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Posted Date: 24 April 2023

doi: 10.20944/preprints202304.0844.v1

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Extended Exton's Triple and Horn's Double Hypergeometric Functions and Associated Bounding Inequalities

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Abstract: This paper introduces extensions $H_{4,p}$ and $X_{8,p}$ of Horn's double hypergeometric function H_4 and Exton's triple hypergeometric function X_8, taking into account recent extensions of Euler's beta function, hypergeometric function, and confluent hypergeometric function. Among the numerous extended hypergeometric functions, the primary rationale for choosing H_4 and X_8 is their comparable extension type. Next, we present various integral representations of Euler and Laplace type, Mellin and inverse Mellin transforms, Laguerre polynomial representation, transformation formulae and a recurrence relation for the extended functions. In particular, we provide a generating function for the $X_{8,p}$ and several bounding inequalities for the $H_{4,p}$ and $X_{8,p}$.

Keywords: extended Beta function; extended hypergeometric function; extended confluent hypergeometric function; extended Appell function; Mellin transforms; inverse Mellin transforms; H-functions; Laguerre polynomials; transformation formulas; recurrence relation; generating function; bounding inequalities

MSC: 33B20; 33C20; 33B15; 33C05

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1. Introduction and Preliminaries

The generalized hypergeometric function with r numerator and s denominator parameters, as the series, reads

$$ {}_rF_s(\tau_1,\cdots,\tau_r;\upsilon_1,\cdots,\upsilon_s;z) = {}_rF_s(\tau_r;\upsilon_s;z) := \sum_{m=0}^{\infty} \frac{(\tau_1)_m\cdots(\tau_r)_m}{(\upsilon_1)_m\cdots(\upsilon_s)_m}\frac{z^m}{m!}, \tag{1} $$

where $(\mu)_n = \mu(\mu+1)\cdots(\mu+n-1)$ $(n\in\mathbb{N})$ and $(\mu)_0 = 1$ signify the Pochhammer symbol, $\tau_j\in\mathbb{C}$, and $\upsilon_j\in\mathbb{C}\setminus\mathbb{Z}_{\leqslant 0}$, $j\in\overline{1,s}:=\{1,2,\ldots,s\}$ for some $s\in\mathbb{N}$. The symbol $(\mu)_m$ is represented by $(\mu)_m = \Gamma(\mu+m)/\Gamma(\mu)$ $(\mu\in\mathbb{C}\setminus\mathbb{Z}_{\leqslant 0},\ m\in\mathbb{N}_0)$, Γ being the familiar Gamma function whose acquainted integral is

$$ \Gamma(\mu) = \int_0^\infty e^{-t}\,t^{\mu-1}\,dt \quad (\Re(\mu)>0). \tag{2} $$

In this and other instances, the sets of positive integers, integers, real numbers, and complex numbers will be denoted by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$, respectively. Also let $\mathbb{N}_0:=\mathbb{N}\cup\{0\}$ and $\mathbb{Z}_{\leqslant 0}:=\mathbb{Z}\setminus\mathbb{N}$. The series in (1) converges for all $z\in\mathbb{C}$ if $r\leqslant s$. It is divergent for all $z\in\mathbb{C}\setminus\{0\}$ when $r>s+1$, unless at least one numerator parameter is in $\mathbb{Z}_{\leqslant 0}$ in which case (1) is a polynomial. For the remaining case $r=s+1$, the series converges on the unit circle $|z|=1$ under the constraints $\tau_j\in\mathbb{C}\setminus\mathbb{Z}_{\leqslant 0}$ $(j\in\overline{1,r})$ and

$$ \Re\left(\sum_{j=1}^{s}\upsilon_j - \sum_{j=1}^{r}\tau_j\right) > 0. $$

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For the noted particular cases, ${}_2F_1$ is called the (Gauss's) hypergeometric function and ${}_1F_1$, which is also denoted by Φ, is referred to as the confluent (Kummer's) hypergeometric function.

In 1997, Chaudhry et al. [1, p. 20, Eq. (1.7)] introduced and explored the p-extended Beta integral:

$$\mathrm{B}(\eta,\,\xi\,;\,p) := \int_0^1 t^{\eta-1}\,(1-t)^{\xi-1}\,\exp\left(-\frac{p}{t(1-t)}\right)\,dt \quad (\Re(p)>0), \tag{3}$$

from which follows a series of investigations of generalized incomplete gamma functions and their applications (see [2–4]; see also [5]). The p-extended Beta integral in (3) is turned out to be connected to the Macdonald, error and Whittaker functions. The case $p=0$ of (3) becomes the classical Beta function given by (see, for example, [6, p. 8, Eq. (43)]):

$$\mathrm{B}(\eta,\,\xi) = \begin{cases} \displaystyle\int_0^1 t^{\eta-1}(1-t)^{\xi-1}\,dt & (\Re(\eta)>0,\ \Re(\xi)>0) \\ \dfrac{\Gamma(\eta)\,\Gamma(\xi)}{\Gamma(\eta+\xi)} & (\eta,\,\xi\in\mathbb{C}\setminus\mathbb{Z}_{\leqslant 0})\,. \end{cases} \tag{4}$$

Making use of the subsequent transformation

$$\frac{(v)_m}{(\phi)_m} = \frac{\mathrm{B}(v+m,\phi-v)}{\mathrm{B}(v,\phi-v)} \quad (\Re(\phi)>\Re(v)>0,\ m\in\mathbb{N}_0)\,, \tag{5}$$

in which the numerator Beta function is replaced by the p-extended Beta function in (3), Chaudhry et al. [7] introduced the p-Gauss's hypergeometric function and the p-Kummer's confluent hypergeometric function which are, respectively, given as follows:

$$F_p\,(\tau,\,v;\,\phi;\,z) := \sum_{m=0}^{\infty}(\tau)_m\,\frac{\mathrm{B}(v+m,\,\phi-v\,;\,p)}{\mathrm{B}(v,\,\phi-v)}\,\frac{z^m}{m!} \tag{6}$$

$$(p\geqslant 0,\ |z|<1\,;\,\Re(\phi)>\Re(v)>0)$$

and

$$\Phi_p\,(v;\,\phi;\,z) := \sum_{m=0}^{\infty}\frac{\mathrm{B}(v+m,\,\phi-v\,;\,p)}{\mathrm{B}(v,\,\phi-v)}\,\frac{z^m}{m!} \quad (p\geqslant 0\,;\,\Re(\phi)>\Re(v)>0). \tag{7}$$

The functions were studied by Chaudhry et al. [7], who revealed numerous intriguing identities and formulas. These include integral representations, differentiation properties, Mellin transforms, transformations, recurrence relations, summation formulas, and asymptotic formulas.

Özarslan and Özergin [8] introduced and investigated the p-extensions of two variable Appell's hypergeometric functions F_1 and F_2 and three variable Lauricella's hypergeometric function $F_D^{(3)}$ (see, for example, [9, Chapter 1]), among which, the p-extended F_2 function is recalled:

$$\begin{aligned} &F_2(\tau,v,v';\phi,\phi';x,y;p) \\ &\quad= \sum_{m,n=0}^{\infty}(\tau)_{m+n}\,\frac{\mathrm{B}(v+m,\phi-v;p)\mathrm{B}(v'+n,\phi'-v';p)}{\mathrm{B}(v,\phi-v)\mathrm{B}(v',\phi'-v')}\,\frac{x^m}{m!}\,\frac{y^n}{n!} \end{aligned} \tag{8}$$

$$(|x|+|y|<1;\ \Re(p)\geqslant 0).$$

They [8] also introduced a new extended Riemann–Liouville fractional derivative to present several intriguing generating relations for the p-Gauss's hypergeometric function (6).

Like (3), the p-extensions (6), (7) and (8) when $p=0$ return to Gauss's hypergeometric function ${}_2F_1$, the confluent hypergeometric function ${}_1F_1$ and Appell's hypergeometric function F_2 of two variables, respectively.

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Our investigation is primarily motivated by the vast range of potential applications of extended Gauss's hypergeometric, confluent hypergeometric, and Appell functions in various fields of mathematical, physical, engineering, and statistical sciences (as detailed in [1,7,8]) and the references therein). In this study, we undertake a systematic exploration of the extended Horn's double hypergeometric function $H_{4,p}$ and extended Exton's triple hypergeometric functions $X_{8,p}$. Specifically, we aim to present various integral representations of Euler and Laplace type, as well as certain integral representations involving Bessel and modified Bessel functions, Mellin transform, Laguerre polynomial representation, transformation formula, and recurrence relation. Additionally, we provide a generating function for the $X_{8,p}$ and several bounding inequalities for the$H_{4,p}$ and $X_{8,p}$.

2. Extended Horn's Double Hypergeometric Function

In terms of the extended Beta function $B(\eta, \xi; p)$ in (3), this section introduces the following extended Horn's double hypergeometric function $H_{4,p}$: For τ, $\upsilon \in \mathbb{C}$ and $\phi, \phi' \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0}$,

$$H_{4,p}[\tau, \upsilon; \phi, \phi'; x, y] := \sum_{k,m=0}^{\infty} \frac{(\tau)_{2k+m}}{(\phi)_k} \frac{\mathrm{B}(\upsilon + m, \phi' - \upsilon; p)}{\mathrm{B}(\upsilon, \phi' - \upsilon)} \frac{x^k}{k!} \frac{y^m}{m!} \qquad (9)$$
$$\left(p > 0;\ 2\sqrt{r_1} + r_2 < 1,\ |x| \leqslant r_1,\ |y| \leqslant r_2 \ \text{ when } \ p = 0\right),$$

where $\Re(\phi') > \Re(\upsilon) > 0$. In light of (5), the case $p = 0$ in (9) gives the classical Horn's double hypergeometric function H_4 (see, for example, [9, p. 24 and p. 59], [10]): For τ, $\upsilon \in \mathbb{C}$ and $\phi, \phi' \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0}$,

$$H_4[\tau, \upsilon; \phi, \phi'; x, y] = \sum_{k,m=0}^{\infty} \frac{(\tau)_{2k+m} (\upsilon)_m}{(\phi)_k (\phi')_m} \frac{x^k}{k!} \frac{y^m}{m!} \qquad (10)$$
$$\left(2\sqrt{r_1} + r_2 < 1,\ |x| \leqslant r_1,\ |y| \leqslant r_2\right).$$

2.1. Integral Representations

Theorem 1. *The following integral representation for $H_{4,p}$ in (9) holds true:*

$$H_{4,p}[\tau, \upsilon; \phi, \phi'; x, y] = \frac{1}{\mathrm{B}(\upsilon, \phi' - \upsilon)} \int_0^1 \frac{u^{\upsilon-1}(1-u)^{\phi'-\upsilon-1}}{(1-yu)^{\tau}}$$
$$\times\ {}_2F_1\left[\frac{\tau}{2}, \frac{\tau}{2} + \frac{1}{2}; \phi; \frac{4x}{(1-yu)^2}\right] \exp\left(-\frac{p}{u(1-u)}\right) du \qquad (11)$$
$$\left(\Re(p) > 0;\ \Re(\phi') > \Re(\upsilon) > 0 \ \textit{when} \ p = 0\right).$$

Proof. By making use of the identity

$$(\tau)_{2k+m} = (\tau)_{2k} (\tau + 2k)_m$$

and the extended Gauss's hypergeometric function (6), the extended Horn's double hypergeometric function (9) can be expressed as a single series:

$$H_{4,p}[\tau, \upsilon; \phi; \phi'; x, y] = \sum_{k=0}^{\infty} \frac{(\tau)_{2k}}{(\phi)_k} F_p \left[\begin{array}{r} \tau + 2k, \upsilon; \\ \phi'; \end{array} y \right] \frac{x^k}{k!}. \qquad (12)$$

Applying the integral representation of the extended Gauss's hypergeometric function [7, p. 592, Eq. (3.2)]

$$F_p(\tau, \upsilon; \phi; z) = \frac{1}{\mathrm{B}(\upsilon, \phi - \upsilon)} \int_0^1 u^{\upsilon-1}(1-u)^{\phi-\upsilon-1}(1-zu)^{-\tau} \exp\left(-\frac{p}{u(1-u)}\right) du \qquad (13)$$

 doi:10.20944/preprints202304.0844.v1

$$\big(\Re(p) > 0;\ p = 0 \text{ and } |\arg(1-z)| < \pi;\ \Re(\phi) > \Re(v) > 0\big)$$

to (12), one finds

$$\begin{aligned} H_{4,p}[\tau, v; \phi; \phi'; x, y] = \sum_{k=0}^{\infty} \int_0^1 & \frac{u^{v-1}(1-u)^{\phi'-v-1}}{\mathrm{B}(v, \phi'-v)\,(1-yu)^{\tau}} \\ & \times \exp\left(-\frac{p}{u(1-u)}\right) \frac{(\tau)_{2k}}{(\phi)_k\, k!} \left\{\frac{x}{(1-yu)^2}\right\}^k du. \end{aligned} \tag{14}$$

Changing the order of summation and integration in (14), which is guaranteed under the restrictions, and using the identity

$$(\tau)_{2k} = 2^{2k} \left(\frac{\tau}{2}\right)_k \left(\frac{\tau+1}{2}\right)_k \quad (\tau \in \mathbb{C},\ k \in \mathbb{N}_0) \tag{15}$$

and the Gauss's hypergeometric function ${}_2F_1$, we get the desired integral representation (11). □

The following corollary is obtained by setting $p = 0$ in Theorem 1.

Corollary 1. *The following integral representation for H_4 holds true:*

$$\begin{aligned} H_4[\tau, v; \phi, \phi'; x, y] = & \frac{1}{\mathrm{B}(v, \phi'-v)} \int_0^1 \frac{u^{v-1}(1-u)^{\phi'-v-1}}{(1-yu)^{\tau}} \\ & \times {}_2F_1\left[\frac{\tau}{2}, \frac{\tau}{2} + \frac{1}{2}; \phi; \frac{4x}{(1-yu)^2}\right] du \end{aligned} \tag{16}$$

$$\big(\Re(\phi') > \Re(v) > 0\big),$$

where the additional restrictions for the other parameters and variables would follow from those in (10).

Theorem 2. *The following Laplace type integral representation for $H_{4,p}$ in (9) holds true:*

$$H_{4,p}[\tau, v; \phi, \phi'; x, y] = \frac{1}{\Gamma(\tau)} \int_0^{\infty} t^{\tau-1} e^{-t}\, {}_0F_1(-; \phi; xt^2)\, \Phi_p(v; \phi'; yt)\, dt \tag{17}$$

$$\big(\Re(p) > 0;\ \Re(\tau) > 0 \text{ when } p = 0\big).$$

Proof. Using the integral representation

$$(\tau)_n := \frac{1}{\Gamma(\tau)} \int_0^{\infty} t^{\tau+n-1}\, e^{-t}\, dt \quad \big(\Re(\tau) > 0,\ n \in \mathbb{N}_0\big) \tag{18}$$

for the Pochhammer symbol $(\tau)_{2k+m}$ in (9) and interchanging the order of summations and integral, we get

$$\begin{aligned} & H_{4,p}[\tau, v; \phi, \phi'; x, y] \\ & = \frac{1}{\Gamma(\tau)} \sum_{k,m=0}^{\infty} \int_0^{\infty} t^{\tau-1}\, e^{-t}\, \frac{1}{(\phi)_k} \frac{\mathrm{B}(v+m, \phi'-v; p)}{\mathrm{B}(v, \phi'-v)} \frac{(xt^2)^k}{k!} \frac{(yt)^m}{m!}\, dt \\ & = \frac{1}{\Gamma(\tau)} \int_0^{\infty} t^{\tau-1}\, e^{-t} \left(\sum_{k=0}^{\infty} \frac{1}{(\phi)_k} \frac{(xt^2)^k}{k!}\right) \left(\sum_{m=0}^{\infty} \frac{\mathrm{B}(v+m, \phi'-v; p)}{\mathrm{B}(v, \phi'-v)} \frac{(yt)^m}{m!}\right) dt. \end{aligned}$$

Now, using (1) and (7) in each summation enclosed in parentheses yields the desired result (17). □

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Remark 1. *The Bessel function $J_\nu(z)$ and the modified Bessel function $I_\nu(z)$ are expressible in terms of hypergeometric functions as follows* (see, for example, [11,12]; see also [13, p. 265, Eq. (3.2)], [14], [15]); in particular, [16]):

$$J_\nu(z) = \frac{(\frac{z}{2})^\nu}{\Gamma(\nu+1)}\, {}_0F_1\left(\text{——}\,; \nu+1; -\frac{1}{4}z^2\right) \qquad (\nu \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant -1}) \tag{19}$$

and

$$I_\nu(z) = \frac{(\frac{z}{2})^\nu}{\Gamma(\nu+1)}\, {}_0F_1\left(\text{——}\,; \nu+1; \frac{1}{4}z^2\right) \qquad (\nu \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant -1}), \tag{20}$$

where $\mathbb{Z}_{\leqslant -1} := \mathbb{Z} \setminus \mathbb{N}_0$ *and* $z \in \mathbb{C} \setminus (-\infty, 0]$.

Now, applying the relationships (19) and (20) to (17), we can deduce certain interesting integral representations for the extended Horn's double hypergeometric function in (9) asserted by Corollary 2 below. We state here the resulting integral representations.

Corollary 2. *Each of the following integral representations holds true:*

$$\begin{aligned} &H_{4,p}[\tau, v; \phi, \phi'; -x, y] \\ &\quad = \frac{\Gamma(\phi)\, x^{\frac{1-\phi}{2}}}{\Gamma(\tau)} \int_0^\infty t^{\tau-\phi} e^{-t} J_{\phi-1}(2\sqrt{xt})\; \Phi_p(v; \phi'; yt)\; dt \end{aligned} \tag{21}$$

and

$$\begin{aligned} &H_{4,p}[\tau, v; \phi, \phi'; x, y] \\ &\quad = \frac{\Gamma(\phi)\, x^{\frac{1-\phi}{2}}}{\Gamma(\tau)} \int_0^\infty t^{\tau-\phi} e^{-t} I_{\phi-1}(2\sqrt{xt})\; \Phi_p(v; \phi'; yt)\; dt. \end{aligned} \tag{22}$$

Here all parameters and variables would be restricted so that the representations can be meaningful and convergent: For example, $\Re(\tau - \phi) > -1$, $\phi \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0}$ *and* $x \in \mathbb{C} \setminus (-\infty, 0]$.

2.2. Transformation Formula

Theorem 3. *The following transformation formula for $H_{4,p}$ holds true:*

$$\begin{aligned} &H_{4,p}[\tau, v; \phi, \phi'; x, y] \\ &\quad = (1-y)^{-\tau} H_{4,p}\left[\tau, \phi' - v; \phi, \phi'; \frac{x}{(1-y)^2}, \frac{y}{y-1}\right]. \end{aligned} \tag{23}$$

Proof. If we first apply extended Kummer's transformation formula (see, for example, [7, p. 596, Eq. (6.3)]):

$$\Phi_p(v; \phi; z) = e^z\; \Phi_p(\phi - v; \phi; -z) \tag{24}$$

to (17) and then set

$$t(1-y) = u \qquad \text{and} \qquad du = (1-y)dt$$

in the resulting integral, we get the transformation formula (23). □

2.3. Recurrence Relation

The following lemma gives a recurrence relation for ${}_0F_1$ which is deducible from [17, p. 19, Eq. (2.2.2)] or [17, p. 20, Eq. (2.2.7)].

Lemma 1. *The following contiguous relation for the function ${}_0F_1$ holds true:*

$$ {}_0F_1(-;\phi-1;x) - {}_0F_1(-;\phi;x) - \frac{x}{\phi(\phi-1)}\,{}_0F_1(-;\phi+1;x) = 0. \tag{25} $$

Proof. Recall a contiguous relation for the function ${}_1F_1$ (see [17, p. 19, Eq. (2.2.2)]):

$$ v(v-1)\,{}_1F_1(\tau;v-1;x) - v(v-1+x)\,{}_1F_1(\tau;v;x) + (v-\tau)x\,{}_1F_1(\tau;v+1;x) = 0. \tag{26} $$

Substituting $\frac{x}{\tau}$ for x in (26) and taking the limit in the resulting identity as $|\tau| \to \infty$ with the aid of

$$ \lim_{|\tau|\to\infty} \left\{(\tau)_n \left(\frac{x}{\tau}\right)^n\right\} = x^n \quad (n \in \mathbb{N}_0,\ |x| < \infty), \tag{27} $$

and replacing v by ϕ in the final identity, we obtain the contiguous relation for the function ${}_0F_1$ in (25).

Additionally, it is worth mentioning that (25) can be proven through a straightforward computation. □

Theorem 4. *The following recurrence relation for $H_{4,p}$ holds true:*

$$ \begin{aligned} H_{4,p}[\tau,v;\phi,\phi';x,y] &= H_{4,p}[\tau,v;\phi-1,\phi';x,y] \\ &\quad + \frac{\tau(\tau+1)x}{\phi(1-\phi)} H_{4,p}[\tau,v;\phi+1,\phi';x,y]. \end{aligned} \tag{28} $$

Proof. By utilizing (25) on the integral form given in (17), we arrive at the relation (28). □

2.4. Mellin Transform and Inverse Mellin Transform

The Mellin transform of a function $f(t)$ with index s is defined by

$$ \mathcal{M}\{f(\tau) : \tau \to s\} := \int_0^\infty \tau^{s-1} f(\tau)\, d\tau, \tag{29} $$

provided that the improper integral exists (see, for example, [13,18]).

Theorem 5. *The following Mellin transform representation of $H_{4,p}$ in (9) holds true:*

$$ \begin{aligned} F(s) :&= \mathcal{M}\{H_{4,p}[\tau,v;\phi,\phi';x,y] : p \to s\} \\ &= \frac{\Gamma(s)\mathrm{B}(v+s,\phi'-v+s)}{\mathrm{B}(v,\phi'-v)} H_4[\tau,v+s;\phi,\phi'+2s;x,y] \end{aligned} \tag{30} $$

$$ (\Re(s) > 0,\ \Re(\phi') > \Re(v) > 0). $$

Also the restrictions of the other parameters and variables would follow from those in (9).

Proof. Using the Mellin transform (29) in (9), and interchanging the order of integral and summations, which is guaranteed under the restrictions, we get

$$ \begin{aligned} &\mathcal{M}\{H_{4,p}[\tau,v;\phi,\phi';x,y] : p \to s\} \\ &= \frac{1}{\mathrm{B}(v,\phi'-v)} \sum_{k,m=0}^{\infty} \frac{(\tau)_{2k+m}}{(\phi)_k} \frac{x^k}{k!} \frac{y^m}{m!} \int_0^\infty p^{s-1}\mathrm{B}(v+m,\phi'-v;p)\, dp. \end{aligned} \tag{31} $$

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Applying the known result (see [1, p. 21, Eq. (2.1)]):

$$\int_0^\infty p^{s-1}\mathrm{B}(x, y; p)\, dp = \Gamma(s)\, \mathrm{B}(x+s, y+s) \\ (\Re(s) > 0,\ \Re(x+s) > 0,\ \Re(y+s) > 0) \tag{32}$$

to the improper integral in (31), we obtain

$$\mathcal{M}\left\{H_{4,p}[\tau, \upsilon; \phi, \phi'; x, y] : p \to s\right\} \\ = \Gamma(s) \sum_{k,m=0}^{\infty} \frac{(\tau)_{2k+m}}{(\phi)_k} \frac{\mathrm{B}(\upsilon+m+s, \phi'-\upsilon+s)}{\mathrm{B}(\upsilon, \phi'-\upsilon)} \frac{x^k}{k!} \frac{y^m}{m!},$$

which, upon using (10), yields the desired representation (30). □

Theorem 6. *The following Mellin-Barnes type integral holds true: For a fixed $\mu > 0$,*

$$H_{4,p}[\tau, \upsilon; \phi, \phi'; x, y] = \frac{1}{\mathrm{B}(\upsilon, \phi'-\upsilon)} \sum_{k,m=0}^{\infty} \frac{(\tau)_{2k+m}}{(\phi)_k} \frac{x^k}{k!} \frac{y^m}{m!} \\ \times \frac{1}{2\pi i} \int_{\mu-i\infty}^{\mu+i\infty} \frac{\Gamma(s)\,\Gamma(\phi'-\upsilon+s)\,\Gamma(\upsilon+m+s)}{\Gamma(\phi'+m+2s)} p^{-s} ds \\ = \frac{1}{\mathrm{B}(\upsilon, \phi'-\upsilon)} \sum_{k,m=0}^{\infty} \frac{(\tau)_{2k+m}}{(\phi)_k} H_{1,3}^{3,0}\left[p \,\middle|\, \begin{matrix} (\phi'+m, 2) \\ (0,1), (\phi'-\upsilon, 1), (\upsilon+m, 1) \end{matrix} \right] \frac{x^k}{k!} \frac{y^m}{m!} \tag{33}$$

$$\left(p > 0,\ \Re(\phi') > \Re(\upsilon) > 0,\ 2\sqrt{r_1} + r_2 < 1,\ |x| \leqslant r_1,\ |y| \leqslant r_2\right),$$

where $i = \sqrt{-1}$ and $H_{1,3}^{3,0}$ denotes the H-function (see, for example, [19, Section 1.2]).

Proof. It follows from (9) that

$$f(p) := H_{4,p}[\tau, \upsilon; \phi, \phi'; x, y] \\ = \frac{1}{\mathrm{B}(\upsilon, \phi'-\upsilon)} \sum_{k,m=0}^{\infty} \frac{(\tau)_{2k+m}}{(\phi)_k} \frac{x^k}{k!} \frac{y^m}{m!} \int_0^1 t^{\upsilon+m-1}(1-t)^{\phi'-\upsilon-1} e^{-\frac{p}{t(1-t)}} dt.$$

The following asymptotic conditions are satisfied:

$$f(p) = H_4[\tau, \upsilon; \phi, \phi'; x, y] = O(1) = O\left(p^0\right) \quad \text{as } p \to 0,$$

and $f(p) = o(1) = O\left(p^{-\mu}\right)$ for every $\mu > 0$, as $p \to \infty$. Therefore one finds (see, for example, [18, p. 559]) that the Mellin transform $F(s) = \mathcal{M}\{f(p) : p \to s\}$ is analytic in the strip $0 < \Re(p) < \infty$, and the inverse Mellin transform is given as

$$f(p) = \frac{1}{2\pi i} \int_{\mu-i\infty}^{\mu+i\infty} F(s)\, p^{-s} ds \quad (0 < \mu < \infty). \tag{34}$$

Using (30) for $F(s)$ in (34), we obtain the first equality in (33). The second equality in (33) is found by employing the H-function. □

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Remark 2. *The case of* (30) *when $s = 1$ yields a relation between the extended Horn's double hypergeometric $H_{4,p}$ and the classical Horn's double hypergeometric H_4 as follows:*

$$\int_0^\infty H_{4,p}[\tau, v; \phi, \phi'; x, y]\, dp = \frac{v(\phi' - v)}{\phi'(\phi' + 1)}\, H_4[\tau, v + 1; \phi, \phi' + 2; x, y]. \tag{35}$$

Also, setting $s = 1$ in (32) *gives*

$$\int_0^\infty \mathrm{B}(x, y; p)\, dp = \mathrm{B}(x + 1, y + 1) \quad (\Re(x + 1) > 0,\ \Re(y + 1) > 0). \tag{36}$$

The Mellin-Barnes type integral in (33) *converges* (see [19, Section 1.2]). *Using the duplication formula for the Gamma function* (see, for example, [6, p. 6, Eq. (29)]) *in the Mellin-Barnes type integral in* (33), *we obtain*

$$\begin{aligned} &H^{3,0}_{1,3}\left[p \,\middle|\, \begin{matrix} (\phi' + m, 2) \\ (0, 1), (\phi' - v, 1), (v + m, 1) \end{matrix}\right] \\ &= \frac{\sqrt{\pi}}{2^{\phi' + m - 1}}\, H^{3,0}_{2,3}\left[4p \,\middle|\, \begin{matrix} \left(\frac{\phi' + m}{2}, 1\right), \left(\frac{\phi' + m + 1}{2}, 1\right) \\ (0, 1), (\phi' - v, 1), (v + m, 1) \end{matrix}\right]. \end{aligned} \tag{37}$$

Comparing (9) *with the first equality in* (33) *reveals that $\mathrm{B}(v + m, \phi' - v; p)$, as a function in p, is the inverse Mellin transform of the function:*

$$\frac{\Gamma(s)\, \Gamma(\phi' - v + s)\, \Gamma(v + m + s)}{\Gamma(\phi' + m + 2s)}.$$

2.5. Laguerre Polynomial Representation

Theorem 7. *The following Laguerre polynomial representation for $H_{4,p}$ holds true: For $\Re(p) > 0$ and $\Re(\phi' - v) > 0$,*

$$\begin{aligned} H_{4,p}[\tau, v; \phi, \phi'; x, y] = \frac{e^{-2p}}{\mathrm{B}(v, \phi' - v)} \sum_{m,n=0}^{\infty} \mathrm{B}(v + m + 1, \phi' - v + n + 1) \\ \times H_4[\tau, v + m + 1; \phi, \phi' + m + n + 2; x, y]\, L_m(p)\, L_n(p), \end{aligned} \tag{38}$$

where $L_n(p)$ are Laguerre polynomials (see, for example, [12, Chapter 12]).

Proof. Using the identity due to Miller [20, p. 30, Eq. (3.5)]:

$$\exp\left(-\frac{p}{t(1 - t)}\right) = e^{-2p} \sum_{m,n=0}^{\infty} L_m(p)\, L_n(p)\, t^{m+1}\, (1 - t)^{n+1} \tag{39}$$

in (11), we have

$$\begin{aligned} H_{4,p}[\tau, v; \phi, \phi'; x, y] = \frac{e^{-2p}}{\mathrm{B}(v, \phi' - v)} \int_0^1 u^{v-1}(1 - u)^{\phi' - v - 1}(1 - uy)^{-\tau} \\ \times\ {}_2F_1\left[\frac{\tau}{2}, \frac{\tau}{2} + \frac{1}{2}; \phi; \frac{4x}{(1 - yu)^2}\right] \left\{\sum_{m,n=0}^{\infty} L_m(p)\, L_n(p)\, u^{m+1}\, (1 - u)^{n+1}\right\} du. \end{aligned} \tag{40}$$

Now, changing summations and integral in (40) and using (16), we obtain the desired identity (38). □

 doi:10.20944/preprints202304.0844.v1

3. Extended Exton's Triple Hypergeometric Function

In terms of the extended beta function $\mathrm{B}(x,\, y\,;\, p)$ in (3), this section introduces the following extended Exton's triple hypergeometric function $X_{8,p}$: For $\tau,\ \upsilon,\ \upsilon' \in \mathbb{C}$ and $\phi_1,\ \phi_2,\ \phi_3 \in \mathbb{C}\setminus\mathbb{Z}_{\leqslant 0}$,

$$\begin{aligned} &X_{8,p}[\tau,\upsilon,\upsilon';\phi_1,\phi_2,\phi_3;x,y,z] \\ &:= \sum_{k,m,n=0}^{\infty} \frac{(\tau)_{2k+m+n}}{(\phi_1)_k} \frac{\mathrm{B}(\upsilon+m,\phi_2-\upsilon;p)}{\mathrm{B}(\upsilon,\phi_2-\upsilon)} \frac{\mathrm{B}(\upsilon'+n,\phi_3-\upsilon';p)}{\mathrm{B}(\upsilon',\phi_3-\upsilon')} \frac{x^k}{k!}\frac{y^m}{m!}\frac{z^n}{n!} \end{aligned} \tag{41}$$

$$(p \geqslant 0;\ 2\sqrt{r_1}+r_2+r_3<1,\ |x|\leqslant r_1,\ |y|\leqslant r_2,\ |z|\leqslant r_3,\ \text{ when }\ p=0).$$

Setting $p=0$ in (41) yields the Exton's triple hypergeometric function X_8 (see, for example, [9, p. 84, 41a and p. 101]): For $\tau,\ \upsilon,\ \upsilon' \in \mathbb{C}$ and $\phi_1,\ \phi_2,\ \phi_3 \in \mathbb{C}\setminus\mathbb{Z}_{\leqslant 0}$,

$$X_8[\tau,\upsilon,\upsilon';\phi_1,\phi_2,\phi_3;x,y,z] = \sum_{k,m,n=0}^{\infty} \frac{(\tau)_{2k+m+n}(\upsilon)_m(\upsilon')_n}{(\phi_1)_k(\phi_2)_m(\phi_3)_n} \frac{x^k}{k!}\frac{y^m}{m!}\frac{z^n}{n!} \tag{42}$$

$$(2\sqrt{r_1}+r_2+r_3<1,\ |x|\leqslant r_1,\ |y|\leqslant r_2,\ |z|\leqslant r_3).$$

Remark 3. *It is observed that if $\upsilon'=\phi_3$ and $z=0$ is put in (42), the Exton's triple hypergeometric function X_8 reduces to the Horn's double hypergeometric function (10). In this sense, two extensions share a common type, and their respective p-extensions $H_{4,p}$ (9) and $X_{8,p}$ (41) are interrelated as depicted in (43).*

3.1. Integral Representations

This section explores certain integral representations for the extended Exton's triple hypergeometric function in (41) of Euler and Laplace type. Integral representations incorporating Bessel and modified Bessel functions are provided as corollaries.

Theorem 8. *The following Euler type integral representation for $X_{8,p}$ in (41) holds true:*

$$\begin{aligned} X_{8,p}[\tau,\upsilon,\upsilon';\phi_1,\phi_2,\phi_3;x,y,z] &= \frac{1}{\mathrm{B}(\upsilon',\phi_3-\upsilon')}\int_0^1 \frac{v^{\upsilon'-1}(1-v)^{\phi_3-\upsilon'-1}}{(1-zv)^{\tau}} \\ &\times H_{4,p}\left[\tau,\upsilon;\phi_1,\phi_2;\ \frac{x}{(1-zv)^2},\ \frac{y}{1-zv}\right] \exp\left(-\frac{p}{v(1-v)}\right)\, dv \end{aligned} \tag{43}$$

$$\big(\Re(p)>0;\ \Re(\phi')>\Re(\upsilon')>0 \textit{ when } p=0\big).$$

Proof. The extended Exton's triple hypergeometric function in (41) can be expressed as a double series involving the extended Gauss's hypergeometric function in (6) by making use of the Pochhammer symbol identity $(\tau)_{2k+m+n}=(\tau)_{2k+m}(\tau+2k+m)_n$:

$$\begin{aligned} &X_{8,p}[\tau,\upsilon,\upsilon';\phi_1,\phi_2,\phi_3;x,y,z] \\ &= \sum_{k,m=0}^{\infty} \frac{(\tau)_{2k+m}}{(\phi_1)_k} \frac{\mathrm{B}(\upsilon+m,\phi_2-\upsilon;p)}{\mathrm{B}(\upsilon,\phi_2-\upsilon)}\, F_p\left[\begin{matrix}\tau+2k+m,\upsilon'; \\ \phi_3;\end{matrix}\ z\right] \frac{x^k}{k!}\frac{y^m}{m!}, \end{aligned} \tag{44}$$

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Employing the integral representation for the extended Gauss's hypergeometric function in (13) (see also [7, p. 592, Eq. (3.2)]) in (44), we get

$$\begin{aligned}
&X_{8,p}[\tau, v, v'; \phi_1, \phi_2, \phi_3; x, y, z] \\
&= \frac{1}{\mathrm{B}(v', \phi_3 - v')} \sum_{k,m=0}^{\infty} \int_0^1 v^{v'-1}(1-v)^{\phi_3 - v' - 1}(1-vz)^{-\tau} \exp\left(-\frac{p}{v(1-v)}\right) \\
&\quad \times \frac{(\tau)_{2k+m}}{(\phi_1)_k} \frac{\mathrm{B}(v+m, \phi_2 - v; p)}{\mathrm{B}(v, \phi_2 - v) k!\, m!} \left\{\frac{x}{(1-vz)^2}\right\}^k \left(\frac{y}{1-vz}\right)^m dv.
\end{aligned} \tag{45}$$

Changing the order of summations and integration in (45) and using the extended Horn's function in (9), we get the desired integral representation (43). □

Theorem 9. *The following Euler type integral representation for $X_{8,p}$ in (41) holds true:*

$$\begin{aligned}
X_{8,p}[\tau, v, v'; \phi_1, \phi_2, \phi_3; x, y, z] &= \int_0^1 \int_0^1 \frac{u^{v-1} v^{v'-1} (1-u)^{\phi_2 - v - 1} (1-v)^{\phi_3 - v' - 1}}{\mathrm{B}(v, \phi_2 - v)\, \mathrm{B}(v', \phi_3 - v')(1 - yu - zv)^{\tau}} \\
\times\, {}_2F_1\left[\frac{\tau}{2}, \frac{\tau}{2} + \frac{1}{2}; \phi_1; \frac{4x}{(1-yu-zv)^2}\right] &\exp\left(-\frac{p}{u(1-u)} - \frac{p}{v(1-v)}\right) du\, dv
\end{aligned} \tag{46}$$

$$\left(\Re(p) > 0;\ \Re(\phi_2) > \Re(v) > 0,\ \Re(\phi_3) > \Re(v') > 0 \text{ when } p = 0\right).$$

Proof. The extended Exton's triple hypergeometric function in (41) is expressed as a single series involving the extended second Appell's hypergeometric function in (8) by making use of the Pochhammer symbol identity $(\tau)_{2k+m+n} = (\tau)_{2k}(\tau + 2k)_{m+n}$:

$$X_{8,p}[\tau, v, v'; \phi_1, \phi_2, \phi_3; x, y, z] = \sum_{k=0}^{\infty} \frac{(\tau)_{2k}}{(\phi_1)_k} F_2[\tau + 2k, v, v'; \phi_2, \phi_3; y, z; p] \frac{x^k}{k!}. \tag{47}$$

Employing the integral representation integral representation of the extended second Appell's hypergeometric function (see [8, Theorem 2.2]):

$$\begin{aligned}
&F_2(a, b, b'; c, c'; x, y; p) \\
&= \frac{1}{\mathrm{B}(b, c-b)\mathrm{B}(b', c'-b')} \int_0^1 \int_0^1 \frac{u^{b-1}(1-u)^{c-b-1}\, v^{b'-1}(1-v)^{c'-b'-1}}{(1-xu-yv)^a} \\
&\qquad \times \exp\left(-\frac{p}{u(1-u)}\right) \exp\left(-\frac{p}{v(1-v)}\right) du\, dv
\end{aligned} \tag{48}$$

$$\left(\Re(p) > 0;\ p = 0 \text{ and } |x| + |y| < 1;\ \Re(c) > \Re(b) > 0,\ \Re(c') > \Re(b') > 0,\ \Re(a) > 0\right)$$

in (47), we obtain

$$\begin{aligned}
&X_{8,p}[\tau, v, v'; \phi_1, \phi_2, \phi_3; x, y, z] \\
&= \sum_{k=0}^{\infty} \int_0^1 \int_0^1 \frac{u^{v-1} v^{v'-1} (1-u)^{\phi_2 - v - 1} (1-v)^{\phi_3 - v' - 1}}{\mathrm{B}(v, \phi_2 - v)\, \mathrm{B}(v', \phi_3 - v')(1 - yu - zv)^{\tau}} \\
&\quad \times \exp\left(-\frac{p}{u(1-u)}\right) \exp\left(-\frac{p}{v(1-v)}\right) \frac{(\tau)_{2k}}{(\phi_1)_k\, k!} \left\{\frac{x}{(1-yu-zv)^2}\right\}^k du\, dv.
\end{aligned} \tag{49}$$

Interchanging the order of summation and integrations and using the identity (15) with the choice of ${}_2F_1$ from (1) in (49), we get the desired integral representation (46). □

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Theorem 10. *The following Laplace type integral representation for $X_{8,p}$ in (41) holds true:*

$$X_{8,p}[\tau,v,v';\phi_1,\phi_2,\phi_3;x,y,z] = \frac{1}{\Gamma(\tau)}\int_0^\infty t^{\tau-1}e^{-t}\,{}_0F_1(-;\phi_1;xt^2)\,\Phi_p(v;\phi_2;yt)\,\Phi_p(v';\phi_3;zt)\,dt \tag{50}$$

$$\big(\Re(p)>0;\ \Re(\tau)>0\ \textit{when}\ \ p=0\big).$$

Proof. Applying the integral representations for the Pochhammer symbol $(\tau)_{2k+m+n}$ in (18) to (41), and interchanging the order of summations and integral, we get

$$\begin{aligned} &X_{8,p}[\tau,v,v';\phi_1,\phi_2,\phi_3;x,y,z] \\ &= \frac{1}{\Gamma(\tau)}\int_0^\infty t^{\tau-1}\,e^{-t}\left(\sum_{k=0}^\infty \frac{1}{(\phi_1)_k}\frac{(xt^2)^k}{k!}\right)\left(\sum_{m=0}^\infty \frac{\mathrm{B}(v+m,\phi_2-v;p)}{\mathrm{B}(v,\phi_2-v)}\frac{(yt)^m}{m!}\right) \\ &\quad\times\left(\sum_{n=0}^\infty \frac{\mathrm{B}(v'+n,\phi_3-v';p)}{\mathrm{B}(v',\phi_3-v')}\frac{(zt)^n}{n!}\right)dt. \end{aligned} \tag{51}$$

Then, using the generalized hypergeometric function (1) (with $r=0$ and $s=1$) and extended confluent hypergeometric function (7) in (51), we obtain the desired result (50). □

Likewise, as in Corollary 2, we can deduce integral expressions for the extended triple hypergeometric function of Exton in (41) by utilizing (19) and (20) on (50). This is stated in Corollary 3, and we present the resulting integral representations here, without demonstrating their derivation.

Corollary 3. *Each of the following integral representations holds true:*

$$\begin{aligned} X_{8,p}[\tau,v,v';\phi_1,\phi_2,\phi_3;-x,y,z] &= \frac{\Gamma(\phi_1)\,x^{\frac{1-\phi_1}{2}}}{\Gamma(\tau)} \\ &\times\int_0^\infty t^{\tau-\phi_1}e^{-t}J_{\phi_1-1}(2\sqrt{x}t)\,\Phi_p(v;\phi_2;yt)\,\Phi_p(v';\phi_3;zt)dt \end{aligned} \tag{52}$$

and

$$\begin{aligned} X_{8,p}[\tau,v,v';\phi_1,\phi_2,\phi_3;x,y,z] &= \frac{\Gamma(\phi_1)\,x^{\frac{1-\phi_1}{2}}}{\Gamma(\tau)} \\ &\times\int_0^\infty t^{\tau-\phi_1}e^{-t}I_{\phi_1-1}(2\sqrt{x}t)\,\Phi_p(v;\phi_2;yt)\,\Phi_p(v';\phi_3;zt)\,dt, \end{aligned} \tag{53}$$

Here all parameters and variables would be restricted so that the representations can be meaningful and convergent: For example, $\Re(\tau-\phi_1)>-1$, $\phi_1\in\mathbb{C}\setminus\mathbb{Z}_{\leqslant 0}$ and $x\in\mathbb{C}\setminus(-\infty,0]$.

3.2. Transformation Formulas

This subsection derives transformation formulas for the extended Exton's triple hypergeometric functions $X_{8,p}$. One can consult [21] for transformations of certain hypergeometric functions of three variables.

Theorem 11. *Each of the following transformation formulas for $X_{8,p}$ holds true:*

$$\begin{aligned} &X_{8,p}[\tau,v,v';\phi_1,\phi_2,\phi_3;x,y,z] \\ &= (1-y)^{-\tau}X_{8,p}\left[\tau,\phi_2-v,v';\phi_1,\phi_2,\phi_3;\frac{x}{(1-y)^2},\frac{y}{y-1},\frac{z}{1-y}\right]; \end{aligned} \tag{54}$$

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$$
\begin{aligned}
&\mathrm{X}_{8,p}[\tau, v, v'; \phi_1, \phi_2, \phi_3; x, y, z] \\
&\quad = (1-z)^{-\tau}\mathrm{X}_{8,p}\left[\tau, v, \phi_3 - v'; \phi_1, \phi_2, \phi_3; \frac{x}{(1-z)^2}, \frac{y}{1-z}, \frac{z}{z-1}\right];
\end{aligned} \tag{55}
$$

$$
\begin{aligned}
&\mathrm{X}_{8,p}[\tau, v, v'; \phi_1, \phi_2, \phi_3; x, y, z] = (1-y-z)^{-\tau} \\
&\quad \times \mathrm{X}_{8,p}\left[\tau, \phi_2 - v, \phi_3 - v'; \phi_1, \phi_2, \phi_3; \frac{x}{(1-y-z)^2}, \frac{y}{y+z-1}, \frac{z}{y+z-1}\right].
\end{aligned} \tag{56}
$$

Proof. Applying the extended Kummer's transformation formula (24) to $\Phi_p(v; \phi_2; yt)$ in (50) and then setting

$$
t(1-y) = u \qquad \text{and} \qquad du = (1-y)dt
$$

in the resulting integral, we get the first transformation formula (54). A similar argument $\Phi_p(v'; \phi_3; zt)$ will establish the second transformation formula (55). Finally, using the extended Kummer's transformation formula (24), simultaneously, for both $\Phi_p(v; \phi_2; yt)$ and $\Phi_p(v'; \phi_3; zt)$, we obtain the third transformation formula (56). $\square$

3.3. Recurrence Relation and Generating Function

This subsection investigates a recurrence relation and a generating function for the extended Exton's triple hypergeometric functions $\mathrm{X}_{8,p}$. One can see [22] for contiguous relations between certain hypergeometric functions of three variables.

Theorem 12. *The following recurrence relation for $\mathrm{X}_{8,p}$ holds true:*

$$
\begin{aligned}
\mathrm{X}_{8,p}[\tau, v, v'; \phi_1, \phi_2, \phi_3; x, y, z] &= \mathrm{X}_{8,p}[\tau, v, v'; \phi_1 - 1, \phi_2, \phi_3; x, y, z] \\
&\quad + \frac{\tau(\tau+1)x}{\phi_1(1-\phi_1)}\mathrm{X}_{8,p}[\tau, v, v'; \phi_1 + 1, \phi_2, \phi_3; x, y, z].
\end{aligned} \tag{57}
$$

Proof. Applying the contiguous relation for the function ${}_0F_1$ in (25) to the integral representation (50), we obtain the desired result. $\square$

Theorem 13. *The following generating function for $\mathrm{X}_{8,p}(x, y; z)$ in (41) holds true:*

$$
\begin{aligned}
&\sum_{r=0}^{\infty} \frac{(\lambda)_r}{r!}\, \mathrm{X}_{8,p}[\lambda + r, v, v'; \phi_1, \phi_2, \phi_3; x, y, z]\; t^r \\
&\quad = (1-t)^{-\lambda}\, \mathrm{X}_{8,p}\left(\lambda, v, v'; \phi_1, \phi_2, \phi_3; \frac{x}{(1-t)^2}, \frac{y}{1-t}, \frac{z}{1-t}\right)
\end{aligned} \tag{58}
$$

$$
(\Re(p) \geqq 0,\ \lambda \in \mathbb{C} \ \textit{and} \ |t| < 1).
$$

Proof. Let $\mathcal{L}$ be the left-handed member of (58). Using binomial theorem:

$$
(1-t)^{-\lambda} = \sum_{r=0}^{\infty} (\lambda)_r \frac{t^r}{r!} \quad (|t| < 1) \tag{59}
$$

and (41), we have

$$\begin{aligned}\mathcal{L} &= \sum_{r=0}^{\infty} \frac{(\lambda)_r t^r}{r!} \left(\sum_{k,m,n=0}^{\infty} \frac{(\lambda+r)_{2k+m+n}}{(\phi_1)_k} \frac{\mathrm{B}(v+m,\phi_2-v;p)}{\mathrm{B}(v,\phi_2-v)} \frac{\mathrm{B}(v'+n,\phi_3-v';p)}{\mathrm{B}(v',\phi_3-v')} \frac{x^k}{k!} \frac{y^m}{m!} \frac{z^n}{n!} \right) \\ &= \sum_{k,m,n=0}^{\infty} \sum_{r=0}^{\infty} \frac{(\lambda)_r(\lambda+r)_{2k+m+n}}{(\phi_1)_k} \frac{\mathrm{B}(v+m,\phi_2-v;p)}{\mathrm{B}(v,\phi_2-v)} \frac{\mathrm{B}(v'+n,\phi_3-v';p)}{\mathrm{B}(v',\phi_3-v')} \frac{x^k}{k!} \frac{y^m}{m!} \frac{z^n}{n!} \frac{t^r}{r!}.\end{aligned}$$

Employing the relation $(\lambda)_r(\lambda+r)_{2k+m+n} = (\lambda+2k+m+n)_r(\lambda)_{2k+m+n}$, we obtain

$$\begin{aligned}\mathcal{L} &= \sum_{k,m,n=0}^{\infty} \sum_{r=0}^{\infty} \frac{(\lambda+2k+m+n)_r(\lambda)_{2k+m+n}}{(\phi_1)_k} \frac{\mathrm{B}(v+m,\phi_2-v;p)}{\mathrm{B}(v,\phi_2-v)} \frac{\mathrm{B}(v'+n,\phi_3-v';p)}{\mathrm{B}(v',\phi_3-v')} \\ &\quad\times \frac{x^k}{k!} \frac{y^m}{m!} \frac{z^n}{n!} \frac{t^r}{r!} \\ &= \sum_{k,m,n=0}^{\infty} \frac{(\lambda)_{2k+m+n}}{(\phi_1)_k} \frac{\mathrm{B}(v+m,\phi_2-v;p)}{\mathrm{B}(v,\phi_2-v)} \frac{\mathrm{B}(v'+n,\phi_3-v';p)}{\mathrm{B}(v',\phi_3-v')} \\ &\quad\times \frac{x^k}{k!} \frac{y^m}{m!} \frac{z^n}{n!} \left(\sum_{r=0}^{\infty} (\lambda+2k+m+n)_r \frac{t^r}{r!} \right).\end{aligned}$$

Using the binomial theorem

$$\sum_{r=0}^{\infty} (\lambda+2k+m+n)_r \frac{t^r}{r!} = (1-t)^{-\lambda-2k-m-n} \quad (|t|<1),$$

we find

$$\begin{aligned}\mathcal{L} = (1-t)^{-\lambda} \sum_{k,m,n=0}^{\infty} &\frac{(\lambda)_{2k+m+n}}{(\phi_1)_k} \frac{\mathrm{B}(v+m,\phi_2-v;p)}{\mathrm{B}(v,\phi_2-v)} \frac{\mathrm{B}(v'+n,\phi_3-v';p)}{\mathrm{B}(v',\phi_3-v')} \\ &\times \frac{1}{k!} \left(\frac{x}{(1-t)^2} \right)^k \frac{1}{m!} \left(\frac{y}{1-t} \right)^m \frac{1}{n!} \left(\frac{z}{1-t} \right)^n,\end{aligned}$$

which, in terms of (41), corresponds to the right-handed member of (58). □

4. Bounding Inequalities for $H_{4,p}$ and $X_{8,p}$

This section explores bounding inequalities for the extended Horn's double hypergeometric function $H_{4,p}$ and the extended Exton's triple hypergeometric function $X_{8,p}$. The first auxiliary lemma is a simple but sharp estimate [23, p. 224, Eq. (5.78)]:

$$\max_{0<t<1} e^{-\frac{p}{t(1-t)}} = e^{-4p} \quad (p \geqslant 0), \tag{60}$$

which can be proven by noticing that the function $g(t) = -\frac{1}{t(1-t)}$ has the maximum value -4 at $t = \frac{1}{2}$ on the interval $0 < t < 1$.

The following lemma provides an inequality which is readily verifiable using (3) and the observation (60).

Lemma 2. *Let $p \geqslant 0$ and $\eta, \xi \in \mathbb{R}_{>0}$. Then*

$$\mathrm{B}(\eta,\xi;p) \leq e^{-4p}\,\mathrm{B}(\eta,\xi). \tag{61}$$

Let $\mathbb{R}_{>0}$ stand for the set of positive real numbers, both here and in other contexts.

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4.1. Bounds for the Extended Functions

The following theorem provides bounding inequalities for the extended Gaussian hypergeometric F_p, the extended Kummer's confluent hypergeometric Φ_p, the extended second Appell's F_2, the extended Horn's double hypergeometric $H_{4,p}$, and the extended Exton's triple hypergeomtric function $X_{8,p}$, by using their series representations.

Theorem 14. *Let $p \geqslant 0$. Also let the numerator parameters be nonnegative real numbers and the denominator parameters be positive real numbers. Further let the variables be nonnegative real numbers. Then*

$$\begin{gathered} F_p(\tau, \upsilon; \phi; z) \leqslant e^{-4p}\, {}_2F_1(\tau, \upsilon; \phi; z) \\ (z < 1,\, \phi > \upsilon;\; z = 1,\, \phi > \tau + \upsilon)\,; \end{gathered} \tag{62}$$

$$\Phi_p(\upsilon; \phi; z) \leqslant e^{-4p}\, \Phi(\upsilon; \phi; z); \tag{63}$$

$$\begin{gathered} F_2(\tau, \upsilon, \upsilon'; \phi, \phi'; x, y; p) \leqslant e^{-8p}\, F_2(\tau, \upsilon, \upsilon'; \phi, \phi'; x, y) \\ (x + y < 1,\, \phi > \upsilon,\, \phi' > \upsilon')\,; \end{gathered} \tag{64}$$

$$\begin{gathered} H_{4,p}[\tau, \upsilon; \phi, \phi'; x, y] \leqslant e^{-4p}\, H_4[\tau, \upsilon; \phi, \phi'; x, y] \\ (2\sqrt{x} + y < 1,\, \phi' > \upsilon)\,; \end{gathered} \tag{65}$$

$$\begin{gathered} X_{8,p}[\tau, \upsilon, \upsilon'; \phi_1, \phi_2, \phi_3; x, y, z] \leqslant e^{-8p}\, X_8[\tau, \upsilon, \upsilon'; \phi_1, \phi_2, \phi_3; x, y, z] \\ (2\sqrt{x} + y + z < 1,\, \phi_2 > \upsilon,\, \phi_3 > \upsilon')\,. \end{gathered} \tag{66}$$

Each equality of the inequalities holds when $p = 0$.

Proof. We prove only (62). Applying (61) to the extended Gaussian hypergeometric function (6), we have

$$\begin{aligned} F_p(\tau, \upsilon; \phi; z) &\leqslant e^{-4p} \sum_{n \geqslant 0} (\tau)_n \frac{\mathrm{B}(\upsilon + n, \phi - \upsilon)}{\mathrm{B}(\upsilon, \phi - \upsilon)} \frac{z^n}{n!} \\ &= e^{-4p}\, {}_2F_1(\tau, \upsilon; \phi; z). \end{aligned}$$

This proves (62). The other inequalities can be verified using an argument similar to the one presented in the proof of (62). However, the specifics are omitted. □

4.2. Bounds Obtained via Integral Representations

In this subsection, we investigate the bounds of the extended Horn's double hypergeometric function $H_{4,p}$ and extended Exton's triple hypergeometric function $X_{8,p}$, which were introduced in Sections 2 and 3, respectively. To accomplish this, we review and recall certain inequalities pertaining to the generalized hypergeometric function, Bessel function, and modified Bessel function as follows:

- For $b_j \geqslant \tau_j > 0$ ($j \in \overline{1, r}$) and $x \in \mathbb{R}_{>0}$, the following Luke's two-sided inequalities for ${}_rF_r$ hold true (see [24, Theorem 16, Eq. (5.6)]):

$$e^{\theta x} < {}_rF_r\big(\tau_r;\, \upsilon_r;\, x\big) < 1 - \theta\big(1 - e^x\big), \tag{67}$$

  where

$$\theta = \frac{\max\limits_{1 \leqslant j \leqslant r} \tau_j}{\min\limits_{1 \leqslant j \leqslant r} \upsilon_j}. \tag{68}$$

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For $\phi \geqslant v > 0$, the two-sided inequalities for Kummer's confluent hypergeometric function $\Phi(v;\phi;x) = {}_1F_1(v;\phi;x)$ easily follows:

$$e^{\frac{v}{\phi}x} < \Phi(v;\, \phi;\, x) < 1 - \frac{v}{\phi}(1 - e^x). \tag{69}$$

- Bounding inequalities for J_ν and I_ν:

(i) Lommel's bounds (see, for example, [14, pp. 31 and 406], [25], [26, pp. 548–549]);

$$|J_\nu(t)| \leqslant 1, \qquad |J_{\nu+1}(t)| \leqslant \frac{1}{\sqrt{2}} \quad (\nu \in \mathbb{R}_{>0},\, t \in \mathbb{R}) \tag{70}$$

(ii) Minakshisundaram and Szász bound (see [27, Eq. (1.8)]; see also [28, pp. 36-37]; cf. [14, p. 16]);

$$|J_\nu(t)| \leqslant \frac{1}{\Gamma(\nu+1)} \left(\frac{|t|}{2}\right)^\nu \quad (\nu \geqslant 0,\ t \in \mathbb{R}) \tag{71}$$

(iii) For $\nu \geqslant 0$ and $t \in \mathbb{R}$, Landau bounds [29]

$$|J_\nu(t)| \leqslant b_L\, \nu^{-1/3}, \qquad b_L := \sqrt[3]{2} \sup_{t \geqslant 0} \mathrm{Ai}(t), \tag{72}$$

$$|J_\nu(t)| \leqslant c_L\, |t|^{-1/3}, \qquad c_L := \sup_{t \geqslant 0} t^{1/3} J_0(t)\,, \tag{73}$$

where $\mathrm{Ai}(\cdot)$ stands for the Airy function

$$\mathrm{Ai}(t) := \frac{\pi}{2}\sqrt{\frac{t}{3}} \left(J_{-1/3}\left\{2(t/3)^{3/2}\right\} + J_{-1/3}\left\{2(t/3)^{3/2}\right\}\right). \tag{74}$$

(iv) Olenko bound [30, Theorem 2.1]

$$\sup_{t \geqslant 0} \sqrt{t}|J_\nu(t)| \leqslant b_L \sqrt{\nu^{1/3} + \frac{\tau_1}{\nu^{1/3}} + \frac{3\tau_1^2}{10\nu}} =: d_O, \qquad \nu > 0\,, \tag{75}$$

where τ_1 is the smallest positive zero of the Airy–function Ai in (74) and b_L is the Landau's constant in (72). This bound is asymptotically precise, and the constant b_L is the best possible.

(v) Luke [24, Eq. (6.25)] gave the following inequality for the modified Bessel function I_μ:

$$I_\mu(t) < \frac{(\frac{t}{2})^\mu}{\Gamma(\mu+1)} \cosh t \quad \left(t > 0,\ \mu > -\frac{1}{2}\right). \tag{76}$$

The following theorem states and proves our second set of findings for bounded inequalities of $H_{4,p}$.

Theorem 15. *The following inequalities hold true:*

$$\begin{aligned} \left|H_{4,p}[\tau, v; \phi, \phi'; -x, y]\right| \leqslant & \frac{\Gamma(\phi)\,\Gamma(\tau-\phi+1)\,|x|^{\frac{1-\phi}{2}}}{\Gamma(\tau)\, e^{4p}} \\ & \times \left[1 - \frac{v}{\phi'}\{1 - (1-y)^{-\tau+\phi-1}\}\right] \end{aligned} \tag{77}$$

$$\left\{\begin{matrix} p > 0,\, \tau+1 > \phi > 1,\, \phi' \geqslant v > 0,\, x \geqslant 0,\, y < 1; \\ p = 0,\, \tau+1 > \phi > 1,\, \phi' > v > 0,\, 0 \leqslant y < 1,\, 2\sqrt{|x|}+y < 1 \end{matrix}\right\};$$

$$\begin{aligned} \left|H_{4,p}[\tau, v; \phi, \phi'; -x, y]\right| \leq & \frac{\Gamma(\phi)\,\Gamma(\tau-\phi+1)\, b_L'\, |x|^{\frac{1-\phi}{2}}}{\sqrt[3]{\phi-1}\,\Gamma(\tau)\, e^{4p}} \\ & \times \left[1 - \frac{v}{\phi'}\left(1-(1-y)^{-\tau+\phi-1}\right)\right] \end{aligned} \tag{78}$$

$$\left\{\begin{matrix} p > 0,\, \tau+1 > \phi > 1,\, \phi' \geqslant v > 0,\, x \geqslant 0,\, y < 1; \\ p = 0,\, \tau+1 > \phi > 1,\, \phi' > v > 0,\, 0 \leqslant y < 1,\, 2\sqrt{|x|}+y < 1 \end{matrix}\right\},$$

where

$$b_L' := \sqrt[3]{2} \sup_{t \geqslant 0} \mathrm{Ai}(2\sqrt{xt});$$

$$\left|H_{4,p}[\tau, v; \phi, \phi'; x, y]\right| \leq e^{-4p}\left\{\frac{1-\frac{v}{\phi'}}{(1-2\sqrt{x})^{\tau}} + \frac{\frac{v}{\phi'}}{(1-2\sqrt{x}-y)^{\tau}}\right\} \tag{79}$$

$$\left\{p \geqslant 0,\, \tau > 0,\, \phi > \frac{1}{2},\, \phi' \geqslant v > 0,\, 0 < x < \frac{1}{4},\, 2\sqrt{x}+y < 1\right\}.$$

Proof. Applying the estimate (63) in Theorem 14 to the integral representations (21) and (22), respectively, we obtain

$$\begin{aligned} R_1 := & \left\{\begin{matrix} \left|H_{4,p}[\tau, v; \phi, \phi'; -x, y]\right| \\ \left|H_{4,p}[\tau, v; \phi, \phi'; x, y]\right| \end{matrix}\right\} \\ \leqslant & \frac{\Gamma(\phi)\, e^{-4p}}{\Gamma(\tau)\, |x|^{\frac{\phi-1}{2}}} \int_0^\infty e^{-t}\, t^{\tau-\phi} \left\{\begin{matrix} |J_{\phi-1}(2\sqrt{xt})| \\ |I_{\phi-1}(2\sqrt{xt})| \end{matrix}\right\} \Phi(v; \phi'; yt)\, dt. \end{aligned} \tag{80}$$

Employing Luke's upper bound (69) in (80) gives the following estimate:

$$R_1 \leqslant \frac{\Gamma(\phi)\, e^{-4p}}{\Gamma(\tau)\, |x|^{\frac{\phi-1}{2}}} \int_0^\infty e^{-t}\, t^{\tau-\phi} \left\{\begin{matrix} |J_{\phi-1}(2\sqrt{xt})| \\ |I_{\phi-1}(2\sqrt{xt})| \end{matrix}\right\} \left[1 - \frac{v}{\phi'}\left\{1-e^{yt}\right\}\right] dt =: R_2. \tag{81}$$

Using the first one in (70) in the first one of R_2 in (81), we find

$$R_2 \leqslant \frac{\Gamma(\phi)\, e^{-4p}}{\Gamma(\tau)\, |x|^{\frac{\phi-1}{2}}} \int_0^\infty e^{-t}\, t^{\tau-\phi} \left[1 - \frac{v}{\phi'}\left\{1-e^{yt}\right\}\right] dt,$$

which, upon employing (2) and the following integral formula:

$$\int_0^\infty e^{-\mu t}\, t^{\tau-1}\, dt = \frac{\Gamma(\tau)}{\mu^{\tau}} \quad (\Re(\tau) > 0,\, \mu > 0) \tag{82}$$

to evaluate the right sided integral, and combining the resulting inequality into (81), yields the desired inequality (77).

By utilizing the first Landau's result ((72), we can derive the inequality (78) in a manner similar to obtaining inequality (77).

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Applying the inequality: $\cosh t \leqslant e^t$ $(t \in \mathbb{R})$ to (76) offers the following inequality:

$$I_\mu(t) < \frac{\left(\frac{t}{2}\right)^\mu}{\Gamma(\mu+1)}\, e^t \qquad \left(t > 0,\ \mu > -\frac{1}{2}\right),$$

which gives

$$|I_{\phi-1}(2\sqrt{xt})| < \frac{x^{\frac{\phi-1}{2}}\, t^{\phi-1}}{\Gamma(\phi)}\, e^{2\sqrt{xt}} \qquad \left(x > 0,\, t > 0,\, \phi > \frac{1}{2}\right). \tag{83}$$

Employing (83) at the second inequality of (80), using similar process as in the proof of (77), we get the inequality (79). The involved details are omitted. □

The following theorem states and proves our third set of findings for bounded inequalities of $H_{4,p}$.

Theorem 16. *The following inequalities hold true:*
For $\tau > 0$, $v > 0$, $\phi > 0$ and $x > 0$, $y < 1$, we have

$$\left|H_{4,p}[\tau, v, v'; \phi, \phi'; -x, y]\right| \leq \mathrm{e}^{-4p}\left\{1 - \frac{v}{\phi'}\left(1 - (1-y)^{-\tau}\right)\right\}. \tag{84}$$

For $\tau > 0$, $v > 0$, $\phi > 1$ and $x > 0$, $y < 1$, we get

$$\begin{aligned}\left|H_{4,p}[\tau, v; \phi, \phi'; -x, y]\right| &\leq \frac{\Gamma(\phi)\,\mathrm{e}^{-4p}}{\Gamma(\tau)} \\ &\times \begin{cases} \dfrac{c_L\, x^{-\frac{\phi}{2}+\frac{1}{3}}}{\sqrt[3]{2}}\,\Gamma\left(\tau - \phi + \frac{2}{3}\right)\left\{1 - \frac{v}{\phi'}\left(1 - (1-y)^{-\tau+\phi-\frac{2}{3}}\right)\right\}, \\ \dfrac{d_O\, x^{-\frac{\phi}{2}+\frac{1}{4}}}{\sqrt{2}}\,\Gamma\left(\tau - \phi + \frac{1}{2}\right)\left\{1 - \frac{v}{\phi'}(1 - (1-y)^{-\tau+\phi-\frac{1}{2}})\right\}, \end{cases}\end{aligned} \tag{85}$$

where the first bound needs additional restriction $3\tau - 3\phi + 1 > 0$, while the second one needs additional restriction $2\tau - 2\phi + 1 > 0$. Also, in view of (9), when $p = 0$, we assume that $2\sqrt{|x|} + |y| < 1$.

Proof. Here we first see that the estimates of Bessel function in (71), (73) and (75) are of the magnitude $|J_{\phi-1}(t)| \leq \mathfrak{C}\, t^\kappa$ where $\mathfrak{C} \in \{\frac{1}{2^{\phi-1}\Gamma(\phi)}, c_L, d_O\}$ and $\kappa \in \{\phi - 1, -\frac{1}{3}, -\frac{1}{2}\}$, respectively. Now applying the estimate (71) in Theorem 14 to the integral representation (21), denoted by R_1', gives

$$\begin{aligned} R_1' &:= \left|H_{4,p}[\tau, v; \phi, \phi'; -x, y]\right| \\ &\leqslant \frac{\Gamma(\phi)\, e^{-4p}}{\Gamma(\tau)\, |x|^{\frac{\phi-1}{2}}} \int_0^\infty e^{-t}\, t^{\tau-\phi}\, \left|J_{\phi-1}(2\sqrt{xt})\right|\, \Phi(v; \phi'; yt)\, dt =: R_2'. \end{aligned} \tag{86}$$

$$\begin{aligned} R_2' &\leq \frac{\mathfrak{C} e^{-4p}\, \Gamma(\phi)\, |x|^{\frac{1-\phi+\kappa}{2}}}{\Gamma(\tau)} \int_0^\infty \mathrm{e}^{-t}\, t^{\tau+\kappa-\phi}\left\{1 - \frac{v}{\phi'}\left(1 - \mathrm{e}^{yt}\right)\right\} \mathrm{d}t \\ &= \frac{\mathfrak{C} e^{-4p}\, \Gamma(\phi)\, |x|^{\frac{1-\phi+\kappa}{2}}}{\Gamma(\tau)} \left\{\left(1 - \frac{v}{\phi'}\right) \int_0^\infty \mathrm{e}^{-t} t^{\tau+\kappa-\phi}\, \mathrm{d}t + \frac{v}{\phi'} \int_0^\infty \mathrm{e}^{-(1-y)t} t^{\tau+\kappa-\phi}\, \mathrm{d}t\right\} \\ &= \frac{\mathfrak{C} e^{-4p}\, \Gamma(\phi)\, |x|^{\frac{1-\phi+\kappa}{2}}}{\Gamma(\tau)}\, \Gamma\left(\tau + \kappa - \phi + 1\right) \left\{1 - \frac{v}{\phi'} + \frac{v'}{\phi'}\, \frac{1}{(1-y)^{\tau+\kappa-\phi}}\right\} \\ &= \frac{\mathfrak{C} e^{-4p}\, \Gamma(\phi)\, |x|^{\frac{1-\phi+\kappa}{2}}\, \Gamma\left(\tau + \kappa - \phi + 1\right)}{\Gamma(\tau)} \left\{1 - \frac{v}{\phi'}\left(1 - (1-y)^{-\tau-\kappa+\phi}\right)\right\}. \end{aligned}$$

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Then, choosing $\mathfrak{C} = \frac{1}{2^{\phi-1}\Gamma(\phi)}$, c_L, d_O and $\kappa = \phi - 1$, $\kappa = -\frac{1}{3}, -\frac{1}{2}$, we realize the bounds affiliated to the Minakshisundaram and Szász, the second Landau's and Olenko's estimates, respectively, given in Theorem 16. $\square$

Bounding inequalities for $X_{8,p}$ can be obtained using an argument similar to the one used in Theorem 15. The following theorem presents the first two results in parallel with those in Theorem 15.

Theorem 17. *The following inequalities hold true:*

$$\begin{aligned}
&\left|X_{8,p}[\tau,v,v';\phi_1,\phi_2,\phi_3;-x,y,z]\right| \leqslant \frac{\Gamma(\phi_1)\,\Gamma(\tau-\phi_1+1)\,|x|^{\frac{1-\phi_1}{2}}}{\Gamma(\tau)\,e^{4p}} \\
&\times\left[1-\frac{v}{\phi_2}\left(1-(1-y)^{-\tau+\phi_2-1}\right)-\frac{v}{\phi_3}\left(1-(1-z)^{-\tau+\phi_2-1}\right)\right. \\
&\left.+\frac{vv'}{\phi_2\phi_3}\left(1-(1-y)^{-\tau+\phi_2-1}-(1-z)^{-\tau+\phi_2-1}-(1-y-z)^{-\tau+\phi_2-1}\right)\right]
\end{aligned} \tag{87}$$

$$\left\{\begin{matrix} p>0,\ \tau+1>\phi>1,\ \phi'\geqslant v>0,\ x\geqslant 0,\ y<1; \\ p=0,\ \tau+1>\phi>1,\ \phi'>v>0,\ 0\leqslant y<1,\ 2\sqrt{|x|}+y<1\end{matrix}\right\};$$

$$\begin{aligned}
&\left|X_{8,p}[\tau,v,v';\phi_1,\phi_2,\phi_3;-x,y,z]\right| \leqslant \frac{\Gamma(\phi_1)\,\Gamma(\tau-\phi_1+1)\,b'_L\,|x|^{\frac{1-\phi_1}{2}}}{\sqrt[3]{\phi-1}\,\Gamma(\tau)\,e^{4p}} \\
&\times\left[1-\frac{v}{\phi_2}\left(1-(1-y)^{-\tau+\phi_2-1}\right)-\frac{v}{\phi_3}\left(1-(1-z)^{-\tau+\phi_2-1}\right)\right. \\
&\left.+\frac{vv'}{\phi_2\phi_3}\left(1-(1-y)^{-\tau+\phi_2-1}-(1-z)^{-\tau+\phi_2-1}-(1-y-z)^{-\tau+\phi_2-1}\right)\right]
\end{aligned} \tag{88}$$

$$\left\{\begin{matrix} p>0,\ \tau+1>\phi>1,\ \phi'\geqslant v>0,\ x\geqslant 0,\ y<1; \\ p=0,\ \tau+1>\phi>1,\ \phi'>v>0,\ 0\leqslant y<1,\ 2\sqrt{|x|}+y<1\end{matrix}\right\},$$

where

$$b'_L := \sqrt[3]{2}\sup_{t\geqslant 0}\mathrm{Ai}(2\sqrt{xt}).$$

Proof. Applying the estimate (63) in Theorem 14 to the integral representation (52), we obtain

$$\begin{aligned}
R''_1 &:= \left|X_{8,p}[\tau,v,v';\phi_1,\phi_2,\phi_3;-x,y,z]\right| \\
&\leqslant \frac{\Gamma(\phi_1)\,e^{-4p}}{\Gamma(\tau)\,|x|^{\frac{\phi_1-1}{2}}}\int_0^\infty e^{-t}\,t^{\tau-\phi_1}\left|J_{\phi_1-1}(2\sqrt{xt})\right|\ \Phi_p(v;\phi_2;yt)\ \Phi_p(v';\phi_3;zt)\,dt.
\end{aligned} \tag{89}$$

Employing Luke's upper bound (69) in (89) gives the following estimate:

$$\begin{aligned}
R''_1 \leqslant &\frac{\Gamma(\phi_1)\,e^{-4p}}{\Gamma(\tau)\,|x|^{\frac{\phi_1-1}{2}}}\int_0^\infty e^{-t}\,t^{\tau-\phi_1} \\
&\times\left|J_{\phi_1-1}(2\sqrt{xt})\right|\left[1-\frac{v}{\phi_2}\left\{1-e^{yt}\right\}\right]\left[1-\frac{v'}{\phi_3}\left\{1-e^{zt}\right\}\right]dt =: R''_2.
\end{aligned} \tag{90}$$

Using the first one in (70) in the first one of R''_2 in (90), we find

$$R''_2 \leqslant \frac{\Gamma(\phi_1)\,e^{-4p}}{\Gamma(\tau)\,|x|^{\frac{\phi_1-1}{2}}}\int_0^\infty e^{-t}\,t^{\tau-\phi_1}\left[1-\frac{v}{\phi_2}\left\{1-e^{yt}\right\}\right]\left[1-\frac{v'}{\phi_3}\left\{1-e^{zt}\right\}\right]dt,$$

which, upon employing (2) and the following integral formula (82) to evaluate the right sided integral, and combining the resulting inequality into (90), yields the desired inequality (87).

By utilizing the first Landau's result (72), we can derive the inequality (88) in a manner akin to obtaining the inequality (87). □

5. Concluding Remarks

In 1997, Chaudhry et al. [1, p. 20, Eq. (1.7)] introduced and explored the p-extended Beta integral (3) of the classical Beta function (4). The p-extended Beta integral (3) is proved to be connected to the Macdonald, error and Whittaker functions. Since then, a number of such p-extensions of the hypergeometric function and its various generalizations of one and several variables have been presented and investigated (see, for example, [7], [8], [23]).

This paper explored extensions $H_{4,p}$ (9) and $X_{8,p}$ (41) of Horn's double hypergeometric function H_4 (10) and Exton's triple hypergeometric function X_8 (42), taking into account recent extensions of Euler's beta function, hypergeometric function, confluent hypergeometric function, two variable Appell's hypergeometric functions F_1 and F_2 and three variable Lauricella's hypergeometric function $F_D^{(3)}$. Out of the many extended hypergeometric functions (see, for example, [9, Chapters 2 and 3]), the primary rationale for selecting H_4 and X_8 is their comparable extension type (see Remark 3). We presented various integral representations of Euler and Laplace type, Mellin transforms, Laguerre polynomial representation, transformation formulae and a recurrence relation for the extended functions. In particular, we provided a generating function for the $X_{8,p}$ and several bounding inequalities for the $H_{4,p}$ and $X_{8,p}$.

We conclude this paper by providing a differential equation and posing a question as follows:

- The following differential equation is derivable from (3):

$$\frac{d^n}{dp^n}\mathrm{B}(x,y;p) = (-1)^n\,\mathrm{B}(x-n,y-n;p) \quad (p>0,\ n\in\mathbb{N}_0)\,. \tag{91}$$

- Can other bounding inequalities for the $H_{4,p}$ and $X_{8,p}$ be given?

Author Contributions: Writing—original draft, R.K.P., J.C., S.S.; Writing—review and editing, R.K.P., J.C., S.S. All authors have read and agreed to the published version of the manuscript.

Conflicts of Interest: The authors have no conflict of interest.

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