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Article

Criteria on $v(t)$-Incremental Stability of Dynamical Systems with Time Delay

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Abstract: Incremental stability analysis for time-delay systems has attracted more and more attentions for its contemporary applications in transportation processes, population dynamics, economics, satellite positions, etc. This paper researches criteria on $v(t)$-incremental stability, which is defined to demonstrate the convergence rate for incremental stability, for time-delay systems. Firstly, the sufficient conditions for $v(t)$-incremental stability for time-delay systems with continuous right-hands are studied, and several corollaries for specific cases are provided. As for time-delay systems with discontinuous right-hands, after expounding the relevant conditions for the existence and uniqueness of the Filippov solution, by using approximation methods, sufficient conditions for $v(t)$-incremental stability are obtained. The conclusions are applied to linear switched time-delay systems and Hopfield neural network system with composite right-hand afterwards.

Keywords: Time-delay System; $v(t)$-Incremental Stability; Discontinuous Right-hands; Filippov solutions

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1. Introduction

Stability analysis in mathematics mainly refers to the relevant research on the long-term performance of the dynamical system's steady state. With more and more application of neural networks and complex systems, stability analysis of differential equations, including time-delay dynamical systems has attracted more and more attention from academia and industry. Incremental stability [2,9] has been presented to be a perfect instrument for stability analysis, which is able to address problems of synchronization of coupled systems.

Incremental stability means that as time approaches infinity, the solutions of the dynamical system in different initial states will approach each other, that is, the state variables of the system with different initial states will gradually converge to the same trajectory. This property has a very wide range of applications in different fields of academia and industry. In recent years, due to the increasing potential application value in many frontier fields, such as PI controlled missiles [11] as well as the synchronization problem of network dynamics [12–14], there are already a lot of literature available on incremental stability, e.g., [4,5] provided a systematic exposition and discussion of related issues, and [3] provided specific examples of incremental stability related applications.

In dynamical system analysis, the theoretical research on differential equations with discontinuous right-hands has also been highly valued because of its wide application. In some fields such as mechanical engineering, electronic engineering and automatic control theory, many problems rely on relevant theories of these 'discontinuous' differential equations [20]. Among them, switched systems, as a type of differential equation with discontinuous right-hand functions, are particularly commonly used in the field of automatic control, thus driving the development of related theories [21–24].

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In 1964, Filippov [46] studied the motion of Coulomb friction oscillators and proposed a differential equation with discontinuous right-hand. In order to study the trajectory of the solution, 'differential inclusion' and set-valued mapping was introduced, and the existence and uniqueness of the solution of the discontinuous differential equation were discussed. A detailed discussion on this type of discontinuous differential equation can be found in reference [7]. Before we study the compressibility of the system, we need to first ensure the existence and uniqueness of the Carathéodory solution of the system. Filippov's theory mainly focuses on the existence and uniqueness of the solution of the nonsmooth dynamical system. The relevant conclusions have been listed in [7].

Researches on time-delay differential equations first began in the early 20th century in Volterra [28, 29]. The ordinary differential equationa with time delay are commonly used in contemporary applications, gradually pushing the relevant theories of time-delay systems to integrity and maturity, and producing rich results. There are many important achievements emerging one after another. It is worth mentioning Hale and Verduyn Lunel's comprehensive work [30], which discussed in detail the properties of the solutions for some time-delay differential equations, e.g., uniqueness, continuous dependence of parameters, continuity and compactness of solutions, stability and invariance, etc. At the same time, some concrete analysis and methods on the properties of these solutions for time-delay systems were proposed in literature [31–35], among which [32] includes an introductory chapter that provides detailed examples of time-delay differential equations used to control computer systems, transportation processes, population dynamics, economics, satellite positions, urban transportation, and so on.

Set-valued dynamical systems, also named Filippov systems, whose right-hand are set-valued mappings, are widely used in these applications mentioned above. These set-valued dynamical systems are perfect instruments to represent the time-delay differential equations with discontinuous right-hands [32,36–40] or control systems with time delay [32,41,42]. Therefore, naturally, a large amount of literatures discussed the problems along these lines. One of the most important achievements in this field is [43], in which Haddad focused on upper semicontinuous dynamics, elaborated on the existence and compactness of the solution set, also proved the upper semicontinuity of the solution. Haddad's work [43] is given under functional differential inclusion, where the corresponding time-delay term acts on the infinite dimensional space of continuous function.

In recent years, there have been many related analytical studies and achievements on incremental stability of Filippov systems. In the case that local Lipschisz condition is satisfied, [17] provided a sufficient condition for the local stability of Filippov solutions. [15] used the concept of Filippov solutions to analyze a class of time-delay dynamical systems with discontinuous right-hands. In the sense of Filippov solution, [16] proposed the conditions for global asymptotic stability of the error system of the time-delay neural network with discontinuous activation function. While [10] put forward an approximation method, and gave the specific sufficient conditions for the exponential incremental stability of the switched system.

In this paper, first, several preliminary definitions and the definition of '$v(t)$-incremental stability' are given in Section 2. In Section 3, we research the criteria on $v(t)$-incremental stability of the solutions for time-delay systems with continuous right-hands, involving several specific cases, and relevant corollaries are provided. Then, in Section 4, under the hypothesis that system has a unique solution, we extend the sufficient conditions for $v(t)$-incremental stability to the time-delay system with discontinuous right-hand in the sense of Filippov solution, by using a sequence of 'continuous systems' to approach the corresponding Filippov system. In this section, we also provide the conditions for existence and uniqueness of the solution for the time-delay dynamical system before stability analysis in Section 4.1. The applications on linear switched systems and Hopfield neural network systems with time delay are given in Section 5 respectively, and corresponding numerical examples is given afterwards in Section 6.

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Table 1. Notations.

Notations	Definitions
$\lvert\cdot\rvert_{\chi(t)}$	Vector norm with subscript $\chi(t)$
$\lVert\cdot\rVert_{\chi(t)}$	Matrix norm induced by $\lvert\cdot\rvert_{\chi(t)}$
$\nu_{\chi(t)}$	Matrix measure induced by $\lvert\cdot\rvert_{\chi(t)}$
$\chi(t)$	A right-continuous staircase function w.r.t t, with switching points belonging to $\{t_j\}$
$r(t)$	A piecewise right-continuous function w.r.t t, with switching points $\{t_j\}$
t_0	The initial time
$\overline{\tau}$	The upper bound of τ_k: $\max_k sup_{t\in[t_0,\infty)}\tau_k(t)=\overline{\tau}$
$\underline{\tau}$	The lower bound of τ_k: $\min_k inf_{t\in[t_0,\infty)}\tau_k(t)=\underline{\tau}$
$N(t)$	$N(t)=\#\{j:t\geq t_j, j=1,2,...\}$

2. Preliminaries

Here we first introduce some primary definitions, including matrix measure and multiple norms, incremental stability property, $v(t)$-incremental stability and so on.

Definition 1 (Definition 1 in [10]). *For any real matrix $A\in\mathbb{R}^{n\times n}$ and a given norm $\lVert\cdot\rVert$, we define the corresponding matrix measure $\nu(A)$ as*

$$\nu(A,\lVert\cdot\rVert)=\lim_{h\to 0^+}\frac{\lVert I+hA\rVert-1}{h}.$$

The matrix measure above can be considered as the one-sided directional derivative of the induced matrix norm function $\lVert\cdot\rVert$, evaluated at the point I, in the direction of A.

In following parts, we will study the incremental stability property of time-delay systems under multiple norms. Here we also list the definitions of multiple norms with subscript $\chi(t)$ and the corresponding measures. Note that the function $\chi(t)$ is a piecewise right-continuous function.

Definition 2. *For real matrix $A\in\mathbb{R}^{n\times n}$ and the matrix norm $\lVert\cdot\rVert_{\chi(t)}$, (the corresponding vector measure $\lvert\cdot\rvert_{\chi(t)}$), we here define the corresponding matrix measure $\mu_{\chi(t)}(A)$ as follows:*

$$\mu_{\chi(t)}(A)=\lim_{h\to 0^+}\frac{\lVert I+hA\rVert_{\chi(t)}-1}{h}.$$

If $\lim_{h\to 0^+}\lvert x\rvert_{\chi(t+h)}(\lim_{h\to 0^-}\lvert x\rvert_{\chi(t+h)})$ exists, then we denote the right(left) limit of the norm $\lvert\cdot\rvert_{\chi(t)}$ at time point t by $\lvert\cdot\rvert_{\chi(t\pm)}$. We say the norm $\lvert\cdot\rvert_{\chi(t)}$ is continues at t, if and only if $\lvert\cdot\rvert_{\chi(t+)}=\lvert\cdot\rvert_{\chi(t-)}=\lvert\cdot\rvert_{\chi(t)}$, that is, $\lvert\cdot\rvert_{\chi(t)}$ is right-continuous and left-continuous as well. If there exists $D>0$, such that $\lvert x\rvert_{\chi(t)}<D\lvert x\rvert_{\chi(s)}$ holds for all $t,s\in\mathbb{R}^+$, we say $\lvert\cdot\rvert_{\chi(t)}$ is uniformly equivalent.

Then, we extend the definition for matrix measure in the sense of multiple norms. Consider the time-varying nature of $\chi(t)$, as follows:

Definition 3 (Definition 3 in [27]). *If the following limits exists, the switched matrix measure w.r.t. vector norm $\lvert\cdot\rvert_{\chi(t)}$ is defined as follows:*

$$\nu_{\chi(t)}(A)=\overline{\lim_{h\to 0^+}}\frac{1}{h}\sup_{\lvert x\rvert_{\chi(t)}=1}\lvert(I_n+hA)x\rvert_{\chi(t+h)}-1$$

where $\overline{\lim}$ stands for the upper superior.

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Remark 1. *The definition of multiple matrix norms $\|\cdot\|_{\chi(t)}$ can be thought of as matrix norms induced by vector norm $|\cdot|_{\chi(t)}$: for a matrix A,*

$$\|A\|_{\chi(t)} = \sup_{|x|_{\chi(t)}=1} |Ax|_{\chi(t)}$$

It implies that, if $\chi(t)$ is constant over an internal $[a, a+\delta)$, then it holds that $\nu_{\chi(t)}(A) = \mu_{\chi(t)}(A)$ in $[a, a+\delta)$.

According to the existence of the switched matrix measure, the definition is as follows,

Definition 4 ([27]). *Define the partial differential of the switched norm $|\cdot|_{\chi(t)}$ as follows,*

$$\bar{\partial}_t(|\cdot|_{\chi(t)}) = \overline{\lim_{h\to 0^+}} \sup_{|x|_{\chi(t)}=1} \frac{|x|_{\chi(t+h)} - 1}{h}$$

If $\bar{\partial}_t(|\cdot|_{\chi(t)})$ exists at t, we say the multiple norm $|\cdot|_{\chi(t)}$ is right regular at time t.

According to the 'right regular' property, we have the following proposition [27]:

Proposition 1 ([27]). *If the multiple norm $|\cdot|_{\chi(t)}$ is right regular, then*

1. *the multiple norm $|\cdot|_{\chi(t)}$ is right-continuous at time t;*
2. *$\nu_{\chi(t)}(\cdot)$ exists at time t.*

For clearer statement in the following part, we define a transaction function between norms $|\cdot|_{\chi}$ and $|\cdot|_{\chi'}$,

Definition 5. *Function $C(\chi, \chi') > 0$ is the transaction function between norms $|\cdot|_{\chi}$ and $|\cdot|_{\chi'}$, satisfying that*

$$|\cdot|_{\chi} \le C(\chi, \chi')|\cdot|_{\chi'}.$$

Definition 6. *If the function $C(\chi(t), \chi(t'))$ is well-defined for all t and t′, we say the multiple norm $|\cdot|_{\chi(t)}$ is equivalent for all t.*

Here we consider the following dynamical time-delay system:

$$\begin{cases} \dot{x} = f(x, x_{\tau_1(t)}, ..., x_{\tau_m(t)}, r(t)),\ t \ge t_0 \\ x(s) = \phi(s),\ s \in [t_0 - \bar{\tau}, t_0] \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$, $x_{\tau_k(t)} = x(t - \tau_k(t))(k = 1, ..., m)$ is the time-delay term, each τ_k is a bounded function, $\max_k sup_{t\in[t_0,\infty)}\tau_k(t) = \bar{\tau}$. function $r(t) : [t_0, +\infty) \to \mathbb{R}$ is a upper continuous staircase function, $f = (f_1, ..., f_n) : \mathbb{R}^{n\times(m+1)} \times [t_0, +\infty) \to \mathbb{R}^n$. The function $\phi(\cdot)$ represents the initial value function, and $\phi(\cdot) \in C^1([t_0 - \bar{\tau}, t_0], \mathbb{R}^n)$. Let $x(t; \phi, r_t)$ be the solution of system (1).

Then, we will research the sufficient conditions for incremental stability for time-delay dynamical system (1) afterwards, which is defined as follows.

We have the following definition for several types of incremental stability (IES):

Definition 7. *Let $\phi(t; t_0, x_0)$ be solution of system (1) with the initial time t_0 and initial value function $x_0(\cdot)$. If there exist a function $\beta(s, t)$ of class $\mathcal{KL}$, and some norm $|\cdot|^{\infty}_{[t_0-\bar{\tau},t_0]}$ induced by vector norm $|\cdot|$, defined as $|\varphi|^{\infty}_{[t_0-\bar{\tau},t_0]} = \sup_{s\in[t_0-\bar{\tau},t_0]} |\varphi(s)|$, such that for any initial function $x_0(\cdot), y_0(\cdot)$,*

$$|\phi(t + t_0; x_0, t_0) - \phi(t + t_0; y_0, t_0)| \le \beta(|x_0 - y_0|^{\infty}_{[t_0-\bar{\tau},t_0]}, t),$$

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then, we say that system (1) is Incrementally Asymptotically Stable(δAS) in the region $\Sigma \subset \mathbb{R}^n$. If $\beta(s,t)$ is independent of initial time t, then we say system (1) is Incrementally Uniformly Asymptotically Stable(δUAS). If function $\beta(s,t)$ is of class $\mathcal{EKL}$, then we say system (1) is Incrementally Uniformly Exponentially Asymptotically Stable($\delta UEAS$)

Moreover, if there exist a constant M and a continuous function $v(t)$ such that $\beta(\cdot,\cdot)$ of class $\mathcal{KL}$ satisfies (2) with some norm $|\cdot|$, then system (1) is said to be $v(t)$-incrementally stable.

$$\beta(|x_0 - y_0|^{\infty}_{[t_0-\overline{\tau},t_0]}, t) = \frac{M}{v(t)}|x_0 - y_0|^{\infty}_{[t_0-\overline{\tau},t_0]} \tag{2}$$

More specifically, if $v(t) = e^{ct}$, then system (1) is said to be exponentially incrementally stable. And if $v(t) = t^c$, then system (1) is said to be power-rate incrementally stable.

3. Contraction Theory for Time-Delay Systems

Here we list the following hypothesis, denoted by Assumption 1, including the Carathéodory conditon to guarrantee the existence and uniqueness of the solution of time-delay dynamical system (1) with continuous right-hand.

Assumption 1. *Dynamical time-delay system (1) satisfies the following conditions:*

1. *$f(x, x_{\tau_1(t)}, ..., x_{\tau_m(t)}, r(t))$ is continuously differentiable w.r.t x, and continuous w.r.t. $(x, x_{\tau_1(t)}, ..., x_{\tau_m(t)}, r)$ except for the switching time points $\{t_1, ..., t_j, ...\}$.*
2. *$\tau_k(t)$ is upper bounded and has a positive lower bound for each k, and $\max_k \sup_{t\in[t_0,\infty)} \tau_k(t) = \overline{\tau}$, $\min_k \inf_{t\in[t_0,\infty)} \tau_k(t) = \underline{\tau}$.*
3. *$f_i : \mathbb{R}^{n\times(m+1)} \times [t_0, +\infty) \to \mathbb{R}$ $(i = 1, ..., n)$ is locally Lipschitz.*

Thus, under Assumption 1, system (1) has a unique solution. (Refer to the reference [19] for details.)

Then, here we try to research the contraction property for time-delay system (1). First, we prove the following lemma:

Lemma 1. *Assume that $|\cdot|_{\chi(t)}$ is right regular, $\chi(t)$ is a right-continuous staircase function with discontinuities in $\{t_1, ..., t_j, ...\}$ and $|\cdot|_{\chi(t)} \le C_0|\cdot|_{\chi(t')}$ holds for any $t, t' \ge t_0$. $x(t)$ is continuous, t_0 is the initial time, and $\overline{\tau} = \max_k \sup_{s\ge t_0} \tau_k(s)$. For the following time-delay system:*

$$\dot{x}(t) = A(t)x(t) + \sum_{k=1}^{n} B_k(t)x(t - \tau_k(t))$$

where $x(t) \in \mathbb{R}^n$, $A(t), B_k(t) \in \mathbb{R}^{n\times n}(k = 1, 2, ..., m)$ is piecewise continuous w.r.t. t, and the discontinuities belong to $\{t_1, t_2, ..., t_i, ...\}$, which is a countable set. If there exists a piecewise right-continuous function $m(t) > 0$ whose discontinuities belong to $\{t_1, t_2, ..., t_i, ...\}$, and matrix-valued functions $B^{(1)}_k(t), B^{(2)}_k(t)$ such that

$$v_{\chi(t)}(A(t) + \sum_{k=1}^{m} B^{(1)}_k(t)) + \sum_{k=1}^{m} ||B^{(2)}_k(t)||_{\chi(t)} \frac{m(t)}{m(t - \tau_k(t))} + \sum_{k=1}^{m} \tau_k(t)||B^{(1)}_k(t)||_{\chi(t)}(\tilde{A}^k(t) + \tilde{B}^k(t)) \le -\frac{D^+ m(t)}{m(t)} \tag{3}$$

where D^+ represents the Dini derivative, and

$$\tilde{A}^k(t) = \sup_{t-\tau_k(t)\le s\le t} ||A(s)||_{\chi(s)} \frac{m(t)}{m(s)},$$
$$\tilde{B}^k(t) = \sup_{t-\tau_k(t)\le s\le t} ||B_k(s)||_{\chi(s)} \frac{m(t)}{m(s - \tau_k(s))}.$$

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Then for each $t \in [t_j, t_{j+1})$, we have

$$|x(t)|_{\chi(t_j)} m(t) \le \sup_{t_0-\overline{\tau}\le s\le t} |x(s)|_{\chi(t_j)} m(s) \le C_0 \sup_{t_0-\overline{\tau}\le s\le t_j} |x(s)|_{\chi(t_j-)} m(s)$$

Proof. Let $V(t) = \sup_{t_0-\overline{\tau}\le\theta\le t} |x(\theta)|_{\chi(\theta)} m(\theta)$. Assume that $V(t)$ is strictly increasing at time point t^*, it implies that $V(t^*) = |x(t^*)|_{\chi(t^*)} m(t^*)$. There exists j such that $t^* \in [t_j, t_{j+1})$, and here we calculate the Dini derivative of $|x(t)|_{\chi(t_j)} m(t)$,

$$D^+[|x(t)|_{\chi(t_j)} m(t)] = \dot{m}(t)|x(t)|_{\chi(t_j)} + m(t) \lim_{h\to 0} \frac{|x(t+h)|_{\chi(t_j)} - |x(t)|_{\chi(t_j)}}{h},$$

in which $\dot{x}(t)$ can be rewritten as

$$\begin{aligned}
\dot{x}(t) &= A(t)x(t) + \sum_{k=1}^{m} B_k(t)x(t-\tau_k(t)) \\
&= [A(t) + \sum_{k=1}^{m} B_k^{(1)}(t)]x(t) + \sum_{k=1}^{m} B_k^{(1)}(t) \int_t^{t-\tau_k(t)} \dot{x}(s)ds + \sum_{k=1}^{m} B_k^{(2)}(t)x(t-\tau_k(t))
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\overline{\lim}_{h\to 0} \frac{|x(t+h)|_{\chi(t_j)} - |x(t)|_{\chi(t_j)}}{h} \\
=&\overline{\lim}_{h\to 0} \frac{1}{h}\Big[\Big|x(t) + h(A(t) + \sum_{k=1}^{m} B_k^{(1)}(t))x(t) + h\sum_{k=1}^{m} B_k^{(1)}(t) \int_t^{t-\tau_k(t)} \dot{x}(s)ds \\
&+ h\sum_{k=1}^{m} B_k^{(2)}(t)x(t-\tau_k(t))\Big|_{\chi(t_j)} - |x(t)|_{\chi(t_j)}\Big] \\
=&\overline{\lim}_{h\to 0} \frac{1}{h}|x(t)|_{\chi(t_j)}\Big[\|I + h(A(t) + \sum_{k=1}^{m} B_k^{(1)}(t))\|_{\chi(t_j)} - 1\Big] + \sum_{k=1}^{m} \|B_k^{(1)}(t)\|_{\chi(t_j)} \int_{t-\tau_k(t)}^{t} |\dot{x}(s)|_{\chi(t_j)} ds \\
&+ \sum_{k=1}^{m} \|B_k^{(2)}(t)\|_{\chi(t_j)} |x(t-\tau_k(t))|_{\chi(t_j)} \\
=&\nu_{\chi(t_j)}\Big(A(t) + \sum_{k=1}^{m} B_k^{(1)}(t)\Big)|x(t)|_{\chi(t_j)} + \sum_{k=1}^{m} \|B_k^{(1)}(t)\|_{\chi(t_j)} \int_{t-\tau_k(t)}^{t} |\dot{x}(s)|_{\chi(t_j)} ds \\
&+ \sum_{k=1}^{m} \|B_k^{(2)}(t)\|_{\chi(t_j)} |x(t-\tau(t))|_{\chi(t_j)}
\end{aligned}$$

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that is,

$$
\begin{aligned}
&D^+[|x(t)|_{\chi(t_j)}m(t)] \\
=&\frac{\dot{m}(t)}{m(t)}|x(t)|_{\chi(t_j)}m(t)+\nu_{\chi(t_j)}(A(t)+\sum_{k=1}^{m}B_k^{(1)}(t))|x(t)|_{\chi(t_j)}m(t) \\
&+\sum_{k=1}^{m}\|B_k^{(2)}(t)\|_{\chi(t_j)}|x(t-\tau_k(t))|_{\chi(t_j)} \\
&+\sum_{k=1}^{m}\|B_k^{(1)}(t)\|_{\chi(t_j)}m(t)\int_{t-\tau_k(t)}^{t}|A(s)x(s)+\sum_{k=1}^{m}B_k(s)x(s-\tau_k(s))|_{\chi(t_j)}ds \\
=&\frac{\dot{m}(t)}{m(t)}|x(t)|_{\chi(t_j)}m(t)+\nu_{\chi(t_j)}(A(t)+\sum_{k=1}^{m}B_k^{(1)}(t))|x(t)|_{\chi(t_j)}m(t) \\
&+\sum_{k=1}^{m}\|B_k^{(2)}(t)\|_{\chi(t_j)}|x(t-\tau_k(t))|_{\chi(t_j)}m(t-\tau_k(t))\frac{m(t)}{m(t-\tau_k(t))} \\
&+\sum_{k=1}^{m}\|B_k^{(1)}(t)\|_{\chi(t_j)}\int_{t-\tau_k(t)}^{t}\|A(s)\|_{\chi(t_j)}|x(s)|_{\chi(t_j)}m(s)\frac{m(t)}{m(s)} \\
&+\sum_{k=1}^{m}\|B_k(s)\|_{\chi(t_j)}|x(s-\tau_k(s))|_{\chi(t_j)}m(s-\tau_k(s))\frac{m(t)}{m(s-\tau_k(s))}ds
\end{aligned}
$$

Let

$$
\begin{aligned}
\tilde{A}^k(t) &= \sup_{t-\tau_k(t)\le s\le t}\|A(s)\|_{\chi(t_j)}\frac{m(t)}{m(s)} \\
\tilde{B}^k(t) &= \sup_{t-\tau_k(t)\le s\le t}\|B_k(s)\|_{\chi(t_j)}\frac{m(t)}{m(s-\tau_k(s))}
\end{aligned}
$$

then, for $t=t^*$, we have

$$
\begin{aligned}
D^+V(t)\le&\Bigg[\frac{\dot{m}(t)}{m(t)}+\nu_{\chi(t_j)}(A(t)+\sum_{k=1}^{m}B_k^{(1)}(t))+\sum_{k=1}^{m}\tau_k(t)\|B_k^{(1)}(t)\|_{\chi(t_j)}(\tilde{A}^k(t)+\tilde{B}^k(t)) \\
&+\sum_{k=1}^{m}\|B_k^{(2)}(t)\|_{\chi(t_j)}\frac{m(t)}{m(t-\tau_k(t))}\Bigg]V(t)\le 0
\end{aligned}
$$

which is contradictory with the hypothesis that $V(t)$ is strictly increasing at t^*. Therefore, $V(t)$ is decreasing for $t\in[t_j,t_{j+1})$, that is,

$$V(t)\le V(t_j)$$

that is,

$$|x(t)|_{\chi(t_j)}m(t)\le\sup_{t_0-\overline{\tau}\le s\le t_j}|x(s)|_{\chi(t_j)}m(s)\le C_0\sup_{t_0-\overline{\tau}\le s\le t_j}|x(s)|_{\chi(t_j-)}m(s)$$

The lemma is proved. □

With the conclusion in Lemma 1, we can select a proper function $m(t)>0$, and prove the following theorem for $v(t)$-incremental stability property of system (1).

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Theorem 1. *Suppose that Assumption 1 holds for time-delay system (1). Let $v(t)$ be a positive continuous function satisfying that $\lim_{t\to\infty} v(t) = \infty$. $\chi(t)$ is a right-continuous staircase function with discontinuities also in $\{t_1, ..., t_j, ...\}$. Let $N(t) = \#\{j : t \geq t_j, j = 1, 2, ...\}$, and there exist that*

1. *positive piecewise right-continuous function $m(t) > 0$,*
2. *a constant $T_0 > 0$,*
3. *positive constants $\alpha_k > 0 (k = 0, 1, ...)$,*
4. *$C(\chi(t), \chi(t')) \leq C_0$ for any $t, t' \geq t_0$,*

such that the following conditions hold:

$$m(t_k) \leq \alpha_k m(t_k-), \tag{4}$$

Let $A(t) = (\partial f/\partial x)(x, x_{\tau_1}, ..., x_{\tau_m}, r(t))$, $B_k(t) = (\partial f/\partial x_{\tau_k})(x, x_{\tau_1}, ..., x_{\tau_m}, r(t))$, $k = 1, ..., m$ are piecewise continuous w.r.t. t, and the discontinuities belong to $\{t_1, t_2, ..., t_i, ...\}$. There exist matrix-valued functions $B_k^{(1)}(t)$, $B_k^{(2)}(t)$, $B_k^{(1)}(t) + B_k^{(2)}(t) = B_k(t)$ such that

$$\nu_{\chi(t)}(A(t) + \sum_{k=1}^{m} B_k^{(1)}(t)) + \sum_{k=1}^{m} \tau_k(t)\|B_k^{(1)}(t)\|_{\chi(t)}(\tilde{A}^k(t) + \tilde{B}^k(t)) + \sum_{k=1}^{m} \|B_k^{(2)}(t)\|_{\chi(t)} \frac{m(t)}{m(t-\tau_k(t))} \leq -\frac{D^+ m(t)}{m(t)} \tag{5}$$

where

$$\tilde{A}_t^k = \sup_{t-\tau_k(t)\leq s\leq t} \|A(s)\|_{\chi(t)} \frac{m(t)}{m(s)}$$
$$\tilde{B}_t^k = \sup_{t-\tau_k(t)\leq s\leq t} \|B_k(s)\|_{\chi(t)} \frac{m(t)}{m(s-\tau_k(s))}.$$

and for any $t > T_0$,

$$\frac{\sup_{\theta\in[t_0-\bar{\tau},t_0]} m(\theta)}{m(t)} \prod_{k=1}^{N(t)} \alpha_k \beta_k \leq \frac{1}{v(t)}. \tag{6}$$

where $\beta_k = C(\chi(t_k), \chi(t_k-))$, then system (1) is $v(t)$-incremental stable.

Proof. For any initial state x_0, y_0, denote the corresponding initial function by $x_0(\cdot), y_0(\cdot), x_0(\cdot), y_0(\cdot) \in C([t_0 - \bar{\tau}, t_0], \mathbb{R}^m)$. Here we define a function $\varphi_\lambda(\cdot)$ as $\varphi_\lambda(s) = (1-\lambda)x_0(s) + \lambda y_0(s), \lambda \in [0,1], s \in [t_0 - \bar{\tau}, t_0]$, which is the initial value function of the initial state φ_λ. Let $\varphi_0 = x_0, \varphi_1 = y_0$. For $f(x, x_{\tau_1}, ..., x_{\tau_m}, r(t))$ is continuous with respect to $(x, x_{\tau_1}, ..., x_{\tau_m}, r)$ except for the switching time points $\{t_j\}$, and is continuously differentiable with respect to x, so we have the solution $\psi(t, \lambda) = x(t; \varphi_\lambda, r_t)$ is continuously differentiable with respect to φ_λ. Let $\omega = \partial\psi/\partial\lambda$, thus we can conclude that ω is well defined and continuous. From chain rule, $\omega(t, \lambda)$ is the solution of the following system:

$$\begin{cases} \dot{\omega} = \frac{\partial^2\psi}{\partial\lambda\partial t} = \frac{\partial f}{\partial \lambda} = \frac{\partial f}{\partial x}\omega + \sum_{k=1}^{m} \frac{\partial f}{\partial x_{\tau_k}}\omega_{\tau_k} \\ \omega(s, \lambda) = \frac{\partial \varphi_\lambda}{\partial \lambda}(s), s \in [t_0 - \bar{\tau}, t_0] \end{cases}$$

For $\partial\psi/\partial\lambda$ is well defined and continuous, with condition (5), from the proof of Lemma 1, select a piecewise right-continuous function $m(t) > 0$ whose discontinuities belong to $\{t_1, t_2, ..., t_i, ...\}$, we have

$$m(t)|\omega(t, \lambda)|_{\chi(t)} \leq V(t) \leq V(t_j) = \sup_{t_0-\bar{\tau}\leq s\leq t_j} m(s)|\omega(s, \lambda)|_{\chi(t_j)}$$

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holds for any $t \in [t_j, t_{j+1})$. Thus, under the condition (4) and (6), we have

$$\begin{aligned} m(t)|\omega(t,\lambda)|_{\chi(t)} \leq V(t) \leq V(t_j) = \sup_{t_0-\bar{\tau}\leq s\leq t_j} m(s)|\omega(s,\lambda)|_{\chi(t_j)} \\ \leq \alpha_j \cdot \beta_j \sup_{t_0-\bar{\tau}\leq s\leq t_j} m(s)|\omega(s,\lambda)|_{\chi(t_j-)} \\ \leq \alpha_j \cdot \beta_j \sup_{t_0-\bar{\tau}\leq s\leq t_{j-1}} m(s)|\omega(s,\lambda)|_{\chi(t_{j-1})} \\ \leq \prod_{j=1}^{N(t)} \alpha_j \cdot \beta_j \cdot \sup_{t_0-\bar{\tau}\leq s\leq t_0} m(s)|\omega(s,\lambda)|_{\chi(t_0)} \\ \leq \prod_{j=1}^{N(t)} \alpha_j \cdot \beta_j \cdot \sup_{\theta\in[t_0-\bar{\tau},t_0]} m(\theta) \sup_{t_0-\bar{\tau}\leq s\leq t_0} |\omega(s,\lambda)|_{\chi(t_0)} \end{aligned}$$

then, we have

$$|\omega(t,\lambda)|_{\chi(t)} \leq \frac{\sup_{\theta\in[t_0-\bar{\tau},t_0]} m(\theta)}{m(t)} \prod_{k=1}^{N(t)} \alpha_k\beta_k |\omega(t_0,\lambda)|_{\chi(t_0)} \leq \frac{1}{v(t)} \sup_{t_0-\bar{\tau}\leq s\leq t_0} |\omega(t_0,\lambda)|_{\chi(t_0)}.$$

Therefore, together with the condition (6), we conclude that the time-delay system (1) is incrementally uniformly asymptotically stable:

$$\begin{aligned} |x(t;y_0,r_t) - x(t;x_0,r_t)|_{\chi(t_0)} \leq C_0|x(t;y_0,r_t) - x(t;x_0,r_t)|_{\chi(t)} \\ \leq C_0 \Big| \int_0^1 \frac{\partial\psi(t,\lambda)}{\partial\lambda} d\lambda \Big|_{\chi(t)} \leq C_0 \int_0^1 |\omega(t,\lambda)|_{\chi(t)} d\lambda \\ \leq C_0 \frac{\sup_{\theta\in[t_0-\bar{\tau},t_0]} m(\theta)}{m(t)} \prod_{k=1}^{N(t)} \alpha_k\beta_k \int_0^1 \sup_{t_0-\bar{\tau}\leq s\leq t_0} |\omega(t_0,\lambda)|_{\chi(t_0)} d\lambda \\ \leq \frac{C_0}{v(t)} \sup_{\theta\in[t_0-\bar{\tau},t_0]} |x_0(\theta) - y_0(\theta)|_{\chi(t_0)} \end{aligned}$$

□

Actually, the key thought of Theorem 1 is replacing the time-delay term $B_k(t)x_{\tau_k}(t)$ with $B_k^{(1)}(t)x_{\tau_k}(t) + B_k^{(2)}(t)x(t) + B_k^{(2)}(t)\int_t^{t-\tau_k(t)} \dot{x}(s)ds$. In some special cases, we can set $B_k^{(1)}(t) = 0$ or $B_k^{(2)}(t) = 0$ for $k = 1, 2, ..., m$, which infers the following corollaries.

Corollary 1. *Suppose that Assumption 1 holds. Let $v(t)$ be a positive continuous function satisfying that $\lim_{t\to\infty} v(t) = +\infty$. $\chi(t)$ is a right-continuous staircase function with discontinuities in $\{t_1, ..., t_j, ...\}$. Let $N(t) = \#\{j : t \geq t_j, j = 1, 2, ...\}$, and there exist that*

1. *positive piecewise right-continuous function $m(t) > 0$,*
2. *a constant $T_0 > 0$,*
3. *positive constants $\alpha_k > 0 (k = 0, 1, ...)$,*
4. *$C(\chi(t), \chi(t')) \leq C_0$ for any $t, t' \geq t_0$,*

such that the following conditions hold:

$$m(t_k) \leq \alpha_k m(t_k-), \tag{7}$$

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Let $A(t) = (\partial f/\partial x)(x, x_{\tau_1}, ..., x_{\tau_m}, r(t))$, $B_k(t) = (\partial f/\partial x_{\tau_k})(x, x_{\tau_1}, ..., x_{\tau_m}, r(t))$, $k = 1, ..., m$ are piecewise continuous w.r.t. t, and the discontinuities belong to $\{t_1, t_2, ..., t_i, ...\}$,

$$\nu_{\chi(t)}(A(t)) + \sum_{k=1}^{m} \|B_k(t)\|_{\chi(t)} \frac{m(t)}{m(t-\tau_k(t))} \leq -\frac{D^+ m(t)}{m(t)}$$

and for any $t > T_0$,

$$\frac{\sup_{\theta\in[t_0-\overline{\tau},t_0]} m(\theta)}{m(t)} \prod_{k=1}^{N(t)} \alpha_k \beta_k \leq \frac{1}{v(t)}.$$

where $\beta_k = C(\chi(t_k), \chi(t_k-))$, then system (1) is $v(t)$-incremental stable.

Corollary 2. *Suppose that Assumption 1 holds. Let $v(t)$ be a positive continuous function satisfying that $\lim_{t\to\infty} v(t) = +\infty$. $\chi(t)$ is a right-continuous staircase function with discontinuities in $\{t_1, ..., t_j, ...\}$. Let $N(t) = \#\{j : t \geq t_j, j = 1, 2, ...\}$, and there exist that*

1. *positive piecewise right-continuous function $m(t) > 0$,*
2. *a constant $T_0 > 0$,*
3. *positive constants $\alpha_k > 0 (k = 0, 1, ...)$,*
4. *$C(\chi(t), \chi(t')) \leq C_0$ for any $t, t' \geq t_0$,*

such that the following conditions hold:

$$m(t_k) \leq \alpha_k m(t_k-),$$

Let $A(t) = (\partial f/\partial x)(x, x_{\tau_1}, ..., x_{\tau_m}, r(t))$, $B_k(t) = (\partial f/\partial x_{\tau_k})(x, x_{\tau_1}, ..., x_{\tau_m}, r(t))$, $k = 1, ..., m$, there exist matrix-valued functions $B_k^{(1)}(t)$, $B_k^{(2)}(t)$, $B_k^{(1)}(t) + B_k^{(2)}(t) = B_k(t)$ such that

$$\nu_{\chi(t)}(A(t) + \sum_{k=1}^{m} B_k(t)) + \sum_{k=1}^{m} \tau_k(t) \|B_k(t)\|_{\chi(t)} (\tilde{A}^k(t) + \tilde{B}^k(t)) \leq -\frac{D^+ m(t)}{m(t)}$$

where

$$\tilde{A}_t^k = \sup_{t-\tau_k(t)\leq s\leq t} \|A(s)\|_{\chi(t)} \frac{m(t)}{m(s)}$$

$$\tilde{B}_t^k = \sup_{t-\tau_k(t)\leq s\leq t} \|B_k(s)\|_{\chi(t)} \frac{m(t)}{m(s-\tau_k(s))}$$

and for any $t > T_0$,

$$\frac{\sup_{\theta\in[t_0-\overline{\tau},t_0]} m(\theta)}{m(t)} \prod_{k=1}^{N(t)} \alpha_k \beta_k \leq \frac{1}{v(t)}.$$

where $\beta_k = C(\chi(t_k), \chi(t_k-))$, then system (1) is $v(t)$-incremental stable.

Furthermore, if we take $m(t) = e^{\gamma_{N(t)} t}$ and $v(t) = e^{ct}$, we have the following corollary for exponential incremental stability.

Corollary 3. *Suppose that Assumption 1 holds, $\chi(t)$ is a right-continuous staircase function with discontinuities in $\{t_1, ..., t_j, ...\}$. Let $N(t) = \#\{j : t \geq t_j, j = 1, 2, ...\}$, and there exist*

1. *a positive constant $T_0 > 0$,*
2. *a constant sequence $\gamma_k (k = 0, 1, ...)$,*
3. *for any $t, t' \geq t_0$, it holds that $C(\chi(t), \chi(t')) \leq C_0$,*
4. *a positive constant $c > 0$,*

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Let $A(t) = (\partial f/\partial x)(x, x_{\tau_1}, \ldots, x_{\tau_m}, r(t))$, $B_k(t) = (\partial f/\partial x_{\tau_k})(x, x_{\tau_1}, \ldots, x_{\tau_m}, r(t))$, $k = 1, \ldots, m$, there exist $B_k^{(1)}(t), B_k^{(2)}(t)$, such that $B_k^{(1)}(t) + B_k^{(2)}(t) = B_k(t)$ and

$$\nu_{\chi(t)}(A(t) + \sum_{k=1}^{m} B_k^{(1)}(t)) + \sum_{k=1}^{m} \tau_k(t)\|B_k^{(1)}(t)\|_{\chi(t)}(\tilde{A}_t^k + \tilde{B}_t^k) + \sum_{k=1}^{m} \|B_k^{(2)}(t)\|_{\chi(t)} \frac{\exp\left((\gamma_{N(t)} - \gamma_{N(t-\tau_k(t))})t\right)}{\exp\left(\gamma_{N(t-\tau_k(t))}\tau_k(t)\right)} \leq -\gamma_{N(t)},$$

where

$$\tilde{A}_t^k = \sup_{t-\tau_k(t)\leq s\leq t} \|A(s)\|_{\chi(t)} \exp\left(\gamma_{N(t)}t - \gamma_{N(s)}s\right),$$
$$\tilde{B}_t^k = \sup_{t-\tau_k(t)\leq s\leq t} \|B_k(s)\|_{\chi(t)} \exp\left(\gamma_{N(t)}t - \gamma_{N(s-\tau_k(s))}(s - \tau_k(s))\right),$$

and for any $t > T_0$,

$$\frac{1}{t}\left[\sum_{i=0}^{N(t)-1} [(\gamma_{i+1} - \gamma_i)t_{i+1} + \log \beta_{i+1}] + \gamma_0 t_0 - \gamma_{N(t)}t\right] \leq -c$$

where $\beta_j = C(\chi(t_j), \chi(t_j-))$, then the system (1) is exponentially incrementally uniformly stable, and the exponential convergence rate can be estimated as $O(e^{-ct})$.

If we take $m(t) = t^{\alpha_{N(t)}}$, and $v(t) = t^c$, we have the following corollary for power-rate incremental stability.

Corollary 4. *Suppose that Assumption 1 holds, $\chi(t)$ is a right-continuous staircase function with discontinuities in $\{t_1, \ldots, t_j, \ldots\}$. Let $N(t) = \#\{j : t \geq t_j, j = 1, 2, \ldots\}$, and there exist*

1. *a positive constant $T_0 > 0$,*
2. *a constant sequence $\alpha_k (k = 0, 1, \ldots)$,*
3. *for any $t, t' \geq t_0$, it holds that $C(\chi(t), \chi(t')) \leq C_0$,*
4. *a positive constant $c > 0$,*

Let $A(t) = (\partial f/\partial x)(x, x_{\tau_1}, \ldots, x_{\tau_m}, r(t))$, $B_k(t) = (\partial f/\partial x_{\tau_k})(x, x_{\tau_1}, \ldots, x_{\tau_m}, r(t))$, $k = 1, \ldots, n$, and $A(t), B_k(t)$ are piecewise continuous w.r.t. t, and the discontinuities belong to $\{t_1, t_2, \ldots, t_i, \ldots\}$. There exist $B_k^{(1)}(t), B_k^{(2)}(t)$, such that $B_k^{(1)}(t) + B_k^{(2)}(t) = B_k(t)$ and

$$\nu_{\chi(t)}(A(t) + \sum_{k=1}^{m} B_k^{(1)}(t)) + \sum_{k=1}^{m} \tau_k(t)\|B_k^{(1)}(t)\|_{\chi(t)}(\tilde{A}_t^k + \tilde{B}_t^k) + \sum_{k=1}^{m} \|B_k^{(2)}(t)\|_{\chi(t)} \frac{t^{\alpha_{N(t)}}}{(t - \tau_k(t))^{\alpha_{N(t-\tau_k(t))}}} \leq -\alpha_{N(t)}t^{-1},$$

where

$$\tilde{A}_t^k = \sup_{t-\tau_k(t)\leq s\leq t} \|A(s)\|_{\chi(t)} \frac{t^{\alpha_{N(t)}}}{s^{\alpha_{N(s)}}}$$
$$\tilde{B}_t^k = \sup_{t-\tau_k(t)\leq s\leq t} \|B_k(s)\|_{\chi(t)} \frac{t^{\alpha_{N(t)}}}{(s - \tau_k(s))^{\alpha_{N(s-\tau_k(s))}}}$$

and for any $t > T_0$,

$$\frac{1}{\ln t}\left[\sum_{i=1}^{N(t)} [(\alpha_i - \alpha_{i-1}) \ln t_i + \ln \beta_i] + \alpha_0 \ln t_0\right] - \alpha_{N(t)} \leq -c$$

where $\beta_j = C(\chi(t_j), \chi(t_j-))$, then the system (1) is power-rate incrementally uniformly stable, and the convergence rate can be estimated as $O(t^{-c})$.

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4. Incremental Stability for Time-Delay Dynamical Systems with Discontinuous Right-Hands

Here we consider the time-delay dynamical systems[25] with discontinuous right-hands using multiple norms, formulated as follows:

$$\dot{x} = f(x(t), x_{\tau_1}(t), ..., x_{\tau_m}(t), r(t)) \tag{8}$$

where $x, x_{\tau_1}, ..., x_{\tau_m} \in \mathbb{R}^n$, $x_{\tau_k}(t) = x(t - \tau_k(t)), (k = 1, ..., m)$ represents the time-delay term. The right-hand function f may be discontinuous w.r.t. $(x, x_{\tau_1}, ..., x_{\tau_m})$. The solution of the system (8) can be defined as a solution of the following differential inclusion, which is named a (time-delay) Filippov system,

$$\begin{cases} \dot{x} \in F(x(t), x_{\tau_1}(t), ..., x_{\tau_m}(t), r(t)), \ \ t \in [0, T] \\ x(s) = \phi(s), s \in [t_0 - \overline{\tau}, t_0] \end{cases} \tag{9}$$

where $x_{\tau_k}(t) = x(t - \tau_k(t))$, $\tau_k(t)$ is non-negative for $t \in [t_0, T]$, $T \in (t_0, +\infty]$, $\overline{\tau} = \max_k \sup_{s \geq t_0} \tau_k(s)$, the initial function $\phi(\cdot)$ is defined on $[t_0 - \overline{\tau}, t_0]$, and F is a set-valued mapping defined as follows,

$$\begin{aligned} F(x(t), x_{\tau_1}(t), ..., x_{\tau_m}(t), r(t)) &= \mathcal{K}[f](x(t), x_{\tau_1}(t), ..., x_{\tau_m}(t), r(t)) \\ &= \bigcap_{\epsilon > 0} \bigcap_{\mu(P)=0} \overline{co}\{f(B((x, x_{\tau_1}, ..., x_{\tau_m}), \epsilon) \setminus P, r(t))\}, \end{aligned}$$

where $\mu(\cdot)$ stands for the Lebesgue measure, $B((x, x_{\tau_1}, ..., x_{\tau_m}), \epsilon) = \{(y, y_{\tau_1}, ..., y_{\tau_m}) : |(y^\top, y_{\tau_1}^\top, ..., y_{\tau_m}^\top)^\top - (x^\top, x_{\tau_1}^\top, ..., x_{\tau_m}^\top)^\top| \leq \epsilon\}$ represents the ϵ-neighborhood of $(x, x_{\tau_1}, ..., x_{\tau_m})$ with the given vector norm $|\cdot|$, and $\overline{co}$ represents convex closure.

4.1. Existence and Uniqueness of the Solution

Before the main theorem, the existence and uniqueness of the Cauchy problem of the Filippov system (9) should be proved first. Herein, several the existing result on dynamical systems without time-delay terms are presented as follows. Readers are referred to [7,18,19] for the details. For some dynamical system without time-delay terms, formulated as follows,

$$\dot{x} = f(x, t)$$

in which $x \in \mathbb{R}^n$, similarly we have the corresponding Filippov system,

$$\dot{x} = F(x, t) = \mathcal{K}[f](x, t) = \bigcap_{\epsilon > 0} \bigcap_{\mu(P)=0} \overline{co}\{f(B(x, \epsilon) \setminus P, t)\} \tag{10}$$

Then, from Definition 4 and 5 in [18], We have conclusion that, under the following Assumption 2, it can be guaranteed that system (10) has at least one solution.

Assumption 2 ([18])**.** *The set-valued mapping $F : \mathbb{R}^n \times \mathbb{R}^+ \rightrightarrows \mathbb{R}^n$ satisfies that for all $(x, t) \in \mathbb{R}^n \times \mathbb{R}^+$, $F(x, t)$ is non-empty, bounded, convex and closed, and F is upper semicontinuous at (x, t).*

In Assumption 2, 'Upper semicontinuity' for the set-valued mapping F is defined as follows.

Definition 8 (Sec. 1, Chap. 2 in [19])**.** *A set-valued mapping $F : \mathbb{R}^n \times \mathbb{R}^+ \rightrightarrows Y$ is called upper semicontinuous at $(x, t) \in \mathbb{R}^n \times \mathbb{R}^+$ if and only if for any neighbourhood $\mathcal{U}$ of $F(x, t)$, $\exists \delta > 0$, such that $\forall (\tilde{x}, \tilde{t}) \in B((x, t), \delta)$, $F(\tilde{x}, \tilde{t}) \subset \mathcal{U}$.*

Assumption 3. *With respect to a given Euclid norm $|\cdot|$ defined on a n-dimensional space, we have the following hypothesis:*

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1. *For any $(x, y_1, ..., y_m, r) \in \mathbb{R}^{n\times(m+1)} \times \mathbb{R}$, $F(x, y_1, ..., y_m, r(t))$ is nonempty, convex, closed in $\mathbb{R}^n$, and set-valued mappingF is upper semicontinuous w.r.t. $(x, y_1, ..., y_m, r)$.*
2. *(Linearly increasing) There exists $\alpha > 0$ such that*

$$\sup\{|v| : v \in F(x, y_1, ..., y_m, r)\} \le \alpha(|x| + |y_1| + ... + |y_m| + |r| + 1)$$

holds for any $(x, y_1, ..., y_m, r) \in \mathbb{R}^{n\times(m+1)} \times \mathbb{R}$. With Gronwall inequality[26], it can be easily seem equivalent to: there exists $\Theta > 0$ such that

$$\sup\{|v| : v \in F(x, y_1, ..., y_m, r)\} \le \Theta$$

holds for any $(x, y_1, ..., y_m, r) \in \mathbb{R}^{n\times(m+1)} \times \mathbb{R}$.

3. *Function $\tau_k : [t_0, T] \to [0, \infty)$ $(k = 1, ..., m)$ is continuously differentiable and bounded, with its upper bound $\overline{\tau} := \max\{\tau_k(t) : t \in [t_0, T], k = 1, ..., m\}$ and lower bound $\underline{\tau} = \min\{\tau_k(t) : t \in [t_0, T], k = 1, ..., m\} > 0$.*
4. *The initial function $\phi(\cdot) \in L^\infty([t_0 - \overline{\tau}, t_0], \mathbb{R}^n)$ is measurable.*
5. *For any $(x, y_1, ..., y_m, r) \in \mathbb{R}^{n\times(m+1)} \times \mathbb{R}^n \times \mathbb{R}^+$, there exists continuous function $h(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$, such that $|F(x_1, y_1, ..., y_m, r(t)) - F(x_2, y_1, ..., y_m, r(t))| \le h(|x_1 - x_2|)$ holds.*

First we fix a continuous initial function $\phi(\cdot) \in L^\infty([t_0 - \overline{\tau}, t_0], \mathbb{R}^n)$, then select a measurable function $\psi : [t_0 - \overline{\tau}, t_0] \to \mathbb{R}^n$ such that $\psi(s) \in \mathcal{K}[\phi](s)$ holds for $s \in [t_0 - \overline{\tau}, t_0]$ almost everywhere. For $t \in [t_0, t_0 + \underline{\tau}]$, consider the following differential inclusion[47]:

$$\begin{cases} \dot{x}(t) \in F(x(t), \psi(t - \tau_1(t)), ..., \psi(t - \tau_m(t)), r(t)) \\ x(0) = \phi(0) \end{cases}, \tag{11}$$

From Assumption 3, together with Assumption 2 and the conclusion for existence of the solution of system (11), the inclusion (9) has at least one solution defined in $[t_0, t_0 + \underline{\tau}]$.

Therefore, similarly, the solution can be extended to $[t_0, +\infty)$. Assume, as inductive step, that the solution x is defined on $[t_0 - \overline{\tau}, t_0 + N\underline{\tau}]$, for some $N = 1, 2, ...$ Then, one can consider the vector $\alpha = x(t_0 + N\underline{\tau})$ as the initial state of the following differential inclusion,

$$\begin{cases} \dot{x}(t) \in F(x(t), \psi(t - \tau_1(t)), ..., \psi(t - \tau_m(t)), r(t)) \\ x(0) = \phi(0) \end{cases} \text{a.a.}, t \in [t_0 + N\underline{\tau}, t_0 + (N+1)\underline{\tau}].$$

Then one can extend $x(t), t \in [t_0 - \overline{\tau}, t_0 + N\underline{\tau}]$ to a right neighborhood of $t_0 + N\underline{\tau}$, the interval $[t_0 - \overline{\tau}, t_0 + (N+1)\underline{\tau}]$. That is, we have the following lemma,

Lemma 2. *If the time-delay Filippov system (9) satisfies Assumption 3, then, the system (9) has at least on solution in $[t_0, T]$, $T \in (0, +\infty]$.*

Then, when it comes to the problem of uniqueness, enlightened by Chapter 2, Section 10, Theorem 1 in [7], we similarly research the conditions for uniqueness of the solution of system (9).

Theorem 2. *Suppose that function $f(x(t), x_{\tau_1}(t), ..., x_{\tau_m}(t), r(t))$ defined on region $D \subset \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^+$ is discontinuous on zero measure set M, and there exists a integral function $l(t)$ such that $|f(x, x_{\tau_1}, ..., x_{\tau_m}, r)| \le l(t)$ holds for any $(x, x_{\tau_1}, ..., x_{\tau_m}, r) \in D$, and $l(t) < \infty$ almost everywhere. Let $\epsilon_0 > 0$, for any $(x_0, x_{\tau_1}, ..., x_{\tau_m}, r)$, $(y_0, y_{\tau_1}, ..., y_{\tau_m}, r)$ satisfying that $|x_0 - y_0| < \epsilon_0$ and $|x_{\tau_k} - y_{\tau_k}| \le |x_0 - y_0|$, $(k = 1, ..., m)$,*

$$(x_0 - y_0) \cdot (f(x_0, x_{\tau_1}, ..., x_{\tau_m}, t) - f(y_0, y_{\tau_1}, ..., y_{\tau_m}, r(t))) \le l(t)|x_0 - y_0|^2 \tag{12}$$

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Then, under the simplest convex definition (Page 50 in [7]), equation $\dot{x} = f(x, x_{\tau_1}, ..., x_{\tau_m}, r(t))$ is right-unique on D.

Proof. If for all $t \in \mathbb{R}^+$, any $\epsilon_0 > 0$, $x(\cdot)$ and $y(\cdot)$ satisfying that $|x(\cdot) - y(\cdot)|^{\infty}_{[t-\bar{\tau},t]} \leq |x(t) - y(t)| < \epsilon_0$, on the basis of (12), we have

$$(x(t) - y(t)) \cdot (f(x(t), x_{\tau_1}(t), ..., x_{\tau_m}(t), r(t)) - f(y(t), y_{\tau_1}(t), ..., y_{\tau_m}(t), r(t))) \leq l(t)|x(t) - y(t)|^2,$$

holds. Thus, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|x(t) - y(t)|^2 &= (x(t) - y(t))(\dot{x}(t) - \dot{y}(t)) \\ &= (x(t) - y(t)) \cdot (f(x(t), x_{\tau_1}(t), ..., x_{\tau_m}(t), r(t)) - f(y(t), y_{\tau_1}(t), ..., y_{\tau_m}(t), r(t))) \\ &\leq l(t)|x(t) - y(t)|^2.\end{aligned}$$

Therefore, similar to the proof of Theorem 1, in Section 10, Chapter 2 in [7], it can be seem that

$$\frac{d}{dt}\left(|x(t) - y(t)|^2 e^{-L(t)}\right) \leq 0$$

where $L(t) = \int_{t_0}^{t} l(s)ds$, that is, $|x(t) - y(t)|^2 e^{-L(t)}$ is decreasing w.r.t. time t. If $|x(\cdot) - y(\cdot)|^{\infty}_{[t_0-\bar{\tau},t_0]} = \sup\{|x(t) - y(t)| : t \in [t_0 - \bar{\tau}, t_0]\} = 0$, then $|x(t) - y(t)| = 0$ holds for $t > t_0$, then we obtained the right uniqueness of the solution of system (9). □

If system (8) is a switched system, according to Theorem 2, one can prove the right uniqueness of the solution.

Here formulate a switched system with time delay, that is, the right-hand function is switched w.r.t. $(x, x_{\tau_1}, ..., x_{\tau_m})$.

$$f(x, t) = f_i(x, x_{\tau_1}, ..., x_{\tau_m}, r(t)),\ (x, x_{\tau_1}, ..., x_{\tau_m}, t) \in R_i \tag{13}$$

in which $f_i : \mathbb{R}^{n\times(m+1)} \times \mathbb{R}^+ \to \mathbb{R}^n$, regions $R_i \subset \mathbb{R}^{n\times(m+1)} \times \mathbb{R}^+$, $i = 1, ..., K$. All of the regions R_i have nonempty interior. The discontinuities compose of several smooth hypersurfaces of dimension $d(d < (m+1)n + 1)$. Suppose that $\{S_i\}_{i=1}^{N}$ is a sequence $((m+1)n)$-dimentional smooth hypersurfaces, $S_i = \{(x, x_{\tau_1}, ..., x_{\tau_m}, t) \in \mathbb{R}^{n\times(m+1)} \times \mathbb{R}^+ \mid \phi_i(x, x_{\tau_1}, ..., x_{\tau_m}, t) = 0\}$, in which $\phi_i(x, x_{\tau_1}, ..., x_{\tau_m}, t) \in C^1(\mathbb{R}^{n\times(m+1)} \times \mathbb{R}^+, \mathbb{R})$, the continuous region of function f is a sequence of connected regions, whose boundaries are the switching surfaces. Suppose that the switching surfaces never intersect each other. Denote one of the connected continuous region of f by $G_i^+(G_i^-)$, then it satisfies that

1. $\partial G_i^+(\partial G_i^-) \subset \bigcup_k S_k$;
2. $f(x, x_{\tau_1}, ..., x_{\tau_m}, r(t))$ is continuous in $G_i^+(G_i^-)$;
3. $\phi_i(x, x_{\tau}, ..., x_{\tau_m}, t) > 0(< 0)$ holds in $G_i^+(G_i^-)$.

in which G_i^+ and G_i^- are two different regions with their common boundary on S_i.

Take a switched system defined as (13) with $K = 2, N = 1$ as an example.In the domain G, consider one of the switching hypersurface S, $f^+(x, x_{\tau_1}, ..., x_{\tau_m}, r(t))$ and $f^-(x, x_{\tau_1}, ..., x_{\tau_m}, r(t))$ represent the limiting values of the function $f(x, x_{\tau_1}, ..., x_{\tau_m}, r(t))$ at point $(x, x_{\tau_1}, ..., x_{\tau_m}, t)$ from the regions G^+ and G^- respectively. $f_N^+(x, x_{\tau_1}, ..., x_{\tau_m}, r(t))$ and $f_N^-(x, x_{\tau_1}, ..., x_{\tau_m}, r(t))$ represent the projections of the vectors $f^+(x, x_{\tau_1}, ..., x_{\tau_m}, r(t))$ and $f^-(x, x_{\tau_1}, ..., x_{\tau_m}, r(t))$ onto the normal vector to S directed from G^+ to G^- respectively at the point $(x, x_{\tau_1}, ..., x_{\tau_m}, t)$.

Together with Theorem 2, according to bimodal time-delay systems, we have the following conlusion:

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Theorem 3. *Under the notations defined above, for all $t \in [t_0, +\infty)$ and point $(x, x_{\tau_1}, ..., x_{\tau_m}, t) \in S$, if the inequality $f_N^-(x, x_{\tau_1}, ..., x_{\tau_m}, r(t)) - f_N^+(x, x_{\tau_1}, ..., x_{\tau_m}, r(t)) > 0$ is fulfilled, then right uniqueness of Filippov solution for the bimodal system (13)($K = 2$, $N = 1$) occurs. (possibly different inequalities for different $(x, x_{\tau_1}, ..., x_{\tau_m})$ and t).*

Proof. We use the conclusion in Theorem 2 to prove Theorem 3 above.

For any point $(z, z_{\tau_1}, ..., z_{\tau_m}, t)$ on the switching surface S, and $(x, x_{\tau_1}, ..., x_{\tau_m}, t) \in R_1$, $(y, y_{\tau_1}, ..., y_{\tau_m}, t) \in R_2$, which satisfy that $|y_{\tau_k} - z_{\tau_k}| \le |y - z|$, $|x_{\tau_k} - z_{\tau_k}| \le |x - z|$, $|x_{\tau_k} - y_{\tau_k}| \le |y - x|$. Since $\partial f_1/\partial x$, $\partial f_2/\partial x$ and $\partial f_1/\partial x_{\tau_i}, \partial f_2/\partial x_{\tau_i}$ $(i = 1, ..., m)$ are bounded, it implies that there exist $l, k_i > 0$ $(i = 1, ..., m)$ such that

$$\begin{aligned} &|f(x, x_{\tau_1}, ..., x_{\tau_m}, r(t)) - f_1(z, z_{\tau_1}, ..., z_{\tau_m}, r(t))| \le l|x - z| + \sum_{i=1}^{m} k_i |x_{\tau_i} - z_{\tau_i}|, \\ &|f(y, y_{\tau_1}, ..., y_{\tau_m}, r(t)) - f_2(z, z_{\tau_1}, ..., z_{\tau_m}, r(t))| \le l|y - z| + \sum_{i=1}^{m} k_i |y_{\tau_i} - z_{\tau_i}|. \end{aligned} \tag{14}$$

Similar to the proof of Theorem 2 in Chapter 2 in [7], if $f_N^-(x, x_{\tau_1}, ..., x_{\tau_m}, r(t)) - f_N^+(x, x_{\tau_1}, ..., x_{\tau_m}, r(t)) > 0$ holds, then

$$(x - z)(f_1(z, z_{\tau_1}, ..., z_{\tau_m}, r(t)) - f_2(z, z_{\tau_1}, ..., z_{\tau_m}, r(t))) \le 0. \tag{15}$$

The inequality (15) still holds if vector $x - y$ is substituted with vector $x - z$, which is on the same direction. Together with (14) and (15), it infers that

$$\begin{aligned} &(x - y)(f(x, x_{\tau_1}, ..., x_{\tau_m}, r) + f_2(z, z_{\tau_1}, ..., z_{\tau_m}, r) - f_1(z, z_{\tau_1}, ..., z_{\tau_m}, r) - f(y, y_{\tau_1}, ..., y_{\tau_m}, r)) \\ \le &|y - x| \cdot \Big(|f(x, x_{\tau_1}, ..., x_{\tau_m}, r) - f_1(z, z_{\tau_1}, ..., z_{\tau_m}, r)| + |f(y, y_{\tau_1}, ..., y_{\tau_m}, r) - f_2(z, z_{\tau_1}, ..., z_{\tau_m}, r)| \Big) \\ < &|y - x| \cdot \Big(l|x - z| + \sum_{i=1}^{m} k_i |x_{\tau_i} - z_{\tau_i}| + l|y - z| + \sum_{i=1}^{m} k_i |y_{\tau_i} - z_{\tau_i}| \Big). \end{aligned}$$

That is, if $|y_{\tau_i} - z_{\tau_i}| \le |y - z|$, $|x_{\tau_i} - z_{\tau_i}| \le |x - z|$, $|x_{\tau_i} - y_{\tau_i}| \le |y - x|$ $(i = 1, ..., m)$, it holds that

$$(x - y)(f(x, x_{\tau_1}, ..., x_{\tau_m}, r(t)) - f(y, y_{\tau_1}, ..., y_{\tau_m}, r(t))) < 2(l + \sum_{i=1}^{m} k_i)|y - x|^2.$$

Thus, together with Theorem 2, at any point $(z, z_{\tau_1}, ..., z_{\tau_m}, t)$ in the domain, right uniqueness of Filippov solution for system (13) $(K = 2, N = 1)$ occurs for $t \in [t_0, +\infty)$. □

4.2. Criteria for Incremental Stability for Filippov Systems with Time Delay

Here, with the conclusions above, we then research the conditions of incremental stability for time-delay systems with discontinuous right-hands.

$$\dot{x}(t) = f(x, x_{\tau_1}, ..., x_{\tau_m}, r(t)) \tag{16}$$

where $x_{\tau_k}(t) = x(t - \tau_k(t))$, $\tau_k(t)$ $(k = 1, 2, ..., m)$ is a bounded function with respect to time t, the function $f(x, x_{\tau_1}, ..., x_{\tau_m}, r(t)) = (f_1, f_2, ..., f_n) : \mathbb{R}^{n \times (m+1)} \times \mathbb{R}^+ \rightarrow \mathbb{R}^n$ is discontinuous with respect to $(x, x_{\tau_1}, ..., x_{\tau_m}, t)$ on a zero measure set. $r(t)$ is a piecewise right-continuous switched function, with its discontinuities in $\{t_1, t_2, ..., t_j, ...\}$.

Then according to the discontinuous right-hand $f(x, x_{\tau_1}, ..., x_{\tau_m}, r(t))$, with corresponding set-valued mapping $F(x, x_{\tau_1}, ..., x_{\tau_m}, r(t)) = \mathcal{K}[f](x, x_{\tau_1}, ..., x_{\tau_m}, r(t))$, we construct a sequence of

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functions $\{f^p(x, x_{\tau_1}, ..., x_{\tau_m}, r(t))\}_{p=1}^{\infty}$, satisfying the following conditions, denoted by Condition $C_{time-delay}(\Sigma)$, where $\Sigma \in \mathbb{R}^{n\times(m+1)}$:

1. $f^p(x, x_{\tau_1}, ...x_{\tau_m}, r(t)) = (f_1^p, f_2^p, ..., f_m^p) : \mathbb{R}^{n\times(m+1)} \times \mathbb{R}^+ \rightarrow \mathbb{R}^n$ is continuous and continuously differentiable w.r.t. $(x, x_{\tau_1}, ...x_{\tau_m})$, and continuous w.r.t. $(x, x_{\tau_1}, ...x_{\tau_m}, r)$. And $f_i^p, i = 1, ..., m$ satisfies local Lipschitz conditions for $(x, x_{\tau_1}, ...x_{\tau_m}) \in \Sigma$.
2. For each $t \geq t_0$ and compact set $\Sigma \subset \mathbb{R}^{n\times(m+1)}$,
$$\lim_{m\rightarrow\infty} d_H\{Graph(f^p(\Sigma, r(t))), Graph(F(\Sigma, r(t)))\} = 0$$
holds, where $F(x, x_{\tau_1}, ...x_{\tau_m}, r(t)) = \mathcal{K}[f](x, x_{\tau_1}, ...x_{\tau_m}, r(t))$ and d_H represents the Hausdorff metric. $Graph(F)$ and $Graph(f^p)$ are considered on $\mathbb{R}^{m\times(n+1)} \times [t_0, +\infty)$, where $Graph(F(x, x_{\tau_1}, ...x_{\tau_m}, r(t))) = \{(x, t, y) : y \in F(x, x_{\tau_1}, ...x_{\tau_m}, r(t)), (x, x_{\tau_1}, ...x_{\tau_m}) \in \Sigma, t \geq t_0\}$, and so it is with $Graph(f^p(x, x_{\tau_1}, ...x_{\tau_m}, r(t)))$.
3. for any compact set $\Sigma \subset \mathbb{R}^{m\times(n+1)}$, there exists measure $w(\cdot)$, defined as $w(\Sigma) = q(\lambda(\Sigma))$, in which λ represents the Lebesgue measure, q is a measurable function mapping $\mathbb{R}^+$ to $\mathbb{R}^+$, such that $|f^p(x, x_{\tau_1}, ...x_{\tau_m}, t)| \leq w(\Sigma)$ holds for each $(x, x_{\tau_1}, ...x_{\tau_m}) \in \Sigma$ and $t \geq t_0$.

Thus, we have the following conclusion on incremental stability of time-delay systems with discontinuous right-hands:

Theorem 4. *Suppose that system (16) has a unique solution, and there exists a sequence of functions* $\{f^p\}_{p=1}^{\infty}$ *satisfying Condition* $C_{time-delay}(\Sigma)$*. Let* $v(t)$ *be a positive continuous function satisfying that* $\lim_{t\rightarrow\infty} v(t) = \infty$ *and* $\chi(t)$ *is a right-continuous staircase function with discontinuities in* $\{t_1, ..., t_j, ...\}$*. Let* $N(t) = \#\{j : t \geq t_j, j = 1, 2, ...\}$*, and suppose that there exist*

1. *positive piecewise right-continuous function* $m(t) > 0$,
2. *a constant* $T_0 > 0$,
3. *positive constants* $\alpha_k > 0 (k = 0, 1, ...)$,
4. $C(\chi(t), \chi(t')) \leq C_0$ *for any* $t, t' \geq t_0$,

such that the following conditions hold:
$$m(t_k) \leq \alpha_k m(t_k-),$$
Let $A^p(t) = (\partial f^p/\partial x)(x, x_{\tau_1}, ..., x_{\tau_m}, r(t))$, $B_k^p(t) = (\partial f^p/\partial x_{\tau_k})(x, x_{\tau_1}, ..., x_{\tau_m}, r(t))$, $k = 1, ..., m$*, there exist matrix-valued functions* $B_{1k}^p(t)$, $B_{2k}^p(t)$, $B_{1k}^p(t) + B_{2k}^p(t) = B_k^p(t)$ *such that*

$$\nu_{\chi(t)}(A^p(t) + \sum_{k=1}^m B_{1k}^p(t)) + \sum_{k=1}^m \tau_k(t)\|B_{1k}^p(t)\|_{\chi(t)}(\tilde{A}_k^p(t) + \tilde{B}_k^p(t)) + \sum_{k=1}^m \|B_{2k}^p(t)\|_{\chi(t)} \frac{m(t)}{m(t-\tau_k(t))} \leq -\frac{D^+m(t)}{m(t)}$$

where

$$\tilde{A}_k^p(t) = \sup_{t-\tau_k(t)\leq s\leq t} \|A^p(s)\|_{\chi(t)} \frac{m(t)}{m(s)}$$
$$\tilde{B}_k^p(t) = \sup_{t-\tau_k(t)\leq s\leq t} \|B_k^p(s)\|_{\chi(t)} \frac{m(t)}{m(s-\tau_k(s))}.$$

and for any $t > T_0$,

$$\frac{\sup_{\theta\in[t_0-\overline{\tau}, t_0]} m(\theta)}{m(t)} \prod_{k=1}^{N(t)} \alpha_k\beta_k \leq \frac{1}{v(t)}.$$

where $\overline{\tau} = \max_k \sup_{\theta\geq t_0} \tau_k(\theta)$, $\beta_k = C(\chi(t_k), \chi(t_k-))$*, then system (9) is* $v(t)$*-incremental stable in* Σ *for* $t \in [t_0, +\infty)$.

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Proof. First, we construct a sequence of functions satisfying Condition $C_{time-delay}(\Sigma)$, and the corresponding sequence of time-delay system:

$$\dot{x}(t) = f^p(x, x_{\tau_1}, ...x_{\tau_m}, r(t)) \tag{17}$$

in which, according to Condition $C_{time-delay}(\Sigma)$, $f_i, i = 1, ..., m$ satisfy local Lipschitz consitions, so it can be seem that (17) has a unique solution for $t \in \mathbb{R}^+$.Therefore, from Theorem 1, for each p, system (17) is $v(t)$-incremental stable.

Denote any two of the solutions of (9) with different initial function $x_0(\cdot)$ and $y_0(\cdot)$ by $x(t)$ and $y(t)$. Then here try to approximate $x(t)$ and $y(t)$ by two sequences of solutions of (17), denoted by x^p and y^p respectively. The initial functions $x_0(s)$ and $y_0(s)$ here are defined for $s \in [t_0 - \overline{\tau}, t_0]$, where $\overline{\tau} = \max_k \sup_{s\in[t_0,+\infty)} \tau_k(s)$.

With f^p satisfying condition $C_{time-delay}(\Sigma)$, for each given T, $t_0 \le t \le T$, we have $x^p(t)$ and $\dot{x}^p(t)$ are bounded regarding $(x_0(\cdot), T)$. It is the same with $y^p(t)$. Thus $x^p(t)$ $(y^p(t))$ is uniformly bounded on $[t_0, T]$, and $\dot{x}^p(t)$ $(\dot{y}^p(t))$ is uniformly bounded on $[t_0, T]$ as well.

Because of Condition 1 in $C_{time-delay}(\Sigma)$, $x^p(t)$ $(y^p(t))$ is continuous with respect to t. Together with Condition 3 in $C_{time-delay}(\Sigma)$, we conclude that $x^p(t)$ $(y^p(t))$, $p \in \mathbb{N}^+$ are equicontinuous for $t \in [t_0, T]$.

Here we present the Arzela-Ascoli lemma (similar to Theorem 2.2 in [8]):

Lemma 3. *(Arzela-Ascoli Lemma) X is a compact set on $\mathbb{R}^n$. If a sequence $\{f_n\}_1^\infty$ in $C(X)$ is bounded and equicontinuous, then it has a uniformly convergent subsequence.*

From Lemma 3, one can find a sub-sequence of $\{x^p(t)\}_{p\in\mathbb{N}^+}$ and $\{y^p(t)\}_{p\in\mathbb{N}^+}$ (still denoted by $x^m(t)$ and $y^m(t)$) satisfying that $x^p(t)$ $(y^p(\mathrm{t}))$ uniformly converges to a continuous function $x^*(t)$ $(y^*(t))$ on $[t_0, T]$, $T \in (t_0, +\infty]$. For all $k \in \mathbb{N}^+$, one can find a subsequence $\{x^{kj}(t)\}_{j\in\mathbb{N}^+}$ of $\{x^p(t)\}_{p\in\mathbb{N}^+}$ such that $|x^{kj}(t) - x^*(t)| < \frac{1}{j}$ holds on the interval $[t_0, t_0 + k + 1]$. Then by diagonal selection principle, select a new subsequence $\{x^{kk}(t)\}_{k\in\mathbb{N}^+}$ such that $\{x^{kk}(t)\}_{k\in\mathbb{N}^+}$ uniformly converges to a continuous function $x^*(t)$ on $[t_0, T]$, $T \in (t_0, +\infty]$. It is the same with $x^*_{\tau_k}(t)$, for $k = 1, ..., m$.

The system $\dot{x}^p(t) = f^p(x, x_{\tau_1}, ..., x_{\tau_m}, r(t))$ has a unique solution. $x^p(t)$ satisfies Lipschitz condition:

$$|x^p(t) - x^p(t')|_{\mathcal{X}(t_0)} \le L|t - t'|_{\mathcal{X}(t_0)}$$

where $t, t' \in [t_0, T]$, $L > 0$. Because of norm equivalence, Lipschitz condition above also holds with other norms defined in $\mathbb{R}^n$. So $x^*(t)$ $(y^*(t))$ also satisfies Lipschitz condition, that is, $\dot{x}^*(t)$ $(\dot{y}^*(t))$ exists and is bounded and measurable for $[t_0, T)$, $T \in (t_0, +\infty]$.

We then have conclusion that $\dot{x}^p(t)$ $(\dot{y}^p(t))$ weakly converges to $\dot{x}^*(t)$ $(\dot{y}^*(t))$ on the space $L_1([t_0, T], \mathbb{R}^n)$, the demonstrations are as follows.

$C_0^\infty([t_0, T], \mathbb{R}^n)$ is dense in the Banach space $L^\infty([t_0, T], \mathbb{R}^n)$, which is the conjugate space $L_1([t_0, T], \mathbb{R}^n)$. Therefore, the following equation

$$\int_{t_0}^{T} \langle \dot{x}^p(t) - \dot{x}^*, q(t)\rangle dt = -\int_{t_0}^{T} \langle \dot{q}(t), x^p(t) - x^*\rangle dt.$$

holds for each $q(t) \in C_0^\infty([t_0, T], \mathbb{R}^n)$. Since $\{\dot{x}^p(t)\}$ is bounded for each p, from Lebesgue-dominant convergence theorem, we have

$$\lim_{p\to\infty} \int_{t_0}^{T} \langle \dot{x}^p(t) - \dot{x}^*, q(t)\rangle dt = -\int_{t_0}^{T} \langle \dot{q}(t), \lim_{p\to\infty} x^p(t) - x^*(t)\rangle dt = 0.$$

That is, $\{\dot{x}^p(t)\}$ weakly converges to $\dot{x}^*(t)$ on the space $L_1([t_0, T], \mathbb{R}^n)$.

By the Mazur's convexity theorem [8], one can find a_l^n (b_l^n) with $\sum_{l=1}^m a_l^m = 1$ $(\sum_{l=1}^m b_l^m = 1)$ such that $\dot{\tilde{x}}^p$ converges to $\dot{x}^*(t)$ almost everywhere on $[t_0, T]$, where $\tilde{x}^p(t) = \sum_{l=1}^p a_l^p x^p$. Notice that $\tilde{x}^p$ is in

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the convex closure of $\{x^p\}$, $\dot{\tilde{x}}^p$ converges to $\dot{x}^*$ uniformly. So it is with $\dot{\tilde{y}}^p(t)$ with $\tilde{y}^p(t) = \sum_{l=1}^{p} b_l^p y^p$. And it is the same with $\{\dot{x}_{\tau_k}^p(t)\}$ and $\{\dot{x}_{\tau_k}^*(t)\}$.

Recall Condition 3 in $C_{time-delay}(\Sigma)$. For $\Sigma \in \mathbb{R}^n$, it holds that

$$\lim_{p\to\infty} d_H \{Graph(f^p(\Sigma,t)), Graph(F(\Sigma,t))\} = 0, \forall t \geq t_0,$$

With $\dot{\tilde{x}}^p(t)$ in the convex closure of $\{f^p(x^p, x_{\tau_1}^p, ..., x_{\tau_m}^p, r(t))\}$, for any $\epsilon > 0$, there exists $N > 0$ such that $\dot{x}^p(t) \in B(F(x^*, x_{\tau_1}^*, ..., x_{\tau_m}^*, r(t)), \epsilon)$ for all $p > N$ and $(x^*, x_{\tau_1}^*, ..., x_{\tau_m}^*), (x^p, x_{\tau_1}^p, ..., x_{\tau_m}^p) \in \Sigma$, $t \in [t_0, T]$.

Since ϵ can be arbitrarily small, it can be seem that $\dot{x}^*(t) \in F(x^*, x_{\tau_1}^*, ..., x_{\tau_m}^*, r(t))$ with $x^* \in \Sigma$, which infers that the solution of (9) equals to $x^*(t)$ in Σ almost everywhere on $[t_0, T]$. For $x(t)$ and x^* are both continuous because $\tilde{x}^p$ converges to x^* uniformly on $[t_0, T]$, it can be seem that x^* is the solution of (9). So it is with $y^*(t)$.

Because system (9) has a unique solution, $x(t) = x^*(t)$ and $y(t) = y^*(t)$ almost everywhere for $t \geq t_0$. That is, x^p converges to $x(t)$ uniformly in $[t_0, T]$, $T \in (t_0, +\infty]$. Similar proof can be applied to $y^p(t)$ and $y(t)$.

For $\{f^p(x, x_{\tau_1}, ..., x_{\tau_m}, r(t))\}$ is $v(t)$-incremental stable from Theorem 1, there exists some $M > 0$,

$$|x^p(t) - y^p(t)|_{\chi(t_0)} \leq Me^{-\alpha(t-t_0)} \sup_{s\in[t_0-\overline{\tau},t_0]} |x_0(s) - y_0(s)|_{\chi(t_0)}$$

for all $p \in \mathbb{N}^+$ and $t \geq t_0$. For each given $t \geq t_0$, let $\epsilon(t) = 3Me^{-\alpha(t-t_0)} \sup_{s\in[t_0-\overline{\tau},t_0]} |x_0(s) - y_0(s)|_{\chi(t_0)}$, there exists some $p_0(\epsilon, t)$ with which $|x^p(t) - x(t)|_{\chi(t_0)} \leq \epsilon/3$ and $|y^p(t) - y(t)|_{\chi(t_0)} \leq \epsilon/3$ hold for $p \geq p_0(\epsilon, t)$, which implies that

$$\begin{aligned} |x(t) - y(t)|_{\chi(t_0)} &\leq |x(t) - x^p(t)|_{\chi(t_0)} + |y(t) - y^p(t)|_{\chi(t_0)} + |x^p(t) - y^p(t)|_{\chi(t_0)} \\ &\leq \epsilon = 3Me^{-\alpha(t-t_0)} \sup_{s\in[t_0-\overline{\tau},t_0]} |x_0(s) - y_0(s)|_{\chi(t_0)}. \end{aligned}$$

This completes the proof.

□

Remark 2. *Similar to Theorem 4, Corollary 1, Corollary 2, Corollary 3 and Corollary 4 can also be extended to discontinuous cases.*

5. Applications

In this section, with conclusion in Theorem 4, the applications to switched time-delay system and Hopfield time-delay sytems are given.

5.1. Linear Switched Time-Delay System

Consider the following linear switched time-delay system:

$$\dot{x}(t) = f(x, x_{\tau_1}, ..., x_{\tau_m}, t) = \begin{cases} A_1(t)x(t) + \sum_{k=1}^{m} B_{1k}(t)x(t-\tau_k(t)) + J_1(t), & \phi(x, x_{\tau_1}, ..., x_{\tau_m}) > 0 \\ A_2(t)x(t) + \sum_{k=1}^{m} B_{2k}(t)x(t-\tau_k(t)) + J_2(t), & \phi(x, x_{\tau_1}, ..., x_{\tau_m}) < 0 \end{cases}. \quad (18)$$

in which $x(t) \in \mathbb{R}^n$, $A_1(t), A_2(t) \in \mathbb{R}^{n\times n}$, $B_{1k}(t), B_{2k}(t)(k = 1, ..., m) \in \mathbb{R}^{n\times n}$ is piecewise continuous, bounded matrix-valued functions, whose discontinuities are in $\{t_1, t_2....t_i, ...\}$. Take a simple bimodal system as example, with the switching surface $S = \{(x, x_{\tau_1}, ..., x_{\tau_m}) : \phi(x, x_{\tau_1}, ..., x_{\tau_m}) = 0\}$.

In order to guarantee the uniqueness of the Filippov solution for system (18), the linear time-delay system satisfies the following hypothesis:

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Assumption 4. *The linear time-delay system (18) satisfies:*

1. *the right-hand function of system (18) satisfies Assumption 3.*
2. *For each point $(x, x_{\tau_1}, ..., x_{\tau_m}) \in S$, the time-delay system (18) satisfies*

$$\frac{\partial \phi}{\partial x}(x, x_{\tau_1}, ..., x_{\tau_m}) \cdot \left((A_1(t) - A_2(t))x + \sum_{k=1}^{m} (B_{1k}(t) - B_{2k}(t))x_{\tau_k} + J_1(t) - J_2(t) \right) < 0$$

Under the definitions and Assumption 4 above, from Lemma 2 and Theorem 3, it can be obtained that system (18) has a unique solution. Thus, together with Theorem 4, we have the following corollary on incremental stability for bimodal linear time-delay systems.

Corollary 5. *Let $v(t)$ be a positive continuous function satisfying that $\lim_{t\to\infty} v(t) = \infty$. Let $N(t) = \#\{j : t \geq t_j, j = 1, 2, ...\}$, $\chi(t)$ is a right-continuous staircase function with its discontinuous points belong to $\{t_1, ..., t_j, ...\}$, and suppose that system (18) satisfies Assumption 4, and there exist*

1. *positive piecewise right-continuous function $m(t) > 0$,*
2. *a constant $T_0 > 0$,*
3. *positive constants $\alpha_k > 0 (k = 0, 1, ...)$,*
4. *$C(\chi(t), \chi(t')) \leq C_0$ for any $t, t' \geq t_0$,*
5. *matrix $B_{1k}^{(1)}, B_{1k}^{(2)}, B_{2k}^{(1)}, B_{2k}^{(2)} \in \mathbb{R}^{n\times n}$ satisfying that $B_{1k}^{(1)}(t) + B_{1k}^{(2)}(t) = B_{1k}(t)$ and $B_{2k}^{(1)}(t) + B_{2k}^{(2)}(t) = B_{2k}(t)$,*

such that the following conditions are satisfied:

$$m(t_k) \leq \alpha_k m(t_k-), \quad \forall k$$

$$\begin{aligned} L_1(t) =& v_{\chi(t)}(A_1(t) + \sum_{k=1}^{m} B_{1k}^{(1)}(t)) + \sum_{k=1}^{m} \tau_k(t)(\tilde{A}_{1k}(t) + \tilde{B}_{1k}(t))\|B_{1k}^{(1)}(t)\|_{\chi(t)} \\ &+ \sum_{k=1}^{m} \|B_{1k}^{(2)}(t)\|_{\chi(t)} \frac{m(t)}{m(t - \tau_k(t))} \leq -\frac{D^+ m(t)}{m(t)} \\ L_2(t) =& v_{\chi(t)}(A_2(t) + \sum_{k=1}^{m} B_{2k}^{(1)}(t)) + \sum_{k=1}^{m} \tau_k(t)(\tilde{A}_{2k}(t) + \tilde{B}_{2k}(t))\|B_{2k}^{(1)}(t)\|_{\chi(t)} \\ &+ \sum_{k=1}^{m} \|B_{2k}^{(2)}(t)\|_{\chi(t)} \frac{m(t)}{m(t - \tau_k(t))} \leq -\frac{D^+ m(t)}{m(t)} \end{aligned}$$

in which, for $i = 1, 2$,

$$\tilde{A}_{ik}(t) = \sup_{t-\tau_k(t)\leq s\leq t} \|A_i(s)\|_{\chi(t)} \frac{m(t)}{m(s)}$$

$$\tilde{B}_{ik}(t) = \sup_{t-\tau_k(t)\leq s\leq t} \|B_{ik}(s)\|_{\chi(t)} \frac{m(t)}{m(s - \tau_k(s))}$$

and for all $t > T_0$,

$$\frac{\sup_{\theta\in[t_0-\bar{\tau}, t_0]} m(\theta)}{m(t)} \prod_{k=1}^{N(t)} \alpha_k \beta_k \leq \frac{1}{v(t)}. \tag{19}$$

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where $\beta_k = C(\chi(t_k), \chi(t_k-))$. Moreover, in the neighbourhood of switching surface S, that is, when $-\frac{\delta}{2} < \phi(x, x_{\tau_1}, ..., x_{\tau_m}) < \frac{\delta}{2}$, let $w = (A_1(t) - A_2(t))x(t) + \sum_{k=1}^m (B_{1k}(t) - B_{2k}(t))x(t - \tau_k(t)) + J_1(t) - J_2(t)$, if it is satisfied that

$$\begin{aligned}
&pL_1(t) + (1-p)L_2(t) + \frac{1}{\delta}\nu_{\chi(t)}\Big(w \cdot \frac{\partial \phi}{\partial x}\Big) + \frac{1}{\delta}\sum_{k=1}^m \Big\|w \cdot \frac{\partial \phi}{\partial x_{\tau_k}}\Big\|_{\chi(t)} \frac{m(t)}{m(t-\tau_k(t))} \\
&+ p(1-p)\sum_{k=1}^m \tau_k(t)\Big[(\|B_{1k}^{(1)}\|_{\chi(t)} - \|B_{2k}^{(1)}\|_{\chi(t)})(\tilde{A}_2^k(t) - \tilde{A}_1^k(t) + \tilde{B}_{2k}(t) - \tilde{B}_{1k}(t)) \\
&+ \frac{1}{\delta}\|pB_{1k}^{(1)} + (1-p)B_{2k}^{(1)}\|_{\chi(t)}\Big(\sup_{t-\tau_k(t)\le s\le t}\Big\|w \cdot \frac{\partial \phi}{\partial x}\Big\|_{\chi(t)} \frac{m(t)}{m(s)} + \Big\|w \cdot \frac{\partial \phi}{\partial x_{\tau_k}}\Big\|_{\chi(t)} \frac{m(t)}{m(s-\tau_k(s))}\Big)\Big] \\
&\le -\frac{D^+ m(t)}{m(t)}
\end{aligned} \tag{20}$$

holds for $-\frac{\delta}{2} < \phi(x, x_{\tau_1}, ..., x_{\tau_m}) < \frac{\delta}{2}$ and $p \in [0,1]$, then the linear switched time-delay system (18) is $v(t)$-incremental stable.

Proof. With conclusion in Theorem 4, by constructing a sequence of time-delay systems with continuous right-hands satisfying Condition $C_{time-delay}(\Sigma)$ as follows, we can prove the $v(t)$-incremental stability property of system (18):

$$\begin{aligned}
\dot{x} = f^\delta =& \sigma\Big(\frac{\phi(x, x_{\tau_1}, ..., x_{\tau_m})}{\delta}\Big)[A_1(t)x(t) + \sum_{k=1}^m B_{1k}(t)x(t-\tau_k(t)) + J_1(t)] \\
&+ \Big(1 - \sigma\Big(\frac{\phi(x, x_{\tau_1}, ..., x_{\tau_m})}{\delta}\Big)\Big)[A_2(t)x(t) + \sum_{k=1}^m B_{2k}(t)x(t-\tau_k(t)) + J_2(t)]
\end{aligned} \tag{21}$$

in which $\sigma(\cdot)$ is defined as

$$\sigma(\rho) = \begin{cases} 1, & \rho > 1/2, \\ \rho + 1/2, & \rho \in [-1/2, 1/2], \\ 0, & \rho < -1/2. \end{cases}$$

let $w = (A_1(t) - A_2(t))x(t) + \sum_{k=1}^m (B_{1k}(t) - B_{2k}(t))x(t - \tau_k(t)) + J_1(t) - J_2(t)$ and $p = \sigma(\frac{\phi(x, x_{\tau_1}, ..., x_{\tau_m})}{\delta})$, the partial derivative of the right-hand function f^δ w.r.t. x and x_{τ_k} are as follows,

$$\begin{aligned}
\frac{\partial f^\delta(x, x_{\tau_1}, ..., x_{\tau_m}, r(t))}{\partial x} =& \sigma\Big(\frac{\phi(x, x_{\tau_1}, ..., x_{\tau_m})}{\delta}\Big)A_1(t) + \Big[1 - \sigma\Big(\frac{\phi(x, x_{\tau_1}, ..., x_{\tau_m})}{\delta}\Big)\Big]A_2(t) + \frac{1}{\delta}w \cdot \frac{\partial \phi}{\partial x} \\
\frac{\partial f^\delta(x, x_{\tau_1}, ..., x_{\tau_m}, r(t))}{\partial x_{\tau_k}} =& \sigma\Big(\frac{\phi(x, x_{\tau_1}, ..., x_{\tau_m})}{\delta}\Big)B_{1k}(t) + \Big[1 - \sigma\Big(\frac{\phi(x, x_{\tau_1}, ..., x_{\tau_m})}{\delta}\Big)\Big]B_{2k}(t) + \frac{1}{\delta}w \cdot \frac{\partial \phi}{\partial x_{\tau_k}}
\end{aligned}$$

Thus, it needs to satisfy that

$$\begin{aligned}
L(f^\delta, t) =& \nu_{\chi(t)}\Big(pA_1 + (1-p)A_2 + \frac{1}{\delta}w \cdot \frac{\partial \phi}{\partial x} + \sum_{k=1}^m [pB_{1k}^{(1)} + (1-p)B_{2k}^{(1)}]\Big) \\
&+ \sum_{k=1}^m \tau_k(t)(\tilde{U}_k(t) + \tilde{V}_k(t))\Big\|pB_{1k}^{(1)} + (1-p)B_{2k}^{(1)}\Big\|_{\chi(t)} \\
&+ \sum_{k=1}^m \Big\|pB_{1k}^{(2)} + (1-p)B_{2k}^{(2)} + \frac{1}{\delta}w \cdot \frac{\partial \phi}{\partial x_{\tau_k}}\Big\|_{\chi(t)} \frac{m(t)}{m(t-\tau_k(t))} \le -\frac{D^+ m(t)}{m(t)}
\end{aligned}$$

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in which

$$\tilde{U}_k(t) = \sup_{t-\tau_k(t)\le s\le t} \|\frac{\partial f^\delta}{\partial x}(x(s), x_{\tau_1}(s), ..., x_{\tau_m}(s), s)\|_{\chi(t_j)} \frac{m(t)}{m(s)}$$
$$\tilde{V}_k(t) = \sup_{t-\tau_k(t)\le s\le t} \|\frac{\partial f^\delta}{\partial x_\tau}(x(s), x_{\tau_1}(s), ..., x_{\tau_m}(s), s)\|_{\chi(t_j)} \frac{m(t)}{m(s-\tau_k(s))}$$

where $t \in [t_j, t_{j+1})$. That is,

$$\begin{aligned} L(f^\delta, t) \le & pL_1(t) + [1-p]L_2(t) + \frac{1}{\delta}\nu_{\chi(t)}\Big(w\cdot\frac{\partial\phi}{\partial x}\Big) + \frac{1}{\delta}\sum_{k=1}^m \Big\|w\cdot\frac{\partial\phi}{\partial x_{\tau_k}}\Big\|_{\chi(t)} \frac{m(t)}{m(t-\tau_k(t))} \\ & + p(1-p)\sum_{k=1}^m \tau_k(t)\Big[(\|B_{1k}^{(1)}\|_{\chi(t)} - \|B_{2k}^{(1)}\|_{\chi(t)})(\tilde{A}_2^k(t) - \tilde{A}_1^k(t) + \tilde{B}_{2k}(t) - \tilde{B}_{1k}(t)) \\ & + \frac{1}{\delta}\|pB_{1k}^{(1)} + (1-p)B_{2k}^{(1)}\|_{\chi(t)}\Big(\sup_{t-\tau_k(t)\le s\le t}\Big\|w\cdot\frac{\partial\phi}{\partial x}\Big\|_{\chi(t)}\frac{m(t)}{m(s)} + \Big\|w\cdot\frac{\partial\phi}{\partial x_{\tau_k}}\Big\|_{\chi(t)}\frac{m(t)}{m(s-\tau_k(s))}\Big)\Big] \end{aligned}$$

Together with (20), it infers that $L(f^\delta, t) \le -\frac{D^+m(t)}{m(t)}$, that is, system (21) is $v(t)$-incremental stable for each $\delta = \frac{1}{m}, m \in \mathbb{N}^+$. From Theorem 4, it can be proved that time-delay system is $v(t)$-incremental stable. □

Here, let $B_{1k}^{(1)} = B_{2k}^{(1)} = 0, (k = 1, ..., m)$, we have the following corollary, which is a variant of Corollary 5.

Corollary 6. *Let $v(t)$ be a positive continuous function satisfying that $\lim_{t\to\infty} v(t) = \infty$. Let $N(t) = \#\{j : t \ge t_j, j = 1, 2, ...\}$, and suppose that system (18) stasifies Assumption 4, and there exist*

1. *positive piecewise right-continuous function $m(t) > 0$,*
2. *a constant $T_0 > 0$,*
3. *positive constants $\alpha_k > 0 (k = 0, 1, ...)$,*
4. *$C(\chi(t), \chi(t')) \le C_0$ for any $t, t' \ge t_0$,*

such that the following conditions are satisfied:

$$m(t_k) \le \alpha_k m(t_k-),$$

$$L_1(t) = \nu_{\chi(t)}(A_1(t)) + \sum_{k=1}^m \|B_{1k}(t)\|_{\chi(t)} \frac{m(t)}{m(t-\tau_k(t))} \le -\frac{D^+m(t)}{m(t)}$$
$$L_2(t) = \nu_{\chi(t)}(A_2(t)) + \sum_{k=1}^m \|B_{2k}(t)\|_{\chi(t)} \frac{m(t)}{m(t-\tau_k(t))} \le -\frac{D^+m(t)}{m(t)}$$

and for all $t > T_0$,

$$\frac{\sup_{\theta\in[t_0-\bar{\tau}, t_0]} m(\theta)}{m(t)} \prod_{k=1}^{N(t)} \alpha_k\beta_k \le \frac{1}{v(t)}.$$

where $\beta_k = C(\chi(t_k), \chi(t_k-))$. Moreover, in the neighbourhood of switching surface S, that is, when $-\frac{\delta}{2} < \phi(x, x_{\tau_1}, ..., x_{\tau_m}) < \frac{\delta}{2}$, let $w = (A_1(t) - A_2(t))x(t) + \sum_{k=1}^m (B_{1k}(t) - B_{2k}(t))x(t-\tau_k(t)) + J_1(t) - J_2(t)$, if it is satisfied that

$$pL_1(t) + (1-p)L_2(t) + \frac{1}{\delta}\nu_{\chi(t)}\Big(w\cdot\frac{\partial\phi}{\partial x}\Big) + \frac{1}{\delta}\sum_{k=1}^m \Big\|w\cdot\frac{\partial\phi}{\partial x_{\tau_k}}\Big\|_{\chi(t)} \frac{m(t)}{m(t-\tau_k(t))} \le -\frac{D^+m(t)}{m(t)}$$

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holds for $-\frac{\delta}{2} < \phi(x, x_{\tau_1}, ..., x_{\tau_m}) < \frac{\delta}{2}$ *and* $p = \phi(x, x_{\tau_1}, ..., x_{\tau_m})/\delta \in [0,1]$*, then the linear switched time-delay system* (18) *is* $v(t)$*-incremental stable.*

5.2. Hopfield Neural Network Systems with Time Delay

Consider the following Hopfield neural network system with time delay:

$$\dot{x}(t) = -D(t)x(t) + T(t)g(x(t)) + S(t)u(x_\tau(t))) + J(t) \tag{22}$$

where $x = (x_1, x_2, ..., x_n)^\top$ is the state variable, the time-delay term $x_\tau(t) = x(t-\tau(t))$. For any $t \in \mathbb{R}^+$, $D(t) = diag\{d_1(t), ..., d_n(t)\}$, $T(t) = (T_{ij}(t)) \in \mathbb{R}^{n,n}$, $S(t) = (S_{ij}(t)) \in \mathbb{R}^{n,n}$, $J = (J_1, J_2, ..., J_n) \in \mathbb{R}^n$ is the input vector, $g(x) = (g_1(x_1), g_2(x_2), ..., g_n(x_n))^\top$, $u(x) = (u_1(x_1), u_2(x_2), ..., u_n(x_n))^\top$.

Here we list the following hypothesis, denoted by Condition C_2:

1. There exists $\tilde{D} = diag\{D_1, D_2, ..., D_n\}$, $D_i > 0$, such that $d_i(\cdot)$ is continuous and $\frac{d_i(\zeta_1)-d_i(\zeta_2)}{\zeta_1-\zeta_2} \geq D_i$ holds for $i = 1,2,...,n$ and $\zeta_1 \neq \zeta_2$.
2. $g_i(\cdot)$ is non-decreasing and non-trivial in any compact set in $\mathbb{R}$, and each $g_i(\cdot)$ has only finite discontinuous points. Therefore, in any compact set in $\mathbb{R}$, except a finite points ρ_k, where there exist finite right and left limits $g_i(\rho_k^+)$ and $g_i(\rho_k^-)$ with $g_i(\rho_k^+) > g_i(\rho_k^-)$, $g_i(\cdot)$ is continuous.
3. $u_i(\cdot)$ is non-decreasing and non-trivial in any compact set in $\mathbb{R}$, and each $u_i(\cdot)$ has only finite discontinuous points. Therefore, in any compact set in $\mathbb{R}$, except a finite points η_k, where there exist finite right and left limits $u_i(\eta_k^+)$ and $u_i(\eta_k^-)$ with $u_i(\eta_k^+) > u_i(\eta_k^-)$, $u_i(\cdot)$ is continuous.

Here define a matrix measure $\nu_{\xi,1}(A) = \max_j[a_{jj} + \sum_{i\neq j} |\xi_i \xi_j^{-1} a_{ij}|]$ for matrix $A = (a_{ij})$, w.r.t. vector norm $|\cdot|_{\xi,1}$ and matrix norm $\|A\|_{\xi,1} = \|\xi A \xi^{-1}\|_1$, where $\xi = diag\{\xi_1, ..., \xi_n\}$. So we have the following corollary:

Corollary 7. *Suppose the system* (22) *has a unique solution for* $t \in \mathbb{R}^+$*, and satisfies Condition* C_2 *above. And there exists a positive diagonal matrix* $\xi(r(t)) = diag\{\xi_1(r(t)), ..., \xi_n(r(t))\}$ *such that*

$$\nu_{\xi(r(t)),1}(T) = \max_j\{T_{jj}(t) + \sum_{i\neq j} |\xi_i(r(t))\xi_j(r(t))^{-1} T_{ij}(t)|\} \leq 0$$

holds for $t \in \mathbb{R}^+$*. Let* $v(t)$ *be a positive continuous function satisfying that* $\lim_{t\to\infty} v(t) = \infty$*. Let* $N(t) = \#\{j : t \geq t_j, j = 1,2,...\}$*, and there exists*

1. *positive piecewise right-continuous function* $m(t) > 0$*,*
2. *a constant* $T_0 > 0$*,*
3. *positive constants* $\alpha_k > 0 (k = 0,1,...)$*,*
4. $C(r(t), r(t')) \leq C_0$ *for any* $t, t' \geq t_0$*,*

such that

$$\nu_{\xi(r(t)),1}\Big(-D(t) + T(t)\frac{\partial g}{\partial x}(x, x_\tau)\Big) + \frac{m(t)}{m(t-\tau(t))}\|S(t)\frac{\partial u}{\partial x_\tau}(x, x_\tau)\|_{\xi(r(t)),1} \leq -\frac{D^+ m(t)}{m(t)} \tag{23}$$

Let $K_i = \min_k[g_i(\rho_k+) - g_i(\rho_k-)]$*,* $M_i = \max_k[u_i(\gamma_k+) - u_i(\gamma_k-)]$*, and* $K = \min_i K_i$*,* $M = \max_i M_i$*,*

$$K \cdot \nu_{\xi(r(t)),1}(T(t)) + M \cdot \frac{m(t)}{m(t-\tau(t))}\|S(t)\|_{\xi(r(t)),1} < 0 \tag{24}$$

For any $t > T_0$*,*

$$\frac{\sup_{\theta\in[t_0-\bar{\tau},t_0]} m(\theta)}{m(t)} \prod_{k=1}^{N(t)} \alpha_k \beta_k \leq \frac{1}{v(t)},$$

where $\beta_k = C(r(t_k), r(t_k-))$*, then system* (22) *is* $v(t)$*-incremental stable.*

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Proof. Suppose that system (22) has a unique solution for $t \in \mathbb{R}^+$, then we construct a sequence of 'continuous systems' as follows,

$$\dot{x}(t) = f^{\delta}(x, x_\tau, t) = -D(t)x(t) + T(t)\tilde{g}(x(t)) + S(t)\tilde{u}(x_\tau(t)))$$

where $\tilde{g}(x) = (\tilde{g}_1(x_1), \tilde{g}_2(x_2), ..., \tilde{g}_n(x_n))^\top$, $\tilde{u}(x) = (\tilde{u}_1(x_1), \tilde{u}_2(x_2), ..., \tilde{u}_n(x_n))^\top$. For each i, denote one of the discontinuous point of $g_i(x)$ by ρ_i, and one of the discontinuous point of $u_i(x)$ by η_i.

Function $\tilde{g}^{\delta}(x)$ is formulated as follows, if $x_i \notin [\rho_i - \frac{\delta}{2}, \rho_i + \frac{\delta}{2}]$, $\tilde{g}_i^{\delta}(x_i) = g_i(x_i)$, and if $x_i \in [\rho_i - \frac{\delta}{2}, \rho_i + \frac{\delta}{2}]$,

$$\tilde{g}_i{}^{\delta}(x_i) = \frac{g_i(\rho_i + \frac{\delta}{2}) - g_i(\rho_i - \frac{\delta}{2})}{\delta}[x_i - \rho_i + \frac{\delta}{2}] + g_i(\rho_i - \frac{\delta}{2}).$$

Function $\tilde{u}(x)$ is similarly constructed. It can be seem that when $\delta \to 0$, the function sequence $f^{\delta}(x, x_\tau, t)$ converge to the Filippov differential inclusion of the right-hand of (22), that is, $\{f^{\delta}(x, x_\tau, t)\}$ satisfies Condition $C_{time-delay}(\mathbb{R}^{n\times 2})$. Let $\bar{g}_i(\rho_i, \delta) = \frac{g_i(\rho_i + \frac{\delta}{2}) - g_i(\rho_i - \frac{\delta}{2})}{\delta}$, $\bar{u}_i(\eta_i, \delta) = \frac{u_i(\eta_i + \frac{\delta}{2}) - u_i(\eta_i - \frac{\delta}{2})}{\delta}$. Here denote the Jacobi matrix of functions $\tilde{g}^{\delta}(\cdot)$ and $\tilde{u}^{\delta}(\cdot)$ by matrix G^{δ} and U^{δ}. In the neighbourhood of the discontinuous point,

$$G^{\delta} = \frac{\partial \tilde{g}(x)}{\partial x} |_{x=(\rho_1, \rho_2, ..., \rho_n)} = diag\{\bar{g}_1(\rho_1, \delta), \bar{g}_2(\rho_2, \delta), ..., \bar{g}_n(\rho_n, \delta)\}$$
$$U^{\delta} = \frac{\partial \tilde{u}(x_\tau)}{\partial x_\tau} |_{x_\tau=(\eta_1, \eta_2, ..., \eta_n)} = diag\{\bar{u}_1(\eta_1, \delta), \bar{u}_2(\eta_2, \delta), ..., \bar{u}_n(\eta_n, \delta)\}$$

that is, we have

$$\frac{\partial f^{\delta}(x, x_\tau, t)}{\partial x} = -D(t) + T(t)G^{\delta},$$
$$\frac{\partial f^{\delta}(x, x_\tau, t)}{\partial x_\tau} = S(t)U^{\delta},$$

In the continuous regions, the function sequence $\{f^{\delta}(x, x_\tau, t)\}$ satisfies (23), and $\nu_{\xi(r(t)),1}(T(t)) < 0$. Meanwhile, in the neighbourhood of the discontinuities, together with the condition (24), it holds that,

$$\begin{aligned}
&\nu_{\xi(r(t)),1}(-D(t) + T(t)G^{\delta}) + \frac{m(t)}{m(t-\tau(t))}\|S(t)U^{\delta}\|_{\xi(r(t)),1} \\
\leq& \nu_{\xi(r(t)),1}(-D(t)) + \nu_{\xi(r(t)),1}(T(t)G^{\delta}) + \frac{m(t)}{m(t-\tau(t))}\|S(t)U^{\delta}\|_{\xi(r(t)),1} \\
\leq& \nu_{\xi(r(t)),1}(-D(t)) + \min_i \bar{g}_i(\rho_i, \delta)\nu_{\xi(r(t)),1}(T(t)) + \max_i \bar{u}_i(\eta_i, \delta)\frac{m(t)}{m(t-\tau(t))}\|S(t)\|_{\xi(r(t)),1} \\
\leq& -\frac{D^+ m(t)}{m(t)}
\end{aligned}$$

for $\delta \to 0$. Therefore, if there exists a large enough number $m_0 > 0$, such that for each $m > m_0$, $m \in \mathbb{N}^+$,

$$\nu_{\xi(r(t)),1}(-D(t) + T(t)G^{\delta}) + \frac{m(t)}{m(t-\tau(t))}\|S(t)U^{\delta}\|_{\xi(r(t)),1} \leq -\frac{D^+ m(t)}{m(t)}$$

holds for $\delta = 1/m$. According to Theorem 4 and Theorem 1, the time-delay system (22) is $v(t)$-incremental stable.

□

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6. Numerical Experiments

6.1. Linear Time-Delay System

Consider a linear switched time-delay system formulated as follows:

$$\dot{x}(t) = \begin{cases} A_1 x(t) + B_1 x(t-\tau(t)) + J_1, & z^\top x > 0 \\ A_2 x(t) + B_2 x(t-\tau(t)) + J_2, & z^\top x < 0 \end{cases} \tag{25}$$

where $x, x_\tau \in \mathbb{R}^2$. The switching surface of system (25) is $\{x : \phi(x) = z^\top x = 0\}$ where $z = [1,3]^\top$.

Let $\tau(t) = 1$ and

$$A_1 = \begin{bmatrix} -6 & 1 \\ 0 & -7 \end{bmatrix},\ A_2 = \begin{bmatrix} -7 & -2 \\ -1 & -10 \end{bmatrix}, B_1 = B_2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},\ J_1 = \begin{bmatrix} -1 \\ -2 \end{bmatrix},\ J_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Here first prove that system (25) has a unique solution for $t \in [1, +\infty)$. On the switching surface $\{x : \phi(x) = z^\top x = 0\}$, we have

$$\frac{d\phi}{dx}(x) \cdot \Big((A_1 - A_2)x + J_1 - J_2 \Big) = [1,3]^\top \left(\begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix} x(t) + \begin{bmatrix} -1 \\ -3 \end{bmatrix} \right) = -4,$$

According to Assumption 4, the uniqueness of Filippov solution for (25) occurs for $t \in [0, +\infty)$. Let $m(t) = e^{\alpha_{N(t)} t}$, where $\alpha_{N(t)} = 1 - 1/2^{[t]}$ and the number $[t]$ represents the floor of time t. So, for $t > 1$,

$$\begin{aligned} L_1(t) =& \nu_2(A_1) + \|B_1\|_2 \frac{e^{\alpha_{N(t)} t}}{e^{\alpha_{N(t)-1}(t-1)}} = -5.75 + \frac{e^{(1-1/2^{[t]})t}}{e^{(1-1/2^{[t-1]})(t-1)}} < -3.5 < -1 + \frac{1}{2^{[t]}} \\ L_2(t) =& \nu_2(A_2) + \|B_2\|_2 \frac{e^{\alpha_{N(t)} t}}{e^{\alpha_{N(t)-1}(t-1)}} = -6.35 + \frac{e^{(1-1/2^{[t]})t}}{e^{(1-1/2^{[t-1]})(t-1)}} < -4 < -1 + \frac{1}{2^{[t]}} \end{aligned}$$

where $\|\cdot\|_2$ stands for 2-norm. Then for $-\delta/2 < \phi(x) < \delta/2$ and $p = \phi(x)/\delta + 1/2 \in [0,1]$,

$$pL_1(t) + (1-p)L_2(t) + \frac{1}{\delta}\nu_2\left(\left(\begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix} x(t) + \begin{bmatrix} -1 \\ -3 \end{bmatrix} \right) \cdot [1,3] \right) < -1.5 < -\frac{D^+ m(t)}{m(t)},$$

and for all $t > T_0$,

$$\frac{1}{t}\left[\sum_{i=0}^{N(t)-1} [(\alpha_{i+1} - \alpha_i) t_{i+1} + \log \beta_{i+1}] + \alpha_0 t_0 - \alpha_{N(t)} t \right] \le \frac{1}{t}\left[\sum_{i=0}^{N(t)-1} [\frac{1}{2^{i+1}}(i+1)] - (1 - \frac{1}{2^{[t]}})t \right] \le -0.5$$

Thus, from Corollary 6, the switched time-delay system (25) is exponentially incrementally asymptotically stable.With the initial state $x_0(s) = [5 \times (0.5+s)^2, 6 \times (1-s) - 1.5]$ and $y_0(s) = [-3 \times (1+s)^3 - 2, -3 \times (1-s)]$ for $s \in [0,1]$, the corresponding solution of system (25) are $x(t) = (x_1(t), x_2(t))$ and $y(t) = (y_1(t), y_2(t))$ respectively.

Figure 1 shows the dynamical trajectories of two of the solutions with initial function defined as $x_0(\cdot)$ and $y_0(\cdot)$ for system (25). And Figure 2 shows the errors between the two dynamical trajectories of their segments.

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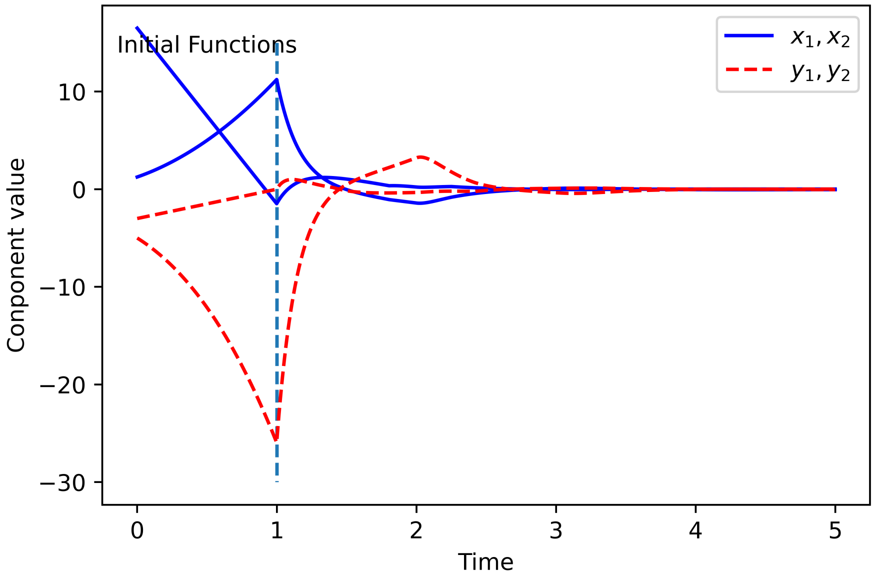

Figure 1. Dynamical trajectories of the solutions for time-delay system (25).

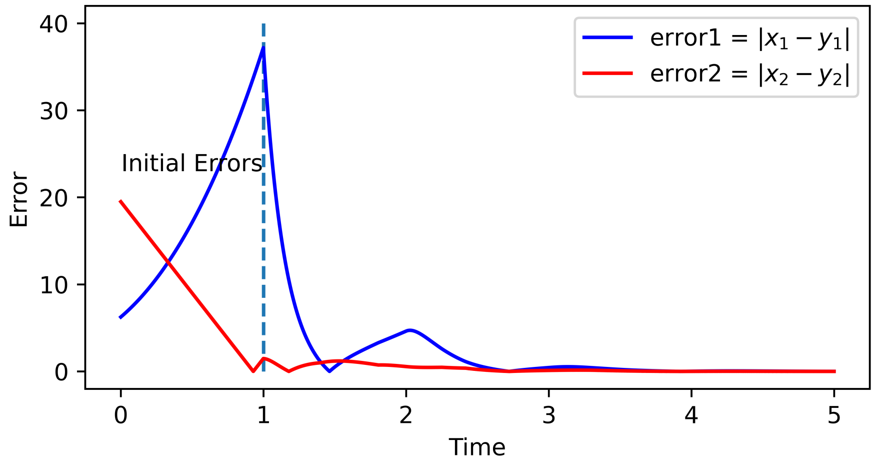

Figure 2. A diagram for exponential incremental uniform stability (the error of each segment) of time-delay system (25).

6.2. Hopfield Neural Network with Time Delay

Here we take an example of Hopfield system with time delay for illustration. The system is formulated as follows,

$$\dot{x}(t) = D(t)x(t) + T(t)g(x(t)) + \sigma(t)S(t)u(x_\tau(t))) + J(t) \tag{26}$$

where $x = [x_1, x_2]^\top$ is the state vector, $\sigma(t)$ is the switched function with respect to time t, takes value between 0 and 1. The parameter matrix are

$$\sigma(t) = \begin{cases} 0 & t \in [kT_0, kT_0 + \frac{1}{2}T_0) \\ 1 & t \in [kT_0 + \frac{1}{2}T_0, (k+1)T_0) \end{cases}, \quad \tau(t) = \begin{cases} t & t \le 0.2 \\ 0.1 \times |\sin(\pi t/0.4)| + 0.1 & t > 0.2 \end{cases}$$

$$D(t) = \begin{bmatrix} -0.3 + 0.1 \times \sin(t) & 0 \\ 0 & -0.3 + 0.1 \times \cos(t) \end{bmatrix}, \quad J(t) = \begin{bmatrix} 10\sin(2t) \\ -10\sin(2t) \end{bmatrix},$$

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$$T(t) = \begin{bmatrix} -5-\sin(t) & 2.5+\cos(t) \\ 2.5-\cos(t) & -5-\sin(t) \end{bmatrix}, S(t) = \begin{bmatrix} 0.1\times\sin(t) & 0.1+0.1\times\cos(t) \\ 0.1-0.1\times\cos(t) & 0.1\times\sin(t) \end{bmatrix}$$

$$g_i(x) = \begin{cases} x+2.5 & x>0 \\ x-2.5 & x<0 \end{cases}, \quad u_i(x) = \begin{cases} x+1 & x>0 \\ x-1 & x<0 \end{cases}, \quad i=1,...,n$$

where $k \in \mathbb{N}_{\geq 0}$, $T_0 = 1$. According to the uniqueness conditions for Filippov solution of the time-delay system, Lemma 2 and Theorem 3, it can be seem that (26) has a unique solution in compact set $\Sigma \in \mathbb{R}^2$.

Here define the norm with subscript $\sigma(t)$ as $|x|_0 = 0.8|x_1| + |x_2|$, $|x|_1 = |x_1| + |x_2|$, $\beta_{01} = 1.25$, $\beta_{10} = 1$, then $\nu_0(T) < -0.5$, $\nu_1(T) = -2.5 + \sqrt{2} < -1$, $\|S\|_1 < 0.3$, $\|S\|_0 < 0.25$, therefore, let $m(t) = e^{\alpha_{N(t)}t}$, where $N(t) = j, t \in [t_{j-1}, t_j)$, $\alpha_j = \max\{1/4, (1/2) - (1/2)^j\}$, $t_j = (T_0/2)\cdot j$, $j = 1, 2, ...$, such that in the continuous region of the right-hand of system (26), it holds that

$$\alpha_{N(t)} + \nu_0(-D(t) + T(t)) < 0$$

$$\alpha_{N(t)} + \nu_1(-D(t) + T(t)) + \frac{\exp\left(\alpha_{N(t)}t\right)}{\exp\left(\alpha_{N(t-\tau(t))}(t-\tau(t))\right)}\|S(t)\|_1 < 0.5 - 1 + 0.25 < 0$$

Together with (24), where $K = 5$, $M = 2$, conditon (24) holds for $\sigma(t) = 0, 1$. And for $t > 10$, we have

$$\sum_{i=0}^{N(t)-1} [(\alpha_{i+1} - \alpha_i)t_{i+1} + \log\beta_{i+1}] + \alpha_0 t_0 - \alpha_{N(t)}t \leq 1.75 + \log(1.25)t - 0.49t < -0.25t$$

Together with Corollary 7, the switched time-delay system (26) is exponentially incrementally asymptotically stable.With the initial state $x_0(s) = [5\times(0.5+s)^2, 6\times(1-s) - 1.5]$ and $y_0(s) = [-2*(1+s)^2 - 2, -3\times(1-s)]$ for $s \in [0, 0.5]$, the corresponding solution of system (26) are $x(t) = (x_1(t), x_2(t))$ and $y(t) = (y_1(t), y_2(t))$ respectively. When $t > 0.5$, $\sigma(t) = 1$ and time lag occurs.

Figure 3 shows the dynamical trajectories of two of the solutions with initial function defined as $x_0(\cdot)$ and $y_0(\cdot)$ for system (26). And Figure 4 shows the errors between the two dynamical trajectories of their segments.

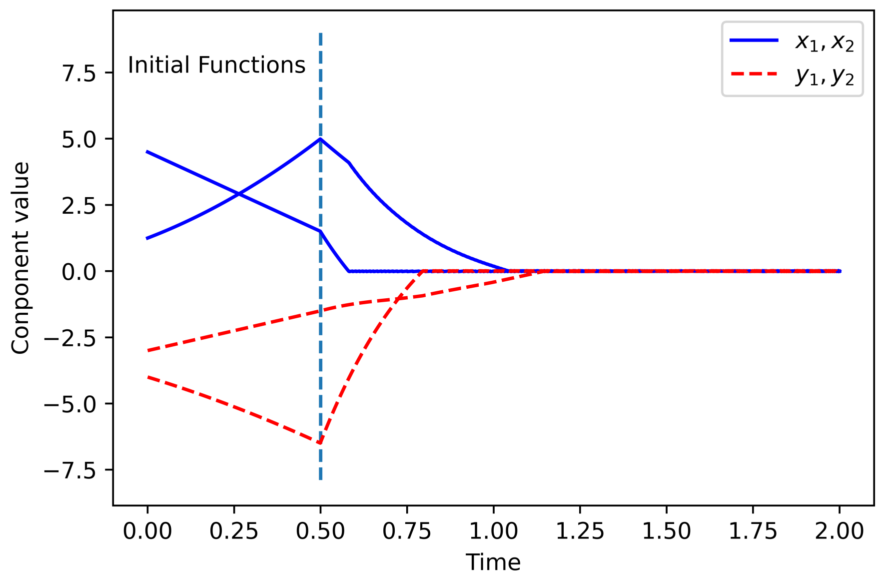

Figure 3. Dynamical trajectories of the solutions for time-delay system (26).

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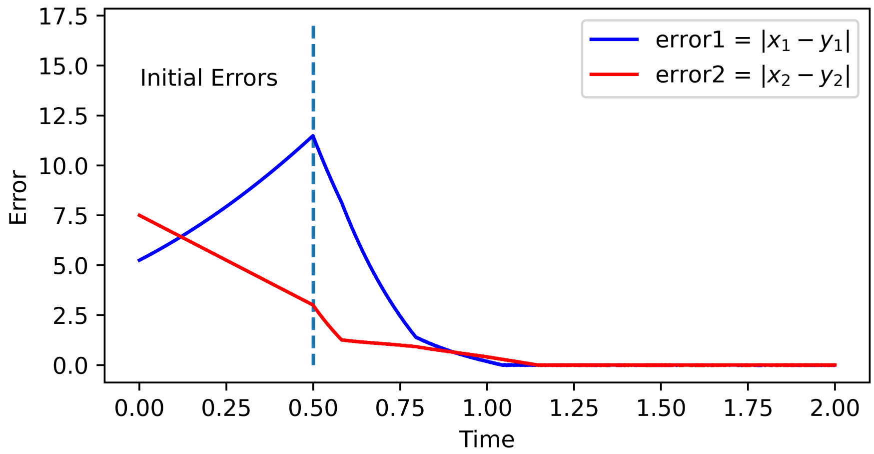

Figure 4. A diagram for exponential incremental uniform stability (the error of each segment) for system (26).

7. Conclusions

This paper mainly researches criteria on incremental stability for time-delay dynamical systems, including systems with continuous right-hands and systems with discontinuous right-hands respectively. For clearer illustration, a type of incremental stability property, named $v(t)$-incremental stability, is defined, and the corresponding sufficient conditions for $v(t)$-incremental stability of solutions for time-delay dynamical systems with continuous right-hands are proposed and proved. Before studying sufficient conditions for incremental stability of the systems with discontinuous right-hands, we first provide the conditions for existence and uniqueness of the Filippov solution. Then, by constructing a sequence of systems with continuous right-hands and using approximation method, sufficient conditions for $v(t)$-incremental stability of the systems with discontinuous right-hands are obtained.

There still needs much further work. Our theorem can be helpful for applications in other more complex scenarios and we may propose more corollaries for some more complex systems in the future. Furthermore, we may seek for some other approaches to construct "continuous systems" to approximate discontinuous systems.

8. Patents

This section is not mandatory, but may be added if there are patents resulting from the work reported in this manuscript.

Author Contributions: Conceptualization, W.L.; methodology, Y.L. and W.L.; software, Y.L.; validation, Y.L. and W.L.; formal analysis, Y.L.; investigation, Y.L.; resources, W.L.; data curation, W.L.; writing—original draft preparation, Y.L.; writing—review and editing, W.L.; visualization, Y.L.; supervision, W.L.; project administration, W.L.; funding acquisition, W.L. All authors have read and agreed to the published version of the manuscript.

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Sample Availability: Samples of the compounds ... are available from the authors.

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