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Vladimir N. Chernega [‡] and Olga V. Man'ko [*,‡]

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Dynamics of System States in the Probability Representation of Quantum Mechanics

Vladimir N. Chernega [1,†,‡] and Olga V. Man'ko [2,†,‡]

1 Russian University of Transport (MIIT), Moscow, Russia
2 Lebedev Physical Institute, Russian Academy of Sciences, Leninskii Prospect 53, Moscow 119991, Russia; mankoov@lebedev.ru
* Correspondence: mankoov@lebedev.ru; Tel.: +007-499-132-61-97
† Current address: Lebedev Physical Institute, Russian Academy of Sciences, Leninskii Prospect 53, Moscow 119991, Russia
‡ These authors contributed equally to this work.

Abstract: The evolution of even and odd Schrödinger cat states of the inverted oscillator is obtained in the center-of-mass tomographic probability description of the two-mode oscillator.The notion of entangled probability distribution is reviewed. Evolution equations describing the time-dependence of probability distributions identified with quantum system states are discussed. The connection with the Schrödinger equation and the von Neumann equation is clarified.

Keywords: probability distributions; entanglement; quantizer operator; dequantizer operator; symplectic tomography; center-of-mass tomography, even and odd cat states

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1. Introduction

In conventional formulation of quantum mechanics [1] the states of the quantum systems are identified either with vectors $|\psi\rangle$ in Hilbert space [2] or with density operators $\hat{\rho}$ acting in the Hilbert space [3]. The vectors in the Hilbert space are associated with wave functions $\psi(x)$ of pure quantum states and the density operators are associated with pure or mixed states described by density matrices [4] or matrix elements of the density operators in some representation. There were constructed different other representations of quantum states like e.g. Wigner functions $W(q,p)$ which are quasiprobability distributions [5,6] which have some properties of probabilistic distributions. In classical mechanics the system states are identified with probability distribution functions and their properties are described by conventional probability theory [7]. The probability theory is used also to study different aspects of quantum system properties [8] as well as in connection with quantum mechanical methods applications to other areas of science [9]. Some new aspects of quantum system correlation properties like entanglement phenomena were discussed in [10,11]. The entanglement phenomenon in quantum physics provides possibility to apply this notion also in classical probability theory [12]. The conventional probability distributions determining the quantum system states were considered in [13] and this representation is called probability representation of quantum mechanics (see also [14–16]).

The tomograms and the entanglement phenomenon in the two mode squeezed states and two-mode even and odd coherent states were considered in [17]. Stimulated Raman scattering and stimulated Brillouin scattering of light were considered in the frame of symplectic tomography scheme in [18,19]. Also the entanglement phenomenon at the processes of stimulated light scattering of different types and its connection with the probability distribution functions determining the states of photon and phonon modes was discussed [18–21]. In [22] the evolution of different kinds of states in the Kerr medium including maximally entangled states were threoretically stidued in the frame of optical tomography scheme (which is partial case of symplectic tomography scheme). The unstable of reconstructed tomogram determining the state was considered in [23] in the connection with the Radon transform properties. In [24] it was shown that in classical mechanics can be introduced the Hermitian

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operators and the concepts of classical mechanics can be formulated in the language analogous to quantum mechanics language. New fundamental aspects of quantum mechanics based on groupoid approach are investigated in [25]. In [26] the evolution of states of system containing quantum and classical parts was studied. The cosmology features were considered in the frame of probability representation of quantum states in [27,28]. The density matrix properties using the symplectic representation of quantum mechanics are given in [29]. Some tomographic methods, quantization based on associative star-product of functions, applications of these approaches to different kinds of experiments were discussed in [30–39].

The idea to construct the probability representation of quantum states is based on the method of mapping operators onto functions called symbols of operators. This method is the same method which is used to construct the Wigner function [5] and other quasidistributions like the Husimi function [40], the Glauber–Sudarshan function [41,42].

The aim of the paper is to study properties of the probability representation and to consider the probability distributions describing the quantum states in the case of continious variables. We will consider dynamics of the quantum oscillator states as dynamics of the probability distributions including the superpositions of the wave functions and the superposition principle. Also some examples of the probability distributions for continious variables (called tomographic probability distributions) will be studied for quantum oscillator systems.

The paper is organized as follows. The notion of entangled probability distributions describing the quantum states in probability representation of quantum mechanics is discussed in the Section 2. Specific example of entangled probability distribution for a two-mode oscillator is considered in the Section 3. The time-dependence of states in different representations of quantum mechanics is described in the Section 4. Probability representation of quantum states is described in the Section 5 using the method of quantizer-dequantizer operators as well as the evolution equation for the probability distributions and other functions corresponding to quasiprobability representations of system states. Symplectic tomography of oscillator system states is discussed in the Section 6 and dynamics of operator symbols for the Hamiltonians which are quadratic forms of position and momentum operators are considered in the Section 7. The center-of-mass tomography and dynamics of the Schrödinger cat states of the ordinary and inverted two-mode oscillators including explicit expressions for time evolution of the center-of-mass tomography are obtained in the Section 8. The conclusions and prospectives of the probability representation pf quantum mechanics for studying entanglement and dynamics of quantum system states are presented in the Section 9.

2. Entangled Probability Distributions of Random Variables

Following [12] we introduce concept of separable and entangled probability distributions using the notion of entangled states in quantum mechanics and introduced in [13,14,43] notion of probability representation of quantum states. In this representation for a quantum system the density operators of separable states can be written as convex sum of tensor products of the density operators of the subsystems. Using the probability representation of the density operators we formulate the new notion in conventional probability theory on example of probability distribution of two random variables which are obtained using the invertable map of the density operators onto the probability distributions. Definition: the conditional probability distribution $P(X_1, X_2|a_1, a_2)$ is called separable if it can be represented as convex sum of the probability distributions $P^{(k)}(X_1|a_1)$ and $P^{(k)}(X_2|a_2)$ of the form

$$P(X_1, X_2|a_1, a_2) = \sum_k \mathcal{P}_k P_1^{(k)}(X_1|a_1) P_2^{(k)}(X_2|a_2). \tag{1}$$

Here $P(X_1, X_2|a_1, a_2) \geq 0$, $P_1(X_1|a_1) \geq 0$, $P_2(X_2|a_2) \geq 0$, coefficients $\mathcal{P}_k \geq 0$, $\sum_k \mathcal{P}_k = 1$ and

$$\int P(X_1, X_2|a_1, a_2) dX_1 dX_2 = 1. \tag{2}$$

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The probability distribution $P(X_1, X_2|a_1, a_2)$ is called entangled probability distribution if it cannot be presented as the convex sum of the form (1), i.e.

$$P(X_1, X_2|a_1, a_2) \neq \sum_k \mathcal{P}_k P_1^{(k)}(X_1|a_1) P_2^{(k)}(X_2|a_2). \tag{3}$$

For separable probability distribution

$$\int P(X_1, X_2|a_1, a_2) dX_2 = \sum_k \mathcal{P}_k P_1^{(k)}(X_1|a_1) \tag{4}$$

and

$$\int P(X_1, X_2|a_1, a_2) dX_1 = \sum_k \mathcal{P}_k P_2^{(k)}(X_2|a_2). \tag{5}$$

For entangled probability distributions we have the probability distribution $\Pi(X_1|a_1)$ as the integral (4)

$$\int P(X_1, X_2|a_1, a_2) dX_2 = \Pi(X_1|a_1), \tag{6}$$

and it cannot be presented as a convex sum like in (4).

3. Examples of the Entangled Probability Distributions

The entangled probability distribution can be related to probability distributions realized by using the superposition principle of quantum state wave functions, for example, the superposition of Fock states like state of two–mode oscillator with the wave function. We consider the very simple model of state $\psi(x_1, x_2)$ of the form

$$\psi(x_1, x_2) = \frac{1}{\sqrt{2}} (\psi_0(x_1)\psi_1(x_2) + \psi_1(x_1)\psi_0(x_2)) = \frac{x_1 + x_2}{\sqrt{\pi}} \exp\left(-\frac{x_1^2}{2} - \frac{x_2^2}{2}\right). \tag{7}$$

The function (7) is the superposition of wave functions of two-mode oscillator. The functions $|\psi_0(x_1)\rangle$ and $|\psi_0(x_2)\rangle$ are ground states of the first and second oscillators, i.e.

$$\psi_0(x_1) = \frac{e^{-\frac{x_1^2}{2}}}{\pi^{1/4}}, \quad \psi_0(x_2) = \frac{e^{-\frac{x_2^2}{2}}}{\pi^{1/4}}, \tag{8}$$

and the function $\psi_1(x_1)$ is the first excited state of the first oscillator and $\psi_1(x_2)$ is the first excited state of the second oscillator, i.e.

$$\psi_1(x_1) = \frac{\sqrt{2}x_1}{\pi^{1/4}} e^{-\frac{x_1^2}{2}}, \quad \psi_1(x_2) = \frac{\sqrt{2}x_2}{\pi^{1/4}} e^{-\frac{x_2^2}{2}}. \tag{9}$$

Using the relation between the symplectic tomogram and the wave function [44]

$$w(X_1, X_2|\mu_1, \mu_2, \nu_1, \nu_2) = \frac{1}{4\pi^2|\nu_1||\nu_2|} \left| \int \psi(x_1, x_2) \exp\left(\frac{i\mu_1}{2\nu_1}x_1^2 + \frac{i\mu_2}{2\nu_2}x_2^2 - \frac{iX_1x_1}{2\nu_1} - \frac{iX_2x_2}{2\nu_2}\right) dx_1 dx_2 \right|^2, \tag{10}$$

one can obtain the explicit form of the conditional probability distribution $w(X, Y|\mu_1, \nu_1, \mu_2, \nu_2)$, i.e.

$$w(X_1, X_2|\mu_1, \nu_1, \mu_2, \nu_2) = \frac{(\nu_2^2 + \mu_2^2) X_1^2 + 2(\nu_1\nu_2 + \mu_1\mu_2) X_1X_2 + (\nu_1^2 + \mu_1^2) X_2^2}{\pi (\nu_1^2 + \mu_1^2)^{3/2} (\nu_2^2 + \mu_2^2)^{3/2}} \times \exp\left[-\frac{X_1^2}{\mu_1^2 + \nu_1^2} - \frac{X_2^2}{\mu_2^2 + \nu_2^2}\right]. \tag{11}$$

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For particular case $\nu_1 = \nu_2 = 1, \quad \mu_1 = \mu_2 = 0$, one gets

$$w(X_1, X_2|\mu_1 = 0, \nu_1 = 1, \mu_2 = 0, \nu_2 = 1) = \frac{1}{\pi}(X_1 + X_2)^2 \exp\left(-X_1^2 - X_2^2\right). \tag{12}$$

One can check, that the function $w(X_1, X_2|\mu_1, \nu_1, \mu_2, \nu_2)$ (12) satisfies the condition

$$\int\int w(X_1, X_2, |\mu_1, \nu_1, \mu_2, \nu_2)dX_1 dX_2 = 1. \tag{13}$$

As we know, this probability distribution function corresponding to superposition of the wave functions (7) determines the quantum state which is entangled state. Due to this we call this probability distribution entangled probability distribution. In quantum mechanics the wave functions of two–mode oscillators which are obtained by means of superposition of two different wave functions are entangled pure states. In connection with this the tomographic probability distribution is described by the probability distribution function (11) and it cannot be represented in the form of equation (1). On the other hand the integral

$$w(X_1|\mu_1, \nu_1) = \int w(X_1, X_2|\mu_1, \nu_1, \mu_2, \nu_2)dX_2 = \frac{\exp\left(-\frac{X_1^2}{\mu_1^2+\nu_1^2}\right)}{\sqrt{\pi\left(\mu_1^2 + \nu_1^2\right)}}\left[\frac{1}{2} + \frac{X_1^2}{\mu_1^2 + \nu_1^2}\right]. \tag{14}$$

One can check, that

$$\int w(X_1|\mu_1, \nu_1)dX_1 = 1. \tag{15}$$

This function $w(X_1|\mu_1, \nu_1)$ (14) is marginal conditional probability distribution of position X_1 which is the position of the first oscillator and the conditions are labeled by the real parameters μ_1, ν_1. Also, if we repeat analogous calculations for the second oscillator we will get

$$w(X_2|\mu_2, \nu_2) = \int w(X_1, X_2|\mu_1, \nu_1, \mu_2, \nu_2)dX_1 = \frac{\exp\left(-\frac{X_2^2}{\mu_2^2+\nu_2^2}\right)}{\sqrt{\pi\left(\mu_2^2 + \nu_2^2\right)}}\left[\frac{1}{2} + \frac{X_2^2}{\mu_2^2 + \nu_2^2}.\right]. \tag{16}$$

One can check, that

$$\int w(X_2|\mu_2, \nu_2)dX_2 = 1. \tag{17}$$

This function $w(X_2|\mu_2, \nu_2)$ is marginal conditional probability distribution of position X_2 which is the position of the second oscillator and the conditions are labeled by the real parameters μ_2, ν_2.

The function (11) is probability distribution function, it has the form of sum of three functions which contain product of Gaussian function and different terms of position products of X_1 and X_2. The two terms are the probability distribution functions. The third term which is obtained from the integral (10) is not probability distribution function but being added to two terms mentioned above it gives the function which is the probability distribution (tomographic probability distribution).

4. Evolution of States in Different Representations

Let us remind the description of quantum state dynamics in the Hilbert space $\mathcal{H}$ where the pure quantum state is associated with the state vector $|\psi\rangle$ [2] and the other states including the pure states are also described by the density operators $\hat{\rho}$ [3,4] acting on the vectors in the Hilbert space $\mathcal{H}$. The dynamics of the states is described by the Schrödinger equation ($\hbar = 1$)

$$i\frac{\partial|\psi(t)\rangle}{\partial t} = \hat{H}|\psi(t)\rangle, \tag{18}$$

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where $\hat{H}$ is the system Hermitian Hamiltonian ($\hat{H} = \hat{H}^\dagger$).

For time independent Hamiltonian the state vector $|\psi\rangle$ evolves by means of the evolution operator $\hat{u}(t) = \exp(-i\hat{H}t)$ of the form

$$|\psi(t)\rangle = \hat{u}(t)|\psi(0)\rangle, \quad \hat{u}(0) = \hat{1}. \tag{19}$$

For the pure state with the state vector $|\psi(t)\rangle$ the density operator is given by the formula $\hat{\rho}(t) = |\psi(t)\rangle\langle\psi(t)|$ and the Shcrödinger equation (19) provides the equation for the density operator of the form (the von Neumann equation)

$$i\frac{\partial(|\psi(t)\rangle\langle\psi(t)|)}{\partial t} = \hat{H}|\psi(t)\rangle\langle\psi(t)| - |\psi(t)\rangle\langle\psi(t)|\hat{H}. \tag{20}$$

This equation is also valid for mixed state with the Hermitian density operator $\hat{\rho}(t) = \sum_k \lambda_k|\psi_k(t)\rangle\langle\psi_k(t)|$. Here the parameters λ_k are probabilities describing mixed states. The equation can be given in the following form

$$\frac{\partial\hat{\rho}(t)}{\partial t} + i[\hat{H}, \rho(t)] = 0. \tag{21}$$

The solution of this equation corresponding to the solution of the equation (19) reads

$$\hat{\rho}(t) = \hat{u}(t)\hat{\rho}(0)\hat{u}^\dagger(t). \tag{22}$$

The operators like position $\hat{q}$ and momentum $\hat{p}$ operators in the Heisenberg representation, namely, $\hat{q}_H(t)$ and $\hat{p}_H(t)$ are given as follows

$$\hat{q}_H(t) = \hat{u}^\dagger(t)\hat{q}\hat{u}(t), \quad \hat{p}_H(t) = \hat{u}^\dagger(t)\hat{p}\hat{u}(t). \tag{23}$$

The integrals of motion $\hat{q}_0(t)$ and $\hat{p}_0(t)$ which have the initial values $\hat{q}_0(t=0) = \hat{q}$ and $\hat{p}_0(t=0) = \hat{p}$ and satisfying the equation (21) are connected with the Heisenberg position and momentum operators for time independent Hamiltonian by the relations

$$\hat{q}_0(-t) = \hat{q}_H(t), \quad \hat{p}_0(-t) = \hat{p}_H(t). \tag{24}$$

The stationary states of a system $|\psi_E(t)\rangle$ satisfying the Schrödinger equation (18) have the form

$$|\psi_E(t)\rangle = \hat{u}(t)|\psi_E(0)\rangle = \exp(-iEt)|\psi_E(0)\rangle, \tag{25}$$

where the vector $|\psi_E(0)\rangle$ is the eigenvector of the Hamiltonian operator, i.e.

$$\hat{H}|\psi_E(0)\rangle = E|\psi_E(0)\rangle. \tag{26}$$

The eigenvalue parameter E describes the energy level of the system. The superposition principle of quantum states means that the vector $|\psi(t)\rangle$ of the form

$$|\psi(t)\rangle = \sum_k C_k|\psi_{E_k}(t)\rangle, \tag{27}$$

where C_k are complex numbers, is the solution of the Shrödinger equation (18). Also it means that the density operator $\hat{\rho}(t)$ of the form

$$\hat{\rho}(t) = \sum_k\sum_{k'} C_k C^*_{k'}|\psi_{E_k}(t)\rangle\langle\psi_{E_{k'}}(t)| \tag{28}$$

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is the solution of the von Neumann equation (21). Also it means that due to equations (25) and (26) we have

$$\hat{\rho}(t) = \sum_k \sum_{k'} C_k C_{k'}^* \exp\left(i(E_k - E_{k'})t\right) |\psi_{E_k}(t)\rangle\langle\psi_{E_{k'}}(t)|, \tag{29}$$

or

$$\hat{\rho}(t) = \sum_k |C_k|^2 |\psi_{E_k}(0)\rangle\langle\psi_{E_k}(0)| + \sum_k \sum_{k' \neq k} C_k C_{k'}^* \exp\left(i(E_k - E_{k'})t\right) |\psi_{E_k}(0)\rangle\langle\psi_{E_{k'}}(0)|. \tag{30}$$

The dynamics of the state density operator is determined for all the states which can be represented as superpositions of energy level states by the formula (30) since the vectors $|\psi_{E_k}(t)\rangle$ form the complete system of vectors in the Hilbert space $\mathcal{H}$.

5. Probability and Other Representations of System States

Now we consider different representations of quantum states using the formalism of quantizer-dequantizer operators $\hat{D}(\vec{x})$ and $\hat{U}(\vec{x})$ [45], where $\vec{x}$ is a set of parameters $(x_1, x_2, \ldots, x_n)$ such that the density operators $\hat{\rho}$ can be mapped onto the set of functions $f_\rho(\vec{x})$ which are named symbols of operators of the form

$$f_\rho(\vec{x}) = \mathrm{Tr}\hat{\rho}\hat{U}(\vec{x}). \tag{31}$$

The operator $\hat{U}(\vec{x})$ is a dequantizer operator. The density operator can be reconstructed from the symbol of density operator with the help of inverse transform

$$\hat{\rho} = \int f_\rho(\vec{x})\hat{D}(\vec{x})d\vec{x}. \tag{32}$$

The operator $\hat{D}(\vec{x})$ is a quantizer operator. All the state representations like Wigner function [5], Husimi function [40], Glauber–Sudarshan function [41,42] and corresponding symbols of other operators are formulated using corresponding quantizer–dequantizer operators. Thus, the quantum mechanics can be formulated using the formalism of operators acting in the Hilbert space onto its symbols which contain the same information about quantum states. One can transform the quantum mechanics formalism and obtain equations (differential or integral) for the density operator symbols. Important novelty is that the possibility of describing quantum states by conventional probability distributions exists [43,46].

All the known quasidistribution functions are obtained using different pairs of a quantizer operator $\hat{D}(\vec{x})$ and a dequantizer operator $\hat{U}(\vec{x})$, where $\vec{x} = x_1, x_2, \ldots x_n$. These operators give the possibility to map the operators $\hat{A}$ acting in the Hilbert space where position $\hat{q}$ and momentum $\hat{p}$ act, due to following generic map of operators $\hat{A} \to f_A(\vec{x})$ given by the formula for the function $f_A(\vec{x})$ called symbol of the operator $\hat{A}$

$$f_A(\vec{x}) = \mathrm{Tr}\left(\hat{A}\hat{U}(\vec{x})\right). \tag{33}$$

The inverse transform $f_A(\vec{x}) \to \hat{A}$ is given by the formula

$$\hat{A} = \int f_A(\vec{x})\hat{D}(\vec{x})d\vec{x}, \tag{34}$$

where quantizer operators $\hat{D}(\vec{x})$ provide the possibility to reconstruct the operator $\hat{A}$ if its symbol $f_A(\vec{x})$ is known. The map given by Eqs. (33), (34) provides the possibility to introduce the star–product of functions $f_A(\vec{x})$ and $f_B(\vec{x})$ which are symbols of operators $\hat{A}$ and $\hat{B}$. The symbol of operator $\hat{A}\hat{B}$ is given by the formula

$$f_{AB}(\vec{x}) = \mathrm{Tr}\left(\hat{A}\hat{B}\hat{U}(\vec{x})\right). \tag{35}$$

Using the relations (33), (34), (35) the star–product of the functions $f_A(\vec{x})$ and $f_B(\vec{x})$

$$\left(f_A \star f_B\right)(\vec{x}) = f_{AB}(\vec{x}) \tag{36}$$

is presented in the integral form

$$(f_A \star f_B)(\vec{x}) = \int f_A(\vec{x}_1) f_B(\vec{x}_2) K(\vec{x}_1, \vec{x}_2, \vec{x}) d\vec{x}_1 d\vec{x}_2, \tag{37}$$

with the kernel which is easy to express in terms of quantizer–dequantizer

$$K(\vec{x}_1, \vec{x}_2, \vec{x}) = \text{Tr}\left(\hat{D}(\vec{x}_1)\hat{D}(\vec{x}_2)\hat{U}(\vec{x})\right). \tag{38}$$

Since the product of operators is associative, i.e. $\left((\hat{A}\hat{B})\hat{C}\right) = \left(\hat{A}(\hat{B}\hat{C})\right)$ the star–product of symbols of the operators is also associative.

One can use the formalism of quantizer–dequantizer operators to write the evolution equation for symbols of density operators. For example, the von Neumann equation for the oscillator density operator $\hat{\rho}(t)$ is written in the form (we use $m = \omega = \hbar = 1$)

$$\frac{\partial \hat{\rho}}{\partial t} + i\left(\hat{H}(t)\hat{\rho}(t) - \hat{\rho}(t)\hat{H}(t)\right) = 0. \tag{39}$$

Here $\hat{\rho}(t)$ has the symbol $f_\rho(\vec{x}, t)$ and the Hamiltonian operator $\hat{H}(t)$ has the symbol $f_H(\vec{x}, t)$, where we consider the symbols for arbitrary quasidistributions corresponding to quantizer–dequantizer operators. Then Eq. (39) takes the form

$$\frac{\partial f_\rho(\vec{x}, t)}{\partial t} + i\left(f_H \star f_\rho - f_\rho \star f_H\right)(\vec{x}, t) = 0. \tag{40}$$

The equation for evolution of density operator symbol for given Hamiltonian $\hat{H}(t)$ has the general form of integral equation

$$\frac{\partial f_\rho(\vec{x}, t)}{\partial t} + i\int \left(f_H(\vec{x}_1, t) f_\rho(\vec{x}_2, t) - f_\rho(\vec{x}_1, t) f_H(\vec{x}_2, t)\right) K(\vec{x}_1, \vec{x}_2, \vec{x}) d\vec{x}_1 d\vec{x}_2 = 0. \tag{41}$$

Here the symbol of the Hamiltonian $f_H(\vec{x}_1, t) = \text{Tr}\left(\hat{H}(t)\hat{U}(\vec{x}_1)\right)$ and the symbol of density operator $f_\rho(\vec{x}_2, t) = \text{Tr}\left(\hat{\rho}(t)\hat{U}(\vec{x}_2)\right)$. Using (31), (38) and (41) one has

$$\frac{\partial f_\rho(\vec{x}, t)}{\partial t} + i\int \left[\text{Tr}\left(\hat{H}(t)\hat{U}(\vec{x}_1)\right) \text{Tr}\left(\hat{\rho}(t)\hat{U}(\vec{x}_2)\right) - \text{Tr}\left(\hat{\rho}(t)\hat{U}(\vec{x}_1)\right) \text{Tr}\left(\hat{H}(t)\hat{U}(\vec{x}_2)\right)\right]$$
$$\times \text{Tr}\left(\hat{D}(\vec{x}_1)\hat{D}(\vec{x}_2)\hat{U}(\vec{x})\right) d\vec{x}_1 d\vec{x}_2 = 0. \tag{42}$$

The equation (42) can be written in the form of kinetic equation for a probability distribution function

$$\frac{\partial f_\rho(\vec{x}, t)}{\partial t} + i\int f_\rho(\vec{x}_2, t) \mathcal{K}(\vec{x}, \vec{x}_2, t) d\vec{x}_2 = 0, \tag{43}$$

where

$$\mathcal{K}(\vec{x}, \vec{x}_2, t) = \int \left(K(\vec{x}_1, \vec{x}_2, x, t) - K(\vec{x}_2, \vec{x}_1, x, t)\right) f_H(\vec{x}_1, t) d\vec{x}_1, \tag{44}$$

and the symbol of density operator $f_\rho(\vec{x}, t)$ is a probability distribution. For symplectic tomogram the inverse quantum Radon transform reads [47]

$$\hat{\rho} = \frac{1}{2\pi}\int w(X|\mu, \nu) \exp\left(i(X\hat{1} - \mu\hat{q} - \nu\hat{p})\right) dX d\mu d\nu. \tag{45}$$

It means that the quantizer operator for the symplectic tomography method has the form

$$\hat{D}(X|\mu, \nu) = \frac{1}{2\pi} \exp\left(i(X\hat{1} - \mu\hat{q} - \nu\hat{p})\right). \tag{46}$$

Thus, we have $\vec{x} = X, \mu, \nu$ and the dequantizer reads

$$\hat{U}(X|\mu,\nu) = \delta\left(i(X\hat{1} - \mu\hat{q} - \nu\hat{p})\right). \tag{47}$$

The kernel describing the star–product of the operators in symplectic tomography is expressed as follows

$$\begin{aligned} K(X_1,\mu_1,\nu_1,X_2,\mu_2,\nu_2,X,\mu,\nu) &= \frac{1}{4\pi^2}\mathrm{Tr}\left[\exp\left(i(X_1\hat{1} - \mu_1\hat{q} - \nu_1\hat{p})\right)\right. \\ &\left.\times \exp\left(i(X_2\hat{1} - \mu_2\hat{q} - \nu_2\hat{p})\right)\delta\left(i(X\hat{1} - \mu\hat{q} - \nu\hat{p})\right)\right]. \end{aligned} \tag{48}$$

In an explicit form it reads

$$\begin{aligned} K(X_1,\mu_1,\nu_1,X_2,\mu_2,\nu_2,X,\mu,\nu) &= \frac{1}{4\pi^2}\delta\left(\mu(\nu_1+\nu_2) - \nu(\mu_1+\mu_2)\right) \\ &\times \exp\left(\frac{i}{2}(\nu_1\mu_2 - \nu_2\mu_1 + 2X_1 + 2X_2 - 2\frac{\nu_1+\nu_2}{\nu}X)\right). \end{aligned} \tag{49}$$

In the case of a harmonic oscillator in the tomographic probability representation the symbol of density operator $\hat{\rho}(t)$ is given by the probability distribution function, $(\vec{x} = X, \mu, \nu)$,

$$w_\rho(X|\mu,\nu,t) = f_\rho(\vec{x},t) = \mathrm{Tr}\hat{\rho}(t)\delta\left(X\hat{1} - \mu\hat{q} - \nu\hat{p}\right), \tag{50}$$

The Hamiltonian $\hat{H}$ can be mapped onto its symbol

$$f_{\hat{H}}(X,\mu,\nu) = \mathrm{Tr}\left(\hat{H}\delta(X\hat{1} - \mu\hat{q} - \nu\hat{p})\right). \tag{51}$$

The symplectic tomogram (50) is symbol of the density operator $\hat{\rho}$ and it is the probability distribution of position X [13] depending on extra parameters determining the reference frame in the phase-space where the position X is measured. For symplectic tomography the integral linear equation (43) has the form

$$\frac{\partial w_\rho(X|\mu,\nu,t)}{\partial t} + i\int w_\rho(X_2|\mu_2,\nu_2,t)\mathcal{K}(X,\mu,\nu,X_2,\mu_2,\nu_2,t)dX_2d\mu_2d\nu_2 = 0. \tag{52}$$

Here

$$\begin{aligned} &\mathcal{K}(X,\mu,\nu,X_2,\mu_2,\nu_2,t) = \\ &\int\left[K(X_1,\mu_1,\nu_1,X_2,\mu_2,\nu_2,t) - K(X_2,\mu_2,\nu_2,X_1,\mu_1,\nu_1,t)\right]f_H(X_1,\mu_1,\nu_1,t)dX_1d\mu_1d\nu_1. \end{aligned} \tag{53}$$

The product of operators $\hat{A}\cdot\hat{B}$ is mapped onto star-product of their symbols

$$(\hat{A}\hat{B}) \leftrightarrow (A\star B)(X,\mu,\nu) = \mathrm{Tr}\left(\hat{A}\hat{B}\delta(X\hat{1} - \mu\hat{q} - \nu\hat{p})\right) \tag{54}$$

with the kernel of the star product defined by means of the expression

$$\begin{aligned} &(A\star B)(X,\mu,\nu) = \\ &\int A(X_1,\mu_1\nu_1)B(X_2,\mu_2,\nu_2)K(X_1,\mu_1\nu_1,X_2,\mu_2,\nu_2,X,\mu,\nu)dX_1dX_2d\mu_1d\mu_2d\nu_1d\nu_2. \end{aligned} \tag{55}$$

This formula is application of general formula (38) for the kernel of star-product of symbols. This formula can be used to study the entaglement phenomena of states which are superpositions of two-mode oscillator states.

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6. Symplectic Tomography of Oscillators

One can calculate the tomographic probability distribution $w(X|\mu,\nu)$ called the symplectic tomogram of the state with density operator $\hat{\rho}_{|\psi\rangle} = |\psi\rangle\langle\psi|$ using the formula analogous (10) expressed in terms of wave function $\psi(y)$ of the pure state in position representation which reads [44]

$$w_{|\psi\rangle}(X|\mu,\nu) = \frac{1}{2\pi|\nu|}\left|\int \psi(y)\exp\left(\frac{i\mu}{2\nu}y^2 - \frac{iXy}{\nu}\right)dy\right|^2. \quad (56)$$

The function is nonnegative and satisfies the normalization condition

$$\int w_{|\psi\rangle}(X|\mu,\nu)dX = 1. \quad (57)$$

The physical meaning of the real parameters μ and ν is that they due to using $\delta(X\hat{1} - \mu\hat{q} - \nu\hat{p})$ determining the dequantizer $\hat{U}(x)$ as delta function describe the axes of reference frames in phase space of position $\hat{q}$ and momentum $\hat{p}$ where the position $X\hat{1} = \mu\hat{q} + \nu\hat{p}$ is measured. Thus the tomogram $w(X|\mu,\nu)$ is the conditional probability distribution determining the density operator for the state. If $\mu = 1, \nu = 0$ it is the density matrix diagonal elements $\rho(qq)$ and for $\mu = 0, \nu = 1$ tomogram is diagonal matrix element $\rho(pp)$. It means that if one knows probability distributions of position and momentum in all the reference frames in the phase space the state (state density operator) is known.

In the case of two-mode oscillator the relation between the symplectic tomogram and wave function of the state are done by equation (10) . Using (10) one can obtain the symplectic tomogram of the ground state of two-mode oscillator in the explicit form

$$w_0(X_1,X_2|\mu_1,\mu_2,\nu_1,\nu_2) = \frac{1}{\pi\sqrt{\mu_1^2+\nu_1^2}\sqrt{\mu_2^2+\nu_2^2}}\exp\left(-\frac{X_1^2}{\mu_1^2+\nu_1^2} - \frac{(X_2^2}{\mu_2^2+\nu_2^2}\right), \quad (58)$$

and the tomogram of coherent state of two-mode oscillator in the Gaussian form

$$w_\alpha(X_1,X_2|\mu_1,\mu_2,\nu_1,\nu_2) = \frac{1}{\pi\sqrt{\mu_1^2+\nu_1^2}\sqrt{\mu_2^2+\nu_2^2}}\exp\left(-\frac{(X_1-\bar{X}_1)^2}{\mu_1^2+\nu_1^2} - \frac{(X_2-\bar{X}_2)^2}{\mu_2^2+\nu_2^2}\right), \quad (59)$$

where $\bar{X}_1 = \sqrt{2}\mu_1\mathrm{Re}\alpha + \sqrt{2}\nu_1\mathrm{Im}\alpha$, $\bar{X}_2 = \sqrt{2}\mu_2\mathrm{Re}\alpha + \sqrt{2}\nu_2\mathrm{Im}\alpha$, α is complex number.

7. Dynamics of oPerator Symbols for Hamiltonians Quadratic in Position and Momentum

Let us discuss the probability to find the tomographic probability distribution evolution for the systems with Hamiltonians which are quadratic forms in position and momentum operators. Such systems have integrals of motion which are linear in position and momentum operators. Also the position and momentum operators $\hat{q}_H(t)$ and $\hat{p}_H(t)$ are linear forms of the position $\hat{q}$ and momentum $\hat{p}$ operators with time-dependent coefficient [48]. Due to this we can explicitly obtain the time-dependence of the tomographic probability distributions describing the quantum states and corresponding to solution of the Schrödinger equation for wave function and the von Neumann equation for the density operators. The idea to get the solution of these equations was formulated in [43,46]. It is based on the following observation. Since the system state tomogram is given by the symbol of the density operator $\hat{\rho}(t)$, i.e. (50), where the density operator evolution for the von Neumann equation is described by the evolution operator $\hat{u}(t)$, i.e. $\hat{\rho}(t) = \hat{u}(t)\hat{\rho}(0)\hat{u}^\dagger(t)$ the symbol of the density operator can be rewritten in the form $\mathrm{Tr}\left(\hat{\rho}(0)\delta(X - \mu\hat{u}^\dagger(t)\hat{q}\hat{u}(t) - \nu\hat{u}^\dagger(t)\hat{p}\hat{u}(t)\right)$. Here $\hat{u}^\dagger(t)\hat{q}\hat{u}(t) = \hat{q}_H(t)$ and $\hat{u}^\dagger(t)\hat{p}\hat{u}(t) = \hat{p}_H(t)$ are the Heisenberg position and momentum operators. Such property takes place also for multi-mode systems with the Hamiltonians which are any quadratic

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forms in position and momentum operators, for example, for two-dimensional oscillators both ordinary ones like

$$\hat{H}^{(1)} = \frac{\hat{p}_1^2}{2} + \frac{\hat{p}_2^2}{2} + \frac{\hat{q}_2^2}{2} + \frac{\hat{q}_1^2}{2} \tag{60}$$

and for two-dimensional oscillator both inverted ones like

$$\hat{H}^{(2)} = \frac{\hat{p}_1^2}{2} + \frac{\hat{p}_2^2}{2} - \frac{\hat{q}_2^2}{2} - \frac{\hat{q}_1^2}{2}. \tag{61}$$

The Hamiltonian $\hat{H}_2$ corresponds to the motion of the inverted oscillator. For such Hamiltonians one has time-dependent Heisenberg operators of position and momentum of the form: for the ordinary oscillator with the Hamiltonian (60)

$$\hat{q}_{H^{(1)};1}(t) = \cos t \cdot \hat{q}_1 + \sin t \cdot \hat{p}_1, \quad \hat{q}_{H^{(1)};2}(t) = \cos t \cdot \hat{q}_2 + \sin t \cdot \hat{p}_2, \tag{62}$$

$$\hat{p}_{H^{(1)};1}(t) = -\sin t \cdot \hat{q}_1 + \cos t \cdot \hat{p}_1, \quad \hat{p}_{H^{(1)};2}(t) = -\sin t \cdot \hat{q}_2 + \cos t \cdot \hat{p}_2; \tag{63}$$

for the inverted oscillator with the Hamiltonian (61)

$$\hat{q}_{H^{(2)};1}(t) = \cosh t \cdot \hat{q}_1 + \sinh t \cdot \hat{p}_1, \quad \hat{q}_{H^{(2)};2}(t) = \cosh t \cdot \hat{q}_2 + \sinh t \cdot \hat{p}_2, \tag{64}$$

$$\hat{p}_{H^{(2)};1}(t) = \sinh t \cdot \hat{q}_1 + \cosh t \cdot \hat{p}_1, \quad \hat{p}_{H^{(2)};2}(t) = \sinh t \cdot \hat{q}_2 + \cosh t \cdot \hat{p}_2. \tag{65}$$

Developed formalism provides the possibility to obtain the description of time evolution for all the multi-mode systems with time-dependent quadratic Hamiltonians. For such systems the Heisenberg position and momentum operators are linear forms with time dependent coefficients of usual positions and momenta.

8. Center-of-Mass Tomography

Let us introduce a dequantizer operator for two-mode oscillator $\hat{U}(X_1, X_2, \mu_1, \nu_1, \mu_1, \nu_2)$. Then the symplectic tomogram reads

$$w(X_1, X_2|\mu_1, \nu_1, \mu_2, \nu_2) = \mathrm{Tr}\left(\hat{\rho}\delta(X_1\hat{1} - \mu_1\hat{q}_1 - \nu_1\hat{p}_1)\delta(X_2\hat{1} - \mu_2\hat{q}_2 - \nu_2\hat{p}_2)\right). \tag{66}$$

The dequantizer operator $\hat{U}(\vec{x})$ in the case of the symplectic tomogram is $\delta(X_1\hat{1} - \mu_1\hat{q}_1 - \nu_1\hat{p}_1)\delta(X_2\hat{1} - \mu_2\hat{q}_2 - \nu_2\hat{p}_2)$. The density operator can be reconstructed from the symplectic tomogram with the help of the quantizer operator $\hat{D}(X_1, X_2, \mu_1, \mu_2, \nu_1, \nu_2) = \frac{1}{4\pi^2}\exp(iX_1\hat{1} - \mu_1\hat{q}_1 - \nu_1\hat{p}_1)\exp(iX_2\hat{1} - \mu_2\hat{q}_2 - \nu_2\hat{p}_2)$. Then the symplectic tomogram of the first mode of oscillator is related to (66) as

$$w(X_1|\mu_1, \nu_1) = \int w(X_1, X_2|\mu_1, \nu_1, \mu_2, \nu_2)dX_2. \tag{67}$$

There is another type of tomography, named the center-of-mass tomography. It was introduced in [49] and developed in [50,51]. In the center-of-mass tomography the state is determined by the center-of-mass tomogram. The center-of-mass tomogram is a symbol of the density operator

$$w_{cm}(X|\mu_1, \nu_1, \mu_2, \nu_2) = \mathrm{Tr}\left(\hat{\rho}\delta(X\hat{1} - \mu_1\hat{q}_1 - \nu_1\hat{p}_1 - \mu_2\hat{q}_2 - \nu_2\hat{p}_2)\right). \tag{68}$$

The random variable X we named center-of-mass coordinate which is measured in phase space in rotated and scaled reference frames which are determined by parameters $\mu_1, \nu_1, \mu_2, \nu_2$. The dequantizer operator in the center-of-mass tomography is

$$\hat{U}(X, \mu_1, \nu_1, \mu_2, \nu_2) = \delta(X\hat{1} - \mu_1\hat{q}_1 - \nu_1\hat{p}_1 - \mu_2\hat{q}_2 - \nu_2\hat{p}_2). \tag{69}$$

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The density operator can be reconstructed from the center-of-mass tomogram with the help of the quantizer operator $\hat{D}(X, \mu_1, \nu_1, \mu_1, \nu_2)$, i.e.

$$\hat{\rho} = \frac{1}{4\pi^2} \int w_{cm}(X|\mu_1, \nu_1, \mu_2.\nu_2) \exp\left(i\left(X\hat{1} - \mu_1\hat{q} - \nu_1\hat{p}_1 - \mu_2\hat{q}_2 - \nu_2\hat{p}_2\right)\right) dXd\mu_1 d\nu_1 d\mu_2 d\nu_2. \quad (70)$$

The center-of-mass tomogram of odd and even coherent states is of the form

$$\begin{aligned} &w_{cm,\alpha}(X|\mu_1, \nu_1, \mu_2, \nu_2) = \\ &\frac{1}{\sqrt{\pi}\sqrt{\sigma}N_{\pm}^2(\alpha)}\Big[\exp\left(-(X - \sqrt{2}\mathrm{Re}\alpha_1\mu_1 - \sqrt{2}\mathrm{Re}\alpha_2\mu_2 - \sqrt{2}\mathrm{Im}\alpha_1\nu_1 - \sqrt{2}\mathrm{Im}\alpha_2\nu_2)^2/\sigma\right) \\ &\pm\exp\left(-2|\alpha_1| - 2|\alpha_2| - (X - i\sqrt{2}\mathrm{Im}\alpha_1\mu_1 - i\sqrt{2}\mathrm{Im}\alpha_2\mu_2 + i\sqrt{2}\mathrm{Re}\alpha_1\nu_1 + i\sqrt{2}\mathrm{Re}\alpha_2\nu_2)^2/\sigma\right) \\ &\pm\exp\left(-2|\alpha_1| - 2|\alpha_2| - (X + i\sqrt{2}\mathrm{Im}\alpha_1\mu_1 + i\sqrt{2}\mathrm{Im}\alpha_2\mu_2 - i\sqrt{2}\mathrm{Re}\alpha_1\nu_1 - i\sqrt{2}\mathrm{Re}\alpha_2\nu_2)^2/\sigma\right) \\ &+\exp\left(-(X + \sqrt{2}\mathrm{Re}\alpha_1\mu_1 + \sqrt{2}\mathrm{Re}\alpha_2\mu_2 + \sqrt{2}\mathrm{Im}\alpha_1\nu_1 + \sqrt{2}\mathrm{Im}\alpha_2\nu_2)^2/\sigma\right)\Big], \end{aligned} \quad (71)$$

where $\sigma = \mu_1^2 + \mu_2^2 + \nu_1^2 + \nu_2^2$ and $N_{\pm}^2(\alpha) = 2\left(1 \pm \exp(-2|\alpha_1|^2 - 2|\alpha_2|^2)\right)$. These tomograms (71) are the images of the nonclassical even and odd coherent states in the probability representation of quantum mechanics.

Following the method described in Section 7 we obtain the time-dependent center-of-mass tomogram of Schrödinger cat states. It means that in formula (71) we have to replace $\mu_1, \nu_1, \mu_2, \nu_2$ by time dependent Heisenberg parameters in the case of evolution with the Hamiltonian of ordinary oscillator (60) of the form

$$\begin{aligned} \mu_{H^{(1)};1} &= \mu_1\cos t - \nu_1\sin t, \quad \mu_{H^{(1)};2} = \mu_2\cos t - \nu_2\sin t, \\ \nu_{H^{(1)};1} &= \mu_1\sin t + \nu_1\cos t, \quad \nu_{H^{(1)};2} = \mu_2\sin t + \nu_2\cos t. \end{aligned} \quad (72)$$

So, one has for the initial center-of-mass tomogram of odd and even state given by (71) after the evolution with the Hamiltonian (60) the explicit expression

$$\begin{aligned} &w_{cm,\alpha}(X, \mu_1, \nu_1, \mu_2, \nu_2, t) = \\ &\frac{1}{\sqrt{\pi}\sqrt{\sigma}N_{\pm}^2(\alpha)}\Big[\exp\Big(-(X - \sqrt{2}\mathrm{Re}\alpha_1(\mu_1\cos t - \nu_1\sin t) - \sqrt{2}\mathrm{Re}\alpha_2(\mu_2\cos t - \nu_2\sin t) \\ &-\sqrt{2}\mathrm{Im}\alpha_1(\mu_1\sin t + \nu_1\cos t) - \sqrt{2}\mathrm{Im}\alpha_2(\mu_2\sin t + \nu_2\cos t))^2/\sigma\Big) \\ &\pm\exp\Big(-2|\alpha_1| - 2|\alpha_2| - (X - i\sqrt{2}\mathrm{Im}\alpha_1(\mu_1\cos t - \nu_1\sin t) \\ &-i\sqrt{2}\mathrm{Im}\alpha_2(\mu_2\cos t - \nu_2\sin t) + i\sqrt{2}\mathrm{Re}\alpha_1(\mu_1\sin t + \nu_1\cos t) \\ &+i\sqrt{2}\mathrm{Re}\alpha_2(\mu_2\sin t + \nu_2\cos t))^2/\sigma\Big) \\ &\pm\exp\Big(-2|\alpha_1| - 2|\alpha_2| - (X + i\sqrt{2}\mathrm{Im}\alpha_1(\mu_1\cos t - \nu_1\sin t) \\ &+i\sqrt{2}\mathrm{Im}\alpha_2(\mu_2\cos t - \nu_2\sin t) - i\sqrt{2}\mathrm{Re}\alpha_1(\mu_1\sin t + \nu_1\cos t) \\ &-i\sqrt{2}\mathrm{Re}\alpha_2(\mu_2\sin t + \nu_2\cos t))^2/\sigma\Big) \\ &+\exp\Big(-(X + \sqrt{2}\mathrm{Re}\alpha_1(\mu_1\cos t - \nu_1\sin t) + \sqrt{2}\mathrm{Re}\alpha_2(\mu_2\cos t - \nu_2\sin t) \\ &+\sqrt{2}\mathrm{Im}\alpha_1(\mu_1\sin t + \nu_1\cos t) + \sqrt{2}\mathrm{Im}\alpha_2(\mu_2\sin t + \nu_2\cos t))^2/\sigma\Big)\Big], \end{aligned} \quad (73)$$

where $\sigma = \mu_1^2 + \mu_2^2 + \nu_1^2 + \nu_2^2$.

For inverted oscillator with Hamiltonian (61) the initial center-of-mass tomogram given by (71) takes the form of conditional probability distribution of one random variable X

$$
\begin{aligned}
& w_{cm,\alpha}(X,\mu_1,\nu_1,\mu_2,\nu_2,t) = \\
& \frac{1}{\sqrt{\pi}\sqrt{\sigma}N^2_{\pm}(\alpha)}\Big[\exp\Big(-(X-\sqrt{2}\mathrm{Re}\alpha_1(\mu_1\cosh t+\nu_1\sinh t)-\sqrt{2}\mathrm{Re}\alpha_2(\mu_2\cosh t+\nu_2\sinh t) \\
& -\sqrt{2}\mathrm{Im}\alpha_1(\mu_1\sinh t+\nu_1\cosh t)-\sqrt{2}\mathrm{Im}\alpha_2(\mu_2\sinh t+\nu_2\cosh t))^2/\sigma\Big) \\
& \pm\exp\Big(-2|\alpha_1|-2|\alpha_2|-(X-i\sqrt{2}\mathrm{Im}\alpha_1(\mu_1\cosh t+\nu_1\sinh t) \\
& -i\sqrt{2}\mathrm{Im}\alpha_2(\mu_2\cosh t+\nu_2\sinh t)+i\sqrt{2}\mathrm{Re}\alpha_1(\mu_1\sinh t+\nu_1\cosh t) \\
& +i\sqrt{2}\mathrm{Re}\alpha_2(\mu_2\sinh t+\nu_2\cosh t))^2/\sigma\Big) \\
& \pm\exp\Big(-2|\alpha_1|-2|\alpha_2|-(X+i\sqrt{2}\mathrm{Im}\alpha_1(\mu_1\cosh t+\nu_1\sinh t) \\
& +i\sqrt{2}\mathrm{Im}\alpha_2(\mu_2\cosh t+\nu_2\sinh t)-i\sqrt{2}\mathrm{Re}\alpha_1(\mu_1\sinh t+\nu_1\cosh t) \\
& -i\sqrt{2}\mathrm{Re}\alpha_2(\mu_2\sinh t+\nu_2\cosh t))^2/\sigma\Big) \\
& +\exp\Big(-(X+\sqrt{2}\mathrm{Re}\alpha_1(\mu_1\cosh t+\nu_1\sinh t)+\sqrt{2}\mathrm{Re}\alpha_2(\mu_2\cosh t+\nu_2\sinh t) \\
& +\sqrt{2}\mathrm{Im}\alpha_1(\mu_1\sinh t+\nu_1\cosh t)+\sqrt{2}\mathrm{Im}\alpha_2(\mu_2\sinh t+\nu_2\cosh t))^2/\sigma\Big)\Big],
\end{aligned}
\tag{74}
$$

where $\sigma=\cosh 2t(\mu_1^2+\mu_2^2+\nu_1^2+\nu_2^2)+2\sinh 2t(\mu_1\nu_1+\mu_2\nu_2)$.

9. Conclusion

To conclude we summarize the main results of our paper. We developed the probability representation of quantum states in which the system states are described by standard probability distribution functions. These functions determine the density operators of the states. For this we considered two different schemes of such construction, namely, symplectic tomography probability distributions [13] and center-of-mass tomographic probability distributions [49]. In our work we considered time evolution of the tomographic probability distributions on example of the Schrödinger cat states of the two-mode oscillator. The main result was to find the time evolution and the explicit expressions of center-of-mass tomographic probability distributions for even and odd coherent states of two-mode oscillator as ordinary one (73) and inverted one (74). The result is that the obtained probability distributions describe the entangled states of two-mode oscillator and its evolution. On an example of such a state we constructed the entangled probability distributions and their dynamics. The entangled probability distributions are the new kinds of standard probability distributions [12]. The possibility to construct such new probability distributions can be studied considering multi-mode oscillators with time dependent parameters. The entangled probability descriptions are new kinds of distributions introduced using the quantum mechanics. There are other new aspects of classical probability theory which can be found and formulated in view of existence of quantum formalism of the Hilbert spaces and operator acting in the Hilbert spaces like Bell inequalities which can be considered as consequences of the entangled probability distributions as well as several entropic inequalities which are obvious in quantum mechanics but these are not well clarified and even were not discussed in classical probability theory. We will consider these problems and entropic properties of such probability distributions in future publications.

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References

1. Landau, L.D.; Lifshitz, E.M. *Quantum Mechanics: Non-Relativistic Theory*, 3rd ed.; Elsevier: UK, 1981; 689p; ISBN: 9780750635394.
2. Dirac, P.A.M. *The Principles of Quantum Mechanics*; Clarendon Press: Oxford, UK, 1981; ISBN: 9780198520115.
3. von Neumann, J. Wahrscheinlichkeitstheoretischer Aufbau der Quantenmechanik. *Gött. Nach.* **1927** , 245–272. Available online: http://eudml.org/doc/59230.
4. Landau, L. Das Dämpfungsproblem in der Wellenmechanik. *Z. Phys.* **1927**, *45*, 430–441. https://doi.org/10.1007/BF01343064
5. Wigner, E. On the Quantum Correction For Thermodynamic Equilibrium. *Phys. Rev.* **1932**, *40*, 749–759. https://doi.org/10.1103/PhysRev.40.749
6. Schleich, W. *Quantum Optics in Phase Space*; Wiley-VCH: Hoboken, NJ, USA; Germany, 2001; ISBN-13: 978-3527294350. https://doi.org/10.1002/3527602976
7. Kolmogoroff, A. *Grundbegriffe der Wahrscheinlichkeitsrechnung*; Part of book series Ergebnisse der Mathematik und Ihrer Grenzgebiete; Springer: Berlin, Germany, 1933; Volume 2. https://doi.org/10.1007/978-3-642-49888-6
8. Holevo, A.S. *Probabilistic and Statistical Aspects of Quantum Theory*; North-Holland Publishing Company: 1982; ISBN:9780444863331/0444863338 .
9. Khrennikov, A. *Probability and Randomness. Quantum versus Classical*; World Scientific: Singapore, 2016; ISBN-10:1783267968/ISBN-13:978-1783267965. *Int. J. Quantum Inf.* **2016**, *14*, 1640009 https://doi.org/10.1142/S0219749916400098
10. Khrennikov, A.; Basieva, I. Entanglement of observables: Quantum conditional probability approach. *arXiv* **2023**, arXiv:2303.12393 [quant-ph].
11. Khrennikov, A.; Basieva, I. Conditional probability framework for entanglement and its decoupling from tensor product structure. *J. Phys. A Math. Theor.* **2022**, *55*, 395302. https://doi.org/10.1088/1751-8121/ac8bb3
12. Chernega V.N.; Man'ko, O.V.; Man'ko, V.I. Entangled probability distributions. *arXiv*, **2023**, arXiv:2302.13065v1 [quant-ph].
13. Mancini, S.; Man'ko, V.I.; Tombesi, P. Symplectic Tomography as Classical Approach to Quantum Systems. *Phys. Lett. A* **1996**, *213*, 1–6. https://doi.org/10.1016/0375-9601(96)00107-7
14. Man'ko, O.V.; Man'ko, V.I. Quantum States in Probability Representation and Tomography. *J. Russ. Laser Res.* **1997**, *18*, 407–444. https://doi.org/10.1007/BF02559668
15. Przhiyalkovskiy, Y.V. Quantum process in probability representation of quantum mechanics. *J. Phys. A Math. Gen.* **2022**, *55*, 085301. https://doi.org/10.1088/1751-8121/ac4b15
16. Asorey, M.; Ibort, A.; Marmo, G.; Ventriglia, F. Quantum Tomography Twenty Years Later. *Phys. Scr.* **2015**, *90*, 074031. https://doi.org/10.1088/0031-8949/90/7/074031
17. Kuznetsov, S.V.; Kyusev, A.V.; Man'ko, O.V. Tomographic and statistical properties of superposition states for two-mode systems. *Proc. SPIE* **2004**, *5402*, International Workshop on Quantum Optics 2003. https://doi.org/10.1117/12.562263
18. Kuznetsov, S.V.; Man'ko, O.V.; Tcherniega, N.V. Photon distribution function, tomograms and entanglement in Stimulated Raman Scattering. *J. Opt. B Quantum Semiclass. Opt.* **2003**, *5*, 5503. https://doi.org/10.1088/1464-4266/5/4/357
19. Man'ko, O.V.; Tcherniega, N.V. Tomographic description of Stimulated Brillouin Scattering. *J. Russ. Laser Res.* **2001**, *22*, 201–218. https://doi.org/10.1023/A:1011304404336
20. Giri, S.K.; Sen, B.; Pathak, A.; Jana, P.C. Higher-order two-mode and multimode entanglement in Raman processes. *Phys. Rev. A* **2016**, *93*, 012340. https://doi.org/10.1103/PhysRevA.93.012340
21. Pathak, A.; Křepelka, J.; Peřina, J. Nonclassicality in Raman scattering: Quantum entanglement, squeezing of vacuum fluctuations, sub-shot noise and joint photon–phonon number and integrated-intensity distributions. *Phys. Lett. A* **2013**, *377*, 2692–2701. https://doi.org/10.1016/j.physleta.2013.07.046

22. Rohith, M.; Sudheesh, C. Signatures of entanglement in an optical tomogram. *JOSA B* **2016**, *33*, 126–133. https://doi.org/10.1364/JOSAB.33.000126
23. Facchi, P.; Ligabó, M.; Solimini, S. Tomography: mathematical aspects and applications. *Phys. Scr.* **2015**, *90*, 074007. https://doi.org/10.1088/0031-8949/90/7/074007
24. Claeyes., P.W.; Polkovnikov, A. Quantum eigenstates from classical Gibbs distributions. *SciPost Phys.* **2021**, *10*, 014. https://doi.org/10.21468/SciPostPhys.10.1.014
25. Ciaglia, F.M.; Di Cosmo, F.; Ibort, A.; Marmo, G. Evolution of Classical and Quantum States in the Groupoid Picture of Quantum Mechanics. *Entropy* **2020**, *22*, 1292. https://doi.org/10.3390/e22111292
26. Elze, H.-T.; Gambarotta, G.; Vallone, F. General Linear Dynamics—Quantum, Classical or Hybrid. *J. Phys. Conf. Ser.* **2011**, *306*, 012010. https://doi.org/10.1088/1742-6596/306/1/012010
27. Stornaiolo, C. Emergent classical universes from initial quantum states in a tomographical description. *Int. J. Geom. Meth. Mod. Phys.* **2020**, *17*, 2050167.https://doi.org/10.1142/S0219887820501674
28. Berra–Montiel, J.; Molgado, A. Tomography in loop quantum cosmology. *Eur. Phys. J. Plus* **2022**, *137*, 283. https://doi.org/10.1140/epjp/s13360-022-02504-1
29. Gosson, M.A. Symplectic Radon Transform and the Metaplectic Representation. *Entropy* **2022**, *24*, 761. https://doi.org/10.3390/e24060761
30. Foukzon, J.; Potapov, A.A.; Menkova, E.; Podosenov, S.A. A New Quantum-Mechanical Formalism Based on the Probability Representation of Quantum States. *viXra* **2016**, viXra:1612.0298.
31. Chernega, V.N.; Belolipetskiy, S.N.; Man'ko, O.V.; Man'ko, V.I. Probability representation of quantum mechanics and star product quantization. *J. Phys. Conf. Ser.* **2019**, *1348*, 012101. https://doi.org/10.1088/1742-6596/1348/1/012101
32. Bazrafkan, M.R.; Nahvifard, E. Stationary perturbation theory in the probability representation of quantum mechanics. *J. Russ. Laser Res.* **2009**, *30*, 392–403. https://doi.org/10.1007/s10946-009-9079-9
33. Filinov, V.S.; Schubert, G.; Levashov, P.; Bonitz, M.; Fehske, H.; Fortov, V. E.; Filinov, A. V. Center-of-mass tomographic approach to quantum dynamics. *Phys. Lett. A* **2008**, *372*, 5064. https://doi.org/10.1016/j.physleta.2008.05.053
34. Plotnitsky, A. Nature Has No Elementary Particles and Makes No Measurements or Predictions: Quantum Measurement and Quantum Theory, from Bohr to Bell and from Bell to Bohr. *Entropy* **2021**, *23*, 1197. https://doi.org/10.3390/e23091197
35. Miroshnichenko, G.P. CQED Quantum Tomography of a Microwave Range. *arXiv* **2015**, arXiv:1510.03155 [quant-ph].
36. Koczor, B.; Zeier, R.; Glaser, S.J. Continuous Phase-Space Representations for Finite-Dimensional Quantum States and their Tomography. *Phys. Rev. A* **2020**, *101*, 022318. https://doi.org/10.1103/PhysRevA.101.022318
37. Toninelli, E.; Ndagano, B.; Valles, A.; Forbes, A. Concepts in quantum state tomography and classical implementation with intense light: a tutorial. *Adv. Opt. Photonics* **2019**, *11*, 67–134. https://doi.org/10.1364/AOP.11.000067
38. Almarashi, A.M.; Abd-Elmougod, G.A.; Raqab, M.Z. Quantum Extropy and Statistical Properties of the Radiation Field for Photonic Binomial and Even Binomial Distributions. *J. Russ. Laser Res.* **2020**, *41*, 334–343. https://doi.org/10.1007/s10946-020-09883-9
39. Leon, R.C.C.; Yang, C.H.; Hwang, J.C.C.; Lemyre, J.C.; Tanttu, T.; Huang, W.; Huang, J.H.; Hudson, F.E.; Itoh, K.M.; Laucht, A.; et al. PBell-state tomography in a silicon many-electron artificial molecule. *Nat. Commun.* **2021**, *12*, 3228. https://doi.org/10.1038/s41467-021-23437-w
40. Husimi, K. Some Formal Properties of the Density Matrix. *Proc. Phys. Math. Soc. Jpn.* **1940**, *22*, 264–314. https://doi.org/10.11429/ppmsj1919.22.4_264
41. Glauber, R.J. Coherent and Incoherent States of the Radiation Field. *Phys. Rev.* **1963**, *131*, 2766–2788. https://doi.org/10.1103/PhysRev.131.2766
42. Sudarshan, E.C.G. Equivalence of Semiclassical and Quantum Mechanical Descriptions of Statistical Light Beams. *Phys. Rev. Lett.* **1963**, *10*, 277–279. https://doi.org/10.1103/PhysRevLett.10.277
43. Man'ko, O.V. ; Man'ko, V. I. Probability Representation of Quantum States. *Entropy* **2021**, *23*, 549. https://doi.org/10.3390/e23050549
44. Man'ko, V.I.; Vilela Mendes, R. Noncommutative Time-Frequency Tomography. *Phys. Lett. A* **1999**, *263*, 53–61. https://doi.org/10.1016/S017375-9601(99)00688-X

 doi:10.20944/preprints202304.0795.v1

45. Man'ko, O.V.; Man'ko, V.I.; Marmo, G.; Vitale, P. Star Products, Duality and Double Lie Algebras. *Phys. Lett. A* **2007**, *360*, 522–532. https://doi.org/10.1016/j.physleta.2006.08.057
46. Chernega, V.N.; Man'ko, O.V.; Man'ko, V.I. Entangled qubit states and linear entropy in the probability representation of quantum mechanics. *Entropy* **2022**, *24*, 527. https://doi.org/10.3390/e24040527
47. Man'ko, O.V.; Man'ko, V.I.; Marmo, G. Alternative Commutation Relations, Star Products and Tomography. *J. Phys. A: Math. Gen.* **2002**, *35*, 699–719. https://doi.org/10.1088/0305-4470/35/3/315
48. Dodonov, V.V.; Man'ko, V.I. Invariants and the Evolution of Nonstationary Quantum Systems. In *Proceedings of the P.N. Lebedev Physical Institute*; Nova Science: Commack, NY, USA, 1989; Volume 183; ISBN: 0-941743-49-7.
49. Arkhipov, A.S.; Lozovik, Y.E.; Man'ko, V.I. Tomography for several particles with one random variable. *J. Russ. Laser Res.* **2003**, *24* , 237–255. https://doi.org/10.1023/A:1024051809262
50. Dudinets, I.V.; Man'ko, V.I. Center-of-mass tomography and Wigner function for multimode photon states. *Int. J. Theor. Phys.* **2018**, *57*, 1631–1644. https://doi.org/10.1007/s10773-018-3690-x
51. Amosov, G.G.; Man'ko, V.I. A classical limit for the center-of-mass tomogram in view of the central limit theorem. *Phys. Scr.* **2009**, *80*, 025006. https://doi.org/10.1088/0031-8949/80/02/025006