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Resolution of the $3n+1$ Problem Using Inequality Relation between Indices of 2 and 3

Gaurav Goyal

Department of Mechanical Engineering, Indian Institute of Technology, Delhi, India
* Correspondence: Gaurav.Goyal@mech.iitd.ac.in

Abstract: Collatz conjecture states that an integer n reduces to 1 when certain simple operations are applied to it. Mathematically, it is written as $2^z = 3^k n + C$, where $z, k, C \geq 1$. Suppose the integer n violates Collatz conjecture by re-appearing, then the equation modifies to $2^z n = 3^k n + C$. The article takes an elementary approach to this problem by stating that the inequality $2^z > 3^k$ must hold for n to violate the Collatz conjecture. It leads to the inequality $z > 2k$ that helps obtain the relations $3^k/2^z = 3/4 - p$ and $2^z - 3^k = 2^z/4 + q$, where p, q are some variables. The values of p, q are determined by substitution in the $2^z n = 3^k n + C$, and the solution found is $(n, k, z, p, q) = (1, 1, 2, 0, 0)$

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1. Introduction

Collatz conjecture, or the $3n+1$ problem, is a simple arithmetic function applied to positive integers. If the integer is odd, triple it and add one. It is called the odd step. If the integer is even, it is divided by two and is denoted as the even step. It is conjectured that every integer will eventually reach the number 1. Much work has been done to prove or disprove this conjecture [1–4].
The problem is easy to understand, and since it has attracted much attention from the general public and experts alike, the literature is endless. Still, the efforts made to tackle the $3n+1$ problem can generally be categorized under the following headings:

- Experimental or computational method: This method uses computational optimizations to verify Collatz conjecture by checking numbers for convergence [5–7]. Numbers as large as 10^{20} have shown no divergence from the conjecture.
- Arguments based on probability: The heuristic argument suggests that, on average, the sequence of numbers tends to shrink in size so that divergence does not occur. On average, each odd number is 3/4 of the previous odd integer [8].
- Evaluation of stopping times: Many researchers seem to work on the $3n+1$ problem from this approach [9–13]. In essence, it is sought to prove that the Collatz conjecture yields a number smaller than the starting number.
- Mathematical induction: It is perhaps the most common method to "prove" the Collatz conjecture. The literature involving this particular method seems endless [14,15].

The issue is that the Collatz conjecture is a straightforward arithmetic operation, while the methods used are not. The mismatch is created because the problem has attracted the attention of brilliant people in mathematics who are used to dealing with complex issues with equally complex tools. Therefore, an elementary analysis of the problem might be lacking.
This article takes a rudimentary approach to the Collatz conjecture and treats it as a problem of inequality between indices of 2 and 3. The inequality relation will be turned into equality using variables. The values of these variables will be investigated, and it will be shown that the Collatz conjecture does not need complex analysis.

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2. Prerequisite

Consider that n is an odd integer, and the following function f is applied.

$$f(n) = \begin{cases} 3n+1, & \text{if n is odd} \\ \frac{n}{2} & \text{if n is even} \end{cases}$$

A sequence is formed by performing this operation repeatedly, taking the result at each step as the input for the next. Collatz conjecture states that, for all n, $f^k(n) = 1$ for some non-negative integer k, where the function is applied to n exactly k times. Let the sequence of integers obtained be:

$$1^{st} \textbf{ odd, } z_1 \textbf{ evens, } 2^{nd} \textbf{ odd, } z_2 \textbf{ evens, } \cdots \textbf{, } k^{th} \textbf{ odd, } z_a \text{ evens.}$$

The function $f^k(n)$ is computed to be

$$\begin{aligned} f^k(n) &= \frac{3\left\{\frac{\left\{3\frac{\left\{3\frac{\left\{\frac{3n+1}{2^{z_1}}+1\right\}}{2^{z_2}}+1\right\}}{2^{z_3}}+1\right\}}{\vdots}\right\}+1}{2^{z_a}} \\ f^k(n) &= \frac{3^k n + 3^{k-1} + 3^{k-2}2^{z_1} + \cdots + 3^0 2^{z_1+z_2+\cdots+z_{a-1}}}{2^{z_1+z_2+\cdots+z_{a-1}+z_a}} \\ f^k(n) &= \frac{3^k n + C}{2^z} \end{aligned} \tag{1}$$

where

$$z = z_1 + z_2 + \cdots + z_{a-1} + z_a \tag{2}$$

$$C = 3^{k-1} + 3^{k-2}2^{z_1} + \cdots + 3^0 2^{z_1+z_2+\cdots+z_{a-1}} \tag{3}$$

It is noted that z and C are natural numbers, and C is always odd.
Suppose the integer n violates Collatz conjecture by re-appearing in the sequence, i.e., $f^k(n) = n$. Therefore,

$$2^z n = 3^k n + C \tag{4}$$

3. Relation between z and k

In Eq. (4), C is a non-negative integer such that $C \geq 1$. Therefore, the following inequality is established:

$$2^z > 3^k \tag{5}$$

Several limits on z and k can be established based on Eq. (5). A few of them are calculated here.

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Limit 1

$$
\begin{aligned}
2^z &> 3^k \\
2^z &> (2+1)^k \\
2^z &> 2^k \\
z &> k \qquad\qquad\qquad\qquad k \geq 1
\end{aligned}
$$

Limit 2

$$
\begin{aligned}
2^z &> 3^k \\
2^z &> 9(3)^{k-2} \\
2^z &> (2^3+1)(2+1)^{k-2} \\
2^z &> (2^3)(2^{k-2}) \\
z &> k+1 \qquad\qquad\qquad\qquad k > 2
\end{aligned}
$$

Limit 3

$$
\begin{aligned}
2^z &> 3^k \\
4^{\frac{z}{2}} &> 3^k \\
\frac{z}{2} &\geq \begin{cases} k, & k \geq 1 \\ k-1 & k \geq 5 \\ k-2 & k \geq 10 \\ \text{And so on..} & \end{cases}
\end{aligned}
$$

These inequalities are plotted in Figure 1, where the color-matched shaded area represents the feasible region of each inequality. The overlapping feasible region is bound by $z \geq 2k$ and the Y-axis. Therefore, $z \geq 2k$ is the inequality relation between z and k for $k \geq 1$.

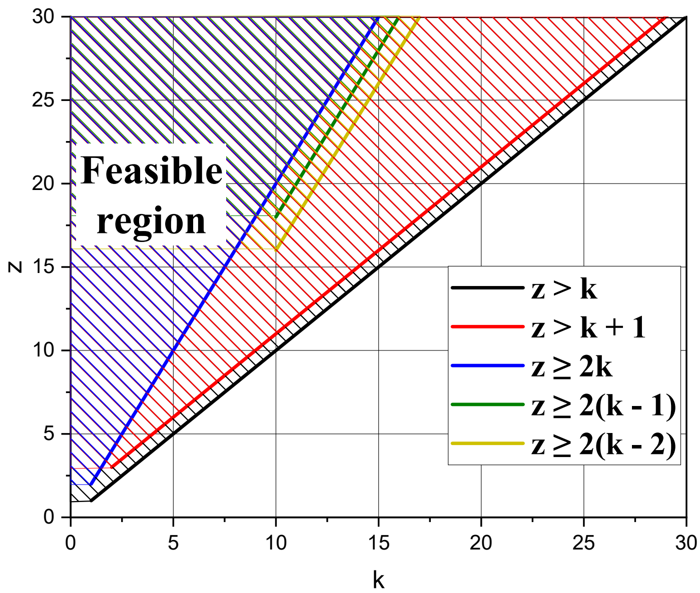

Figure 1. Graphical representation of feasible region obtained by the intersection of various inequalities between z and k.

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4. Calculation of $3^k/2^z$ and $(2^z - 3^k)n$

Equation (4) is modified to the following two forms:

$$n = \frac{3^k}{2^z}n + \frac{C}{2^z} \tag{6}$$
$$C = (2^z - 3^k)n \tag{7}$$

It is intended that the value of C is calculated per Eq. (7) and substituted in Eq. (6) along with the value of $3^k/2^z$ to obtain a generalized equation in n. The value of $3^k/2^z$ is obtained using the inequality $z \geq 2k$.

$$\frac{3^k}{2^z} \leq \frac{3^k}{2^{2k}}$$
$$\frac{3^k}{2^z} \leq \left\{\frac{3}{4}\right\}^k$$

Since $3/4 < 1$, the maximum value of $(3/4)^k$ is obtained for $k = 1$.

$$\frac{3^k}{2^z} \leq \frac{3}{4} \tag{8}$$
$$\frac{3^k}{2^z} = \frac{3}{4} - p \tag{9}$$

where $p \in [0, 3/4)$.
To calculate the value $2^z - 3^k$, take Eq. (8) and modify it as follow

$$1 - \frac{3^k}{2^z} \geq 1 - \frac{3}{4}$$
$$2^z - 3^k \geq \frac{2^z}{4}$$
$$2^z - 3^k = \frac{2^z}{4} + q \tag{10}$$

Where $q \geq 0$.
Combine Eq. (10) and (7) to obtain

$$C = \frac{2^z}{4}n + qn \tag{11}$$

Substitute the value of $3^k/2^z$ from Eq. (9) and C from Eq. (11) in Eq. (6) and simplify

$$n = \left\{\frac{3}{4} - p\right\}n + \frac{\frac{2^z}{4}n + qn}{2^z}$$
$$n = \frac{3n}{4} - np + \frac{n}{4} + \frac{qn}{2^z}$$
$$p - \frac{q}{2^z} = 0 \tag{12}$$

Equation (12) is valid for two cases discussed in the following sections.

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4.1. *Case 1: $p = q = 0$*

For $p = 0$, Equation (9) gives $(k, z) = (1, 2)$.
Similarly, for $q = 0$, Eq. (12) gives $p = 0$, which in turn gives $(k, z) = (1, 2)$ from Eq. (9).

4.2. *Case 2: $p = \frac{q}{2^z}$*

Substitute the value of q in Equation (11) to obtain

$$C = \frac{2^z}{4} n + 2^z np$$
$$C = 2^z n \left\{ \frac{1}{4} + p \right\} \tag{13}$$

Compare Eq. (3) and (13) by rearranging as follow

$$2^z n \left\{ \frac{1}{4} + p \right\} = 3^{k-1} + 3^{k-2} 2^{z_1} + \cdots + 3^0 2^{z_1+z_2+\cdots+z_{a-1}}$$
$$2^z n \left\{ \frac{1}{4} + p \right\} = 2^z \left\{ \frac{3^{k-1}}{2^z} + \frac{3^{k-2} 2^{z_1}}{2^z} + \cdots + \frac{3^0 2^{z_1+z_2+\cdots+z_{a-1}}}{2^z} \right\}$$
$$2^z n \left\{ \frac{1}{4} + p \right\} = 2^z \left\{ \frac{2^{z_1+z_2+\cdots+z_{a-1}}}{2^z} + \frac{3^{k-1}}{2^z} + \frac{3^{k-2} 2^{z_1}}{2^z} + \cdots + \frac{3^1 2^{z_1+\cdots z_{a-2}}}{2^z} \right\}$$
$$2^z n \left\{ \frac{1}{4} + p \right\} = 2^z \left\{ \frac{1}{2^{z_a}} + \textbf{some fraction} \right\} \tag{14}$$

Therefore, $n = 1$ and $z_a = 2$. Value of p and z can not be determined from Eq. (14), but since $n = 1$, the solution is $(n, k, z, p, q) = (1, 1, 2, 0, 0)$.

5. Weak Condition for Divergence

By design, the Collatz sequence has an odd step followed by at least one even step. Two odd steps cannot occur in succession, but several even steps can. And it has been determined in the previous section that the number of even steps is at least twice the number of odd steps. A weak condition for the Collatz sequence to diverge is that precisely one even step follows every odd step. At the end of each odd-even operation, the sequence produces an integer at least 1.5 times the previous one.
The above-mentioned weak condition is met when the odd natural number n is $2^m - 1$. Odd operation is applied in the following manner.

$$\begin{aligned} 3n + 1 &= 3(2^m - 1) + 1 \\ &= (2 + 1)(2^m - 1) + 1 \\ &= 2^{m+1} + 2^m - 2 \end{aligned}$$

The resulting integer is even, and after applying the even step, the final integer is

$$\frac{3n + 1}{2} = 2^m + 2^{m-1} - 1$$

Which is odd, and hence an odd step will follow. Therefore, precisely one even step will follow the odd step, and the sequence diverges. However, the divergence only continues for m odd-even steps. After m odd-even steps are reached, the second last term becomes 2^0, which cancels the -1. The

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resulting integer is even, which leads to two even steps in succession. No other conditions for the divergence have been found yet.

6. Conclusion

Collatz conjecture is a simple algorithm that does not require complex analysis. This article shows that the Collatz conjecture can be treated as an inequality relation between the indices of 2 and 3. Relations such as $3^k/2^z = 3/4 - p$ and $2^z - 3^k = 2^z/4 + q$ are established that are substituted in $2^z n = 3^k n + C$. The value of (n, k, z, p, q) for which Collatz conjecture violates is found to be $(1, 1, 2, 0, 0)$.

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