for any two vertices that are not adjacent $u, v \in V(OD(G)) \setminus \{e\}$ hold $d(u,v) = 2$. By Corollary 2.2, $OD(G)$ is a star graph, then we have

$$\begin{aligned}
H(OD(G)) &= \sum_{u,v \in V(OD(G))} \frac{1}{d(u,v)} \\
&= \sum_{v \in V(OD(G)) \setminus \{e\}} \frac{1}{d(e,v)} + \sum_{u,v \in V(OD(G)) \setminus \{e\}} \frac{1}{d(u,v)} \\
&= (n-1).1 + \binom{n-1}{2}.\frac{1}{2} \\
&= (n-1) + \frac{(n-1)(n-2)}{4} \\
&= (n-1)\left(\frac{n+2}{4}\right).
\end{aligned}$$

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□

Theorem 2.5. Let G be a cyclic group of order n. If n is a prime, then the first Zagreb index of $OD(G)$ is $M_1(OD(G)) = n^2 - n$.

Proof. Let G be a cyclic group of order prime number n. Vertex e is adjacent to all other vertices, so $deg(e) = n - 1$. By Corollary 2.2, $OD(G)$ is a star graph, then we have $deg(v) = 1$ for all $v \in V(OD(G)) \setminus \{e\}$. Hence,

$$\begin{aligned} M_1(OD(G)) &= \sum_{v \in V(OD(G))} (deg(v))^2 \\ &= (deg(e))^2 + \sum_{v \in V(OD(G)) \setminus \{e\}} (deg(v))^2 \\ &= (n-1)^2 + (n-1).1^2 \\ &= n^2 - n. \end{aligned}$$

□

Theorem 2.6. Let G be a cyclic group of order n. If n is a prime, then the second Zagreb index of $OD(G)$ is $M_2(OD(G)) = (n-1)^2$.

Proof. Let G be a cyclic group of order prime number n. Vertex e is adjacent to all other vertices, so $deg(e) = n - 1$. By Corollary 2.2, $OD(G)$ is a star graph, then we have $deg(v) = 1$ for all $v \in V(OD(G)) \setminus \{e\}$. Hence,

$$\begin{aligned} M_2(OD(G)) &= \sum_{uv \in E(OD(G))} deg(u)deg(v) \\ &= \sum_{uv \in E(OD(G))} (n-1).1 \\ &= (n-1)^2. \end{aligned}$$

□

2.2. *For $n = p^k$.*

Theorem 2.7. [9] Let G be a cyclic group of order n. If $n = p^k$ for some prime number p and $k \in \mathbb{N}$, then order divisor graph $OD(G)$ is a complete $(k+1)$–partite graph

$$K_{1,p-1,p(p-1),p^2(p-1),\ldots,p^{k-1}(p-1)}.$$

According to Theorem 2.7, define the partitions $P_0, P_1, P_2, \ldots, P_k$ in $V(OD(G))$ such that

$$|P_0| = 1, |P_1| = p-1, |P_2| = p(p-1), \ldots, |P_k| = p^{k-1}(p-1).$$

The following theorems explain the Wiener index, Harary index, and the first and second Zagreb indices of an order divisor graph of a cyclic group with order $n = p^k$.

Theorem 2.8. Let G be a cyclic group of order n. If $n = p^k$ for some prime number p and $k \in \mathbb{N}$, then the Wiener index of $OD(G)$ is

$$W(OD(G)) = (p^k - 1) + \sum_{i=1}^{k} (p^{i-1}(p-1))(p^{i-1}(p-1) - 1) + \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} p^{i+j-2}(p-1)^2.$$

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Proof. Let G be a cyclic group of order $n = p^k$ for some prime number p and $k \in \mathbb{N}$. By Theorem 2.7, $OD(G)$ is a complete $(k+1)$–partite graph $K_{1,p-1,p(p-1),p^2(p-1),\ldots,p^{k-1}(p-1)}$. Thus, for every $u \in P_i$ and $v \in P_j$ we have $d(u,v) = 2$ if $i = j$ and $d(u,v) = 1$ if $i \neq j$ for all $i,j = 0,1,2,\ldots,k$. Therefore,

$$\begin{aligned}
W(OD(G)) &= \sum_{u,v \in V(OD(G))} d(u,v) \\
&= \sum_{i=1}^{k} \sum_{u,v \in P_i} d(u,v) + \sum_{i=0}^{k-1} \sum_{j=i+1}^{k} \sum_{u \in P_i} \sum_{v \in P_j} d(u,v) \\
&= \sum_{i=1}^{k} \binom{p^{i-1}(p-1)}{2}.2 + \sum_{i=0}^{k-1} \sum_{j=i+1}^{k} |P_i|.|P_j|.1 \\
&= \sum_{i=1}^{k} (p^{i-1}(p-1))(p^{i-1}(p-1)-1) + \sum_{j=1}^{k} |P_0|.|P_j| \\
&\quad + \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} (p^{i-1}(p-1))(p^{j-1}(p-1)) \\
&= (p^k - 1) + \sum_{i=1}^{k} (p^{i-1}(p-1))(p^{i-1}(p-1)-1) + \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} p^{i+j-2}(p-1)^2.
\end{aligned}$$

□

Theorem 2.9. Let G be a cyclic group of order n. If $n = p^k$ for some prime number p and $k \in \mathbb{N}$, then the Harary index of $OD(G)$ is

$$H(OD(G)) = (p^k - 1) + \sum_{i=1}^{k} \frac{(p^{i-1}(p-1))(p^{i-1}(p-1)-1)}{4} + \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} p^{i+j-2}(p-1)^2.$$

Proof. Let G be a cyclic group of order $n = p^k$ for some prime number p and $k \in \mathbb{N}$. By Theorem 2.7, $OD(G)$ is a complete $(k+1)$–partite graph $K_{1,p-1,p(p-1),p^2(p-1),\ldots,p^{k-1}(p-1)}$. Thus, for every $u \in P_i$ and $v \in P_j$ we have $d(u,v) = 2$ if $i = j$ and $d(u,v) = 1$ if $i \neq j$ for all $i,j = 0,1,2,\ldots,k$. Therefore,

$$\begin{aligned}
H(OD(G)) &= \sum_{u,v \in V(OD(G))} \frac{1}{d(u,v)} \\
&= \sum_{i=1}^{k} \sum_{u,v \in P_i} \frac{1}{d(u,v)} + \sum_{i=0}^{k-1} \sum_{j=i+1}^{k} \sum_{u \in P_i} \sum_{v \in P_j} \frac{1}{d(u,v)} \\
&= \sum_{i=1}^{k} \binom{p^{i-1}(p-1)}{2}.\frac{1}{2} + \sum_{i=0}^{k-1} \sum_{j=i+1}^{k} |P_i|.|P_j|.1 \\
&= \sum_{i=1}^{k} \frac{(p^{i-1}(p-1))(p^{i-1}(p-1)-1)}{4} + \sum_{j=1}^{k} |P_0|.|P_j| \\
&\quad + \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} (p^{i-1}(p-1))(p^{j-1}(p-1)) \\
&= (p^k - 1) + \sum_{i=1}^{k} \frac{(p^{i-1}(p-1))(p^{i-1}(p-1)-1)}{4} + \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} p^{i+j-2}(p-1)^2.
\end{aligned}$$

□

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Theorem 2.10. Let G be a cyclic group of order n. If $n = p^k$ for some prime number p and $k \in \mathbb{N}$, then the first Zagreb index of $OD(G)$ is

$$M_1(OD(G)) = (p^k - 1)^2 + \sum_{i=0}^{k-1} p^i(p-1)(p^k - p^i(p-1))^2.$$

Proof. Let G be a cyclic group of order $n = p^k$ for some prime number p and $k \in \mathbb{N}$. By Theorem 2.7, $OD(G)$ is a complete $(k+1)$−partite graph $K_{1,p-1,p(p-1),p^2(p-1),\ldots,p^{k-1}(p-1)}$. Thus, for all $i = 0,1,2,\ldots,k$, we have $deg(v) = p^k - |P_i|$, for every $v \in P_i$. Therefore,

$$\begin{aligned} M_1(OD(G)) &= \sum_{v \in V(OD(G))} (deg(v))^2 \\ &= \sum_{v \in P_0} (deg(v))^2 + \sum_{v \in P_1} (deg(v))^2 + \cdots + \sum_{v \in P_k} (deg(v))^2 \\ &= 1.(p^k - 1)^2 + (p-1)(p^k - (p-1))^2 + \cdots + p^{k-1}(p-1)(p^k - p^{k-1}(p-1))^2 \\ &= (p^k - 1)^2 + \sum_{i=0}^{k-1} p^i(p-1)(p^k - p^i(p-1))^2. \end{aligned}$$

□

Theorem 2.11. Let G be a cyclic group of order n. If $n = p^k$ for some prime number p and $k \in \mathbb{N}$, then the second Zagreb index of $OD(G)$ is

$$M_2(OD(G)) = \sum_{i=0}^{k-1} \sum_{j=i+1}^{k} |P_i||P_j|(p^k - |P_i|)(p^k - |P_j|).$$

Proof. Let G be a cyclic group of order $n = p^k$ for some prime number p and $k \in \mathbb{N}$. By Theorem 2.7, $OD(G)$ is a complete $(k+1)$−partite graph $K_{1,p-1,p(p-1),p^2(p-1),\ldots,p^{k-1}(p-1)}$. Thus, for all $i = 0,1,2,\ldots,k$, we have $deg(v) = p^k - |P_i|$, for every $v \in P_i$. Therefore,

$$\begin{aligned} M_2(OD(G)) &= \sum_{uv \in E(OD(G))} deg(u)deg(v) \\ &= \sum_{u \in P_0} \sum_{v \in V(OD(G)) \setminus P_0} deg(u)deg(v) + \sum_{u \in P_1} \sum_{v \in V(OD(G)) \setminus (P_0 \cup P_1)} deg(u)deg(v) \\ &\quad + \cdots + \sum_{u \in P_{k-1}} \sum_{v \in P_k} deg(u)deg(v) \\ &= \sum_{i=0}^{k-1} \sum_{j=i+1}^{k} |P_i||P_j|(p^k - |P_i|)(p^k - |P_j|). \end{aligned}$$

□

2.3. *For $n = p_1 p_2$.*

Theorem 2.12. Let G be a cyclic group of order n. If $n = p_1 p_2$ with p_1 and p_2 as distinct primes, then order divisor graph $OD(G)$ is complete tripartite graph $K_{1,p_1+p_2-2,(p_1-1)(p_2-1)}$.

Proof. Let G be a cyclic group of order $n = p_1 p_2$ with p_1 and p_2 as distinct primes. Take partitions Q_0, Q_1, Q_2 on $V(OD(G))$ with

$$\begin{aligned} Q_0 &= \{a \in G \mid |a| = 1\}, \\ Q_1 &= \{a \in G \mid |a| = p_1 \text{ atau } |a| = p_2\}, \\ Q_2 &= \{a \in G \mid |a| = p_1 p_2\}. \end{aligned}$$

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Note that G is a cyclic group, then the number of elements of order d where $d \mid n$ is $\phi(d)$ (i.e. Euler phi function of d). Hence $|Q_0| = 1$, $|Q_1| = \phi(p_1) + \phi(p_2) = (p_1 - 1) + (p_2 - 1) = p_1 + p_2 - 2$, and $|Q_2| = \phi(p_1 p_2) = \phi(p_1)\phi(p_2) = (p_1 - 1)(p_2 - 1)$.

Furthermore, take arbitrary $u \in Q_i$ and $v \in Q_j$ with $i, j \in \{0, 1, 2\}$. If $i = j$, then $|u| = |v|$ or $|u| = p_1 \neq p_2 = |v|$, so $uv \notin E(OD(G))$. If $i \neq j$, then $|u| \neq |v|$. Without loss of generality, suppose $i < j$. Thus $o(u) \mid o(v)$, so $uv \in E(OD(G))$. Therefore $OD(G)$ is complete tripartite graph $K_{1,p_1+p_2-2,(p_1-1)(p_2-1)}$. □

The Wiener index, and the Harary index of the order divisor graph of cyclic groups of order $n = p_1 p_2$ are given in the following theorems.

Theorem 2.13. Let G be a cyclic group of order n. If $n = p_1 p_2$ with p_1 and p_2 are distinct primes, then the Wiener index of $OD(G)$ is

$$W(OD(G)) = (p_1 + p_2 - 2)(p_1 p_2 - 2) + (p_1 p_2 - p_1 - p_2 + 1)(p_1 p_2 - p_1 - p_2) + (p_1 p_2 - 1).$$

Proof. Let G be a cyclic group of order $n = p_1 p_2$ with p_1 and p_2 are distinct primes. By Theorem 2.12, $OD(G)$ is a complete tripartite graph $K_{1,p_1+p_2-2,(p_1-1)(p_2-1)}$. Thus, for every $u \in Q_i$ and $v \in Q_j$ we have $d(u, v) = 2$ if $i = j$ and $d(u, v) = 1$ if $i \neq j$ for all $i, j = 0, 1, 2$. Therefore,

$$\begin{aligned}
W(OD(G)) &= \sum_{u,v \in V(OD(G))} d(u,v) \\
&= \sum_{u,v \in Q_1} d(u,v) + \sum_{u,v \in Q_2} d(u,v) + \sum_{v \in Q_1 \cup Q_2} d(e,v) + \sum_{u \in Q_1} \sum_{v \in Q_2} d(u,v) \\
&= \binom{p_1 + p_2 - 2}{2}.2 + \binom{(p_1 - 1)(p_2 - 1)}{2}.2 + (p_1 p_2 - 1) \\
&\quad + (p_1 + p_2 - 2)(p_1 - 1)(p_2 - 1) \\
&= (p_1 + p_2 - 2)(p_1 + p_2 - 3) + (p_1 p_2 - p_1 - p_2 + 1)(p_1 p_2 - p_1 - p_2) \\
&\quad + (p_1 p_2 - 1) + (p_1 + p_2 - 2)(p_1 - 1)(p_2 - 1) \\
&= (p_1 + p_2 - 2)(p_1 p_2 - 2) + (p_1 p_2 - p_1 - p_2 + 1)(p_1 p_2 - p_1 - p_2) + (p_1 p_2 - 1).
\end{aligned}$$

□

Theorem 2.14. Let G be a cyclic group of order n. If $n = p_1 p_2$ with p_1 and p_2 are distinct primes, then the Harary index of $OD(G)$ is

$$H(OD(G)) = \frac{(p_1 + p_2 - 2)(4p_1 p_2 - 3p_1 - 3p_2 + 1)}{4} + \frac{(p_1 p_2 - p_1 - p_2 + 1)(p_1 p_2 - p_1 - p_2)}{4} + (p_1 p_2 - 1).$$

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Proof. Let G be a cyclic group of order $n = p_1 p_2$ with p_1 and p_2 are distinct primes. By Theorem 2.12, $OD(G)$ is a complete tripartite graph $K_{1,p_1+p_2-2,(p_1-1)(p_2-1)}$. Thus, for every $u \in Q_i$ and $v \in Q_j$ we have $d(u,v) = 2$ if $i = j$ and $d(u,v) = 1$ if $i \neq j$ for all $i, j = 0, 1, 2$. Therefore,

$$\begin{aligned}
H(OD(G)) &= \sum_{u,v \in V(OD(G))} \frac{1}{d(u,v)} \\
&= \sum_{u,v \in Q_1} \frac{1}{d(u,v)} + \sum_{u,v \in Q_2} \frac{1}{d(u,v)} + \sum_{v \in Q_1 \cup Q_2} \frac{1}{d(e,v)} + \sum_{u \in Q_1} \sum_{v \in Q_2} \frac{1}{d(u,v)} \\
&= \binom{p_1+p_2-2}{2} . \frac{1}{2} + \binom{(p_1-1)(p_2-1)}{2} . \frac{1}{2} + (p_1p_2 - 1) \\
&\quad + (p_1+p_2-2)(p_1-1)(p_2-1) \\
&= \frac{(p_1+p_2-2)(p_1+p_2-3)}{4} + \frac{(p_1p_2-p_1-p_2+1)(p_1p_2-p_1-p_2)}{4} \\
&\quad + (p_1p_2-1) + (p_1+p_2-2)(p_1-1)(p_2-1) \\
&= \frac{(p_1+p_2-2)(4p_1p_2-3p_1-3p_2+1)}{4} + \frac{(p_1p_2-p_1-p_2+1)(p_1p_2-p_1-p_2)}{4} \\
&\quad + (p_1p_2-1).
\end{aligned}$$

□

Before discussing the first and second Zagreb indices of order divisor graph of cyclic group of order $n = p_1 p_2$, the following lemma is given.

Lemma 2.15. Let G be a cyclic group of order n. If $n = p_1 p_2$ with p_1 and p_2 are distinct primes, then

$$deg(v) = \begin{cases} p_1p_2 - 1, & v \in Q_0 \\ 1 + (p_1-1)(p_2-1), & v \in Q_1 \\ p_1 + p_2 - 1, & v \in Q_2. \end{cases}$$

Proof. Let G be a cyclic group of order $n = p_1 p_2$ with p_1 and p_2 are distinct primes. By Theorem 2.12, $OD(G)$ is a complete tripartite graph $K_{1,p_1+p_2-2,(p_1-1)(p_2-1)}$. There are some conditions as follow.

1. For partition Q_0, we have $Q_0 = \{e\}$, so $deg(e) = n - 1 = p_1p_2 - 1$.
2. For all $v \in Q_1$, $uv \in E(OD(G))$ for every $u \in Q_0 \cup Q_2$, so $deg(v) = 1 + (p_1-1)(p_2-1)$.
3. For all $v \in Q_2$, $uv \in E(OD(G))$ for every $u \in Q_0 \cup Q_1$, so $deg(v) = 1 + p_1 + p_2 - 2 = p_1 + p_2 - 1$.

□

The following theorems explain the first and second Zagreb indices of an order divisor graph of a cyclic group with order $n = p_1 p_2$.

Theorem 2.16. Let G be a cyclic group of order n. If $n = p_1 p_2$ with p_1 and p_2 are distinct primes, then the first Zagreb index of $OD(G)$ is

$$M_1(OD(G)) = (p_1p_2-1)^2 + (p_1+p_2-2)(1+(p_1-1)(p_2-1))^2 + (p_1-1)(p_2-1)(p_1+p_2-1)^2.$$

Proof. Let G be a cyclic group of order $n = p_1 p_2$ with p_1 and p_2 are distinct primes. By Lemma 2.15, we get

$$\begin{aligned}
M_1(OD(G)) &= \sum_{v \in V(OD(G))} (deg(v))^2 \\
&= \sum_{v \in Q_0} (deg(v))^2 + \sum_{v \in Q_1} (deg(v))^2 + \sum_{v \in Q_2} (deg(v))^2 \\
&= (p_1p_2-1)^2 + (p_1+p_2-2)(1+(p_1-1)(p_2-1))^2 + (p_1-1)(p_2-1)(p_1+p_2-1)^2.
\end{aligned}$$

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□

Theorem 2.17. Let G be a cyclic group of order n. If $n = p_1p_2$ with p_1 and p_2 are distinct primes, then the second Zagreb index of $OD(G)$ is

$$M_2(OD(G)) = (p_1 + p_2 - 2)(p_1p_2 - p_1 - p_2 + 2)(p_1p_2 - 1 + p_1p_2(p_1 - 1)(p_2 - 1)(p_1 + p_2 - 1)).$$

Proof. Let G be a cyclic group of order $n = p_1p_2$ with p_1 and p_2 are distinct primes. By Lemma 2.15, we get

$$\begin{aligned} M_2(OD(G)) &= \sum_{uv\in E(OD(G))} deg(u)deg(v) \\ &= \sum_{v\in Q_1} deg(e)deg(v) + \sum_{v\in Q_2} deg(e)deg(v) + \sum_{u\in Q_1}\sum_{v\in Q_2} deg(u)deg(v) \\ &=(p_1 + p_2 - 2)(1 + (p_1 - 1)(p_2 - 1)) + (p_1 - 1)(p_2 - 1)(p_1 + p_2 - 1) \\ &\quad + (p_1 + p_2 - 2)(p_1 - 1)(p_2 - 1)(1 + (p_1 - 1)(p_2 - 1))(p_1 + p_2 - 1) \\ &=(p_1 + p_2 - 2)(p_1p_2 - p_1 - p_2 + 2)(p_1p_2 - 1 + p_1p_2(p_1 - 1)(p_2 - 1)(p_1 + p_2 - 1)). \end{aligned}$$

□

2.4. *For prime* $n = p_1^{k_1}p_2^{k_2}\dots p_m^{k_m}$.

Theorem 2.18. Let G be a cyclic group of order n. If $n = p_1^{k_1}p_2^{k_2}\dots p_m^{k_m}$ for all $i = 1,2,\dots,m$, p_i are distinct primes and $k_i \in \mathbb{N}$, then order divisor graph $OD(G)$ is $(k_1 + k_2 + \cdots + k_m + 1)-$partite graph.

Proof. Let G be a cyclic group of order $n = p_1^{k_1}p_2^{k_2}\dots p_m^{k_m}$ for all $i = 1,2,\dots,m$, p_i are distinct primes and $k_i \in \mathbb{N}$. Take partitions $R_0, R_1, R_2, \dots, R_{k_1+k_2+\dots+k_m}$ on $V(OD(G))$ with R_i being the set of all elements in G whose order is a multiplication of i prime numbers (may be the same) for every $i = 0,1,2,\dots,k_1 + k_2 + \cdots + k_m$. For each $i = 0,1,2,\dots,k_1 + k_2 + \cdots + k_m$, take any $u, v \in R_i$. Then $o(u)$ and $o(v)$ is a multiplication of i prime numbers. Therefore $o(u) \nmid o(v)$ and $o(v) \nmid o(u)$, so $uv \notin E(OD(G))$. Hence order divisor graph $OD(G)$ is $(k_1 + k_2 + \cdots + k_m + 1)-$partite graph. □

Given cyclic group G of order $n = p_1^{k_1}p_2^{k_2}\dots p_m^{k_m}$. Define $D(n)$ as the set of all positive factors of n. Take partitions A_d on $V(OD(G))$ for all $d \in D(n)$ with

$$A_d = \{v \in V(OD(G)) \mid |v| = d\}.$$

Note that G is a cyclic group, so $|A_d| = \phi(d)$ where $\phi(d)$ is Euler phi function of d. Two following theorems explain the Wiener index and the Harary index of an order divisor graph of a cyclic group with order $n = p_1^{k_1}p_2^{k_2}\dots p_m^{k_m}$.

Theorem 2.19. Let G be a cyclic group of order n. If $n = p_1^{k_1}p_2^{k_2}\dots p_m^{k_m}$ for all $i = 1,2,\dots,m$, p_i are distinct primes and $k_i \in \mathbb{N}$, then the Wiener index of $OD(G)$ is

$$W(OD(G)) = \sum_{d\in D(n)} 2\binom{\phi(d)}{2} + \sum_{s\in D(n)}\sum_{t\in D(n)\setminus\{s\}, s|t} \phi(s)\phi(t) + \sum_{s\in D(n)}\sum_{t\in D(n)\setminus\{s\}, s<t, s\nmid t} 2\phi(s)\phi(t).$$

Proof. Let G be a cyclic group of order $n = p_1^{k_1}p_2^{k_2}\dots p_m^{k_m}$ for all $i = 1,2,\dots,m$, p_i are distinct primes and $k_i \in \mathbb{N}$. For every $u \in A_s$ and $v \in A_t$, there are three conditions as follow

1. if $s = t$, then $d(u, v) = 2$;
2. if $s \neq t$ and $s \mid t$ or $t \mid s$, then $d(u, v) = 1$;
3. if $s \neq t$, $s \nmid t$, and $t \nmid s$, then $d(u, v) = 2$

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for all $s,t \in D(n)$. Therefore,

$$\begin{aligned}
W(OD(G)) &= \sum_{u,v\in V(OD(G))} d(u,v) \\
&= \sum_{d\in D(n)} \sum_{u,v\in A_d} d(u,v) + \sum_{s\in D(n)} \sum_{t\in D(n)\setminus\{s\},s|t} \sum_{u\in A_s} \sum_{v\in A_t} d(u,v) \\
&\quad + \sum_{s\in D(n)} \sum_{t\in D(n)\setminus\{s\},s<t,s\nmid t} \sum_{u\in A_s} \sum_{v\in A_t} d(u,v) \\
&= \sum_{d\in D(n)} 2\binom{\phi(d)}{2} + \sum_{s\in D(n)} \sum_{t\in D(n)\setminus\{s\},s|t} \phi(s)\phi(t) + \sum_{s\in D(n)} \sum_{t\in D(n)\setminus\{s\},s<t,s\nmid t} 2\phi(s)\phi(t).
\end{aligned}$$

□

Theorem 2.20. Let G be a cyclic group of order n. If $n = p_1^{k_1} p_2^{k_2} \dots p_m^{k_m}$ for all $i = 1,2,\dots,m$, p_i are distinct primes and $k_i \in \mathbb{N}$, then the Harary index of $OD(G)$ is

$$H(OD(G)) = \sum_{d\in D(n)} \frac{1}{2}\binom{\phi(d)}{2} + \sum_{s\in D(n)} \sum_{t\in D(n)\setminus\{s\},s|t} \phi(s)\phi(t) + \sum_{s\in D(n)} \sum_{t\in D(n)\setminus\{s\},s<t,s\nmid t} \frac{1}{2}\phi(s)\phi(t).$$

Proof. Let G be a cyclic group of order $n = p_1^{k_1} p_2^{k_2} \dots p_m^{k_m}$ for all $i = 1,2,\dots,m$, p_i are distinct primes and $k_i \in \mathbb{N}$. Analogous to the Theorem 2.19, we have

$$\begin{aligned}
H(OD(G)) &= \sum_{u,v\in V(OD(G))} \frac{1}{d(u,v)} \\
&= \sum_{d\in D(n)} \sum_{u,v\in A_d} \frac{1}{d(u,v)} + \sum_{s\in D(n)} \sum_{t\in D(n)\setminus\{s\},s|t} \sum_{u\in A_s} \sum_{v\in A_t} \frac{1}{d(u,v)} \\
&\quad + \sum_{s\in D(n)} \sum_{t\in D(n)\setminus\{s\},s<t,s\nmid t} \sum_{u\in A_s} \sum_{v\in A_t} \frac{1}{d(u,v)} \\
&= \sum_{d\in D(n)} \frac{1}{2}\binom{\phi(d)}{2} + \sum_{s\in D(n)} \sum_{t\in D(n)\setminus\{s\},s|t} \phi(s)\phi(t) + \sum_{s\in D(n)} \sum_{t\in D(n)\setminus\{s\},s<t,s\nmid t} \frac{1}{2}\phi(s)\phi(t).
\end{aligned}$$

□

The following lemma is useful in the theorem that characterises the first and second Zagreb indices of an order divisor graph of a cyclic group with order $n = p_1^{k_1} p_2^{k_2} \dots p_m^{k_m}$.

Lemma 2.21. Let G be a cyclic group of order n. If $n = p_1^{k_1} p_2^{k_2} \dots p_m^{k_m}$ for all $i = 1,2,\dots,m$, p_i are distinct primes and $k_i \in \mathbb{N}$, then

$$deg(v) = \sum_{s\in D(n)\setminus\{d\},s|d} \phi(s) + \sum_{t\in D(n)\setminus\{d\},d|t} \phi(t)$$

for every $v \in A_d$.

Theorem 2.22. Let G be a cyclic group of order n. If $n = p_1^{k_1} p_2^{k_2} \dots p_m^{k_m}$ for all $i = 1,2,\dots,m$, p_i are distinct primes and $k_i \in \mathbb{N}$, then the first Zagreb index of $OD(G)$ is

$$M_1(OD(G)) = \sum_{d\in D(n)} \phi(d)\left(\sum_{s\in D(n)\setminus\{d\},s|d} \phi(s) + \sum_{t\in D(n)\setminus\{d\},d|t} \phi(t)\right)^2.$$

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Proof. Let G be a cyclic group of order $n = p_1^{k_1} p_2^{k_2} \dots p_m^{k_m}$ for all $i = 1, 2, \dots, m$, p_i are distinct primes and $k_i \in \mathbb{N}$. By Lemma 2.21 we have,

$$\begin{aligned} M_1(OD(G)) &= \sum_{v \in V(OD(G))} (deg(v))^2 \\ &= \sum_{d \in D(n)} \sum_{v \in A_d} (deg(v))^2 \\ &= \sum_{d \in D(n)} \phi(d) \left(\sum_{s \in D(n) \setminus \{d\}, s|d} \phi(s) + \sum_{t \in D(n) \setminus \{d\}, d|t} \phi(t) \right)^2 . \end{aligned}$$

□

Theorem 2.23. Let G be a cyclic group of order n. If $n = p_1^{k_1} p_2^{k_2} \dots p_m^{k_m}$ for all $i = 1, 2, \dots, m$, p_i are distinct primes and $k_i \in \mathbb{N}$, then the second Zagreb index of $OD(G)$ is

$$M_2(OD(G)) = \sum_{c \in D(n) \setminus \{n\}} \left(\phi(c) deg(A_c) \sum_{d \in D(n) \setminus \{c\}, c|d} \phi(d) deg(A_d) \right)$$

where

$$deg(A_c) = \sum_{s \in D(n) \setminus \{c\}, s|c} \phi(s) + \sum_{t \in D(n) \setminus \{c\}, c|t} \phi(t).$$

Proof. Let G be a cyclic group of order $n = p_1^{k_1} p_2^{k_2} \dots p_m^{k_m}$ for all $i = 1, 2, \dots, m$, p_i are distinct primes and $k_i \in \mathbb{N}$. By Lemma 2.21 we have,

$$\begin{aligned} M_2(OD(G)) &= \sum_{uv \in E(OD(G))} deg(u) deg(v) \\ &= \sum_{c \in D(n) \setminus \{n\}} \sum_{u \in A_c} \sum_{d \in D(n) \setminus \{c\}, c|d} \sum_{v \in A_d} deg(u) deg(v) \\ &= \sum_{c \in D(n) \setminus \{n\}} \sum_{u \in A_c} \left(deg(u) \sum_{d \in D(n) \setminus \{c\}, c|d} \sum_{v \in A_d} deg(v) \right) \\ &= \sum_{c \in D(n) \setminus \{n\}} \left(\phi(c) deg(A_c) \sum_{d \in D(n) \setminus \{c\}, c|d} \phi(d) deg(A_d) \right) \end{aligned}$$

where

$$deg(A_c) = \sum_{s \in D(n) \setminus \{c\}, s|c} \phi(s) + \sum_{t \in D(n) \setminus \{c\}, c|t} \phi(t).$$

□

3. Conclusion

In this research, we characterize the form of the order divisor graph of cyclic groups based on the group order. We also have found some topological indices of order divisor graphs of cyclic groups, including the Wiener index, Harary index, first Zagreb index, and second Zagreb index. It is very interesting to find the Szeged index and Hosoya index of the order divisor graphs of cyclic groups for further research.

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