Article

Not peer-reviewed version

Classical Solutions for the Generalized Kawahara-KdV System

S. Georgiev , A. Boukarou , K. Bouhali , Kh. Zennir * , H. Elkhair , E. Hassan , A. Alfedeel , A. Alarfaj

Posted Date: 28 April 2023

doi: 10.20944/preprints202304.1109.v1

Keywords: Fractional derivatives; Generalized Kawahara-KdV system; existence; classical solution.

Article

Classical Solutions for the Generalized Kawahara-KdV System

S. G. Georgiev [1], A. Boukarou[2], K. Bouhali [3,4], Kh. Zennir [3], H. M. Elkhair [5], E. I. Hassan [6], A. H. A. alfedeel [6] and A. M. alarfaj [6]

1 Department of Differential Equations, Faculty of Mathematics and Informatics, University of Sofia, Bulgaria. svetlingeorgiev1@gmail.com
2 Faculty of Mathematics, University of Science and Technology Houari Boumediene, Algeria. boukarouaissa@gmail.com
3 Department of Mathematics, College of Sciences and Arts, Qassim University, Ar-Rass, Saudi Arabia, k.bouhali@qu.edu.sa, k.zennir@qu.edu.sa
4 Université 20 août 1955 Skikda Bp 26 Route El-Hadaiek 21000 Skikda, Algeria.
5 Deanship of Scientific Research, Imam Mohammad Ibn Saud Islamic University, P. O. Box 5701, Riyadh 11432, Saudia Arabia, hmdirar@imamu.edu.sa
6 Department of Mathematics and Statistics, Imam Mohammad Ibn Saud Islamic University (IMSIU), Riyadh, Saudi Arabia, eiabdalla@imamu.edu.sa, aaalnadhief@imamu.edu.sa, amalarfaj@imamu.edu.sa
* Correspondence: k.zennir@qu.edu.sa;

Abstract: In this article we investigate the generalized Kawahara-KdV system. A new topological approach is applied to prove the existence of at least one classical solution and at least two nonnegative classical solutions. The arguments are based upon recent theoretical results.

Keywords: Fractional derivatives; Generalized Kawahara-KdV system; existence; classical solution.

MSC: 37C25, 47H10

---

1. Introduction

In this paper, we investigate the Cauchy problem for the generalized Kawahara-KdV system

$$\begin{aligned} \partial_t u + \sum_{k=0}^{N_1}\sum_{l=0}^{N_1-k} \partial_x \left\{ \sum_{m=0}^{N_1-k} \partial_x^m u^p P_{k,l,m}\left(\partial_x^l v\right) \right\} + \sum_{k=1}^{N_2} a_k(t,x)\partial_x^{2k+1} u &= 0 \\ \partial_t v + \sum_{k=0}^{N_3}\sum_{l=0}^{N_3-k} \partial_x^k \left\{ \sum_{m=0}^{N_3-k} \partial_x^m v^p Q_{k,l,m}\left(\partial_x^l u\right) \right\} + \sum_{k=1}^{N_4} b_k(t,x)\partial_x^{2k+1} v &= 0, \\ (t,x) \in [0,\infty)\times\mathbb{R}, \quad u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \quad x \in \mathbb{R}, \end{aligned} \tag{1}$$

where

(Hyp1) $u_0, v_0 \in \mathcal{C}^q(\mathbb{R})$, $0 \le u_0, v_0 \le \mathcal{B}$ on $\mathbb{R}$ for some positive constant $\mathcal{B} > 1$, $a_j, b_k \in \mathcal{C}([0,\infty)\times\mathbb{R})$, $0 \le |a_j|, |b_k| \le \mathcal{B}$ on $[0,\infty)\times\mathbb{R}$, $j = 1,\ldots,N_2$, $k = 1,\ldots,N_4$,

$$\begin{aligned} P_{k,l,m}(z) &= \sum_{r=0}^{N_5} c_{k,l,m,r}(t,x) z^r, \\ Q_{k,l,m}(z) &= \sum_{r=0}^{N_6} d_{k,l,m,r}(t,x) z^r, \quad (t,x) \in [0,\infty)\times\mathbb{R}, \quad z \in \mathbb{R}, \end{aligned}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 28 April 2023 doi:10.20944/preprints202304.1109.v1

$$
c_{k,l,m,j}, d_{k,l,m,r} \in \mathcal{C}([0,\infty), \mathcal{C}^q(\mathbb{R})),
$$

$$
0 \leq \left|\partial_x^{p_1} c_{k,l,m,j}\right|, \quad \left|\partial_x^{p_1} d_{k,l,m,r}\right| \leq \mathcal{B},
$$

on $[0,\infty) \times \mathbb{R}$, $j = 1, \ldots, N_5$, $r = 1, \ldots, N_6$, $p_1 = 1, \ldots, N_1$, $p, N_1, N_2, N_3, N_4, N_5, N_6 \in \mathbb{N}$,

$$
q = \max\{N_1, 2N_2 + 1, N_3, 2N_4 + 1\}.
$$

Kondo and Pes [1] proved this system is local well-posedness in analytic Gevrey spaces $G^{\sigma,s}(\mathbb{R})$ with $s \geq 2N + 1/2$ where $N = \max\{N_2, N_4\}$.

The range of equations that this model encompasses is obviously broad and can represent many physical situations. As examples, we consider the nonlinear term

$$
\sum_{k=0}^{N_1} \sum_{\ell=0}^{N_1-k} \partial_x^k \left\{ \sum_{m=0}^{N_1-k} \partial_x^m u^p P_{k,\ell,m}\left(\partial_x^\ell v\right) \right\}.
$$

If $N_1 = 1$, then we have that this term is equal to

$$
\begin{aligned}
&u^p P_{0,0,0}(v) + \partial_x u^p P_{0,0,1}(v) + u^p P_{0,1,0}(\partial_x v) + \partial_x u^p P_{0,1,1}(\partial_x v) \\
&\quad + \partial_x \left[u^p P_{1,0,0}(v) + \partial_x u^p P_{1,0,1}(v)\right].
\end{aligned} \tag{2}
$$

When $N_1 = N_2 = 1$ and $p = 1$, taking $P_{0,0,0} = P_{0,0,1} = P_{0,1,0} = P_{0,1,1} = P_{1,0,1} \equiv 0, P_{1,0,0}(x) = x^2, a_1 = 1$ and doing the same choices to the polynomial $Q_{k,\ell,m}$ with $N_3 = N_4 = 1$ and $b_1 = 1$, we have a couple system of modified Korteweg-de Vries equations (see [2,3])

$$
\begin{cases}
\partial_t u + \partial_x^3 u + \partial_x\left(uv^2\right) = 0, \\
\partial_t v + \partial_x^3 v + \partial_x\left(vu^2\right) = 0, \\
u(x,0) = u_0(x), v(x,0) = v_0(x).
\end{cases}
$$

Considering in (2) the case when $p = q \in \mathbb{N}$ and $P_{1,0,0}(x) = x^{q+1}$, also the remaining null polynomials, since that $N_1 = N_2 = N_3 = N_4 = 1$, we obtain a more general system, studied in [4], as follows

$$
\begin{cases}
\partial_t u + \partial_x^3 u + \partial_x\left(u^q v^{q+1}\right) = 0, \\
\partial_t v + \partial_x^3 v + \partial_x\left(v^q u^{q+1}\right) = 0, \\
u(x,0) = u_0(x), v(x,0) = v_0(x).
\end{cases}
$$

On the other hand, to find systems more complicated, we can consider $N_1 = 2$ and $p = 1$, then in this case the term nonlinear is more general, as we can see bellow

$$
\begin{aligned}
&uP_{0,0,0}(v) + \partial_x uP_{0,0,1}(v) + \partial_x^2 uP_{0,0,2}(v) \\
&\quad + uP_{0,1,0}(\partial_x v) + \partial_x uP_{0,1,1}(\partial_x v) + \partial_x^2 uP_{0,1,2}(\partial_x v) \\
&\quad + uP_{0,2,0}\left(\partial_x^2 v\right) + \partial_x uP_{0,2,1}\left(\partial_x^2 v\right) + \partial_x^2 uP_{0,2,2}\left(\partial_x^2 v\right) \\
&\quad + \partial_x \Big[uP_{1,0,0}(v) + \partial_x uP_{1,0,1}(v) + \partial_x^2 uP_{1,0,2}(v) \\
&\quad + uP_{1,1,0}(\partial_x v) + \partial_x uP_{1,1,1}(\partial_x v) + \partial_x^2 uP_{1,1,2}(\partial_x v)\Big] \\
&\quad + \partial_x^2 \left[uP_{2,0,0}(v) + \partial_x uP_{2,0,1}(v) + \partial_x^2 uP_{2,0,2}(v)\right].
\end{aligned}
$$

Observe that, changing v by u, and considering again all identical null polynomials, except $P_{0,0,1}(x) = x^k$, we obtain the Kawahara equation [5]

$$
\partial_t u + \partial_x^3 u + \partial_x^5 u + u^k \partial_x u = 0.
$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 28 April 2023 doi:10.20944/preprints202304.1109.v1

The aim of this paper is to investigate the IVP (1) for existence of global classical solutions.

Theorem 1. *Suppose* $((Hyp1)$. *Then the IVP* (1) *has at least one solution*

$$(u,v) \in \left(\mathcal{C}^1([0,\infty),\mathcal{C}^q(\mathbb{R}))\right)^2.$$

Theorem 2. *Suppose* $((Hyp1)$. *Then the IVP* (1) *has at least two nonnegative solutions*

$$(u_1,v_1),(u_2,v_2) \in \left(\mathcal{C}^1([0,\infty),\mathcal{C}^q(\mathbb{R}))\right)^2.$$

The paper is organized as follows. In the next section, we give some auxiliary results. In section 3, we prove Theorem 1. In section 4, we prove Theorem 2. In section 5, we give an example to illustrate our main results.

2. Preliminary Results

To prove our existence result we will use the following fixed point theorem.

Theorem 3. *Let* $\epsilon > 0$, $\mathcal{B} > 0$, $\mathcal{E}$ *be a Banach space and* $\mathcal{X} = \{x \in \mathcal{E} : \|x\| \leq \mathcal{B}\}$. *Let also,* $\mathcal{T}x = -\epsilon x$, $x \in \mathcal{X}$, $S : \mathcal{X} \to \mathcal{E}$ *is a continuous,* $(I-S)(\mathcal{X})$ *resides in a compact subset of* $\mathcal{E}$ *and*

$$\{x \in \mathcal{E} : x = \lambda(I-S)x, \quad \|x\| = \mathcal{B}\} = \emptyset, \forall \lambda \in \left(0, \frac{1}{\epsilon}\right). \tag{3}$$

Then there exists $x^* \in \mathcal{X}$ *so that*

$$\mathcal{T}x^* + Sx^* = 0.$$

Proof. Define

$$r\left(-\frac{1}{\epsilon}x\right) = \begin{cases} -\frac{1}{\epsilon}x & \text{if} \quad \|x\| \leq \mathcal{B}\epsilon \\ \frac{\mathcal{B}x}{\|x\|} & \text{if} \quad \|x\| > \mathcal{B}\epsilon. \end{cases}$$

Then $r\left(-\frac{1}{\epsilon}(I-S)\right) : \mathcal{X} \to \mathcal{X}$ is continuous and compact. Hence and the Schauder fixed point theorem, it follows that there exists $x^* \in \mathcal{X}$ so that

$$r\left(-\frac{1}{\epsilon}(I-S)x^*\right) = x^*.$$

Assume that $-\frac{1}{\epsilon}(I-S)x^* \notin \mathcal{X}$. Then

$$\left\|(I-S)x^*\right\| > \mathcal{B}\epsilon, \quad \frac{\mathcal{B}}{\|(I-S)x^*\|} < \frac{1}{\epsilon},$$

and

$$x^* = \frac{\mathcal{B}}{\|(I-S)x^*\|}(I-S)x^* = r\left(-\frac{1}{\epsilon}(I-S)x^*\right),$$

and hence, $\|x^*\| = \mathcal{B}$. This contradicts with (3). Therefore $-\frac{1}{\epsilon}(I-S)x^* \in \mathcal{X}$ and

$$x^* = r\left(-\frac{1}{\epsilon}(I-S)x^*\right) = -\frac{1}{\epsilon}(I-S)x^*,$$

or

$$-\epsilon x^* + Sx^* = x^*,$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 28 April 2023 doi:10.20944/preprints202304.1109.v1

or

$$\mathcal{T}x^* + \mathcal{S}x^* = x^*.$$

This completes the proof. □

Let $\mathcal{X}$ be a real Banach space.

Definition 1. *A mapping $\mathcal{K} : \mathcal{X} \to \mathcal{X}$ is said to be completely continuous if it is continuous and maps bounded sets into relatively compact sets.*

The concept for l-set contraction is related to that of the Kuratowski measure of noncompactness which we recall for completeness.

Definition 2. *Let $\Omega_{\mathcal{X}}$ be the class of all bounded sets of $\mathcal{X}$. The Kuratowski measure of noncompactness $\alpha : \Omega_{\mathcal{X}} \to [0, \infty)$ is defined by*

$$\alpha(\mathcal{Y}) = \inf\left\{\delta > 0 : \mathcal{Y} = \bigcup_{j=1}^{m} \mathcal{Y}_j \quad and \quad diam(\mathcal{Y}_j) \le \delta, \quad j = 1, \ldots, m\right\},$$

where $diam(\mathcal{Y}_j) = \sup\{\|x - y\|_{\mathcal{X}} : x, y \in \mathcal{Y}_j\}$ is the diameter of $\mathcal{Y}_j$, $j = 1, \ldots, m$.

For the main properties of measure of noncompactness we refer the reader to [6].

Definition 3. *A mapping $\mathcal{K} : \mathcal{X} \to \mathcal{X}$ is said to be l-set contraction if it is continuous, bounded and there exists a constant $l \ge 0$ such that*

$$\alpha(\mathcal{K}(\mathcal{Y})) \le l\alpha(\mathcal{Y}),$$

for any bounded set $\mathcal{Y} \subset \mathcal{X}$. The mapping $\mathcal{K}$ is said to be a strict set contraction if $l < 1$.

Obviously, if $\mathcal{K} : \mathcal{X} \to \mathcal{X}$ is a completely continuous mapping, then $\mathcal{K}$ is 0-set contraction (see [7], pp. 264).

Definition 4. *Let $\mathcal{X}$ and $\mathcal{Y}$ be real Banach spaces. A mapping $\mathcal{K} : \mathcal{X} \to \mathcal{Y}$ is said to be expansive if there exists a constant $h > 1$ such that*

$$\|\mathcal{K}x - \mathcal{K}y\|_{\mathcal{Y}} \ge h\|x - y\|_{\mathcal{X}},$$

for any $x, y \in \mathcal{X}$.

Definition 5. *A closed, convex set $\mathcal{P}$ in $\mathcal{X}$ is said to be cone if*

1. *$\alpha x \in \mathcal{P}$ for any $\alpha \ge 0$ and for any $x \in \mathcal{P}$,*
2. *$x, -x \in \mathcal{P}$ implies $x = 0$.*

Denote $\mathcal{P}^* = \mathcal{P} \backslash \{0\}$.

Lemma 4. *Let $\mathcal{X}$ be a closed convex subset of a Banach space $\mathcal{E}$ and $\mathcal{U} \subset \mathcal{X}$ a bounded open subset with $0 \in \mathcal{U}$. Assume there exists $\varepsilon > 0$ small enough and that $\mathcal{K} : \overline{\mathcal{U}} \to \mathcal{X}$ is a strict k-set contraction that satisfies the boundary condition:*

$$\mathcal{K}x \notin \{x, \ \lambda x\} \ \text{for all} \ x \in \partial\mathcal{U} \ \text{and} \ \lambda \ge 1 + \varepsilon.$$

Then the fixed point index $i\,(\mathcal{K}, \mathcal{U}, \mathcal{X}) = 1$.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 28 April 2023 doi:10.20944/preprints202304.1109.v1

Proof. Consider the homotopic deformation $\mathcal{H} : [0,1] \times \overline{\mathcal{U}} \to \mathcal{X}$ defined by

$$\mathcal{H}(t,x) = \frac{1}{\varepsilon+1} t\mathcal{K}x.$$

The operator $\mathcal{H}$ is continuous and uniformly continuous in t for each x, and the mapping $\mathcal{H}(t,.)$ is a strict set contraction for each $t \in [0,1]$.In addition, $\mathcal{H}(t,.)$ has no fixed point on $\partial\mathcal{U}$. On the contrary,
• If $t = 0$, there exists some $x_0 \in \partial\mathcal{U}$ such that $x_0 = 0$, contradicting $x_0 \in \mathcal{U}$.
• If $t \in (0,1]$, there exists some $x_0 \in \mathcal{P} \cap \partial\mathcal{U}$ such that $\frac{1}{\varepsilon+1} t\mathcal{K}x_0 = x_0$; then $\mathcal{K}x_0 = \frac{1+\varepsilon}{t} x_0$ with $\frac{1+\varepsilon}{t} \geq 1+\varepsilon$, contradicting the assumption. From the invariance under homotopy and the normalization properties of the index, we deduce

$$i\,(\frac{1}{\varepsilon+1}\,\mathcal{K},\mathcal{U},\mathcal{X}) = i\,(0,\mathcal{U},\mathcal{X}) = 1.$$

New, we show that

$$i\,(\mathcal{K},\mathcal{U},\mathcal{X}) = i\,(\frac{1}{\varepsilon+1}\,\mathcal{K},\mathcal{U},\mathcal{X}).$$

We have

$$\frac{1}{\varepsilon+1}\mathcal{K}x \neq x,\ \forall\, x \in \partial\mathcal{U}. \tag{4}$$

Then there exists $\gamma > 0$ such that

$$\|x - \frac{1}{\varepsilon+1}\,\mathcal{K}x\| \geq \gamma,\ \forall\, x \in \partial\mathcal{U}.$$

In other hand, we have $\frac{1}{\epsilon+1}\mathcal{K}x \to \mathcal{K}x$ as $\epsilon \to 0$, for $x \in \overline{\mathcal{U}}$. So for ε small enough

$$\|\mathcal{K}x - \frac{1}{\varepsilon+1}\,\mathcal{K}x\| < \frac{\gamma}{2},\ \forall\, x \in \partial\mathcal{U}.$$

Define the convex deformation $G : [0,1] \times \overline{\mathcal{U}} \to \mathcal{X}$ by

$$G(t,x) = t\mathcal{K}x + (1-t)\frac{1}{\varepsilon+1}\mathcal{K}x.$$

The operator G is continuous and uniformly continuous in t for each x, and the mapping $G(t,.)$ is a strict set contraction for each $t \in [0,1]$ (since $t + \frac{1}{\varepsilon+1}(1-t) < t+1-t = 1$).In addition, $G(t,.)$ has no fixed point on $\partial\mathcal{U}$. In fact, for all $x \in \partial\mathcal{U}$, we have

$$\begin{aligned} \|x - G(t,x)\| &= \|x - t\mathcal{K}x - (1-t)\tfrac{1}{\varepsilon+1}\mathcal{K}x\| \\ &\geq \|x - \tfrac{1}{\varepsilon+1}\mathcal{K}x\| - t\|\mathcal{K}x - \tfrac{1}{\varepsilon+1}\mathcal{K}x\| \\ &> \gamma - \tfrac{\gamma}{2} > \tfrac{\gamma}{2}. \end{aligned}$$

Then our claim follows from the invariance property by homotopy of the index.
□

Proposition 5. *Let $\mathcal{P}$ be a cone in a Banach space $\mathcal{E}$. Let also, $\mathcal{U}$ be a bounded open subset of $\mathcal{P}$ with $0 \in \mathcal{U}$. Assume that $\mathcal{T} : \Omega \subset \mathcal{P} \to \mathcal{E}$ is an expansive mapping with constant $h > 1$, $S : \overline{\mathcal{U}} \to \mathcal{E}$ is a l-set contraction with $0 \leq l < h-1$, and $S(\overline{\mathcal{U}}) \subset (I-\mathcal{T})(\Omega)$. If there exists $\varepsilon \geq 0$ such that*

$$Sx \notin \{(I-\mathcal{T})(x),\quad (I-\mathcal{T})(\lambda x)\}\ \text{for all } x \in \partial\mathcal{U} \cap \Omega \text{ and } \lambda \geq 1+\varepsilon,$$

then the fixed point index $i_*\,(\mathcal{T}+S,\mathcal{U}\cap\Omega,\mathcal{P}) = 1$.

Proof. The mapping $(I-\mathcal{T})^{-1}S : \overline{\mathcal{U}} \to \mathcal{P}$ is a strict set contraction and it is readily seen that the following condition is satisfied

$$(I-\mathcal{T})^{-1}Sx \notin \{x, \lambda x\} \quad \text{for} \quad \text{all} \quad x \in \partial\mathcal{U} \quad \text{and} \quad \lambda \geq 1+\epsilon.$$

Our claim then follows from the definition of i_* and the following Lemma 4. □

The following result will be used to prove our main result.

Theorem 6. *Let $\mathcal{P}$ be a cone of a Banach space $\mathcal{E}$; Ω a subset of $\mathcal{P}$ and $\mathcal{U}_1, \mathcal{U}_2$ and $\mathcal{U}_3$ three open bounded subsets of $\mathcal{P}$ such that $\overline{\mathcal{U}}_1 \subset \overline{\mathcal{U}}_2 \subset \mathcal{U}_3$ and $0 \in \mathcal{U}_1$. Assume that $\mathcal{T} : \Omega \to \mathcal{P}$ is an expansive mapping with constant $h > 1$, $S : \overline{\mathcal{U}}_3 \to \mathcal{E}$ is a k-set contraction with $0 \leq k < h-1$ and $S(\overline{\mathcal{U}}_3) \subset (I-\mathcal{T})(\Omega)$. Suppose that $(\mathcal{U}_2 \setminus \overline{\mathcal{U}}_1) \cap \Omega \neq \emptyset$, $(\mathcal{U}_3 \setminus \overline{\mathcal{U}}_2) \cap \Omega \neq \emptyset$, and there exists $u_0 \in \mathcal{P}^*$ such that the following conditions hold:*

(i) *$Sx \neq (I-\mathcal{T})(x-\lambda u_0)$, for all $\lambda > 0$ and $x \in \partial\mathcal{U}_1 \cap (\Omega + \lambda u_0)$,*
(ii) *there exists $\epsilon \geq 0$ such that $Sx \neq (I-\mathcal{T})(\lambda x)$, for all $\lambda \geq 1+\epsilon$, $x \in \partial\mathcal{U}_2$ and $\lambda x \in \Omega$,*
(iii) *$Sx \neq (I-\mathcal{T})(x-\lambda u_0)$, for all $\lambda > 0$ and $x \in \partial\mathcal{U}_3 \cap (\Omega + \lambda u_0)$.*

Then $\mathcal{T} + S$ has at least two non-zero fixed points $x_1, x_2 \in \mathcal{P}$ such that

$$x_1 \in \partial\mathcal{U}_2 \cap \Omega \text{ and } x_2 \in (\overline{\mathcal{U}}_3 \setminus \overline{\mathcal{U}}_2) \cap \Omega,$$

or

$$x_1 \in (\mathcal{U}_2 \setminus \mathcal{U}_1) \cap \Omega \text{ and } x_2 \in (\overline{\mathcal{U}}_3 \setminus \overline{\mathcal{U}}_2) \cap \Omega.$$

Proof. If $Sx = (I-\mathcal{T})x$ for $x \in \partial\mathcal{U}_2 \cap \Omega$, then we get a fixed point $x_1 \in \partial\mathcal{U}_2 \cap \Omega$ of the operator $\mathcal{T}+S$. Suppose that $Sx \neq (I-\mathcal{T})x$ for any $x \in \partial\mathcal{U}_2 \cap \Omega$. Without loss of generality, assume that $\mathcal{T}x + Sx \neq x$ on $\partial\mathcal{U}_1 \cap \Omega$ and $\mathcal{T}x + Sx \neq x$ on $\partial\mathcal{U}_3 \cap \Omega$, otherwise the conclusion has been proved. By [8, Proposition 2.16] and Proposition 5, we have

$$i_*(\mathcal{T}+S, \mathcal{U}_1 \cap \Omega, \mathcal{P}) = i_*(\mathcal{T}+S, \mathcal{U}_3 \cap \Omega, \mathcal{P}) = 0 \text{ and } i_*(\mathcal{T}+S, \mathcal{U}_2 \cap \Omega, \mathcal{P}) = 1.$$

The additivity property of the index yields

$$i_*(\mathcal{T}+S, (\mathcal{U}_2 \setminus \overline{\mathcal{U}}_1) \cap \Omega, \mathcal{P}) = 1 \text{ and } i_*(\mathcal{T}+S, (\mathcal{U}_3 \setminus \overline{\mathcal{U}}_2) \cap \Omega, \mathcal{P}) = -1.$$

Consequently, by the existence property of the index, $\mathcal{T}+S$ has at least two fixed points $x_1 \in (\mathcal{U}_2 \setminus \mathcal{U}_1) \cap \Omega$ and $x_2 \in (\overline{\mathcal{U}}_3 \setminus \overline{\mathcal{U}}_2) \cap \Omega$. □

3. Proof of Theorem 1

Let $\mathcal{X}_1 = \mathcal{C}^1([0,\infty), \mathcal{C}^q(\mathbb{R}))$ be endowed with the norm

$$\begin{aligned} \|u\|_1 &= \max\{ \sup_{(t,x)\in[0,\infty)\times\mathbb{R}} |u(t,x)|, \quad \sup_{(t,x)\in[0,\infty)\times\mathbb{R}} |\partial_t u(t,x)|, \\ &\quad \sup_{(t,x)\in[0,\infty)\times\mathbb{R}} |\partial_x^j u(t,x)|, \quad j \in \{1,\ldots,q\}\}, \end{aligned}$$

provided it exists. Define $\mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_1$ with the norm

$$\|(u,v)\| = \max\{\|u\|_1, \quad \|v\|_1\}.$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 28 April 2023 doi:10.20944/preprints202304.1109.v1

For $(u,v) \in \mathcal{X}$, define

$$Q_1(u,v)(t,x) = \sum_{k=0}^{N_1} \sum_{l=0}^{N_1-k} \partial_x \left\{ \sum_{m=0}^{N_1-k} \partial_x^m u^p P_{k,l,m}\left(\partial_x^l v\right) \right\},$$

$$Q_2(u,v)(t,x) = \sum_{k=0}^{N_3} \sum_{l=0}^{N_3-k} \partial_x^k \left\{ \sum_{m=0}^{N_3-k} \partial_x^m v^p Q_{k,l,m}\left(\partial_x^l u\right) \right\}, \quad (t,x) \in [0,\infty) \times \mathbb{R}.$$

Then the IVP (1) can be rewritten in the form

$$\begin{aligned} \partial_t u + Q_1(u,v) + \sum_{k=1}^{N_2} a_k(t,x) + \partial_x^{2k+1} u &= 0, \\ \partial_t v + Q_2(u,v) + \sum_{k=1}^{N_4} b_k(t,x) \partial_x^{2k+1} v &= 0, \quad (t,x) \in [0,\infty) \times \mathbb{R}, \\ u(0,x) = u_0(x), \quad v(0,x) &= v_0(x), \quad x \in \mathbb{R}. \end{aligned} \tag{5}$$

Let

$$\begin{aligned} C_1 = &\left\{ \sum_{k=0}^{N_1} \sum_{l=0}^{N_1-k} \sum_{m=0}^{N_1-k} \sum_{r=0}^{k} \binom{k}{r} ((k-r+m)!)^2 p! \mathcal{B}^p \sum_{j=0}^{N_5} \sum_{i=0}^{r} \binom{r}{i} (i!)^2 j! \mathcal{B}^{j+1}, \right. \\ &\left. \sum_{k=0}^{N_3} \sum_{l=0}^{N_3-k} \sum_{m=0}^{N_3-k} \sum_{r=0}^{k} \binom{k}{r} ((k-r+m)!)^2 p! \mathcal{B}^p \sum_{j=0}^{N_6} \sum_{i=0}^{r} \binom{r}{i} (i!)^2 j! \mathcal{B}^{j+1} \right\}. \end{aligned}$$

Lemma 7. *Suppose ((Hyp1). If $(u,v) \in \mathcal{X}$ and $\|(u,v)\| \le \mathcal{B}$, then*

$$|Q_1(u,v)(t,x)|, \quad |Q_2(u,v)(t,x)| \le C_1, \quad (t,x) \in [0,\infty) \times \mathbb{R}.$$

Proof. We have

$$\begin{aligned} &\partial_x \left\{ \sum_{m=0}^{N_1-k} \partial_x^m u^p P_{k,l,m}\left(\partial_x^l v\right) \right\} \\ &= \sum_{m=0}^{N_1-k} \partial_x^k \left(\partial_x^m u^p P_{k,l,m}\left(\partial_x^l v\right) \right) \\ &= \sum_{m=0}^{N_1-k} \sum_{r=0}^{k} \binom{k}{r} \partial_x^{k-r+m} u^p \partial_x^r P_{k,l,m}\left(\partial_x^l v\right) \\ &= \sum_{m=0}^{N_1-k} \sum_{r=0}^{k} \binom{k}{r} \partial_x^{k-r+m} u^p \partial_x^r \sum_{j=0}^{N_5} c_{k,l,m,j} \left(\partial_x^l v\right)^j \\ &= \sum_{m=0}^{N_1-k} \sum_{r=0}^{k} \binom{k}{r} \partial_x^{k-r+m} u^p \sum_{j=0}^{N_5} \sum_{i=0}^{r} \binom{r}{i} \partial_x^{r-i} c_{k,l,m,j} \partial_x \left(\partial_x^l v\right)^j, \end{aligned}$$

$k \in \mathbb{N}, 0 \le k \le N_1$. Since $\mathcal{B} > 1$, we have

$$\left|\partial_x^{r_1} u^{r_2}\right| \le (r_1!)^2 r_2! \mathcal{B}^{r_2},$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 28 April 2023 doi:10.20944/preprints202304.1109.v1

for any $r_1, r_2 \in \mathbb{N}$, $r_1 \leq q$. Then

$$\begin{aligned}
&\left|\partial_x \left\{ \sum_{m=0}^{N_1-k} \partial_x^m u^p P_{k,l,m}\left(\partial_x^l v\right) \right\}\right| \\
&\leq \sum_{m=0}^{N_1-k} \sum_{r=0}^{k} \binom{k}{r} \left|\partial_x^{k-r+m} u^p\right| \sum_{j=0}^{N_5} \sum_{i=0}^{r} \binom{r}{i} \left|\partial_x^{r-i} c_{k,l,m,j}\right| \left|\partial_x^i \left(\partial_x^l v\right)^j\right| \\
&\leq \sum_{m=0}^{N_1-k} \sum_{r=0}^{k} \binom{k}{r} ((k-r+m)!)^2\, p!\mathcal{B}^p \sum_{j=0}^{N_5} \sum_{i=0}^{r} \binom{r}{i} (i!)^2 j! \mathcal{B}^{j+1},
\end{aligned}$$

on $[0,\infty) \times \mathbb{R}$, $0 \leq k \leq N_1$, and then

$$\begin{aligned}
|Q_1(u,v)| &\leq \sum_{k=0}^{N_1} \sum_{l=0}^{N_1-k} \sum_{m=0}^{N_1-k} \sum_{r=0}^{k} \binom{k}{r} ((k-r+m)!)^2\, p! B^p \sum_{j=0}^{N_5} \sum_{i=0}^{r} \binom{r}{i} (i!)^2 j! \mathcal{B}^{j+1} \\
&\leq C_1,
\end{aligned}$$

on $[0,\infty) \times \mathbb{R}$. As above,

$$|Q_2(u,v)| \leq C_1,$$

on $[0,\infty) \times \mathbb{R}$. This completes the proof. □

For $(u,v) \in \mathcal{X}$, define the operators

$$\begin{aligned}
S_1^1(u,v)(t,x) &= u(t,x) - u_0(x) \\
&\quad + \int_0^t \left(Q_1(u,v)(s,x) + \sum_{k=1}^{N_2} a_k(s,x) \partial_x^{2k+1} u(s,x) \right) ds, \\
S_1^2(u,v)(t,x) &= v(t,x) - v_0(x) \\
&\quad + \int_0^t \left(Q_2(u,v)(s,x) + \sum_{k=1}^{N_4} b_k(s,x) \partial_x^{2k+1} v(s,x) \right) ds, \\
S(u,v)(t,x) &= \left(S_1^1(u,v)(t,x), S_1^2(u,v)(t,x) \right),
\end{aligned}$$

$(t,x) \in [0,\infty) \times \mathbb{R}$.

Lemma 8. *Suppose* $((Hyp1)$. *If* $(u,v) \in \mathcal{X}$ *satisfies the equation*

$$S_1(u,v)(t,x) = 0, \quad (t,x) \in [0,\infty) \times \mathbb{R},$$

then (u,v) *is a solution to the IVP* (1).

Proof. We have

$$\begin{aligned} 0 &= u(t,x) - u_0(x) \\ &+ \int_0^t \left(Q_1(u,v)(s,x) + \sum_{k=1}^{N_2} a_k(s,x)\partial_x^{2k+1}u(s,x) \right) ds, \\ 0 &= v(t,x) - v_0(x) \\ &+ \int_0^t \left(Q_2(u,v)(s,x) + \sum_{k=1}^{N_4} b_k(s,x)\partial_x^{2k+1}v(s,x) \right) ds, \end{aligned} \tag{6}$$

$(t,x) \in [0,\infty) \times \mathbb{R}$, which we differentiate with respect to t and we get (5). We put $t = 0$ in (6) and we obtain

$$\begin{aligned} 0 &= u(0,x) - u_0(x) \\ 0 &= v(0,x) - v_0(x), \quad x \in \mathbb{R}. \end{aligned}$$

Thus, (u,v) is a solution to the IVP (1). This completes the proof. □

Let

$$\mathcal{B}_1 = \max\{2\mathcal{B}, C_1 + N_2\mathcal{B}^2, C_1 + N_4\mathcal{B}^2\}.$$

Lemma 9. *Suppose ((Hyp1). If $(u,v) \in \mathcal{X}$ and $\|(u,v)\| \leq \mathcal{B}$, then*

$$\begin{aligned} |S_1^1(u,v)(t,x)| &\leq \mathcal{B}_1(1+t), \\ |S_1^2(u,v)(t,x)| &\leq \mathcal{B}_1(1+t), \quad (t,x) \in [0,\infty) \times \mathbb{R}. \end{aligned}$$

Proof. We have

$$\begin{aligned} |S_1^1(u,v)(t,x)| &= \Bigg| u(t,x) - u_0(x) \\ &+ \int_0^t \left(Q_1(u,v)(s,x) + \sum_{k=1}^{N_2} a_k(s,x)\partial_x^{2k+1}u(s,x) \right) ds \Bigg| \\ &\leq |u(t,x)| + |u_0(x)| \\ &+ \int_0^t \left(|Q_1(u,v)(s,x)| + \sum_{k=1}^{N_2} |a_k(s,x)||\partial_x^{2k+1}u(s,x)| \right) ds \\ &\leq 2\mathcal{B} + \int_0^t (C_1 + N_2\mathcal{B}^2) ds \\ &\leq \mathcal{B}_1(1+t), \quad (t,x) \in [0,\infty) \times \mathbb{R}. \end{aligned}$$

As above,

$$|S_1^2(u,v)(t,x)| \leq \mathcal{B}_1(1+t), \quad (t,x) \in [0,\infty) \times \mathbb{R}.$$

This completes the proof. □

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 28 April 2023 doi:10.20944/preprints202304.1109.v1

Below, suppose that

(Hyp2) there exist a function $g \in \mathcal{C}([0,\infty)\times\mathbb{R})$, $g > 0$ on $(0,\infty)\times(\mathbb{R}\backslash\{0\})$, $g(0,x) = g(t,0) = 0$, $(t,x) \in [0,\infty)\times\mathbb{R}$, and a positive constant $\mathcal{A}$ such that

$$q! \cdot 2^{q+1}(1+t+t^2)(1+|x|+\cdots+|x|^q)\int_0^t \left|\int_0^x g(t_1,x_1)dx_1\right| dt_1 \le \mathcal{A},$$

$(t,x) \in [0,\infty)\times\mathbb{R}$.

In the last section we will give examples for g and $\mathcal{A}$ that satisfy $(Hyp2)$. For $(u,v) \in \mathcal{X}$, define the operators

$$\begin{aligned} S_2^1(u,v)(t,x) &= \int_0^t \int_0^x (t-t_1)(x-x_1)^q g(t_1,x_1) S_1^1(u,v)(t_1,x_1)dx_1dt_1, \\ S_2^2(u,v)(t,x) &= \int_0^t \int_0^x (t-t_1)(x-x_1)^q g(t_1,x_1) S_1^2(u,v)(t_1,x_1)dx_1dt_1, \\ S_2(u,v)(t,x) &= (S_2^1(u,v)(t,x), S_2^2(u,v)(t,x)), \quad (t,x) \in [0,\infty)\times\mathbb{R}. \end{aligned}$$

Lemma 10. *Suppose* $((Hyp1)$ *and* $(Hyp2)$*. If* $(u,v) \in \mathcal{X}$ *satisfies the equation*

$$S_2(u,v)(t,x) = 0, \quad (t,x) \in [0,\infty)\times\mathbb{R},$$

then (u,v) *is a solution to the IVP* (1).

Proof. We differentiate two times in t and $q+1$ times in x the equation (10) and we get

$$g(t,x)S_1^1(u,v)(t,x) = g(t,x)S_1^2(u,v)(t,x) = 0, \quad (t,x) \in [0,\infty)\times\mathbb{R}.$$

Hence,

$$S_1^1(u,v)(t,x) = S_1^2(u,v)(t,x) = 0, \quad (t,x) \in (0,\infty)\times(\mathbb{R}\backslash\{0\}).$$

Since $S_1^1(u,v)(\cdot,\cdot)$ and $S_1^2(u,v)(\cdot,\cdot)$ are continuous functions on $[0,\infty)\times\mathbb{R}$, we have

$$\begin{aligned} 0 &= S_1^1(u,v)(0,x) = S_1^2(u,v)(0,x) \\ &= \lim_{t\to 0} S_1^1(u,v)(t,x) = \lim_{t\to 0} S_1^2(u,v)(t,x) \\ &= \lim_{x\to 0} S_1^1(u,v)(t,x) = \lim_{x\to 0} S_1^2(u,v)(t,x) \\ &= S_1^1(u,v)(t,0) = S_1^2(u,v)(t,0), \quad (t,x) \in [0,\infty)\times\mathbb{R}. \end{aligned}$$

Therefore

$$S_1^1(u,v)(t,x) = S_1^2(u,v)(t,x) = 0, \quad (t,x) \in [0,\infty)\times\mathbb{R}.$$

Now, applying Lemma 8, we get the desired result. □

Lemma 11. *Suppose* $((Hyp1)$ *and* $(Hyp2)$*. If* $(u,v) \in \mathcal{X}$, $\|(u,v)\| \le \mathcal{B}$*, then*

$$\|S_2(u,v)\| \le \mathcal{A}\mathcal{B}_1.$$

Proof. We will use the inequality $(z+w)^r \le 2^r(z^r+w^r)$, $w, z, q \ge 0$. We have

$$\begin{aligned} |S_2^1(u,v)(t,x)| &= \left| \int_0^t \int_0^x (t-t_1)(x-x_1)^q g(t_1,x_1) S_1^1(u,v)(t_1,x_1) dx_1 dt_1 \right| \\ &\le \int_0^t \left| \int_0^x (t-t_1)|x-x_1|^q g(t_1,x_1)|S_1^1(u,v)(t_1,x_1)| dx_1 \right| dt_1 \\ &\le \mathcal{B}_1 \int_0^t \left| \int_0^x (t-t_1)(1+t_1)|x-x_1|^q g(t_1,x_1) dx_1 \right| dt_1 \\ &\le \mathcal{B}_1 t(1+t) 2^{q+1} |x|^q \int_0^t \left| \int_0^x g(t_1,x_1) dx_1 \right| dt_1 \\ &\le \mathcal{A}\mathcal{B}_1, \quad (t,x) \in [0,\infty) \times \mathbb{R}, \end{aligned}$$

and

$$\begin{aligned} |\partial_t S_2^1(u,v)(t,x)| &= \left| \int_0^t \int_0^x (x-x_1)^q g(t_1,x_1) S_1^1(u,v)(t_1,x_1) dx_1 dt_1 \right| \\ &\le \int_0^t \left| \int_0^x |x-x_1|^q g(t_1,x_1)|S_1^1(u,v)(t_1,x_1)| dx_1 \right| dt_1 \\ &\le \mathcal{B}_1 \int_0^t \left| \int_0^x (1+t_1)|x-x_1|^q g(t_1,x_1) dx_1 \right| dt_1 \\ &\le \mathcal{B}_1 (1+t) 2^{q+1} |x|^q \int_0^t \left| \int_0^x g(t_1,x_1) dx_1 \right| dt_1 \\ &\le \mathcal{A}\mathcal{B}_1, \quad (t,x) \in [0,\infty) \times \mathbb{R}, \end{aligned}$$

and

$$\begin{aligned} |\partial_x S_2^1(u,v)(t,x)| &= q\left| \int_0^t \int_0^x (t-t_1)(x-x_1)^{q-1} g(t_1,x_1) S_1^1(u,v)(t_1,x_1) dx_1 dt_1 \right| \\ &\le q \int_0^t \left| \int_0^x (t-t_1)|x-x_1|^{q-1} g(t_1,x_1)|S_1^1(u,v)(t_1,x_1)| dx_1 \right| dt_1 \\ &\le q\mathcal{B}_1 \int_0^t \left| \int_0^x (t-t_1)(1+t_1)|x-x_1|^{q-1} g(t_1,x_1) dx_1 \right| dt_1 \\ &\le q\mathcal{B}_1 t(1+t) 2^q |x|^{q-1} \int_0^t \left| \int_0^x g(t_1,x_1) dx_1 \right| dt_1 \\ &\le \mathcal{A}\mathcal{B}_1, \quad (t,x) \in [0,\infty) \times \mathbb{R}, \end{aligned}$$

and so on. As above,

$$\begin{aligned} |S_2^2(u,v)(t,x)| \le \mathcal{A}\mathcal{B}_1, \quad |\partial_t S_2^2(u,v)(t,x)| &\le \mathcal{A}\mathcal{B}_1, \\ |\partial_x^j S_2^2(u,v)(t,x)| \le \mathcal{A}\mathcal{B}_1, \quad j \in \{1,\dots,q\}. \end{aligned}$$

$(t,x) \in [0,\infty) \times \mathbb{R}$. Thus,

$$\|S_2(u,v)\| \leq \mathcal{A}\mathcal{B}_1.$$

This completes the proof. □

Below, suppose

(Hyp3) $\epsilon \in (0,1)$, $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{B}_1$ satisfy the inequalities $\epsilon\mathcal{B}_1(1+\mathcal{A}) < 1$ and $\mathcal{A}\mathcal{B}_1 < \mathcal{B}$.

Let $\tilde{\tilde{\tilde{\mathcal{Y}}}}$ denote the set of all equi-continuous families in $\mathcal{X}$ with respect to the norm $\|\cdot\|$. Let also, $\tilde{\tilde{\mathcal{Y}}} = \overline{\tilde{\tilde{\tilde{\mathcal{Y}}}}}$ be the closure of $\tilde{\tilde{\tilde{\mathcal{Y}}}}$, $\tilde{\mathcal{Y}} = \tilde{\tilde{\mathcal{Y}}} \cup \{(u_0,v_0)\}$,

$$\mathcal{Y} = \{(u,v) \in \tilde{\mathcal{Y}} : (u,v) \geq 0, \quad \|(u,v)\| \leq \mathcal{B}\}.$$

Note that $\mathcal{Y}$ is a compact set in $\mathcal{X}$. For $(u,v) \in \mathcal{X}$, define the operators

$$\begin{aligned} \mathcal{T}(u,v)(t,x) &= -\epsilon(u,v)(t,x), \\ S(u,v)(t,x) &= (u,v)(t,x) + \epsilon(u,v)(t,x) + \epsilon S_2(u,v)(t,x), \quad (t,x) \in [0,\infty) \times \mathbb{R}. \end{aligned}$$

For $(u,v) \in \mathcal{Y}$, using Lemma 10, we have

$$\begin{aligned} \|(I-S)(u,v)\| &= \|\epsilon(u,v) - \epsilon S_2(u,v)\| \\ &\leq \epsilon\|(u,v)\| + \epsilon\|S_2(u,v)\| \\ &\leq \epsilon\mathcal{B}_1 + \epsilon\mathcal{A}\mathcal{B}_1 \\ &= \epsilon\mathcal{B}_1(1+\mathcal{A}) \\ &< \mathcal{B}. \end{aligned}$$

Thus, $S : \mathcal{Y} \to \mathcal{X}$ is continuous and $(I-S)(\mathcal{Y})$ resides in a compact subset of $\mathcal{X}$. Now, suppose that there is a $(u,v) \in \mathcal{X}$ so that $\|(u,v)\| = \mathcal{B}$ and

$$(u,v) = \lambda(I-S)(u,v),$$

or

$$\frac{1}{\lambda}(u,v) = (I-S)(u,v) = -\epsilon(u,v) - \epsilon S_2(u,v),$$

or

$$\left(\frac{1}{\lambda} + \epsilon\right)(u,v) = -\epsilon S_2(u,v),$$

for some $\lambda \in \left(0, \frac{1}{\epsilon}\right)$. Hence, $\|S_2(u,v)\| \leq \mathcal{A}\mathcal{B}_1 < \mathcal{B}$,

$$\epsilon\mathcal{B} < \left(\frac{1}{\lambda} + \epsilon\right)\mathcal{B} = \left(\frac{1}{\lambda} + \epsilon\right)\|(u,v)\| = \epsilon\|S_2(u,v)\| < \epsilon\mathcal{B},$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 28 April 2023 doi:10.20944/preprints202304.1109.v1

which is a contradiction. Hence and Theorem 3, it follows that the operator $\mathcal{T} + S$ has a fixed point $(u^*, v^*) \in \mathcal{Y}$. Therefore

$$\begin{aligned}(u^*, v^*)(t, x) &= \mathcal{T}(u^*, v^*)(t, x) + S(u^*, v^*)(t, x) \\ &= -\epsilon(u^*, v^*)(t, x) + (u^*, v^*)(t, x) + \epsilon(u^*, v^*)(t, x) + \epsilon S_2(u^*, v^*)(t, x),\end{aligned}$$

$(t, x) \in [0, \infty) \times \mathbb{R}$, whereupon

$$0 = S_2(u^*, v^*)(t, x), \quad (t, x) \in [0, \infty) \times \mathbb{R}.$$

From here and from Lemma 10, it follows that (u^*, v^*) is a solution to the IVP (1). This completes the proof.

4. Proof of Theorem 2

Let $\mathcal{X}$ be the space used in the previous section. Suppose

(Hyp4) Let $m > 0$ be large enough and $\mathcal{A}$, $\mathcal{B}$, r, L, R_1 be positive constants that satisfy the following conditions

$$r < L < R_1 \leq \mathcal{B}, \quad \epsilon > 0, \quad R_1 > \left(\frac{2}{5m} + 1\right) L,$$

$$\mathcal{A}\mathcal{B}_1 < \frac{L}{5}.$$

Let

$$\widetilde{P} = \{(u, v) \in \mathcal{X} : (u, v) \geq 0 \quad \text{on} \quad [0, \infty) \times \mathbb{R}\}.$$

With $\mathcal{P}$ we will denote the set of all equi-continuous families in $\widetilde{P}$. For $(u, v) \in \mathcal{X}$, define the operators

$$\begin{aligned}\mathcal{T}_1(u, v)(t, x) &= (1 + m\epsilon)(u, v)(t, x) - \left(\epsilon\frac{L}{10}, \epsilon\frac{L}{10}\right), \\ S_3(u, v)(t, x) &= -\epsilon S_2(u, v)(t, x) - m\epsilon(u, v)(t, x) - \left(\epsilon\frac{L}{10}, \epsilon\frac{L}{10}\right),\end{aligned}$$

$t \in [0, \infty)$. Note that any fixed point $(u, v) \in \mathcal{X}$ of the operator $\mathcal{T}_1 + S_3$ is a solution to the IVP (1). Define

$$\begin{aligned}\mathcal{U}_1 &= \mathcal{P}_r = \{(u, v) \in \mathcal{P} : \|(u, v)\| < r\}, \\ \mathcal{U}_2 &= \mathcal{P}_L = \{(u, v) \in \mathcal{P} : \|(u, v)\| < L\}, \\ \mathcal{U}_3 &= \mathcal{P}_{R_1} = \{(u, v) \in \mathcal{P} : \|(u, v)\| < R_1\}, \\ R_2 &= R_1 + \frac{\mathcal{A}}{m}\mathcal{B}_1 + \frac{L}{5m}, \\ \Omega &= \overline{\mathcal{P}_{R_2}} = \{(u, v) \in \mathcal{P} : \|(u, v)\| \leq R_2\}.\end{aligned}$$

1. For $(u_1, v_1), (u_2, v_2) \in \Omega$, we have

$$\|\mathcal{T}_1(u_1, v_1) - \mathcal{T}_1(u_2, v_2)\| = (1 + m\epsilon)\|(u_1, v_1) - (u_2, v_2)\|,$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 28 April 2023 doi:10.20944/preprints202304.1109.v1

whereupon $\mathcal{T}_1 : \Omega \to \mathcal{X}$ is an expansive operator with a constant $h = 1 + m\varepsilon > 1$.

2. For $(u, v) \in \overline{\mathcal{P}}_{R_1}$, we get

$$\begin{aligned} \|S_3(u,v)\| &\leq \varepsilon\|S_2(u,v)\| + m\varepsilon\|(u,v)\| + \varepsilon\frac{L}{10} \\ &\leq \varepsilon\Big(\mathcal{A}\mathcal{B}_1 + mR_1 + \frac{L}{10}\Big). \end{aligned}$$

Therefore $S_3(\overline{\mathcal{P}}_{R_1})$ is uniformly bounded. Since $S_3 : \overline{\mathcal{P}}_{R_1} \to \mathcal{X}$ is continuous, we have that $S_3(\overline{\mathcal{P}}_{R_1})$ is equi-continuous. Consequently $S_3 : \overline{\mathcal{P}}_{R_1} \to \mathcal{X}$ is a 0-set contraction.

3. Let $(u_1, v_1) \in \overline{\mathcal{P}}_{R_1}$. Set

$$(u_2, v_2) = (u_1, v_1) + \frac{1}{m}S_2(u_1, v_1) + \left(\frac{L}{5m}, \frac{L}{5m}\right).$$

Note that $S_2 u_1 + \frac{L}{5} \geq 0$, $S_2 v_1 + \frac{L}{5} \geq 0$ on $[0, \infty) \times \mathbb{R}$. We have $u_2, v_2 \geq 0$ on $[0, \infty) \times \mathbb{R}$ and

$$\begin{aligned} \|(u_2,v_2)\| &\leq \|(u_1,v_1)\| + \frac{1}{m}\|S_2(u_1,v_1)\| + \frac{L}{5m} \\ &\leq R_1 + \frac{\mathcal{A}}{m}\mathcal{B}_1 + \frac{L}{5m} \\ &= R_2. \end{aligned}$$

Therefore $(u_2, v_2) \in \Omega$ and

$$-\varepsilon m(u_2, v_2) = -\varepsilon m(u_1, v_1) - \varepsilon S_2(u_1, v_1) - \varepsilon\left(\frac{L}{10}, \frac{L}{10}\right) - \varepsilon\left(\frac{L}{10}, \frac{L}{10}\right)$$

or

$$\begin{aligned} (I - \mathcal{T}_1)(u_2, v_2) &= -\varepsilon m(u_2, v_2) + \varepsilon\left(\frac{L}{10}, \frac{L}{10}\right) \\ &= S_3(u_1, v_1). \end{aligned}$$

Consequently $S_3(\overline{\mathcal{P}}_{R_1}) \subset (I - \mathcal{T}_1)(\Omega)$.

4. Assume that for any $(u_0, v_0) \in \mathcal{P}^*$ there exist $\lambda \geq 0$ and $(u, v) \in \partial\mathcal{P}_r \cap (\Omega + \lambda(u_0, v_0))$ or $v \in \partial\mathcal{P}_{R_1} \cap (\Omega + \lambda(u_0, v_0))$ such that

$$S_3(u, v) = (I - \mathcal{T}_1)((u, v) - \lambda(u_0, v_0)).$$

Then

$$-\epsilon S_2(u, v) - m\epsilon(u, v) - \epsilon\left(\frac{L}{10}, \frac{L}{10}\right) = -m\epsilon((u, v) - \lambda(u_0, v_0)) + \epsilon\left(\frac{L}{10}, \frac{L}{10}\right),$$

or

$$-S_2(u, v) = \lambda m(u_0, v_0) + \left(\frac{L}{5}, \frac{L}{5}\right).$$

Hence,

$$\|S_2 v\| = \left\|\lambda m(u_0, v_0) + \left(\frac{L}{5}, \frac{L}{5}\right)\right\| > \frac{L}{5}.$$

This is a contradiction.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 28 April 2023 doi:10.20944/preprints202304.1109.v1

5. Suppose that for any $\epsilon_1 \geq 0$ small enough there exist a $(u_1, v_1) \in \partial\mathcal{P}_L$ and $\lambda_1 \geq 1 + \epsilon_1$ such that $\lambda_1(u_1, v_1) \in \overline{\mathcal{P}}_{R_1}$ and
$$S_3(u_1, v_1) = (I - \mathcal{T}_1)(\lambda_1(u_1, v_1)). \tag{7}$$
In particular, for $\epsilon_1 > \frac{2}{5m}$, we have $(u_1, v_1) \in \partial\mathcal{P}_L$, $\lambda_1(u_1, v_1) \in \overline{\mathcal{P}}_{R_1}$, $\lambda_1 \geq 1 + \epsilon_1$ and (7) holds. Since $(u_1, v_1) \in \partial\mathcal{P}_L$ and $\lambda_1(u_1, v_1) \in \overline{\mathcal{P}}_{R_1}$, it follows that
$$\left(\frac{2}{5m} + 1\right) L < \lambda_1 L = \lambda_1 \|(u_1, v_1)\| \leq R_1.$$
Moreover,
$$-\epsilon S_2(u_1, v_1) - m\epsilon(u_1, v_1) - \epsilon\left(\frac{L}{10}, \frac{L}{10}\right) = -\lambda_1 m\epsilon(u_1, v_1) + \epsilon\left(\frac{L}{10}, \frac{L}{10}\right),$$
or
$$S_2(u_1, v_1) + \left(\frac{L}{5}, \frac{L}{5}\right) = (\lambda_1 - 1)m(u_1, v_1).$$
From here,
$$2\frac{L}{5} \geq \left\| S_2(u_1, v_1) + \left(\frac{L}{5}, \frac{L}{5}\right) \right\| = (\lambda_1 - 1)m\|(u_1, v_1)\| = (\lambda_1 - 1)mL,$$
and
$$\frac{2}{5m} + 1 \geq \lambda_1,$$
which is a contradiction.

Therefore all conditions of Theorem 2 hold. Hence, the IVP (1) has at least two solutions (u_1, v_1) and (u_2, v_2) so that
$$\|(u_1, v_1)\| = L < \|(u_2, v_2)\| < R_1,$$
or
$$r < \|(u_1, v_1)\| < L < \|(u_2, v_2)\| < R_1.$$

5. Example

Below, we will illustrate our main results. Let $\mathcal{B} = 1$ and
$$R_1 = 10, \quad L = 5, \quad r = 4, \quad m = 10^{50}, \quad \mathcal{A} = \frac{1}{10\mathcal{B}_1}, \quad \epsilon = \frac{1}{5\mathcal{B}_1(1 + \mathcal{A})},$$
$N_j = 5$, $j \in \{1, \ldots, 4\}$, $p = 10$. Then
$$\mathcal{A}\mathcal{B}_1 = \frac{1}{10} < \mathcal{B}, \quad \epsilon\mathcal{B}_1(1 + \mathcal{A}) < 1,$$
i.e., $(Hyp3)$ holds. Next,
$$r < L < R_1, \quad \epsilon > 0, \quad R_1 > \left(\frac{2}{5m} + 1\right) L, \quad \mathcal{A}\mathcal{B}_1 < \frac{L}{5}.$$
i.e., $(Hyp4)$ holds. Take
$$h(s) = \log \frac{1 + s^{q+1}\sqrt{2} + s^{2q+2}}{1 - s^{q+1}\sqrt{2} + s^{2q+2}}, \quad l(s) = \arctan \frac{s^{q+1}\sqrt{2}}{1 - s^{2q+2}}, \quad s \in \mathbb{R}, \quad s \neq \pm 1.$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 28 April 2023 doi:10.20944/preprints202304.1109.v1

Then

$$\begin{aligned} h'(s) &= \frac{2\sqrt{2}(q+1)s^q(1-s^{2q+2})}{(1-s^{q+1}\sqrt{2}+s^{2q+2})(1-s^{q+1}\sqrt{2}+s^{2q+2})}, \\ l'(s) &= \frac{(q+1)\sqrt{2}s^q(1+s^{2q+2})}{1+s^{4q+4}}, \quad s\in\mathbb{R}, \quad s\neq\pm1. \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{s\to\pm\infty}\sum_{r=0}^{q+1}s^r h(s) &= \lim_{s\to\pm\infty}\frac{h(s)}{\frac{1}{\sum_{r=0}^{l+1}s^r}} \\ &= \lim_{s\to\pm\infty}\frac{h'(s)}{-\frac{\sum_{r=0}^{q}(r+1)s^r}{\left(\sum_{r=0}^{q+1}s^r\right)^2}} \\ &= -\lim_{s\to\pm\infty}\frac{2\sqrt{2}(q+1)s^q(1-s^{2q+2})\left(\sum_{r=0}^{q+1}s^r\right)^2}{\left(\sum_{r=0}^{q}(r+1)s^r\right)(1-s^{q+1}\sqrt{2}+s^{2q+2})(1-s^{q+1}\sqrt{2}+s^{2q+2})} \\ &\neq \pm\infty, \end{aligned}$$

and

$$\begin{aligned} \lim_{s\to\pm\infty}\sum_{r=0}^{q+1}s^r l(s) &= \lim_{s\to\pm\infty}\frac{l(s)}{\frac{1}{\sum_{r=0}^{q+1}s^r}} \\ &= \lim_{s\to\pm\infty}\frac{l'(s)}{-\frac{\sum_{r=0}^{q}(r+1)s^r}{\left(\sum_{r=0}^{q+1}s^r\right)^2}} \\ &= -\lim_{s\to\pm\infty}\frac{(q+1)\sqrt{2}s^q(1+s^{2q+2})\left(\sum_{r=0}^{q+1}s^r\right)^2}{(1+s^{4q+4})\left(\sum_{r=0}^{q}(r+1)s^r\right)} \\ &\neq \pm\infty. \end{aligned}$$

Consequently

$$\begin{aligned} -\infty &< \lim_{s\to\pm\infty}\left(\sum_{r=0}^{q+1}s^r\right)h(s) < \infty, \\ -\infty &< \lim_{s\to\pm\infty}\left(\sum_{r=0}^{q+1}s^r\right)l(s) < \infty. \end{aligned}$$

Hence, there exists a positive constant C_2 so that

$$\sum_{r=0}^{q+1}|s|^r\left(\frac{1}{(4q+4)\sqrt{2}}\log\frac{1+s^{q+1}\sqrt{2}+s^{2q+2}}{1-s^{q+1}\sqrt{2}+s^{2q+2}}+\frac{1}{(2q+2)\sqrt{2}}\arctan\frac{s^{q+1}\sqrt{2}}{1-s^{2q+2}}\right) \leq C_2,$$

$s \in \mathbb{R}$. Note that $\lim_{s\to\pm 1} l(s) = \frac{\pi}{2}$ and by [9] (pp. 707, Integral 79), we have

$$\int \frac{dz}{1+z^4} = \frac{1}{4\sqrt{2}} \log \frac{1+z\sqrt{2}+z^2}{1-z\sqrt{2}+z^2} + \frac{1}{2\sqrt{2}} \arctan \frac{z\sqrt{2}}{1-z^2}.$$

Let

$$Q(s) = \frac{s^q}{(1+s^{4q+4})}, \quad s \in \mathbb{R},$$

and

$$g_1(t,x,y) = Q(t)Q(x), \quad t \in [0,\infty), \quad x \in \mathbb{R}.$$

Then there exists a constant $C > 0$ such that

$$2^{q+1}(q+1)!(1+t+t^2)\left(\sum_{r=0}^{q} |x|^r\right) \int_0^t \left| \int_0^x g_1(t_1,x_1)dx_1 \right| dt_1 \le C, \quad (t,x) \in [0,\infty) \times \mathbb{R}.$$

Let

$$g(t,x) = \frac{\mathcal{A}}{C} g_1(t,x), \quad (t,x) \in [0,\infty) \times \mathbb{R}.$$

Then

$$2^{q+1}q!(1+t+t^2)\left(\sum_{r=0}^{q} |x|^r\right) \int_0^t \left| \int_0^x g(t_1,x_1)dx_1 \right| dt_1 \le \mathcal{A}, \quad (t,x) \in [0,\infty) \times \mathbb{R}.$$

i.e., $(Hyp3)$ holds. Therefore for the IVP

$$\begin{aligned}
\partial_t u + \sum_{k=0}^{5}\sum_{l=0}^{5-k} \partial_x \left\{ \sum_{m=0}^{5-k} \partial_x^m u^{10} \partial_x^l v \right\} + \sum_{k=1}^{5} \frac{1}{(1+t^{2k})(1+x^{2k})} \partial_x^{2k+1} u &= 0 \\
\partial_t v + \sum_{k=0}^{5}\sum_{l=0}^{5-k} \partial_x^k \left\{ \sum_{m=0}^{5-k} \partial_x^m v^{10} \partial_x^l u \right\} + \sum_{k=1}^{5} \frac{1}{(2+t^{4k})(3+x^{6k})} \partial_x^{2k+1} v &= 0, \\
(t,x) \in [0,\infty) \times \mathbb{R}, \quad u(0,x) = \frac{1}{1+x^4}, \quad v(0,x) = \frac{1}{3+4x^8}, \quad x \in \mathbb{R},
\end{aligned}$$

are fulfilled all conditions of Theorem 1 and Theorem 2.

Funding: This research received no external funding.

Acknowledgments: The authors extend their appreciation to the Deanship of Scientific Research at Imam Mohammad Ibn Saud Islamic University for funding this work through Research Group no. RG-21-09-51

Conflicts of Interest: Authors declare that they have no conflict of interest.

References

1. C.Kondo and R.Pes. Well-Posedness for a Coupled System of Kawahara/KdV Type Equations. Applied Mathematics & Optimization, 2021.
2. A. Boukarou, K. Guerbati, K. Zennir, S. Alodhaibi and S. Alkhalaf, Well-posedness and time regularity for a system of modified Korteweg-de Vries-type equations in analytic Gevrey spaces, Mathematics, vol. 8, no. 809, pp. 1-16, 2020.
3. X. Carvajal and M. Panthee. Sharp well-posedness for a coupled system of mKdV-type equations, Journal of Evolution Equations, vol. 19, no. 4, pp. 1167–1197, 2019.
4. E.Alarcon, J.Angulo and J. Montenegro. Stability and instability of solitary waves for a nonlinear dispersive system. Nonlinear Anal. 36, 1015-1035 (1999).

 doi:10.20944/preprints202304.1109.v1

5. Y. Jia and Z. Huo. Well-posedness for the fifth-order shallow water equations, J. Differential Equations 246 (2009), 2448–2467.
6. J. Banas and K. Goebel. Measures of Noncompactness in Banach Spaces, Lecture Notes in Pure and Applied Mathematics, **60**. Marcel Dekker, Inc., New York, 1980.
7. P. Drabek and J. Milota. Methods in Nonlinear Analysis, Applications to Differential Equations, Birkhäuser, 2007.
8. S. Djebali and K. Mebarki. Fixed point index theory for perturbation of expansive mappings by *k*-set contractions, Top. Meth. Nonli. Anal., Vol 54, No 2 A(2019), 613–640.
9. A. Polyanin and A. Manzhirov. Handbook of integral equations, CRC Press, 1998.