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Resolution of the 3n+1 Problem Using Inequality Relation Between Indices of 2 and 3

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Resolution of the $3n+1$ problem using inequality relation between indices of 2 and 3

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Abstract: Collatz conjecture states that an integer n reduces to 1 when certain simple operations are applied to it. Mathematically, the Collatz function is written as $f^k(n) = \frac{3^k n + C}{2^z}$, where $z, k, C \geq 1$. Suppose the integer n violates Collatz conjecture by reappearing as $2^i n$, where $i \geq 1$, then the equation modifies to $n\left(1 - \frac{3^k}{2^z 2^i}\right) = \frac{C}{2^z 2^i}$. The article takes an elementary approach to this problem by calculating the bounds on the values of $\frac{C}{2^z 2^i}$ and $1 - \frac{3^k}{2^z 2^i}$. Correspondingly, an upper limit on the integer n is placed that can re-appear in the sequence. The integer n lies in the $(-\infty, 5)$ range, and the limit on the number of odd steps is $k < 3$. Finally, it is shown that no integer chain exists that does not lead to 1.

Keywords: Collatz conjecture; 3n+1; inequality relations

MSC: 41A17; 41A29; 11B25; 40A15; 40A25

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1. Introduction

Collatz conjecture, or the $3n+1$ problem, is a simple arithmetic function applied to positive integers. If the integer is odd, triple it and add one. It is called the odd step. If the integer is even, it is divided by two and is denoted as the even step. It is conjectured that every integer will eventually reach the number 1. Much work has been done to prove or disprove this conjecture [1–4].
The problem is easy to understand, and since it has attracted much attention from the general public and experts alike, the literature is endless. Still, the efforts made to tackle the $3n+1$ problem can generally be categorized under the following headings:

- Experimental or computational method: This method uses computational optimizations to verify Collatz conjecture by checking numbers for convergence [5–7]. Numbers as large as 10^{20} have shown no divergence from the conjecture.
- Arguments based on probability: On average, the sequence of numbers tends to shrink in size so that divergence does not occur. On average, each odd number is $3/4$ of the previous odd integer [8].
- Evaluation of stopping times: Many researchers seem to work on the $3n+1$ problem from this approach [9–13]. In essence, it is sought to prove that the Collatz conjecture yields a number smaller than the starting number.
- Mathematical induction: It is perhaps the most common method to "prove" the Collatz conjecture. The literature involving this particular method seems endless [14,15].

The issue is that the Collatz conjecture is a straightforward arithmetic operation, while the methods used are not. The mismatch is created because the problem has attracted the attention of brilliant people in mathematics who are used to dealing with complex issues with equally complex tools. Therefore, an elementary analysis of the problem might be lacking.
This article takes a rudimentary approach to the Collatz conjecture and treats it as a problem of inequality between indices of 2 and 3. The inequality relation will be turned into equality using variables. The values of these variables will be investigated, and it will be shown that the Collatz conjecture does not need complex analysis.

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2. Prerequisite

Consider that n is an integer, and the following Collatz function f is applied.

$$f(n) = \begin{cases} 3n+1, & \text{if n is odd} \\ \frac{n}{2} & \text{if n is even} \end{cases}$$

A sequence is formed by performing this operation repeatedly, taking the result at each step as the input for the next. Collatz conjecture states that, for all n, $f^k(n) = 1$ for some non-negative integer k, where the function is applied to n exactly k times. Let the sequence of steps be:

$$1^{st} \textbf{ odd step, } 1^{st} \textbf{ even step, } \cdots, (k-1)^{th} \textbf{ odd step, } (k-1)^{th} \textbf{ even step}$$

The sequence ends at the odd term; an odd step is applied to obtain an even integer. This even integer is computed in terms of the function $f^k(n)$:

$$f^k(n) = 3\left\{\frac{\left\{3\frac{\left\{3\frac{\left\{3\frac{3n+1}{2^{z_1}}+1\right\}}{2^{z_2}}+1\right\}}{2^{z_3}}+1\right\}}{\vdots}\right\}+1$$

$$f^k(n) = \frac{3^k n + 3^{k-1} + 3^{k-2}2^{z_1} + \cdots + 3^1 2^{z_1+z_2+\cdots+z_{k-2}} + 2^{z_1+z_2+\cdots+z_{k-2}+z_{k-1}}}{2^{z_1+z_2+\cdots+z_{k-2}+z_{k-1}}}$$

$$f^k(n) = \frac{3^k n + C}{2^z} \tag{1}$$

where $z = z_1 + z_2 + \cdots + z_{k-2} + z_{k-1}$ and

$$C = 3^{k-1} + 3^{k-2}2^{z_1} + \cdots + 3^1 2^{z_1+z_2+\cdots+z_{k-2}} + 2^{z_1+z_2+\cdots+z_{k-2}+z_{k-1}} \tag{2}$$

It is noted that $(z, C) > 0$.

3. Methodology

One of the significant results of the Collatz conjecture is that "almost all orbits of the Collatz map attain almost bounded values [9,10]." In simpler words, suppose the Collatz conjecture is valid up to the integer $n-1$. To test if the integer n complies with the Collatz conjecture, it is enough to show that the Collatz function attains a value smaller than the integer n.
Secondly, for the integer n to repeat and form a closed chain cycle, an integer of the form $2^i n$ must appear in the sequence where $i \geq 1$. Let $f^k(n) = 2^i n$ in Equation (1).

$$2^i n = \frac{3^k n + C}{2^z}$$

$$n\left(1 - \frac{3^k}{2^z 2^i}\right) = \frac{C}{2^z 2^i} \tag{3}$$

Equation (3) tells that the maximum n that can repeat depends on the value of $1 - \frac{3^k}{2^z 2^i}$ and the value of $\frac{C}{2^z 2^i}$.
Therefore, a basic strategy towards resolving the $3n+1$ problem can be outlined as follow:

- Establish the conditions that prevent the Collatz sequence from falling below the starting integer and also allow for the starting integer to re-appear.
- Obtain bounds on $\frac{C}{2^z 2^i}$.
- Obtain bounds on $1 - \frac{3^k}{2^z 2^i}$.

4. Conditions for an Unbounded Collatz Orbit & Repeating Integers

For an unbounded Collatz orbit, it is required that every integer in the Collatz sequence is greater than n. Therefore, the integer obtained after $(k-1)^{th}$ even step obeys the following equation

$$
\begin{aligned}
f^{k-1}(n) &> n \\
\frac{3^{k-1}n + C}{2^z} &> n \\
n\left(1 - \frac{3^{k-1}}{2^z}\right) &< \frac{C}{2^z}
\end{aligned}
$$

The above inequality is valid for only some values of (n, C, k, z); therefore, it cannot be used as a general statement. However, if the value in the parenthesis of RHS is allowed to be a negative integer, then the inequality becomes true for all (n, C). Hence, the necessary condition for an unbounded Collatz orbit is

$$\frac{3^{k-1}}{2^z} > 1 \tag{4}$$

Note that the indices of 3 and 2 are equivalent, i.e., the index of 3 is $k-1$, and there are $k-1$ terms in z.

Similarly, the following relation should be true for n to be a positive integer in Equation (3):

$$\frac{3^k}{2^z 2^i} < 1 \tag{5}$$

Equation (4) and (5) are the two necessary conditions for an unbounded Collatz orbit with a repeating integer n.

5. Bounds on the Value of $\frac{C}{2^z 2^i}$

5.1. Upper Bound

Equation (2) is re-written as

$$\frac{C}{2^z 2^i} = \underbrace{\frac{3^{k-1}}{2^z 2^i}}_{\text{First term}} + \underbrace{\frac{3^{k-2} 2^{z_1}}{2^z 2^i}}_{\text{Second term}} + \cdots + \underbrace{\frac{2^z}{2^z 2^i}}_{\text{Last term}}$$

Analysis of each term starts with the Equation (5).

<u>First term</u>

$$
\begin{aligned}
\frac{3^k}{2^z 2^i} &< 1 \\
\frac{3^{k-1}}{2^z 2^i} &< \frac{1}{3}
\end{aligned}
$$

<u>Second term</u>

$$\frac{3^k}{2^z 2^i} < 1$$
$$\frac{3^{k-2}2^{z_1}}{2^z 2^i} < \frac{2^{z_1}}{3^2}$$

The indices of 2 and 3 on the RHS are not equivalent, i.e., the index of 3 is two while the index of 2 is z_1. Let $(k-1) = 1$ in Equation (4) and manipulate as follow

$$\frac{3^{k-1}}{2^z} > 1$$
$$\frac{3^1}{2^{z_1}} > 1$$
$$\frac{2^{z_1}}{3^1} < 1$$
$$\frac{2^{z_1}}{3^2} < \frac{1}{3}$$

Therefore, the value of the second term is

$$\frac{3^{k-2}2^{z_1}}{2^z 2^i} < \frac{1}{3}$$

Similarly, it is concluded that all terms in the expansion of $\frac{C}{2^z 2^i}$ are less than $\frac{1}{3}$.
Last term

$$\frac{3^k}{2^z 2^i} < 1$$
$$\frac{2^z}{2^z 2^i} < \frac{2^z}{3^k}$$
$$\frac{2^z}{2^z 2^i} < \frac{1}{3}$$

The value of the last term must be investigated further. The last term is simplified to $\frac{1}{2^i} < \frac{1}{3}$. A value less than $\frac{1}{3}$ means $2^i \geq 4$ or $i \geq 2$. In other words, starting from a positive integer n, it is impossible to obtain the integer $2n$ directly after an odd step.
Finally, there are k terms in the expansion of $\frac{C}{2^z 2^i}$ and each term is less than $\frac{1}{3}$. Therefore,

$$\frac{C}{2^z 2^i} < \frac{k}{3} \tag{6}$$

5.2. Lower Bound

Analysis of each term starts with the Equation (4).
First term

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$$\frac{3^{k-1}}{2^z} > 1$$
$$\frac{3^{k-1}}{2^z} > \frac{1}{3}$$
$$\frac{3^{k-1}}{2^z 2^i} > \frac{1}{2^i 3}$$

Second term

$$\frac{3^{k-1}}{2^z} > 1$$
$$\frac{3^{k-2} 2^{z_1}}{2^z} > \frac{2^{z_1}}{3}$$
$$\frac{3^{k-2} 2^{z_1}}{2^z} > \frac{1}{3}$$
$$\frac{3^{k-2} 2^{z_1}}{2^z 2^i} > \frac{1}{2^i 3}$$

Similarly, it is concluded that all terms in the expansion of $\frac{C}{2^z 2^i}$ are greater than $\frac{1}{2^i 3}$. There are k terms in the expansion of $\frac{C}{2^z 2^i}$ and each term is greater than $\frac{1}{2^i 3}$. Therefore,

$$\frac{C}{2^z 2^i} > \frac{k}{2^i 3} \tag{7}$$

6. Bounds on the Value of $\left(1 - \frac{3^k}{2^z 2^i}\right)$

6.1. Upper Bound

Let the following inequality be valid for a repeating integer n

$$n\left(1 - \frac{3^k}{2^z 2^i}\right) > 1$$

Therefore, using Equation (3) and (6)

$$\frac{C}{2^z 2^i} > 1$$
$$\frac{k}{3} > 1$$
$$k > 3$$

The above equation does not include the known repeating cycle of $1, 4, 2, 1$ for which $k = 1$. It is therefore concluded that the following inequality is valid for a repeating integer n

$$n\left(1 - \frac{3^k}{2^z 2^i}\right) < 1 \tag{8}$$

6.2. Lower Bound

Take the Equation (5) and manipulate as follow:

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$$
\begin{aligned}
2^z 2^i &> 3^k \\
2^z &> \frac{3^k}{2^i} \\
3^k - 2^z &< 3^k - \frac{3^k}{2^i} \\
\frac{3^k}{2^z} - 1 &< \frac{3^k}{2^z} - \frac{3^k}{2^z 2^i} \\
\frac{3^k}{2^z} &< 1 + \frac{3^k}{2^z} - \frac{3^k}{2^z 2^i} \\
\frac{3^k}{2^z 2^i} &< \frac{1}{2^i} + \frac{3^k}{2^z 2^i} - \frac{3^k}{2^z 2^i 2^i} \\
1 - \frac{3^k}{2^z 2^i} &> 1 - \frac{1}{2^i} - \frac{3^k}{2^z 2^i} + \frac{3^k}{2^z 2^i 2^i} \\
1 - \frac{3^k}{2^z 2^i} &> -\frac{3^k}{2^z 2^i}\left\{1 - \frac{1}{2^i}\right\} + \left\{1 - \frac{1}{2^i}\right\} \\
1 - \frac{3^k}{2^z 2^i} &> -\frac{3^k}{2^z 2^i}\left\{1 - \frac{1}{2^i}\right\} \\
1 &> \frac{3^k}{2^z 2^i 2^i} \\
2^i &> \frac{3^k}{2^z 2^i} \\
1 - \frac{3^k}{2^z 2^i} &> 1 - 2^i
\end{aligned} \tag{9}
$$

7. Resolution to the $3n + 1$ Problem

7.1. Part 1

Substitute Equation (9) and (6) in Equation (3) to obtain the range of integers that repeat in the Collatz sequence.

$$n < \frac{k}{3}\left(1 - 2^i\right)^{-1} \tag{10}$$

For all values of (k, i), the integer n is a negative integer. For the lowest value of $(k, i) = (1, 1)$, the integer n belongs to the range $(-\infty, 0)$. Thus, all negative integers have a possibility of repeating. The next question regarding which negative integers do repeat is left unanswered.

7.2. Part 2

Substitute Equation (8) and (7) in Equation (3) to obtain

$$
\begin{aligned}
1 &> \frac{k}{2^i 3} \\
k &< 2^i 3
\end{aligned} \tag{11}
$$

Notice that k is independent of n in the above inequality.
When an integer of the form $2^i n$ is obtained, only an even step occurs until the integer n is reached, that is, the value of k does not increase or decrease after that. Since k has become stagnant, changing

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the value of i to $i-1$ does not affect the upper bound of k. Thus, k has also become independent of i, and any arbitrary value of i can be inserted in the Equation (11). Let $i = 0$

$$k < 3$$

7.3. Value of n for $k < 3$

The value of n for $k = 1$ is not calculated as it is easy to show $(n, k) = (1, 1)$. For $k = 2$, the value of n is computed as

$$\begin{aligned} 2^i n &= \frac{3^2 n + 3 + 2^{z_1}}{2^{z_1}} \\ n &= \frac{3 + 2^{z_1}}{2^z 2^i - 9} \end{aligned}$$

The value of z_1 is calculated in accordance to Equation (4) and is found $z_1 = 1$. For $2^z 2^i > 9$, the value of $2^z 2^i - 9 > 1$, therefore,

$$\begin{aligned} n &= \frac{3+2}{2^z 2^i - 9} \\ n &= \frac{5}{2^z 2^i - 9} \\ n &< 5 \end{aligned}$$

Therefore, the integers that repeat in the Collatz sequence lie in the range $(-\infty, 5)$ and the limit on odd steps that occur in a closed chain cycle is $k < 3$.

8. Do All Positive Integers Reach 1?

Let there exist a number chain that does not converge to 1. Since the only closed chain in the $3n + 1$ series is $1, 4, 2, 1$, this n-chain is an open chain. The n-chain converges to n from infinity and then diverges to infinity.

The n-chain contains all terms of the form $2^i n$ where $i \geq 0$. Further, terms arising from the arithmetic function $3n + 1$ are also part of this chain. Every even integer x in the n-chain is connected to a precursor even number and a precursor odd number (iff $3m + 1 = x$ is possible for some m). Similarly, every odd integer is connected to a precursor even number. The branches that arise out of the n-chain are infinite. In short, the n-chain contains every integer greater than n up to infinity.

However, there should exist no linkage between the n-chain and the 1-chain. Because, if there were some linkage, all the integers in the n-chain would converge to 1 using the said linkage.

It is absurd, as shown in Figure 1, as this means the 1-chain ends abruptly below n. It implies that there exists no integer $2x$ in the n-chain such that $x < n$. Conversely, there exists no x in the 1-chain such that $3x + 1 > n$.

It is concluded that a n-chain that does not converge to 1 is impossible.

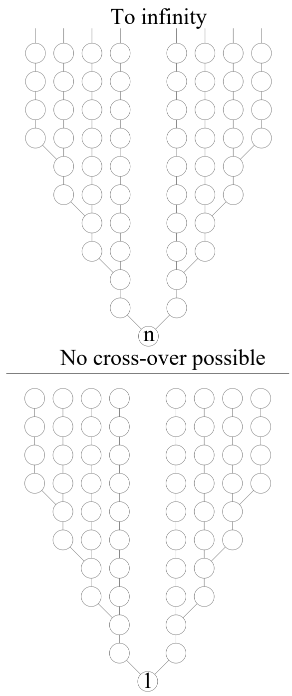

Figure 1. n-chain and 1-chain shown for representation purpose only; may not be factually correct.

9. Conclusions

This article re-writes the Collatz sequence in the form $n\left(1-\frac{3^k}{2^z 2^i}\right) = \frac{C}{2^z 2^i}$, where $\frac{C}{2^z 2^i} = \frac{3^{k-1}}{2^z 2^i} + \frac{3^{k-2} 2^{z_1}}{2^z 2^i} + \cdots + \frac{2^z}{2^z 2^i}$. Conditions for an unbounded Collatz orbit and repeating integers are discovered. It helps in placing bounds on the value of $\left(1-\frac{3^k}{2^z 2^i}\right)$ and $\frac{C}{2^z 2^i}$. Correspondingly, it is found that the integers that repeat in the Collatz sequence lie in the range $(-\infty, 5)$ and the limit on the number of odd steps in a closed chain cycle is $k < 3$.

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