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Solution of the Capacitated Lot-Sizing Problem with Remanufacturing (CLSPR) in a General Way with the Help of Simulation and Relaxation

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Abstract: The capacitated lot-sizing problem with product recovery (CLSP-RM) holds significant importance in reverse logistics but is notoriously complex (NP-hard). In this study, two techniques are introduced to confront this challenge. The first technique entails devising a linear optimization task that eliminates capacity limitations across a wide problem spectrum, yielding a remarkably accurate approximation of the optimal solution. This adaptable approach presents a potent alternative and holds potential for extension to diverse problem categories owing to its versatile nature. The second technique employs a simulation methodology utilizing Halton's uniform random numbers to address the issue. This randomized production search method sidesteps considerations of production costs, inventory expenditures, and production order when determining production batches. The research's novelty lies in its application of these techniques to the problem. The suggested methods undergo evaluation via a benchmark dataset of approximately 4200 instances, with comparison against solutions derived through the Gurobi solver. The results underscore the efficacy and resilience of the introduced methodology in tackling the CLSP-RM predicament (The test instances and solutions are available **here**).

Keywords: capacitated lot-sizing problem; heuristic; simulation based optimization; remanufacturing

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1. Introduction

Two seminal papers have profoundly shaped the landscape of the Lot Sizing Problem (LSP). The inaugural paper, known as the Dynamic Economic Lot Sizing model (DLS), was concurrently introduced by [1,2], and is widely recognized as the Manne-Wagner-Whitin Model. In its "classical" rendition, the DLS addresses a discrete-time, finite-horizon inventory management challenge involving a singular item. Its primary objective is to efficiently meet deterministic time-varying demands through optimal stock management or procurement strategies, with further elucidation provided in ([3]). The second pivotal paper, the Economic Lot Scheduling Problem (ELSP), was brought forth by Elmaghraby ([4]). Both of these works tackle a fundamental issue within production planning determining optimal lot sizes and production schedules for a multitude of items produced on a shared production line or machine. The overarching goal is to minimize expenses associated with production and inventory while simultaneously fulfilling demand requirements and adhering to pertinent production constraints. Supplementary insights can be found in ([4]). Researchers have further expanded upon the work of Elmaghraby and Manne-Wagner-Whitin to tackle variations and more advanced versions of the ELSP and DLS. These advancements include incorporating uncertainties, time windows, and remanufacturing into the models. For a comprehensive overview of the economic lot sizing problem, readers can consult [5,6]. The Economic Lot Scheduling Problem (ELSP) with remanufacturing options (ELSPR) represents an extension of the classical Wagner Whitin model. A notable enhancement is the incorporation of a distinct element: within each time period, predetermined quantities of used products are introduced into the system. These returned items offer the potential for remanufacturing, serving to fulfill demand alongside the conventional manufacturing processes. In recent decades, there has been a growing emphasis on production planning, particularly with regard to incorporating recycling options [7]. The literature outlines two distinct categories of recycling production planning problems.

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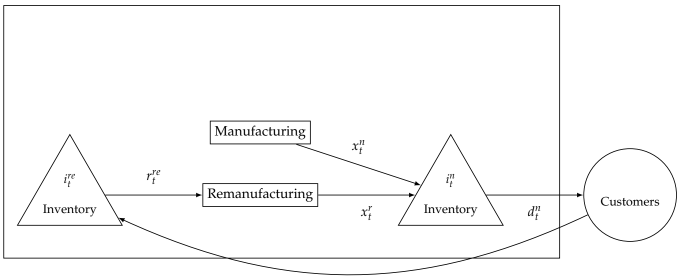
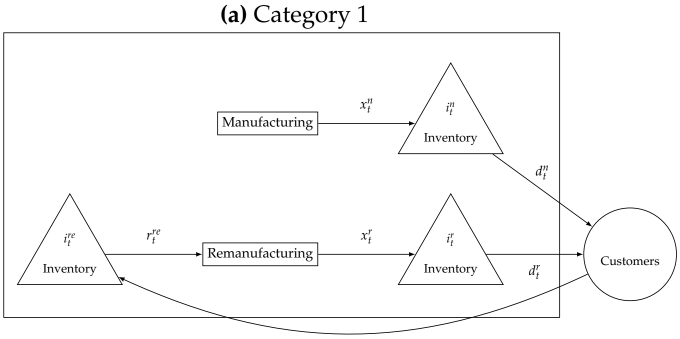
In the first category (Cat. 1), the primary objective is to meet the external demand for products through either remanufactured or newly produced goods Figure 1. This category focuses on regulating two types of inventories: the inventory of new products and the inventory of remanufactured products. It is assumed that both new and remanufactured products are identical and, thus, considered serviceable entities that can fulfill demands. Numerous authors have extensively studied this category [8]. This category also encompasses issues related to pricing and lot sizing, wherein demand is fulfilled through various substitution alternatives.

The second category (Cat. 2) deals with two distinct demands: one for new products and the other for recycled products. Additionally, this category introduces a key feature where a deterministic quantity of returned goods enters the system in each period. These returned items can be remanufactured and used to meet the demand for remanufactured products, complementing the regular production of new items. As a result, the inventory includes three types of stocks: new products, remanufactured products, and returned products Figure 1.

In specific industrial sectors, such as the paper industry, two different types of demand are observed: one for recycled paper and the other for paper made from new fiber. Recycled paper is typically priced lower due to its reduced water and energy consumption during production, compared to virgin fiber papers [9]. Moreover, recycled paper often benefits from shorter transportation distances as it is sourced locally, while virgin fiber and new fiber papers are imported from other countries in larger quantities. This observation extends to other sectors, including photocopiers, tires, and personal computers [see 10,11].



(a) Category 1



(b) Category 2

Figure 1. Dynamic Capacitated Lot-Sizing with Produkt Returns and Remanufacturing.

Publications regarding Category 1: Building upon Elmaghraby's groundwork ([see 4]), Tang and Teunter ([12]) investigated the hybrid production line's multi-product dynamic lot sizing. This involved manufacturing new products and remanufacturing returns, with a single manufacturing and remanufacturing lot per product synchronized within a common cycle. They constructed a Mixed Integer Linear Programming (MILP) problem for precise resolution. The multi-product dynamic lot sizing problem with distinct manufacturing and remanufacturing sources, each operating on separate dedicated lines, was further examined in [13].

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In a study by [14], a multi-item economic lot-sizing problem involving remanufacturing and capacitated production was examined. The study drew upon the concept introduced by Garfinkel and Nemhauser (1969), known as the Set Partitioning Problem (SPP). Another contribution, by [15], delved into the multi-item economic lot-sizing problem with remanufacturing and uncapacitated production. They extended the model initially proposed by [16] and put forward two innovative variations of the Variable Neighborhood Search (VNS) algorithm. These variants aimed at discovering optimal solutions for the ELSR problem. In a separate work, [17] introduced an effective Mixed Integer Programming (MIP)-based matheuristic for the multi-item capacitated lot-sizing problem with remanufacturing (CLSP-RM). Notably, this approach addresses capacity constraints individually for new and remanufactured products.

[18] conducted a comparative analysis of MIP approaches for the economic problem of single-item lot-sizing with remanufacturing (ELSR) and uncapacitated production. They proposed a shortest path formulation. [19] contributed a polynomial-time heuristic within this category. Their work also includes a comprehensive compilation of methods developed by [20–24]. Addressing an extension of the economic problem of single-item lot sizing, [25] introduced considerations for remanufacturing, final disposal, and distinct demand flows for new and remanufactured products. This extension also accounted for a unidirectional substitution, wherein the demand for remanufactured products can be satisfied by new items, but not vice versa. The authors proposed both a network flow formulation and a pseudopolynomial time dynamic programming algorithm. Notably, the model does not incorporate capacity constraints.

Publications regarding category 2: By [26] address the production planning problem within a hybrid manufacturing and remanufacturing system (HMRS). They postulate a multi-objective mathematical model (MIP) whose objective is to determine a production plan taking into account the available capacities in each period, which satisfies the demand for new and remanufactured products and minimizes all costs of production, storage and disposal of new products, as well as the minimization of CO2 emissions generated in production. They introduce a solution method based on a non-dominant sorting algorithm (NSGA-II).

[27] investigates the joint problem of pricing and lot sizing in a hybrid manufacturing and remanufacturing system with a one-way substitution option. Two demand performances, for new and remanufactured items, are considered in this paper. In the case of a shortage of remanufactured items, a one-way substitution option is assumed, so that the demand for these remanufactured items is satisfied with new items. The presented mathematical model is an MIP but without capacity constraints. As a solution method they adapt "the cost and benefit heuristic" (CB-heuristic) introduced by [28] and also a memetic algorithm which is improved with the help of a local search.

Zhang et al. ([29]) investigate the capacitated lot sizing problem in closed-loop suply chain considering setup costs, product returns, and remanufacturing. They present a Lagrange relaxation to solve the problem. Based on Zhang's model we formulate the same problem but taking more general capacity constraints and doing more extensive numerical experiments.

The capacitated lot-sizing problem model is NP-hard. The proof of this statement is to be found in [30,31]. [32] has shown that even the two-item problem with constant capacity is NP-hard. Heuristic methods employed for solving problems within the CLSP class typically rely on a wide range of sorting rules and other criteria derived from factors such as demand, capacity, setup costs, inventory costs, and production costs. Examples of such heuristic methods include those presented in references like [22,23,33,34]. The achieved solution margin tends to vary within the range of 1% to 10%, as reported in references such as [29,34].

1.1. Research Contributions

This study incorporates two significant constraints. The first constraint ensures that the cumulative demand for both new and remanufactured products remains within the cumulative capacity for each period. The second constraint mandates that the cumulative demand for remanufactured products during each period does not surpass the cumulative quantities of used products returned within the same period. No specific relationship is imposed between the existing capacities and the quantities of returned products. This particular characteristic, which is overlooked in certain promoted publications, adds an additional layer of complexity to the problem, rendering it non-standard in nature.

The contribution of this work can be summarized as follows: We examine two overarching problem classes under the framework of the aforementioned capacity constraints. These classes are as follows:

- The first class involves production periods where the demand for both new and remanufactured product exceeds the available capacity within those specific time frames. To address this scenario, we propose a linear program devoid of capacity constraints, providing a highly accurate approximation of the optimal solution.
- In the second problem class, each period experiences demand that falls short of the existing capacity. This specific problem category exhibits an NP-hard nature, leading us to adopt a direct simulation approach. Additionally, the quasi-random numbers $\mathbb{QRN}$, introduced by [35–37], showcase significant properties, as evidenced by [38]. Among these characteristics, their uniform distribution stands out. Accordingly, we present a straightforward simulation utilizing these quasi-random numbers ($\mathbb{QRN}$). Irrespective of the input parameters, we organize production based on both feasibility and randomness. Starting from the initial production period, we define lower and upper limits for viable production ranges of the new product. This production information informs the establishment of production range limits for the remanufactured product. The order in which the new or remanufactured product is produced holds no significance in the algorithm's execution. These defined limits delineate production intervals that strictly adhere to all problem constraints. Within each interval, a random production quantity is generated using uniform random numbers derived from the Halton sequence. It is imperative to emphasize that the random production process significantly influences the subsequent determination of feasible production intervals.
- Moreover, our heuristic approach ensures the generation of a feasible solution whenever the problem is solvable. After simulating N production plans, we select the most cost-effective one. This solution is then compared against numerical solutions obtained through the Gurobi solver. The scheduling process is straightforward, and the results prove to be quite satisfactory given the scale of the presented problems.

1.2. *Outline*

The rest of the paper is organized as follows: section 2 we propose a new mathematical programming formulation for the problem. In section 3 we present a relaxation of the problem without capacity constraints (Model A) and we will see that the solution obtained for the chosen class of tasks is very close to the optimal solution of the original problem. In section 4, we present a heuristic solution method (Model B) for another class of tasks. In section 5, we report the results of computational experiments. In section 6 we present some conclusions and suggested directions for further research.

2. Problem Description

We assume that a factory produces two types of products, one manufactured from raw materials and the other remanufactured from collected used products. The demands for these two products are separate, deterministic, and time varying during a finite planing horizon, and should be satisfied without backlogging. The costs consist of fixed setup cost, linear production cost proportional to the production quantity, and linear inventory holding costs. All cost components are considered for both manufacturing and remanufacturing activities per unit and periode [29]. The following Table 1 summarize the notation used in this paper.

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Table 1. Data and parameters

Name	Paramenter
T	Number of time periods
	$t \in \{1, \cdots, T\}$
i_0^n, i_0^r, i_0^{re}	Initial inventory stocks
	Cost per unit and period t
s_t^n	Setup cost for manufacturing new product
s_t^r	Setup cost for remanufacturing product
p_t^n	production cost of new product
p_t^r	production cost of remanufactured product
h_t^n	holding cost of new product
h_t^r	holding cost of remanufactured product
h_t^{re}	holding cost of returned product
	Demand and return in period t
d_t^n	Demand of new product
d_t^r	Demand of remanufactured product
r_t^{re}	Quantity of returned product
	Available Capacities in period t
c_t	capacities for manufacturing and remanufacturing (capacity requirement for new product and recovery product is set to one).
	$d_t = d_t^n + d_t^r, \; \forall t = 1, \cdots, T$
	$d_{[T]}^n = \sum_{t=1}^T d_t^n$
	$d_{[T]}^r = \sum_{t=1}^T d_t^r$

Table 2. Decision variables

Name	Paramenter
α_t^n	1, if new products are manufactured in period t; 0, otherwise
x_t^n	quantity of new products manufactured in period t
i_t^n	inventory stock of new products at the end of period t
α_t^r	1, if returned products are remanufactured in period t; 0, otherwise
x_t^r	quantity of returned products remanufactured in period t
i_t^r	inventory stock of remanufactured products at the end of period t
i_t^{re}	inventory stock of returned products at the end of period t

We make the following assumptions

1. The demand for new products and remanufactured products are separate and backlog is not allowed.
2. The manufacturing capacity is sufficient to meet the demands in each period, in particularly we have: the capacity can satisfy the demands for new products and remanufactured products simultaneously, i.e.

$$\sum_{j=1}^{t}(d_j^n + d_j^r) \leq \sum_{j=1}^{t} c_j, \quad \forall t = 1, \cdots, T \tag{1}$$

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3. Initial and end inventory stocks are zero, [1] i.e.

$$\begin{aligned} & i_0^n = 0,\ i_0^r = 0,\ i_0^{re} = 0 \\ & i_T^n = 0,\ i_T^r = 0 \end{aligned} \tag{2}$$

The demand will be fully satisfied if the final inventories are zero.

4. The quantity of returned products can satisfy the demand for remanufactured products i.e.

$$\sum_{j=1}^{t} d_j^r \leq \sum_{j=1}^{t} r_j^{re} \quad \forall t = 1, \cdots, T. \tag{3}$$

5. In economic terms, inventory holding cost of returned products is less than that of remanufactured products.

$$\sum_{j=t}^{T} h_j^{re} \leq \sum_{j=t}^{T} h_j^r, \quad \forall t = 1, \cdots, T \tag{4}$$

This hypothesis can be found in [39] too.

Hence, the problem can be formulated as

$$\begin{aligned} f(x^n, x^r) = & \sum_{t=1}^{T} (s_t^n \alpha_t + p_t^n x_t^n + h_t^n i_t^n) + \sum_{t=1}^{T} (s_t^r \alpha_t^r + p_t^r x_t^r + h_t^r i_t^r) \\ & + \sum_{t=1}^{T} (h_t^{re} i_t^{re}) \longrightarrow \min \end{aligned} \tag{5}$$

subject to

$$i_t^n = i_{t-1}^n + x_t^n - d_t^n, \quad \forall t = 1, \cdots, T \tag{6}$$

$$i_t^r = i_{t-1}^r + x_t^r - d_t^r, \quad \forall t = 1, \cdots, T \tag{7}$$

$$i_t^{re} = i_{t-1}^{re} + r_t^{re} - x_t^r, \quad \forall t = 1, \cdots, T \tag{8}$$

$$x_t^n + x_t^r \leq c_t, \quad \forall t = 1, \cdots, T \tag{9}$$

$$x_t^n \leq d_{[T]}^n \alpha_t^n, \quad \forall t = 1, \cdots, T \tag{10}$$

$$x_t^r \leq d_{[T]}^r \alpha_t^r, \quad \forall t = 1, \cdots, T \tag{11}$$

$$x_t^n, x_t^r \geq 0, \quad \forall t = 1, \cdots, T \tag{12}$$

$$\alpha_t^n, \alpha_t^r \in \{0, 1\} \tag{13}$$

The objective function (5) minimizes the sum of setup cost, production cost, and inventory cost for new products and remanufactured products in all periods. Constraints (6),(7) and (8) are the inventory balance constraints for new products, remanufactured products and returned products. Constraints (9) represent capacity constraints for manufacturing and remanufacturing activities. Constraints (10), (11) allow production only with the according setups (i.e $\alpha_t^n = 1, \alpha_t^r = 1$.) (12) and (13) are the standard integrality and non-negative constraints.

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[1] We can always transform a problem with non zero initial or final stock by adapting the demand.

2.1. Rewriting the Optimization Problem

i_t^n, i_t^r, i_t^{re} are replaced in the objective function by

$$i_t^n = \sum_{j=1}^{t}\left(x_j^n - d_j^n\right), \quad i_t^r = \sum_{j=1}^{t}\left(x_j^r - d_j^r\right), \tag{14}$$

$$i_t^{re} = \sum_{j=1}^{t}\left(r_j^{re} - x_j^r\right) \tag{15}$$

And with a bit of algebraic transformations, we obtain

$$\sum_{t=1}^{T}(p_t^n x_t^n + h_t^n i_t^n) = \sum_{t=1}^{T}(p_t^n + \sum_{j=t}^{T} h_j^n) x_t^n - \sum_{t=1}^{T}(h_t^n \sum_{k=1}^{t} d_k^n)$$

$$\sum_{t=1}^{T}(p_t^r x_t^r + h_t^r i_t^r + h_t^{re} i_t^{re}) = \sum_{t=1}^{T}(p_t^r + \sum_{j=t}^{T}(h_j^r - h_j^{re})) x_t^r + \sum_{t=1}^{T}(-h_t^r \sum_{k=1}^{t} d_k^r + h_t^{re} \sum_{k=1}^{t} r_k^{re})$$

With the notation

$$v_t^n = p_t^n + \sum_{j=t}^{T} h_j^n,$$

$$v_t^r = p_t^r + \sum_{j=t}^{T}(h_j^r - h_j^{re})$$

$$K = \sum_{t=1}^{T}\left\{-h_t^n \sum_{k=1}^{t} d_k^n - h_t^r \sum_{k=1}^{t} d_k^r + h_t^{re} \sum_{k=1}^{t} r_k^{re}\right\}.$$

our model is finally

$$\varphi(x^n, x^r) = \sum_{t=1}^{T}[s_t^n \alpha_t^n + v_t^n x_t^n] + [s_t^r \alpha_t^r + v_t^r x_t^r] + K \longrightarrow \min \tag{16}$$

Subject to

$$\sum_{j=1}^{t} x_j^n \geq \sum_{j=1}^{t} d_t^n, \quad \forall t = 1, \cdots, T \tag{17}$$

$$\sum_{j=1}^{t} x_j^r \geq \sum_{j=1}^{t} d_t^r, \quad \forall t = 1, \cdots, T \tag{18}$$

$$\sum_{j=1}^{t} r_j^{re} \geq \sum_{j=1}^{t} x_j^r, \quad \forall t = 1, \cdots, T \tag{19}$$

$$x_t^n + x_t^r \leq c_t, \quad \forall t = 1, \cdots, T \tag{20}$$

$$x_t^n \leq d_{[T]}^n \alpha_t^n, \quad \forall t = 1, \cdots, T \tag{21}$$

$$x_t^r \leq d_{[T]}^r \alpha_t^r, \quad \forall t = 1, \cdots, T \tag{22}$$

$$x_t^n, x_t^r \geq 0, \quad \forall t = 1, \cdots, T \tag{23}$$

$$\alpha_t^n, \alpha_t^r \in \{0, 1\} \tag{24}$$

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3. Model A (Relaxation)

According to our conditions(1) and (3) the general problem has a solution, if the total pro-period demand is less than the pro-period capacity (i.e.$d_t <= c_t$). However there are situations, where conditions (1) and (3) are satisfied but $d_t <= c_t$ is not satisfied e.g. Table 3. We need a feasibility routine which ensures that all demand is satisfied without backlogging. Indeed there are periods (or could be) in which total demand exceeds total capacity. In this case some inventory will have to be build up in earlier periods which slack capacity. We explain how to shift excess demand to earlier periods in which slack capacity is available. We use and complement the idea of [40].

$$\begin{aligned} \tilde{w}_T &= 0 \\ \tilde{w}_t &= \max\{d_{t+1} - c_{t+1} + \tilde{w}_{t+1}; 0\}, \quad t = T-1, \cdots, 1. \end{aligned}$$

We define

$$\begin{aligned} w_1 &= \tilde{w}_1 \\ w_t &= \tilde{w}_t - \tilde{w}_{t-1}, \quad t = 2, \cdots, T. \end{aligned}$$

It is easy to see that the sum $w_{[T]}$ is null.

$$\begin{aligned} w_{[T]} &= w_1 + w_2 + \cdots + w_T \\ &= \tilde{w}_1 + (\tilde{w}_2 - \tilde{w}_1) + \cdots + (\tilde{w}_T - \tilde{w}_{T-1}) \\ &= \tilde{w}_T = 0. \end{aligned}$$

Remark 1. *The vector $w^T = (w_1, \cdots, w_T)$ is very useful. Because w_t gives the amount of stock to accumulate ($w_t > 0$) or reduce ($w_t < 0$) in each period so that production does not exceed available capacity pro period. And allows us to determine a good permissible solution.*

Table 3. Original Data

t	r_t^{re}	d_t^n	d_t^r	d_t	c_t	w
1	198	153	183	336	609	57
2	806	84	302	386	632	246
3	223	100	146	246	101	-145
4	283	100	127	227	295	68
5	500	248	598	846	620	-226
6	500	0	0	0	561	0
Sum	2510	685	1356	2041	2818	0

We see that the demand transformation done in Table 4 is a valid (permissible) solution to the problem (16)-(24). There are many ways to make this transformation, and for this reason to transform the demand in an optimal way we formulate the following problem.

$$\varphi(x^n, x^r) = \sum_{t=1}^{T} [s_t^n \alpha_t^n + v_t^n x_t^n] + [s_t^r \alpha_t^r + v_t^r x_t^r] + K \longrightarrow \min \tag{25}$$

Subject to

$$\sum_{j=1}^{t} x_j^n \geq \sum_{j=1}^{t} d_t^n, \quad \forall t = 1, \cdots, T \tag{26}$$

$$\sum_{j=1}^{t} x_j^r \geq \sum_{j=1}^{t} d_t^r, \quad \forall t = 1, \cdots, T \tag{27}$$

$$\sum_{j=1}^{t} r_j^{re} \geq \sum_{j=1}^{t} x_j^r, \quad \forall t = 1, \cdots, T \tag{28}$$

$$x_t^n + x_t^r - d_t^n - d_t^r = w_t, \quad \forall t = 1, \cdots, T \tag{29}$$

$$x_t^n \leq d_{[T]}^n \alpha_t^n, \quad \forall t = 1, \cdots, T \tag{30}$$

$$x_t^r \leq d_{[T]}^r \alpha_t^r, \quad \forall t = 1, \cdots, T \tag{31}$$

$$x_t^n, x_t^r \geq 0, \quad \forall t = 1, \cdots, T \tag{32}$$

$$\alpha_t^n, \alpha_t^r \in \{0, 1\} \tag{33}$$

If the model (25)-(33) has no solution, it means the model (16)-(24) has no solution too.
We have a problem with no capacity restrictions. We solve this problem and compare it with the optimal solution of the initial problem (16)-(24).

Table 4. Demand Transformation

t	$\tilde{d}_t^n$	$\tilde{d}_t^r$	$\tilde{d}_t$	c_t	$\tilde{d}_{[t]}$	$c_{[t]}$
1	195	198	393	609	393	609
2	42	590	632	632	1025	1241
3	101	0	101	101	1126	1342
4	295	0	295	295	1421	1637
5	52	568	620	620	2041	2257
6	0	0	0	561	2041	2818
Sum	685	1356	2041	2818		

Remark 2. *It is important to clarify that this task only makes sense if the vector w is not equal to the null vector. If the vector w is equal to the null vector it means that in each period $d_t <= c_t$. In this case a relaxation is not possible and the problem (16)-(24) will be solved with a heuristic method (Model B).*

4. Model B (Simulation)

4.1. Low Discrepancy Sequences

The generation of random numbers with a computer is not possible Knuth [41]. As John von Neumann said: *Any one who considers arithmetical methods of producing random digits is, of course, in a state of sin, [42]. An excellent overview of the methods of generating pseudo-random numbers are available [e.g. 41,43,44].*

In this section we explain the number-theoretical concept of discrepancy. Then, we introduce the Halton sequence which is probably the easiest low-discrepancy [2] number generation method to describe.

Definition 1. *Let $\{z_1, \cdots, z_N\}$ a sequence of real numbers with $0 < z_i < 1,\ i = 1, \cdots, N$. The discrepancy D_N for the sequence is defined as*

$$D_N = \sup_{l} |\ S_N(l) - N\,|l|\ | \tag{34}$$

[2] low discrepancy sequences are called quasi-random sequences

where l *is any subinterval* $[a,b) \subseteq [0,1]$, $|l| = b-a$, *and* $S_N(l)$ *denotes the number of elements of the sequence, that belongs to the interval* l.

A measure for how a sequence of real numbers $\{z_1,\cdots,z_N\}$, $a < z_i < b,\ i = 1,\cdots,N$ *is equidistributed on an interval* $[a,b]$ *is the discrepancy* D_N. *Low-discrepancy sequences, also known as* quasirandom *sequences, are numbers that are better equidistributed in a given volume than pseudo-random numbers.*

Remark 3. *A sequence* $\{z_1,\cdots,z_N\}, 0 < z_i < 1, i = 1,\cdots,N$ *of real numbers is said equidistributed on the interval* $[0,1]$ *if* $D_N = o(N), N \to \infty$, *[35].*

The ($\mathbb{QRN}$ *) of Halton, Sobol and Niederreiter have a low discrepancy* $D_N = O(\ln(N)/N)$. *While pseudorandon sequences have a discrepancy* $D_N = O(1/\sqrt{N})$, *[**?**]. Figure 2 uses two-dimensional projection of a pseudorandom sequence and of a low-discrepancy (Halton) sequence to demonstrate the fundamental difference between the two classes of sequences.*

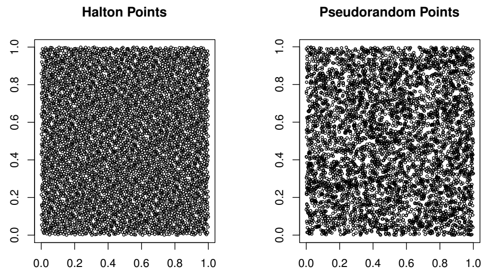

Figure 2. Two-dimensional projection of 5000 Halton and Pseudorandom points

The desirable properties of a sequence of this ($\mathbb{QRN}$ *) may be summarized as follows [see 37]:*

1. *the least period length should be sufficiently large,*
2. *it should have littie intrinsic structure (such as lattice structure),*
3. *it should have good statistical properties,*
4. *the algorithm generating the sequence should be reasonably efficient.*

It's easy to generate sequences of Halton with the following Algorithm 1.

Algorithm 1: Construction of Halton sequences

Input: p *prime*, $n >= 1$ *natural number*
Output: *A Halton number* z_h
$i = 1, z_h = 0$
while $n \neq 0$ **do**
  $r = n \mod p$
  $n = n \backslash p$
  $z_h = z_h + \frac{r}{p^i}$
  $i = i + 1$
end
return z_h

The following Halton sequences of Table 5 are constructed according to Algorithm 1 that uses a prime number as its base.

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Table 5. Halton Numbers

	Prime numbers			
	2	**3**	**5**	**7**
n	Halton numbers			
1	0,5	0,33333333	0,2	0,14285714
2	0,25	0,66666667	0,4	0,28571429
3	0,75	0,11111111	0,6	0,42857143
4	0,125	0,44444444	0,8	0,57142857
5	0,625	0,77777778	0,04	0,71428571

Remark 4. *To generate the n-th Halton point in a sequence consider the base $b-$ary expansion of a $n = \sum_{i=0}^{\infty} a_i\, b^i$ where the $b-$ary coefficients $a_i \in \{0, \cdots, b-1\}$. Then the n-th Halton point is $H(n) = \sum_{i=0}^{\infty} a_i\, b^{-i-1}$. It's easy to build a Halton-sequence with the following observation: If $a_0 < b-1$ then $H(n+1) = H(n) + 1/b$ else if $a_0 = b-1$ then $H(n+1) = H(n) - (1 - b^k - b^{k+1})$ where $k = \min\{i \geq 0 : a_i \neq b-1\}$ (details see [38]). This method is very efficient and will be used in this paper.*

4.2. Notation

We use the following notation $\forall t = 1, \cdots, T$

Name	*Meaning*	
$d^n_{[t]}$ =	$\sum_{j=1}^{t} d^n_j$	$d^n_{[0]}$ = 0
$d^r_{[t]}$ =	$\sum_{j=1}^{t} d^r_j$	$d^r_{[0]}$ = 0
$r^{re}_{[t]}$=	$\sum_{j=1}^{t} r^{re}_j$	$r^{re}_{[0]}$= 0
$x^n_{[t]}$ =	$\sum_{j=1}^{t} x^n_j$	$x^n_{[0]}$ = 0
$x^r_{[t]}$ =	$\sum_{j=1}^{t} x^r_j$	$x^r_{[0]}$ = 0

The notation $x \in [a, b]_p$ means $x = (b-a)z + a,\ \ a \leq b,\ 0 \leq z \leq 1$ and the number z is simulated with the following distribution

$$z = \begin{pmatrix} z_h & 0 & 1 \\ p & q & q \end{pmatrix} \tag{35}$$

$$0 \leq p \leq 1, \quad q = \frac{1}{2}(1-p), \quad z_h \text{ is a Halton number}$$

We generate a pseudorandom number $0 \leq g \leq 1$ to decide, that values take z, see Algorithm 2.

In Model B (section 4) we will investigate the class of problems with the condition

$$d^n_t + d^r_t = d_t \leq c_t, \forall t \tag{36}$$

If this condition is not satisfied, the problem is easily solved with Model A (section 3).

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Algorithm 2: Simulation of z with distribution (35)

if $g \leq p$ **then**
  $z = z_h$;
end
else if $g \leq p + q$ **then**
  $z = 0$;
end
else
  $z = 1$;
end

4.3. Simulation

The simulation is based on the following lemma.

Lemma 1. *Let $x_t = x_t^n + x_t^r$ and $d_t \leq c_t, \forall t = 1, \cdots, T$. Then*

$$d_{[t]} - x_{[t-1]} \leq x_t \leq c_t, \quad \forall t = 1, \cdots, T \tag{37}$$

Proof. The proof proceeds by induction on t. In fact, if $t = 1$ because $x_{[0]} = 0$ and $d_1 \leq c_1$, we can choose x_1 such that $d_1 \leq x_1 \leq c_1$.

$$d_{[t+1]} - x_{[t]} = d_{t+1} - x_t + d_{[t]} - x_{[t-1]}$$

By induction hypothesis

$$\begin{aligned} &\leq d_{t+1} - x_t + x_t \\ &\leq c_{t+1} \end{aligned}$$

Then we can x_{t+1} choose such that $d_{[t+1]} - x_{[t]} \leq x_{t+1} \leq c_{t+1}$. □

Remark 5. *If in Lemma 1 $d_{[t]} - x_{[t-1]} < 0$ then production in period t is $x_t = 0$ because production up to period $t-1$ satisfies demand up to period t.*

Remark 6. *Lemma 1 gives the following lower bound for the production of the products*

$$x_t^n \geq u_t^n = \max\{d_{[t]}^n - x_{[t-1]}^n; 0\} \tag{38}$$
$$x_t^r \geq u_t^r = \max\{d_{[t]}^r - x_{[t-1]}^r; 0\} \tag{39}$$

4.4. Basis of the Simulation

The production plan is created step by step starting from period $t = 1$. To determine the production in period t, we know the selected production until period $t-1$. In each period, using the constraints of the task (16)-(24) for the production of the products, we determine a lower and an upper bound. Then the production quantity is chosen randomly between the lower and upper limits. These production quantities affect the lower and upper bounds of the future period. We continue in this way until the period $t = T$. Then we calculate the value of the cost function(i.e., objective function). We repeat this procedure (N times) and choose the production plan with the lowest cost. The advantages of this method is that we do not have to worry about the inventory, production or setup costs. The method is now presented in more detail.

Proposition 1. *If $u_t^n = 0$ then $x_t^n = 0$, $\alpha_t^n = 0$, else $x_t^n \geq u_t^n$, $\alpha_t^n = 1$.*

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Proof. From (17) we get

$$x^n_{[t]} - d^n_{[t]} = x_t + (x^n_{[t-1]} - d^n_{[t]}) \geq 0 \tag{40}$$

If $(x^n_{[t-1]} - d^n_{[t]}) \geq 0$, then total production of new products up to period $(t-1)$ satisfies total demand up to period t and therefore nothing is produced in period t, i.e. $x^n_t = 0$, $\alpha^n_t = 0$.

If $(x^n_{[t-1]} - d^n_{[t]}) < 0$, then the production x^n_t has a lower bound. From (40) we obtain

$$x^n_t \geq d^n_{[t]} - x^n_{[t-1]}. \tag{41}$$

□

Proposition 2. *If $u^r_t = 0$ then $x^r_t = 0$, $\alpha^r_t = 0$, else $\alpha^r_t = 1$ and*

$$r^{re}_{[t]} - x^r_{[t-1]} \geq x^r_t \geq u^r_t. \tag{42}$$

Proof. From (18) and (19)

$$r^{re}_{[t]} \geq x^r_t + x^r_{[t-1]} \geq d^r_{[t]}. \tag{43}$$

If $(x^r_{[t-1]} - d^r_{[t]}) \geq 0$, then total production up to period $(t-1)$ satisfies total demand up to period t and therefore nothing is produced in period t, i.e. $x^r_t = 0$, $\alpha^r_t = 0$.

If $(x^r_{[t-1]} - d^r_{[t]}) < 0$, then the production x^r_t has an upper and lower bound. From (43) results the assertion. □

Proposition 3. *If $u^n_t > 0$ and $u^r_t > 0$, then*

$$o^n_t = \min\{c_t - u^r_t, d^n_{[T]} - x^n_{[t-1]}\}$$
$$o^r_t = \min\{c_t - x^n_t, d^r_{[T]} - x^r_{[t-1]}, r^{re}_{[t]} - x^r_{[t-1]}\}$$

And

$$u^n_t \leq x^n_t \leq o^n_t \tag{44}$$
$$u^r_t \leq x^r_t \leq o^r_t \tag{45}$$

Proof. From (20) and remark 6

$$c_t \geq x^n_t + x^r_t \geq x^n_t + u^r_t$$
$$c_t - u^r_t \geq x^n_t \tag{46}$$

From (2)

$$d^n_{[T]} = x^n_{[T]} \geq x^n_{[t-1]} + x^n_t$$
$$d^n_{[T]} - x^n_{[t-1]} \geq x^n_t \tag{47}$$

From (46) and (47)

$$x^n_t \leq o^n_t \tag{48}$$

Therefore we simulate the production x^n_t according to (35)

$$x^n_t \in [u^n_t, o^n_t]_p,\ \alpha^n_t = 1. \tag{49}$$

From (20) using x^n_t of (49) we get

$$c_t - x^n_t \geq x^r_t \tag{50}$$

and then from (2) we obtain

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$$
\begin{aligned}
d^r_{[T]} = x^r_{[T]} \geq x^r_{[t]} = x^r_t + x^r_{[t-1]} \\
d^r_{[T]} - x^r_{[t-1]} \geq x^r_t
\end{aligned}
\tag{51}
$$

From (50), (51) and (42) results

$$
x^r_t \leq o^r_t \tag{52}
$$

Therefore we simulate the production x^r_t according to (35)

$$
x^r_t \in [u^r_t, o^r_t]_p, \ \alpha^r_t = 1. \tag{53}
$$

□

4.5. Simulation of the Objective Function

Let R be a matrix with Halton's $\mathbb{QRN}$

$$
R = \begin{pmatrix} R(1,1) & \cdots & R(1,T) \\ R(2,1) & \cdots & R(2,T) \end{pmatrix}
$$

We simulate the production starting in period t=1. The calculation of the objective function is carried out with Algorithm 3 using proposition 1, 2 and 3.

Algorithm 3: Calculation of the objective function

```
Data: R = [2 × T] matrix with Halton-QRN
Result: φ(x^n, x^r)
t = 1, φ = 0, x^n_[0] = x^r_[0] = 0
while t ≤ T do
    if u^n_t > 0 and u^r_t > 0 then
        x_t ∈ [u^n_t, o^n_t]_p using R(1,t)
        x^r_t ∈ [u^r_t, o^r_t]_p using R(2,t)
        α^n_t = 1, α^r_t = 1
    end
    else if u^n_t > 0 and u^r_t = 0 then
        x^n_t ∈ [u^n_t, o^n_t]_p using R(1,t)
        x^r_t = 0,
        α^n_t = 1, α^r_t = 0
    end
    else if u^n_t = 0 and u^r_t > 0 then
        x^n_t = 0
        x^r_t ∈ [u^r_t, o^r_t]_p using R(2,t)
        α^n_t = 0, α^r_t = 1
    end
    else
        x_t = 0, x^r_t = 0
        α^n_t = 0, α^r_t = 0
    end
    φ := φ + s^n_t α^n_t + v^n_t x^n_t + s^r_t α^r_t + v^r_t x^r_t
    x^n_[t] := x^n_[t-1] + x^n_t,  x^r_[t] := x^r_[t-1] + x^r_t
    t = t + 1
end
```

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Further information on the complexity-theoretic approach to randomness can be found in [45,46] and [47].

Remark 7. *1. The generation of the matrix $R[N \times T]$ requires $O(N \times T)$ operations.*
2. Die evaluation of the function $\varphi(x^n, x^r)$ with Algorithm 3 requires $O(T)$ operations
Then, the computational complexity of the simulation with N points and T periods is $O(N \times T)$.

5. Numerical Experiments

5.1. Test Design

We analyse the quality of our solution approach of model A and model B by defining 11 problem classes (PC) by varying the number of periods, see Table 6. The planning horizon T is made very large because in the paper industry planning is done daily. Each PC consists of 200 test instances (TI). In model A, 1824 of the 2200 TI were solvable, in model B 2200. In total, we examined 4024 TI [3]+. Model A and Model B are implemented on a computer with $Intel(R)\ Core(TM) iT - 9700K$, $CPU@3.60GHz$,3600MHz.

Table 6. Problem classes

	PC1	PC2	PC3	PC4	PC5	PC6
T	15	30	60	90	120	150
	PC7	PC8	PC9	PC10	PC11	
T	180	210	240	270	300	

We vary different parameters to define the TI, e.g., the time between orders (TBO) to determine setup costs. The specifications of the parameters are designed in an exaggerated form, which may not occur in practice, to make the TI as difficult as possible.

The parameters for generating data sets (see Table 7) use the following notation $x \in [a; b] \Leftrightarrow x = (b-a)\theta + a, \ \ a \leq b$, where $0 \leq \theta \leq 1$ is a random number, that means the values are uniformly distributed on the interval $[a, b]$.

5.2. Model A

We randomly generate capacities c_t, then with condition (1) randomly generate aggregate demand d_t. This latter is then randomly cut into d_t^n and d_t^r. The remaining parameters according to Table 7.

Table 7. Parameters for Model A

$c_t \in [0;\ 800]$	d_t^n, d_t^r with condition (1)
$p^n \in [15;\ 20]$	$p^r \in [10;\ 15]$
$h_t^n \in [5;\ 10]$	$h_t^r \in [3;\ 8]$
h_t^{re} with condition (4)	r_t^{re} with condition (3)
$s_t^n =$	$\frac{\bar{d}^n\, TBO^2\, h_t^n}{2}$, $TBO \in \{1, 2, 4\}$
$s_t^r =$	$\frac{\bar{d}^r\, TBO^2\, h_t^r}{2}$, $TBO \in \{1, 2, 4\}$
	$\bar{d}^n, \bar{d}^r$ represent the average demand values.

5.2.1. Results of Model A

The solution of problem (25)-(33) (TD) and the optimal solution of problem (16)-(24) (OP) were found with the Gurobi solver version 9.0.3.

We randomly generated 200 instances per PC and usually between 10 and approx. 20% of the instances have no solution (see line Count). This justifies the fact that the problem is not standard. For example, in Table A1 we see that of the 200 instances for problem class $T = 60$, only 167 had a solution.

[3] The test instances and solutions are available **here**

With the following notation, we can better understand the results of Table A1.

T	$\in \{15, 30, \cdots, 300\}$
m	$\in Count = \{172, 155, \cdots, 170\}$
$\varphi_{Ti}(x^n, x^r)$	$i = 1, \cdots, m$ *solution of problem* (25)-(33)
$\varphi^*_{Ti}(x^n, x^r)$	$i = 1, \cdots, m$ *solution of problem* (16)-(24)
μ_T	$= \frac{\sum_{i=1}^{m} \varphi_{Ti}(x^n,x^r)}{m}$
μ^*_T	$= \frac{\sum_{i=1}^{m} \varphi^*_{Ti}(x^n,x^r)}{m}$
CPU_T	$= \frac{\sum_{i=1}^{m} CPU_{Ti}}{m}$
CPU^*_T	$= \frac{\sum_{i=1}^{m} CPU^*_{Ti}}{m}$

In Table A1 the relative error for Total costs (Tc) and CPU-time for every problem class was calculated as

$$RelativeAvg.Error(Tc) = \frac{\mu_T - \mu^*_T}{\mu^*_T}$$

$$RelativeAvg.Error(CPU) = \frac{CPU_T - CPU^*_T}{CPU^*_T}$$

This is exactly how we calculated the relative errors in inventory cost and setup cost.

Attached in Appendix A are the results of the average total cost, average CPU time, average Inventory costs

There is hardly any difference between the cost of relaxation (TD) and the original task (OP). Only the computation time for relaxation is faster. The longer the planning horizon, the smaller the difference between the optimal solutions of the problems TD and original optimization task OP. This feature applies to the stocks of the return and setup costs too. On the other hand, the inventory costs for problem TD are always smaller than the inventory costs of problems OP. For more details, please see Figure A1 and Appendix A.1 for the exact calculations. However, we see beyond doubt that this class of problems can be solved very well either with the relaxation TD or directly with a standard solver (here Gurobi).

5.3. Model B

[30] have shown that several families of CLSP are NP-hard. For the construction of the NP-hard instances (2200 instances) we follow the findings of [31]. They use the following notation $Nr/\alpha/\beta/\gamma/\sigma$, where $Nr, \alpha, \beta, \gamma$ and σ specify respectively the number of items, a special structure for the setup costs, the holding costs, production costs, and capacities. In this paper [31] show that the following class 2/C/G/A/C is NP-hard. For this reason we have created 2200 instances, where the set-up costs per product and capacities per instance are constant. The holding costs do not necessarily follow a specified pattern, the production costs can be chosen arbitrarily. The maximum calculation time for Gurobi is 600 seconds. The heuristic operates according to the number of simulations, which gradually increases with the number of periods. The largest group T300 uses 130 seconds. The parameters for generating data sets (see Table 8) use the following notation $x \in [a; b] \Leftrightarrow x = (b - a)\theta + a,\ \ a \leq b$, where $0 \leq \theta \leq 1$ is a random number, that means the values are uniformly distributed on the interval $[a, b]$.

The important assumption in model B is: the capacities for each TI is constant, the set-up costs in each TI and for each product are constant. These parameters vary between a minimum and a maximum depending on the T parameter.

The capacities for each TI vary according to the parameter T. For example if $T = 15$ the capacities are between 600 and 800. If $T = 300$ the capacities vary between 3000 and 5500. Analogously the other parameters. The remaining parameters according to Table 8.

Table 8. Model B: Parameters

$c \in [200;\ 5500]$	d^n_t, d^r_t with condition (1)
$p^n \in [4;\ 20]$	$p^r \in [2;\ 15]$
$h^n_t \in [0.6;\ 10]$	$h^r_t \in [0.6;\ 8]$
h^{re}_t with condition (4)	r^{re}_t with condition (3)
$sc^n \in [4000;\ 30000]$	$sr^r \in [3000;\ 16000]$

For the simulation we used following parameters:

$$N_T = 2^{15}T$$

N_T is the Halton's numbers used for $T \in \{15, 30, \cdots, 300\}$ periods.

All instances for $T = 15, 30 \cdots, 300$ have the same schema Step 1 until Step 5. We used $\tau = 4$ (see Table 9).

Table 9. Schematic of the simulation

Step 1	Initialisation:φ^*, $k = 1$, $\tau > k$, T, $N = \frac{N_T}{\tau}$
Step 2	$p_k = \frac{k}{\tau}$. If $k = \tau$, stop; otherwise go to Step 3.
Step 3	Using Algorithms 3 and 4 along with p_k calculate the function $\varphi_{p_k} := \min\{\varphi_i(x, y), i = 1, \cdots, N\}$.
Step 4	If $\varphi^* > \varphi_{p_k}$ then $\varphi^* = \varphi_{p_k}$.
Step 5	$k = k + 1$ and go to Step 2.

Algorithm 4: Heuristic: blind search

```
Data: R = [N_T × T] random matrix
Result: min{φ(x,y)}
Initialisation: φ_min
for k ← 1 to N_T − 1; k = k + 2 do
    x ← R[k]
    y ← R[k + 1]
    φ ← φ(x,y)
    if φ < φ_min then
        φ_min ← φ
    end
end
```

5.3.1. Results of Model B

We generated 200 random instances for each problem class (PC) and with a Box-plot we compared the feasible solutions $G_{Ti}(x^n, x^r)$ of the problem (16)-(24) found by Gurobi 9.0.3 with the solution $S_{Ti}(x^n, x^r)$ of the simulation presented in this paper and clearly see the similarity of the results found (see Figures A4 and A5).

With the following notation we present the results (see Table A3).

T	$\in \{15, 30, \cdots, 300\}$
m	$= 200$
$S_{Ti}(x^n, x^r)$	$i = 1, \cdots, m$
$G_{Ti}(x^n, x^r)$	$i = 1, \cdots, m$
μ_T	$= \frac{\sum_{i=1}^m S_{Ti}(x^n, x^r)}{m}$
μ_T^*	$= \frac{\sum_{i=1}^m G_{Ti}(x^n, x^r)}{m}$
CPU_S_T	$= \frac{\sum_{i=1}^m CPU_{Ti}}{m}$ *with Simulation*
CPU_T^*	$= \frac{\sum_{i=1}^m CPU_{Ti}^*}{m}$ *with Gurobi*

In Table A3 the relative error for Total costs (Tc) and CPU-time for every problem class was calculated as

$$Relative Avg.Error(Tc) = \frac{\mu_T - \mu_T^*}{\mu_T^*}$$

$$Relative Avg.Error(CPU) = \frac{CPU\_S_T - CPU_T^*}{CPU_T^*}$$

The graphical comparison is shown in Figure A3.

This is exactly how we calculated the relative errors in inventory cost and setup cost. Attached in Appendix B are the results of the average total cost, average CPU time, average Inventory costs. Figure A3 (average CPU time) clearly shows that the Gurobi admissible solution for the PC from T150 to T300 are not optimal and that the CPU of the simulation is much faster.

The simulation in average determines the setup costs and return inventory costs always higher than Gurobi. On the other hand, the inventory costs of Gurobi are higher than the simulation. What can we say about the quality of the solution of the problems? The simulation could not give a better solution than Gurobi's solution. Gurobi solved the (PC) problems up to T120 in an optimal way. The problems from T150 to T300 were not solved optimally by Gurobi, since it would take too much time due to the NP-hard category of the problem. The simulation found feasible solutions much faster than Gurobi. Here is the advantage of the simulation, the simplicity of its implementation and the speed in finding an acceptable solution.

6. Conclusions and Outlook

We have analyzed a problem that belongs to the NP-hard class. However, the choice of the parameters is very important to obtain a problem that is really NP-hard. With the choice of parameters made in model A, we see that this class of problems is easily solved with a standard solver. By doing a relaxation of the problem, the solution is found more quickly. The error rate is between 0.02% and 2% (taking into account more than 1800 instances). In addressing more intricate CLSP class problems characterized by Model A, the solution derived from Model A can serve as an initial approximation for tackling the problem. This initial approximation can then be seamlessly integrated into various heuristic or metaheuristic approaches.

If we choose the parameters according to model B, Gurobi needs a lot of time to find the optimal solution. In this kind of problem, the presented simulation can help a lot in finding a good solution. We have seen that the error rate is between 1.7% and 3.5% (taking into account more than 2000 instances). On average Gurobi solves the problem with a maximum time of 600 seconds better than the simulation. The great advantage of the simulation is that the calculation is extremely fast and easy.Thanks to the Halton numbers, few simulations are needed to obtain a very good approximation of the solution.

The quality of the solutions can be improved by increasing the number of simulations but it is necessary to have a fairly fast computer. In this work we use at most 10 million simulations. Another parameter that influences the quality of the solutions is the correct choice of the probability p (see (35)). Is there an optimal probability? This is a question for further research.

The simulation possesses a broad nature and can be tailored to examine additional, intricate production issues within the NP-hard category. The benefit is readily apparent: it circumvents the necessity for an extensive array of sorting rules and additional criteria stemming from factors like demand, setup expenses, production outlays, or capacities.

Acknowledgments: *I extend my heartfelt gratitude to Robert W. Grubbström and Chistian Almeder for their invaluable insights and thoughtful comments on this paper.*

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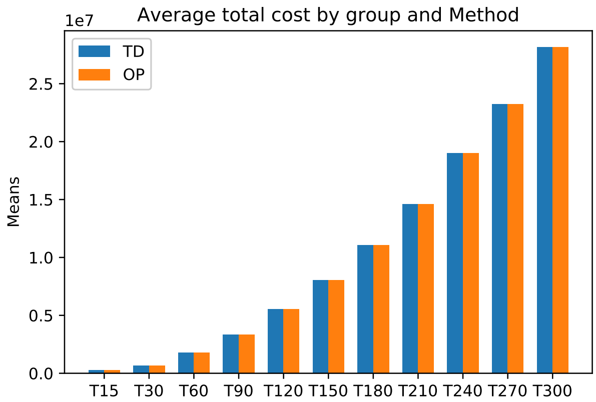
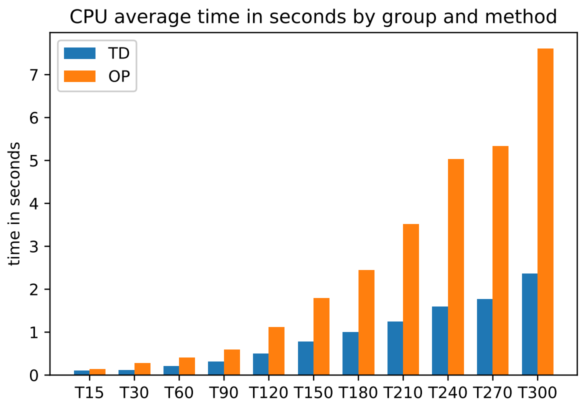
Appendix A. Results Visualization

Appendix A.1. Model A



(a) Totalcosts



(b) CPU-time.

Figure A1. Model A: Average Total cost and CPU Time.

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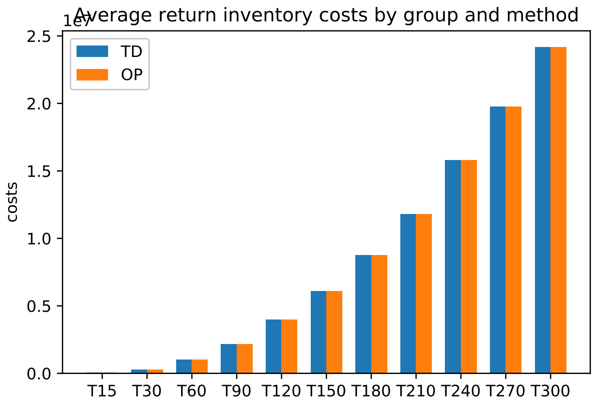
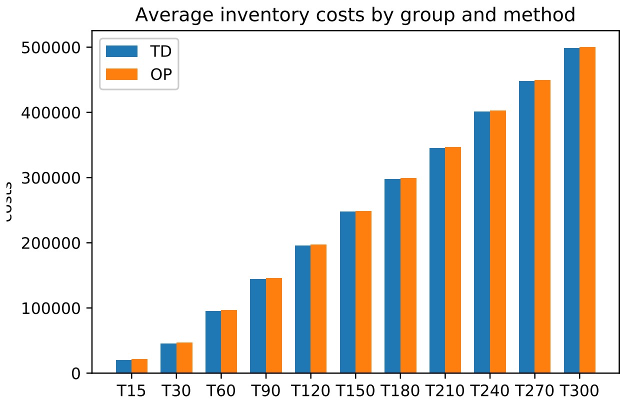


(a) Return Inventory cost.



(b) Inventory cost.

Figure A2. Model A: Average Inventory cost.

Appendix A.2. Model B

Visualization of the results and average costs.

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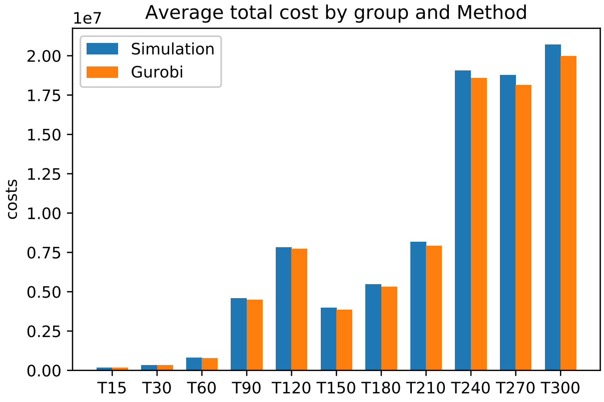
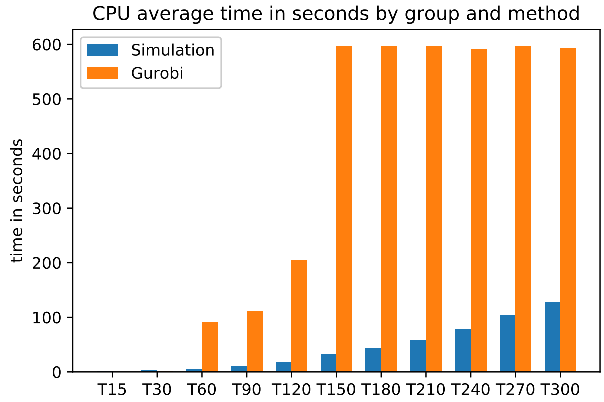


(a) Total costs.



(b) CPU time.

Figure A3. Model B: Average Total cost and CPU time.

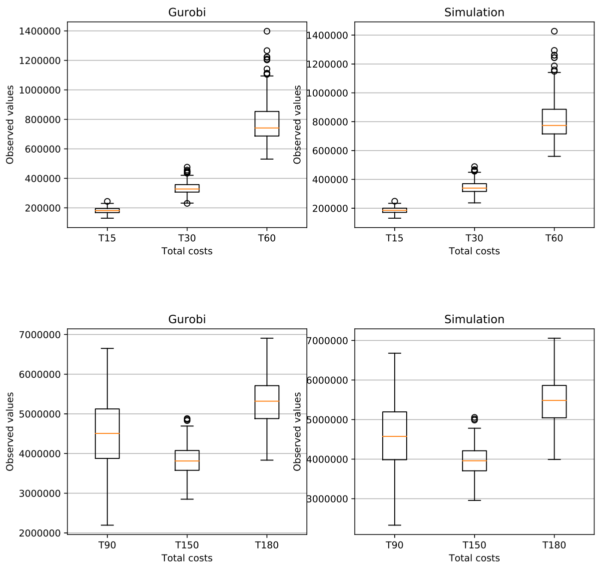


Figure A4. Model B: Average Total cost.

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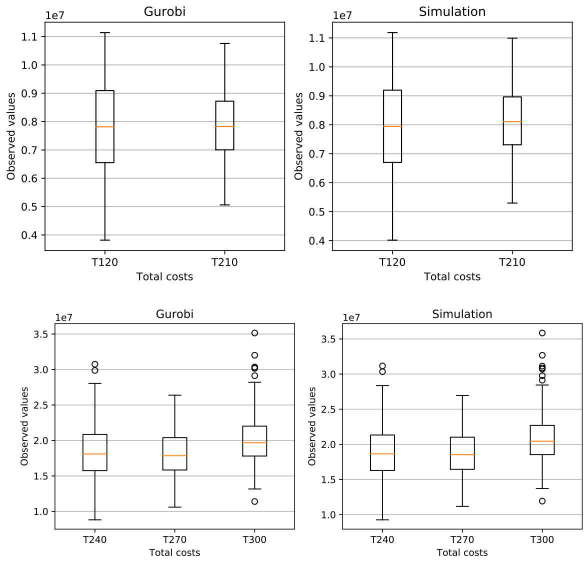

Figure A5. Model B: Average Total cost.

Appendix B. Average Costs

Table A1. Model A: Average Total Costs

Total costs	T	15	30	60	90	120	150
	Count	172	155	167	162	166	164
TD	Mean	274061	665537	1798127	3348945	5552436	8062797
	Std.	87161	151370	389288	660684	906092	1349699
Optimal	Mean	268593	659453	1792494	3344788	5546715	8058311
	Std.	83431	148783	385859	659403	904686	1348457
Relative error	TcError	2.04%	0.92%	0.31%	0.12%	0.10%	0.06%

Total costs	T	150	180	210	240	270	300
	Count	164	164	168	168	170	168
TD	Mean	8062797	11087631	14620895	19015245	23243458	28160753
	Std.	1349699	1811439	2199306	2911556	3384673	3870830
Optimal	Mean	8058311	11081145	14614610	19008049	23237758	28154353
	Std.	1348457	1808995	2198938	2910200	3382841	3868995
Relative error	TcError	0.06%	0.06%	0.04%	0.04%	0.02%	0.02%

Table A2. Model A: Average CPU time

CPU Time	T	15	30	60	90	120	150
	Count	172	155	167	162	166	164
TD	Mean	0.10	0.11	0.21	0.31	0.50	0.78
	Std.	0.06	0.05	0.10	0.16	0.26	0.39
Optimal	Mean	0.14	0.28	0.41	0.59	1.12	1.79
	Std.	0.06	0.11	0.17	0.30	0.51	1.42
Relative error	CPUError	-25.27%	-59.97%	-49.61%	-47.53%	-55.39%	-56.26%

CPU Time	T	150	180	210	240	270	300
	Count	164	164	168	168	170	168
TD	Mean	0.78	1.00	1.25	1.59	1.77	2.36
	Std.	0.39	0.47	0.69	0.82	0,90	1.39
Optimal	Mean	1.79	2.44	3.52	5.03	5.33	7.60
	Std.	1.42	1.73	3.00	3.50	3.98	5.97
Relative error	CPUError	-56.26%	-59.00%	-64.53%	-68.46%	-66.87%	-68,94%

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Table A3. Model B: Average Total Costs

Total costs	**T**	15	30	60	90	120	150
	Count	200	200	200	200	200	200
Simulation	Mean	185971	342933	815220	4576352	7829443	3979678
	Std.	21273	45909	155884	858442	1559305	397507
Gurobi	Mean	182040	332163	784654	4498331	7718552	3845522
	Std.	21125	45150	155097	882996	1594738	393440
Relative error	TcError	2.16%	3.24%	3.90%	1.73%	1.44%	3.49%

Total costs	**T**	150	180	210	240	270	300
	Count	200	200	200	200	200	200
Simulation	Mean	3979678	5477237	8177244	19065355	18786745	20708665
	Std.	397507	593444	1275892	4030464	3201422	3672141
Gurobi	Mean	3845522	5318346	7905438	18570461	18146172	19992493
	Std.	393440	590176	1265172	4055425	3179322	3670651
Relative error	TcError	3.49%	2.99%	3.44%	2.66%	3.53%	3.58%

Table A4. Model B: Average CPU time

CPU Time	**T**	15	30	60	90	120	150
	Count	200	200	200	200	200	200
Simulation	Mean	0.79	2.28	5.94	11.68	19.87	34.12
	Std.	0.13	0.13	0.16	0.18	0.23	0.54
Gurobi	Mean	0.12	1.57	90.38	111.93	204.94	597.03
	Std.	0.08	1.53	156.10	186.04	245.47	36.02
Relative error	CPUError	552.92%	44.87%	-93.43%	-89.56%	-90.31%	-94.28%

CPU Time	**T**	150	180	210	240	270	300
	Count	200	200	200	200	200	200
Simulation	Mean	34.12	46.26	62.84	77.96	101.69	130.17
	Std.	0.54	0.58	1.22	2.63	2.74	3.78
Gurobi	Mean	597.03	596.88	597.47	591.51	596.54	593.22
	Std.	36.02	4.06	3.92	33.19	3.80	7.14
Relative error	CPUError	-94.28%	-92.25%	-89.48%	-86.82%	-82.95%	-78.06%

Table A5. Model B: Inventory costs

Inventory costs	**T**	15	30	60	90	120	150
	Count	200	200	200	200	200	200
Simulation	Mean	28307	52842	112536	253095	340599	260657
	Std.	5341	8170	12391	86118	107587	28744
Gurobi	Mean	27801	57117	132943	509248	790970	318360
	Std.	6907	12243	29889	194930	324335	43350
Relative error	Inv. Error	1.82%	-7.49%	-15.35%	-50.30%	-56.94%	-18.13%

Inventory costs	**T**	150	180	210	240	270	300
	Count	200	200	200	200	200	200
Simulation	Mean	260657	345666	542344	990715	1147482	1282444
	Std.	28744	38613	75292	175805	161861	196713
Gurobi	Mean	318360	507104	835915	1859977	1730800	1959659
	Std.	43350	89630	191367	486708	372666	455506
Relative error	Inv. Error	-18.13%	-31.84%	-35.12%	-46.74%	-33.70%	-34.56%

Table A6. Model B: Return stock cost

Return Inv. costs	**T**	15	30	60	90	120	150
	Count	200	200	200	200	200	200
Simulation	Mean	45867	92667	249888	3690209	6646915	821946
	Std.	18040	35850	137395	1028216	1793396	208315
Gurobi	Mean	41356	74641	195294	3346068	6070949	700634
	Std.	17493	33456	133042	1109603	1957840	207573
Relative error	Ret. Inv. Error	10.91%	24.15%	27.96%	10.28%	9.49%	17.31%

Return Inv. costs	**T**	150	180	210	240	270	300
	Count	200	200	200	200	200	200
Simulation	Mean	821946	1548362	2628642	9347966	7251886	8608929
	Std.	208315	482657	1041268	4406311	2744151	3706917
Gurobi	Mean	700634	1293559	2169556	8142232	6309891	7524531
	Std.	207573	477084	1026872	4293352	2685984	3610684
Relative error	Ret. Inv. Error	17.31%	19.70%	21.16%	14.81%	14.93%	14.41%

Table A7. Model B: Average Setup costs

Setup Costs	**T**	15	30	60	90	120	150
	Count	200	200	200	200	200	200
Simulation	Mean	50033	106876	238570	293071	388685	1521814
	Std.	10663	17638	31424	27563	34664	212705
Gurobi	Mean	55707	114588	248504	307481	408104	1458747
	Std.	10343	17268	30443	27399	37230	201433
Relative error	Setup. Error	-10.19%	-6.73%	-4.00%	-4.69%	-4.76%	4.32%

Setup Costs	**T**	150	180	210	240	270	300
	Count	200	200	200	200	200	200
Simulation	Mean	1521814	1699197	2214098	2828660	3716988	4920930
	Std.	212705	233148	330991	404730	467555	902775
Gurobi	Mean	1458747	1641182	2118141	2669355	3397775	4526298
	Std.	201433	223689	310292	367954	412785	200
Relative error	Setup. Error	4.32%	3.53%	4.53%	5.97%	9.39%	8.72%

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