Article

Group-Frames for Banach Spaces

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Abstract: In the literature, frames generated by unitary representations of groups (known as group-frames) are studied only for Hilbert spaces. We make first study of frames for Banach spaces generated by isometric invertible representations of discrete groups on Banach spaces. These frames are characterized using left regular, right regular, Gram-matrices and group-matrices on classical sequence spaces. A sufficiently large collection of functional-vector pairs using the double commutant of the representation is identified which generate group-frames for Banach spaces. Subsequently, we study Schauder frames generated by time-frequency shift operators on finite dimensional Banach spaces. We derive Moyal formula, fundamental identity of Gabor analysis, Wexler-Raz criterion and Ron-Shen duality in functional form.

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1. Introduction

In their *Memoirs* '**Frames, Bases and Group Representations**' [1], Han and Larson initiated the study of Parseval frames for Hilbert spaces generated by abstract countable groups which sheds light on the structure of unitary representations of the given group. Recall that a countable group with discrete topology is known as a discrete group. The definition of Parseval frame reads as follows.

Definition 1.1. *[1,2] Let G be a countable set. A sequence $\{\tau_g\}_{g \in G}$ in a Hilbert space $\mathcal{H}$ is said to be a **Parseval frame** for $\mathcal{H}$ if*

$$\|h\|^2 = \sum_{g \in G} |\langle h, \tau_g \rangle|^2, \quad \forall h \in \mathcal{H}.$$

Definition 1.1 is equivalent to the following equation:

$$h = \sum_{g \in G} \langle h, \tau_g \rangle \tau_g, \quad \forall h \in \mathcal{H}. \tag{1}$$

One of the main objects in frame theory is to generate Parseval frames from a given vector. In this regard, Han and Larson introduced the notion of frames generated by groups as follows [1].

Definition 1.2. *[1] Let G be a discrete group and $\{\tau_g\}_{g \in G}$ be a Parseval frame for a Hilbert space $\mathcal{H}$. The frame $\{\tau_g\}_{g \in G}$ is said to be a **group-frame** if there exists a unitary representation π of G on $\mathcal{H}$ and a vector $\tau \in \mathcal{H}$ such that*

$$\tau_g = \pi_g \tau, \quad \forall g \in G.$$

*In this case, the representation π is called as **frame representation** and τ is called as a **frame vector**.*

After the introduction of group-frames, several important results appeared which can be classified according to types of groups as follows.

(1) Finite groups [3–12].
(2) Discrete groups [13–16].
(3) locally compact groups [17,18].
(4) compact groups [19].
(5) Lie groups [20,21].
(6) ICC groups [22].

Similar to group-frames for Hilbert spaces, we can ask whether we can generate frames for Banach spaces using groups. In this paper, we are interested in frames for Banach spaces which are generated by discrete groups. Historically, it was Grochenig [23] who introduced frames for Banach spaces known as Banach frames which do not demand the reconstruction of element using series similar to the one given in Equation (1). In 1999, Casazza, Han, and Larson [24] introduced the notion of unconditional Schauder frames (also known as framings) mainly taking the expansion property given in Equation (1). Let $\mathcal{X}$ be a separable Banach space and $\mathcal{X}^*$ be its dual.

Definition 1.3. *[24] Let G be a discrete group. Let $\{\tau_g\}_{g \in G}$ be a sequence in $\mathcal{X}$ and $\{f_g\}_{g \in G}$ be a sequence in $\mathcal{X}^*$. The pair $(\{f_g\}_{g \in G}, \{\tau_g\}_{g \in G})$ is said to be an **unconditional Schauder frame** (we write USF) for $\mathcal{X}$ if*

$$x = \sum_{g \in G} f_g(x)\tau_g, \quad \forall x \in \mathcal{X}, \tag{2}$$

where the series in (2) converges unconditionally.

In this paper, we make a first attempt to understand USF which are generated by representations of discrete groups in Banach spaces. We organize the paper as follows. In Section 2 we start by mentioning the notion of group representation in Banach spaces. Then we introduce the notion of group-frames (Definition 2.2) and show that the notion reduces to Definition 1.2 for Hilbert spaces with unitary representations. Then we introduce a subclass of group-frames for Banach spaces called a group-p-unconditional Schauder frame (group-p-USF) which factor through classical sequence spaces (Definition 2.3).

Theorem 2.8 shows that representations giving group-p-USFs can be obtained by restricting the standard left regular representation on sequence spaces. A connection between group-p-USFs and group-matrices is derived in Theorem 2.10. We show in Theorem 2.11 group-p-USFs can be characterized using algebraic equations. Theorem 2.13 gives a characterization of group-p-USFs using standard left regular representation on sequence spaces. Proposition 2.14 and Theorem 2.17 show that once a pair of functional and a vector which generate group-p-USF is non-empty then the set of all such pairs is very large.

In Section 3 we do finite dimensional Gabor analysis in function form without using the inner product. Even though it is true that finite dimensional Banach spaces can be made a Hilbert space, rather putting an inner product, we believe that it is best to study just by using functional and vectors. First important result of this section is Banach space version of Moyal formula, derived in Theorem 3.3. Using this result we get that we have a large supply of Gabor-Schauder frames for finite dimensional Banach spaces (Corollary 3.4). Then we study Gabor-Schauder frames generated by subgroups of the group of all time-frequency shifts.

Theorem 3.8 derives fundamental identity of Gabor analysis for Banach spaces. Wexler-Raz criterion for Gabor-Schauder frames is derived in Theorem 3.9. A partial Ron-Shen duality for Gabor-Schauder frames is derived in Theorem 3.10.

2. Unconditional Schauder frames generated by groups

Throughout G denotes a group (need not be abelian). We denote the identity element of G by e. Given a Banach space $\mathcal{X}$, $\mathcal{II}(\mathcal{X})$ be the set of all invertible linear isometries on $\mathcal{X}$. The identity

operator on $\mathcal{X}$ is denoted by $I_{\mathcal{X}}$. We use the following definition of representation in Banach spaces throughout.

Definition 2.1. *Let $\mathcal{X}$ be a Banach space and G be a topological group. A map $\pi : G \to \mathcal{II}(\mathcal{X})$ is said to be an **invertible isometric representation** of G if the following condition hold:*

(i) $\pi : G \to \mathcal{II}(\mathcal{X})$ is a group homomorphism, i.e.,

$$\pi_g \pi_h = \pi_{gh}, \quad \forall g, h \in G.$$

(ii) π is continuous in the following sense. For each fixed $x \in \mathcal{X}$, the map

$$G \ni g \mapsto \pi_g x \in \mathcal{X}$$

is continuous.

In this section, we consider only discrete groups. Hence the condition (ii) is always satisfied. Given two invertible isometric representations $\pi : G \to \mathcal{II}(\mathcal{X})$ and $\Delta : G \to \mathcal{II}(\mathcal{Y})$, we say that they are equivalent if there is an invertible operator $T : \mathcal{X} \to \mathcal{Y}$ which intertwines π and Δ. If the intertwining operator is an invertible isometry, then we say that representations are invertible isometric equivalent. For any discrete group G and $p \in [1, \infty)$, we note that G always admit two invertible isometric representations on $\ell^p(G)$ defined as follows. Let $\{\delta_g\}_{g \in G}$ be the standard Schauder basis for $\ell^p(G)$. We denote the coordinate functionals associated to $\{\delta_g\}_{g \in G}$ by $\{\zeta_g\}_{g \in G}$.

(i) p-**left regular representation** $\lambda : G \to \mathcal{II}(\ell^p(G))$ defined on $\{\delta_g\}_{g \in G}$ and extended linearly as

$$\lambda_g \delta_h := \delta_{gh}, \quad \forall g, h \in G.$$

(ii) p-**right regular representation** $\rho : G \to \mathcal{II}(\ell^p(G))$ defined on $\{\delta_g\}_{g \in G}$ and extended linearly as

$$\rho_g \delta_h := \delta_{hg^{-1}}, \quad \forall g, h \in G.$$

With these preliminaries, we now set the following definition.

Definition 2.2. *Let G be a discrete group and $(\{f_g\}_{g \in G}, \{\tau_g\}_{g \in G})$ be an USF for a Banach space $\mathcal{X}$. The USF $(\{f_g\}_{g \in G}, \{\tau_g\}_{g \in G})$ is said to be a **group-USF** if there exist an invertible isometric representation π of G on $\mathcal{X}$, a vector $\tau \in \mathcal{X}$ and a functional $f \in \mathcal{X}^*$ such that*

$$f_g = f \pi_{g^{-1}}, \quad \tau_g = \pi_g \tau, \quad \forall g \in G. \tag{3}$$

We first show that Definition 2.2 truly generalizes Definition 1.2. Let $\{\tau_g\}_{g \in G} = \{\pi_g \tau\}_{g \in G}$ be a group-frame for a Hilbert space $\mathcal{H}$. Let f be the functional on $\mathcal{H}$ defined by τ, i.e., $f(h) = \langle h, \tau \rangle$, for all $h \in \mathcal{H}$. Then

$$f_g(h) = f(\pi_{g^{-1}} h) = \langle \pi_{g^{-1}} h, \tau \rangle = \langle h, \pi_g \tau \rangle = \langle h, \tau_g \rangle, \quad \forall h \in \mathcal{H}.$$

Therefore f_g is determined by τ_g for all $g \in G$.

It seems that we can not give a satisfactory theory for group-USF like that of group-frames for Hilbert spaces. Therefore we study a class of group-USF's defined as follows. Our definition is motivated from the notion of p-approximate Schauder frames defined in [25].

Definition 2.3. *Let $p \in [1, \infty)$. A group-USF $(\{f_g\}_{g \in G}, \{\tau_g\}_{g \in G})$ for a Banach space $\mathcal{X}$ is said to be a **group-p-USF** if the following conditions hold.*

(i) *The map (analysis operator)* $\theta_f : \mathcal{X} \ni x \mapsto \{f_g(x)\}_{g \in G} \in \ell^p(G)$ *is a well-defined isometry.*

(ii) *The map (synthesis operator)* $\theta_\tau : \ell^p(G) \ni \{a_g\}_{g \in G} \mapsto \sum_{g \in G} a_g \tau_g \in \mathcal{X}$ *is a well-defined bounded linear operator.*

*In this case, the representation π is called as **p-USF representation** and the pair (f, τ) is called as a **p-USF functional-vector**.*

We also need following generalization of Definition 2.3.

Definition 2.4. *Let $p \in [1, \infty)$ and G be a discrete group. A USF $(\{f_g\}_{g \in G}, \{\tau_g\}_{g \in G})$ for a Banach space $\mathcal{X}$ is said to be a **p-USF** if the following conditions hold.*

(i) *The map (analysis operator) $\theta_f : \mathcal{X} \ni x \mapsto \{f_g(x)\}_{g \in G} \in \ell^p(G)$ is a well-defined isometry.*

(ii) *The map (synthesis operator) $\theta_\tau : \ell^p(G) \ni \{a_g\}_{g \in G} \mapsto \sum_{g \in G} a_g \tau_g \in \mathcal{X}$ is a well-defined bounded linear operator.*

We begin by recording a characterization result of p-USFs which is motivated from the characterization of Hilbert space frames by Holub [26].

Theorem 2.5. *A pair $(\{f_g\}_{g \in G}, \{\tau_g\}_{g \in G})$ is a p-USF for $\mathcal{X}$, if and only if*

$$f_g = \zeta_g U, \quad \tau_g = V \delta_g, \quad \forall g \in G,$$

where $U : \mathcal{X} \to \ell^p(G)$, $V : \ell^p(G) \to \mathcal{X}$ are bounded linear operators such that $VU = I_\mathcal{X}$ and U is an isometry.

Proof. ($\Leftarrow$) Clearly θ_f and θ_τ are bounded linear operators. Now let $x \in \mathcal{X}$. Then

$$\sum_{g \in G} f_g(x) \tau_g = \sum_{g \in G} \zeta_g(Ux) V \delta_g = V \left(\sum_{g \in G} \zeta_g(Ux) \delta_g \right) = VUx = x. \tag{4}$$

Note that, since U is isometry,

$$\|\theta_f x\| = \left\| \sum_{g \in G} f_g(x) \delta_g \right\| = \left\| \sum_{g \in G} \zeta_g(Ux) \delta_g \right\| = \|Ux\| = \|x\|, \quad \forall x \in \mathcal{X}.$$

($\Rightarrow$) Define $U := \theta_f$, $V := \theta_\tau$. Then $\zeta_g(Ux) = \zeta_g(\theta_f x) = \zeta_g(\{f_k(x)\}_{k \in G}) = f_g(x)$, $\forall x \in \mathcal{X}$, $V \delta_g = \theta_\tau \delta_g = \tau_g$, $\forall g \in G$ and $VU = \theta_\tau \theta_f = I_\mathcal{X}$. Since θ_f is an isometry, we also have U is an isometry. $\square$

We record the following important result from [25].

Theorem 2.6. *[25] Let $(\{f_g\}_{g \in G}, \{\tau_g\}_{g \in G})$ be a p-USF for $\mathcal{X}$. Then*

(i) $I_\mathcal{X} = \theta_\tau \theta_f$.

(ii) $G_{f,\tau} := \theta_f \theta_\tau : \ell^p(G) \to \ell^p(G)$ *is a projection onto $\theta_f(\mathcal{X})$.*

In [1], Han and Larson showed that upto unitary operator, frame representation is a piece of left regular representation. With the help of following lemma we generalize their result for Banach spaces.

Lemma 2.7. *If $(\{f_g\}_{g \in G}, \{\tau_g\}_{g \in G})$ is a group-p-USF for $\mathcal{X}$, then range of its analysis operator is invariant under p-left-regular representation of G, i.e., $\lambda_g(\theta_f(\mathcal{X})) \subseteq \theta_f(\mathcal{X})$ for all $g \in G$.*

Proof. For any $g \in G$,

$$
\begin{aligned}
\lambda_g \theta_f x &= \lambda_g \left(\sum_{h \in G} f_h(x) \delta_h \right) = \sum_{h \in G} f_h(x) \delta_{gh} = \sum_{u \in G} f_{g^{-1}u}(x) \delta_u \\
&= \sum_{u \in G} f(\pi_{u^{-1}g} x) \delta_u = \sum_{u \in G} (f\pi_{u^{-1}})(\pi_g x) \delta_u = \theta_f(\pi_g x) \in \theta_f(\mathcal{X}), \quad \forall x \in \mathcal{X}.
\end{aligned}
$$

$\square$

Theorem 2.8. *Every group-p-USF representation π of G is invertibly isometrically equivalent to a subrepresentation of p-left-regular representation λ of G.*

Proof. Let $(\{f_g\}_{g \in G}, \{\tau_g\}_{g \in G})$ be a group-p-USF for $\mathcal{X}$. Lemma 2.7 says that $\lambda_g(\theta_f(\mathcal{X})) \subseteq \theta_f(\mathcal{X})$ for all $g \in G$. Therefore the map

$$
\Delta_g := \lambda_g|_{\theta_f(\mathcal{X})} : \theta_f(\mathcal{X}) \to \theta_f(\mathcal{X}), \quad \forall g \in G
$$

is a well-defined invertible isometric representation of G. We show that Δ and π are isometrically invertibly equivalent. Note that $\theta_f : \mathcal{X} \to \theta_f(\mathcal{X})$ is an invertible isometry. We are done if we show that $\theta_f : \mathcal{X} \to \theta_f(\mathcal{X})$ intertwines Δ and π. This follows by doing a similar calculation as in the proof of Lemma 2.7. $\square$

Vale and Waldron discovered that for finite groups, groups-frames can be characterized using group-matrices [8]. We show that their result remains valid for Banach spaces. First we recall the definition of group-matrix.

Definition 2.9. *[27] Let G be a discrete group. A matrix $A := [a_{g,h}]_{g,h \in G}$ over $\mathbb{C}$ is said to be a **group-matrix** if there exists a function $v : G \to \mathbb{C}$ such that*

$$
a_{g,h} = v(g^{-1}h), \quad \forall g, h \in G. \tag{5}
$$

Let $(\{f_g\}_{g \in G}, \{\tau_g\}_{g \in G})$ be a group-p-USF for $\mathcal{X}$. Then we note that

$$
f_g(\tau_h) = (f\pi_{g^{-1}})(\pi_h \tau) = f(\pi_{g^{-1}h} \tau), \quad \forall g, h \in G.
$$

Let $G_{f,\tau} := \theta_f \theta_\tau$ be the Gramian of $(\{f_g\}_{g \in G}, \{\tau_g\}_{g \in G})$ whose matrix w.r.t. the standard Schauder basis $\{\delta_g\}_{g \in G}$ for $\ell^p(G)$ is given by

$$
[f_g(\tau_h)]_{g,h \in G}.
$$

Now by defining $v : G \ni g \mapsto v(g) := f(\pi_g \tau) \in \mathbb{C}$, we see from Equation (5) that $G_{f,\tau}$ is a group-matrix. Next theorem shows that converse of this also holds.

Theorem 2.10. *Let G be a discrete group. Then a p-USF $(\{f_g\}_{g \in G}, \{\tau_g\}_{g \in G})$ for $\mathcal{X}$ is a group-p-USF for $\mathcal{X}$ if and only if its Gramian $G_{f,\tau}$ is a group-matrix.*

Proof. As we already derived only if part, we prove if part. Assume that the Gramian $G_{f,\tau}$ is a group-matrix. Then there exists a function $v : G \to \mathbb{C}$ such that

$$
v(g^{-1}h) = f_g(\tau_h), \quad \forall g, h \in G.
$$

Given $g \in G$, define

$$\pi_g : \mathcal{X} \ni x \mapsto \pi_g x := \sum_{h \in G} f_h(x) \tau_{gh} \in \mathcal{X}$$

and

$$\Delta_g : \mathcal{X} \ni x \mapsto \Delta_g x := \sum_{h \in G} f_{gh}(x) \tau_h \in \mathcal{X}.$$

Note that

$$f_{gu}(\tau_{gh}) = \nu((gu)^{-1} gh) = \nu(u^{-1}h) = f_h(\tau_u), \quad \forall u, g, h \in G. \tag{6}$$

Using Equation (6), we get

$$\Delta_g \pi_g x = \Delta_g \left(\sum_{h \in G} f_h(x) \tau_{gh} \right) = \sum_{h \in G} f_h(x) \Delta_g \tau_{gh} = \sum_{h \in G} f_h(x) \sum_{u \in G} f_{gu}(\tau_{gh}) \tau_u$$
$$= \sum_{h \in G} f_h(x) \sum_{u \in G} f_u(\tau_h) \tau_u = \sum_{h \in G} f_h(x) \tau_h = x, \quad \forall x \in \mathcal{X}$$

and

$$\pi_g \Delta_g x = \pi_g \left(\sum_{h \in G} f_{gh}(x) \tau_h \right) = \sum_{h \in G} f_{gh}(x) \pi_g \tau_h = \sum_{h \in G} f_{gh}(x) \sum_{u \in G} f_u(\tau_h) \tau_{gu}$$
$$= \sum_{h \in G} f_{gh}(x) \sum_{u \in G} f_{gu}(\tau_{gh}) \tau_{gu} = \sum_{h \in G} f_{gh}(x) \tau_{gh} = x, \quad \forall x \in \mathcal{X}.$$

Therefore Δ_g is the inverse of π_g. We next show π_g is isometry. For $x \in \mathcal{X}$, using Equation (6) and θ_f is an isometry,

$$\|\pi_g x\|^p = \|\theta_f(\pi_g x)\|^p = \sum_{h \in G} |f_h(\pi_g x)|^p = \sum_{h \in G} \left| f_h \left(\sum_{u \in G} f_u(x) \tau_{gu} \right) \right|^p$$
$$= \sum_{h \in G} \left| \sum_{u \in G} f_u(x) f_h(\tau_{gu}) \right|^p = \sum_{h \in G} \left| \sum_{u \in G} f_u(x) f_{gg^{-1}h}(\tau_{gu}) \right|^p$$
$$= \sum_{h \in G} \left| \sum_{u \in G} f_u(x) f_{g^{-1}h}(\tau_u) \right|^p = \sum_{h \in G} \left| f_{g^{-1}h} \left(\sum_{u \in G} f_u(x) \tau_u \right) \right|^p$$
$$= \sum_{h \in G} |f_{g^{-1}h}(x)|^p = \|x\|^p.$$

To show Equation (3) again using Equation (6),

$$\pi_g \tau_e = \sum_{h \in G} f_h(\tau_e) \tau_{gh} = \sum_{u \in G} f_{g^{-1}u}(\tau_e) \tau_u = \sum_{u \in G} f_{g^{-1}u}(\tau_{g^{-1}g}) \tau_u$$
$$= \sum_{u \in G} f_u(\tau_g) \tau_u = \tau_g, \quad \forall g \in G$$

and

$$(f_e \pi_{g^{-1}})(x) = f_e \left(\sum_{h \in G} f_h(x) \tau_{g^{-1}h} \right) = \sum_{h \in G} f_h(x) f_e(\tau_{g^{-1}h}) = \sum_{h \in G} f_h(x) f_{g^{-1}g}(\tau_{g^{-1}h})$$

$$= \sum_{h \in G} f_h(x) f_g(\tau_h) = f_g \left(\sum_{h \in G} f_h(x) \tau_h \right) = f_g(x), \quad \forall g \in G, \forall x \in \mathcal{X}.$$

We are left with showing that π is a homomorphism. Let $g, h \in G$ and $x \in \mathcal{X}$. Then using Equation (6),

$$\pi_g \pi_h x = \sum_{u \in G} f_u(\pi_h x) \tau_{gu} = \sum_{u \in G} f_u \left(\sum_{v \in G} f_v(x) \tau_{hv} \right) \tau_{gu}$$

$$= \sum_{u \in G} \sum_{v \in G} f_v(x) f_u(\tau_{hv}) \tau_{gu} = \sum_{u \in G} \sum_{v \in G} f_v(x) f_{hh^{-1}u}(\tau_{hv}) \tau_{gu}$$

$$= \sum_{u \in G} \sum_{v \in G} f_v(x) f_{h^{-1}u}(\tau_v) \tau_{gu} = \sum_{u \in G} f_{h^{-1}u} \left(\sum_{v \in G} f_v(x) \tau_v \right) \tau_{gu}$$

$$= \sum_{u \in G} f_{h^{-1}u}(x) \tau_{gu} = \sum_{v \in G} f_v(x) \tau_{(gh)v} = \pi_{gh} x.$$

$\square$

Kaftal, Larson and Zhang showed that group-frames can be characterized by using an algebraic equation and involving inner proved (actually they proved it in the setup of operator-valued frames) [28]. In the following result we generalize the result of Kaftal, Larson and Zhang to Banach spaces.

Theorem 2.11. *Let G be a discrete group and $(\{f_g\}_{g \in G}, \{\tau_g\}_{g \in G})$ be a p-USF for $\mathcal{X}$. Then there is an invertible isometric representation π of G on $\mathcal{X}$ for which*

$$\tau_g = \pi_g \tau_e, \quad f_g = f_e \pi_{g^{-1}}, \quad \forall g \in G \tag{7}$$

if and only if

$$f_{ug}(\tau_{uh}) = f_g(\tau_h), \quad \forall u, g, h \in G. \tag{8}$$

Moreover, the representation can be defined as

$$\pi_g := \theta_\tau \lambda_g \theta_f, \quad \forall g \in G. \tag{9}$$

Proof. $(\Rightarrow)$ Let $u, g, h \in G$. Then we have

$$f_{ug}(\tau_{uh}) = (f_e \pi_{(ug)^{-1}})(\pi_{uh} \tau_e) = (f_e \pi_{g^{-1}u^{-1}})(\pi_{uh} \tau_e) = (f_e \pi_{g^{-1}})(\pi_h \tau_e) = f_g(\tau_h).$$

$(\Leftarrow)$ Let π_g be defined by Equation (9). We are required to show that π is an invertible isometric representation and satisfies Equation (7). We first show that it satisfies Equation (7). To do so, we claim the identity

$$\lambda_g \theta_f \theta_\tau = \theta_f \theta_\tau \lambda_g, \quad \forall g \in G. \tag{10}$$

Using Equation (8) we verify Equation (10). Note that it suffices to verify Equation (10) at the standard basis vectors δ_h, $h \in G$. Consider

$$\theta_f \theta_\tau \lambda_g \delta_h = \theta_f \theta_\tau \delta_{gh} = \theta_f \tau_{gh} = \sum_{u \in G} f_u(\tau_{gh}) \delta_u = \sum_{u \in G} f_{gu}(\tau_{gh}) \delta_{gu} = \sum_{u \in G} f_u(\tau_h) \delta_{gu}$$

$$= \sum_{u \in G} f_u(\tau_{gh}) \lambda_g \delta_u = \lambda_g \left(\sum_{u \in G} f_u(\tau_h) \delta_u \right) = \lambda_g \theta_f \tau_h = \lambda_g \theta_f \theta_\tau \delta_h.$$

Now using Equation (10)

$$\pi_g \tau_e = \theta_\tau \lambda_g \theta_f \tau_e = \theta_\tau \lambda_g \theta_f \theta_\tau \delta_e = \theta_\tau \theta_f \theta_\tau \lambda_g \delta_e = I_\mathcal{X} \theta_\tau \lambda_g \delta_e = \theta_\tau \delta_g = \tau_g, \quad \forall g \in G$$

and

$$f_e(\pi_{g^{-1}} x) = f_e(\theta_\tau \lambda_{g^{-1}} \theta_f x) = f_e \left(\sum_{u \in G} f_u(x) \theta_\tau \lambda_{g^{-1}} \delta_u \right) = f_e \left(\sum_{u \in G} f_u(x) \tau_{g^{-1}u} \right)$$

$$= \sum_{u \in G} f_u(x) f_e(\tau_{g^{-1}u}) = \sum_{u \in G} f_u(x) f_{g^{-1}g}(\tau_{g^{-1}u}) = \sum_{u \in G} f_u(x) f_g(\tau_u)$$

$$= f_g \left(\sum_{u \in G} f_u(x) \tau_u \right) = f_g(x), \quad \forall x \in \mathcal{X}, \forall g \in G.$$

Now we show that π is an invertible isometric representation. First we need to show that it is bijective. We note that, for $g \in G$, the operator $\Delta_g := \theta_\tau \lambda_{g^{-1}} \theta_f$ is the inverse of π_g. In fact, using Equation (10),

$$\pi_g \Delta_g = \theta_\tau \lambda_g \theta_f \theta_\tau \lambda_{g^{-1}} \theta_f = \theta_\tau \theta_f \theta_\tau \lambda_g \lambda_{g^{-1}} \theta_f = I_\mathcal{X},$$

$$\Delta_g \pi_g = \theta_\tau \lambda_{g^{-1}} \theta_f \theta_\tau \lambda_g \theta_f = \theta_\tau \lambda_{g^{-1}} \lambda_g \theta_f \theta_\tau \theta_f = I_\mathcal{X}.$$

To show π_g is an isometry, given $x \in \mathcal{X}$,

$$\|\pi_g x\|^p = \sum_{h \in G} |f_h(\pi_g x)|^p = \sum_{h \in G} |f_h(\theta_\tau \lambda_g \theta_f x)|^p = \sum_{h \in G} \left| (f_h \theta_\tau \lambda_g) \left(\sum_{u \in G} f_u(x) \delta_u \right) \right|^p$$

$$= \sum_{h \in G} \left| f_h \left(\sum_{u \in G} f_u(x) \theta_\tau \lambda_g \delta_u \right) \right|^p = \sum_{h \in G} \left| f_h \left(\sum_{u \in G} f_u(x) \tau_{gu} \right) \right|^p$$

$$= \sum_{h \in G} \left| \sum_{u \in G} f_u(x) f_h(\tau_{gu}) \right|^p = \sum_{h \in G} \left| \sum_{u \in G} f_u(x) f_{gg^{-1}h}(\tau_{gu}) \right|^p$$

$$= \sum_{h \in G} \left| \sum_{u \in G} f_u(x) f_{g^{-1}h}(\tau_u) \right|^p = \sum_{h \in G} \left| f_{g^{-1}h} \left(\sum_{u \in G} f_u(x) \tau_u \right) \right|^p$$

$$= \sum_{h \in G} |f_{g^{-1}h}(x)|^p = \|x\|^p.$$

Finally, we show that π is representation:

$$\pi_g \pi_h = \theta_\tau \lambda_g \theta_f \theta_\tau \lambda_h \theta_f = \theta_\tau \theta_f \theta_\tau \lambda_g \lambda_h \theta_f = I_\mathcal{X} \theta_\tau \lambda_{gh} \theta_f = \pi_{gh}, \quad \forall g, h \in G.$$

□

A careful observation on proof of "if" part of Theorem 2.11 gives the following result.

Theorem 2.12. *Let G be a discrete group and $(\{f_g\}_{g \in G}, \{\tau_g\}_{g \in G})$ be a group-p-USF for $\mathcal{X}$. Then $\lambda_g \theta_f \theta_\tau = \theta_f \theta_\tau \lambda_g$, $\forall g \in G$.*

Next we try to relate frame representation with p-right regular representation. This result was first derived by Mendez, Bodmann, Baker, Bullock and McLaney in the context of binary frame [29].

Theorem 2.13. *Let G be a discrete group and $(\{f_g\}_{g \in G}, \{\tau_g\}_{g \in G})$ be a p-USF for $\mathcal{X}$. Then there is an invertible isometric representation π of G on $\mathcal{X}$ for which*

$$\tau_g = \pi_g \tau_e, \quad f_g = f_e \pi_{g^{-1}}, \quad \forall g \in G$$

if and only if

$$\theta_f \theta_\tau \{a_h\}_{h \in G} = \sum_{g \in G} \eta(g) \rho_g \{a_h\}_{h \in G}, \quad \forall \{a_h\}_{h \in G} \in \ell^p(G),$$

where

$$\eta : G \ni g \mapsto \eta(g) := f_e(\pi_g \tau_e) \in \mathbb{C}. \tag{11}$$

Moreover, the invertible isometric representation can be defined as

$$\pi_g := \theta_\tau \lambda_g \theta_f, \quad \forall g \in G. \tag{12}$$

Proof. ($\Rightarrow$) Let η be the function defined in Equation (11). Now for each δ_h,

$$\theta_f \theta_\tau \delta_h = \theta_f \tau_h = \sum_{u \in G} f_u(\tau_h) \delta_u = \sum_{u \in G} (f \pi_{u^{-1}})(\pi_h \tau) \delta_u$$

$$= \sum_{u \in G} f(\pi_{u^{-1}h} \tau) \delta_u = \sum_{g \in G} f(\pi_g \tau) \delta_{hg^{-1}} = \sum_{g \in G} f(\pi_g \tau) \rho_g \delta_h$$

$$= \left(\sum_{g \in G} f(\pi_g \tau) \rho_g \right) \delta_h = \left(\sum_{g \in G} \eta(g) \rho_g \right) \delta_h.$$

($\Leftarrow$) We note that p-left and p-right regular representations commute. In fact, for any $g, h \in G$ and for each standard basis vector δ_u, we have

$$\lambda_g \rho_h \delta_u = \lambda_g \delta_{uh^{-1}} = \delta_{guh^{-1}} = \rho_h \delta_{gu} = \rho_h \lambda_g \delta_u.$$

We then get

$$\lambda_g \theta_f \theta_\tau \delta_h = \lambda_g \left(\sum_{u \in G} \eta(u) \rho_u \delta_h \right) = \sum_{u \in G} \eta(u) \lambda_g \rho_u \delta_h$$

$$= \left(\sum_{u \in G} \eta(u) \rho_u \right) \lambda_g \delta_h = \theta_f \theta_\tau \lambda_g \delta_h, \quad \forall g \in G.$$

Now we claim the following:

$$f_{ug}(\tau_{uh}) = f_g(\tau_h), \quad \forall u, g, h \in G.$$

Consider

$$f_{ug}(\tau_{uh}) = f_{ug}(\theta_\tau \delta_{uh}) = \zeta_{ug}(\theta_f \theta_\tau \delta_{uh}) = \zeta_{ug} \left(\sum_{v \in G} \eta(v) \rho_v \delta_{uh} \right)$$

$$= \sum_{v \in G} \eta(v) \zeta_{ug} \rho_v \delta_{uh} = \sum_{v \in G} \eta(v) \zeta_{ug} \delta_{uhv^{-1}} = \eta(g^{-1}h) = f_g(\tau_h), \quad \forall u, g, h \in G.$$

Hence claim holds. For each $g \in G$, we now define π_g as in Equation (12). Now by doing a similar calculation as in the converse part of proof of Theorem 2.11 we get that $(\{f_g\}_{g \in G}, \{\tau_g\}_{g \in G})$ is a group-p-USF for $\mathcal{X}$. $\square$

After giving several characterizations for frame representations, we next seek to determine the collection of functionals and vectors which generate group-frames. In the case of Hilbert spaces, Han and Larson completely characterized vectors which generate group-frames using double commutant of image of representation [1]. Even though we are unable to achieve this, we show that certain large sets generate group-frames for Banach spaces. We first set some notations. Let G be a discrete group and $\pi : G \to \mathcal{II}(\mathcal{X})$ be a invertible isometric representation. Assume that there is a vector $\tau \in \mathcal{X}$ and a functional $f \in \mathcal{X}^*$ such that $(\{f_g\}_{g \in G}, \{\tau_g\}_{g \in G})$ is a group-p-USF for $\mathcal{X}$. Define the set of all **group-p-USF vectors** as

$$\mathcal{F}_G(\pi, \mathcal{X}) := \{(f_1, \tau_1) \in \mathcal{X}^* \times \mathcal{X} : (\{f_{1_g}\}_{g \in G}, \{\tau_{1_g}\}_{g \in G}) \text{ is a group-p-USF for } \mathcal{X}\}.$$

By assumption $\mathcal{F}_G(\pi, \mathcal{X}) \neq \emptyset$. We naturally ask what is the structure of $\mathcal{F}_G(\pi, \mathcal{X})$? In the following proposition we show that this set is quite large. We use the following notation in sequel. Given a subset $\mathcal{A}$ of linear operators on $\mathcal{X}$, by $\mathbb{I}^*(\mathcal{A})$ we mean the set of invertible isometric operators $U : \mathcal{X} \to \mathcal{X}$ such that $U \in \mathcal{A}$. Given $f \in \mathcal{X}^*$ and $\tau \in \mathcal{X}$, we define

$$f[\mathbb{I}^*(\mathcal{A})^{-1}, \mathbb{I}^*(\mathcal{A})]\tau := \{(fU^{-1}, U\tau) : U \in \mathbb{I}^*(\mathcal{A})\}.$$

Proposition 2.14. *If an invertible isometric representation $\pi : G \to \mathcal{II}(\mathcal{X})$ admits a functional-frame vector (f, τ), then*

$$f[\mathbb{I}^*(\pi(G)')^{-1}, \mathbb{I}^*(\pi(G)')]\tau \subseteq \mathcal{F}_G(\pi, \mathcal{X}). \tag{13}$$

Proof. Let $(f_1, \tau_1) \in f[\mathbb{I}^*(\pi(G)')^{-1}, \mathbb{I}^*(\pi(G)')]\tau$. Then $f_1 = fU^{-1}$, $\tau_1 = U\tau$ for some invertible isometry $U : \mathcal{X} \to \mathcal{X}$ such that $U \in \mathbb{I}^*(\pi(G)')$. Define $f_{2_g} := f_1\pi_{g^{-1}}$ and $\tau_{2_g} := \pi_g\tau_1$, for all $g \in G$. Now we see that

$$\sum_{g \in G} f_{2_g}(x)\tau_{2_g} = \sum_{g \in G} (f_1\pi_{g^{-1}})(x)\pi_g\tau_1 = \sum_{g \in G} (fU^{-1})(\pi_{g^{-1}}x)\pi_g U\tau$$

$$= \sum_{g \in G} (f\pi_{g^{-1}})(U^{-1}x)U\pi_g\tau = U\left(\sum_{g \in G} (f\pi_{g^{-1}})(U^{-1}x)\pi_g\tau\right)$$

$$= U\left(\sum_{g \in G} f_g(U^{-1}x)\tau_g\right) = UU^{-1}x = x, \quad \forall x \in \mathcal{X}.$$

Therefore $(\{f_{2_g}\}_{g \in G}, \{\tau_{2_g}\}_{g \in G})$ is a group-p-USF for $\mathcal{X}$ and consequently $(f_1, \tau_1) \in \mathcal{F}_G(\pi, \mathcal{X})$. $\square$

We next try to show $\mathcal{F}_G(\pi, \mathcal{X})$ has another large set inside it. For this, we need following two results.

Theorem 2.15. *For any group G, and any $1 \leq p < \infty$,*

$$\lambda(G)' = \rho(G)'' \quad \text{and} \quad \rho(G)' = \lambda(G)''.$$

Proof. We already know from the proof of Theorem 2.13 that p-left and p-right regular representations commute, i.e.,

$$\lambda_g\rho_h = \rho_h\lambda_g, \quad \forall g, h \in G.$$

Hence for every $g \in G$, $\lambda_g \in \rho(G)'$. By varying g, we get $\lambda(G) \subseteq \rho(G)'$. Taking commutant yield $\lambda(G)' \supseteq \rho(G)''$. Now we prove the reverse inclusion. Let $T \in \lambda(G)'$. To show $T \in \rho(G)''$ we need to show that $TS = ST$ for all $S \in \rho(G)'$. So let $S \in \rho(G)'$. Note that to verify $TS = ST$, it suffices to verify $TS\delta_h = ST\delta_h$ for all $h \in G$. Let $h \in G$. Then

$$TS\delta_h = TS\rho_{h^{-1}}\delta_e = T\rho_{h^{-1}}S\delta_e = T\rho_{h^{-1}}\left(\sum_{g \in G} \zeta_g(S\delta_e)\delta_g\right)$$

$$= \sum_{g \in G} \zeta_g(S\delta_e)T\rho_{h^{-1}}\delta_g = \sum_{g \in G} \zeta_g(S\delta_e)T\delta_{gh} = \sum_{g \in G} \zeta_g(S\delta_e)T\lambda_{gh}\delta_e$$

$$= \sum_{g \in G} \zeta_g(S\delta_e)\lambda_{gh}T\delta_e = \sum_{g \in G} \zeta_g(S\delta_e)\lambda_{gh}\left(\sum_{u \in G} \zeta_u(T\delta_e)\delta_u\right)$$

$$= \sum_{g \in G} \zeta_g(S\delta_e) \sum_{u \in G} \zeta_u(T\delta_e)\lambda_{gh}\delta_u = \sum_{g \in G}\sum_{u \in G} \zeta_g(S\delta_e)\zeta_u(T\delta_e)\delta_{ghu}$$

and

$$ST\delta_h = ST\lambda_h\delta_e = S\lambda_h T\delta_e = S\lambda_h\left(\sum_{u \in G} \zeta_u(T\delta_e)\delta_u\right)$$

$$= \sum_{u \in G} \zeta_u(T\delta_e)S\lambda_h\delta_u = \sum_{u \in G} \zeta_u(T\delta_e)S\delta_{hu} = \sum_{u \in G} \zeta_u(T\delta_e)S\rho_{(hu)^{-1}}\delta_e$$

$$= \sum_{u \in G} \zeta_u(T\delta_e)\rho_{(hu)^{-1}}S\delta_e = \sum_{u \in G} \zeta_u(T\delta_e)\rho_{(hu)^{-1}}\left(\sum_{g \in G} \zeta_g(S\delta_e)\delta_g\right)$$

$$= \sum_{u \in G} \zeta_u(T\delta_e) \sum_{g \in G} \zeta_g(S\delta_e)\rho_{(hu)^{-1}}\delta_g = \sum_{u \in G}\sum_{g \in G} \zeta_u(T\delta_e)\zeta_g(S\delta_e)\delta_{ghu}.$$

Therefore $\lambda(G)' = \rho(G)''$. Finally $\lambda(G)'' = \rho(G)''' = \rho(G)'$. $\square$

Theorem 2.16. *For any discrete group G and for every $1 \leq p < \infty$, the map*

$$\Phi : \rho(G)'' \ni A \mapsto \Phi(A) := JAJ \in \lambda(G)''$$

is an algebra isomorphism, where

$$J : \ell^p(G) \ni \{a_g\}_{g \in G} \mapsto J\{a_g\}_{g \in G} := \sum_{g \in G} a_g\delta_{g^{-1}} \in \ell^p(G).$$

Moreover, we have the following.

(i) *If $U \in \rho(G)''$ is invertible (resp. isometry), then $\Phi(U)$ is invertible (resp. isometry).*
(ii) *If $V \in \lambda(G)''$ is invertible (resp. isometry), then $\Phi^{-1}(V)$ is invertible (resp. isometry).*

Proof. We first note that J is an isomorphism and $J^2 = I_{\ell^p(G)}$. Before showing Φ is an isomorphism, we need to show that it is well-defined. Let $A \in \rho(G)''$. We try to show that $JAJ \in \lambda(G)''$ which says Φ is well-defined. By Theorem 2.15, showing $JAJ \in \lambda(G)''$ is same as showing $JAJ \in \rho(G)'$. Let $g \in G$ be arbitrary. We claim that $JAJ\rho_g = \rho_g JAJ$. Since $\{\delta_h\}_{h \in G}$ is a basis for $\ell^p(G)$ it suffices to show $JAJ\rho_g\delta_h = \rho_g JAJ\delta_h$, for all $h \in G$. Now noting $A \in \lambda(G)'$, we get

$$JAJ\rho_g\delta_h = JAJ\delta_{hg^{-1}} = JA\delta_{gh^{-1}} = JA\lambda_g\delta_{h^{-1}} = J\lambda_g A\delta_{h^{-1}}$$

$$= J\lambda_g\left(\sum_{u\in G}\zeta_u(A\delta_{h^{-1}})\delta_u\right) = \sum_{u\in G}\zeta_u(A\delta_{h^{-1}})J\lambda_g\delta_u$$

$$= \sum_{u\in G}\zeta_u(A\delta_{h^{-1}})J\delta_{gu} = \sum_{u\in G}\zeta_u(A\delta_{h^{-1}})\delta_{u^{-1}g^{-1}}$$

and

$$\rho_g JAJ\delta_h = \rho_g JA\delta_{h^{-1}} = \rho_g J\left(\sum_{u\in G}\zeta_u(A\delta_{h^{-1}})\delta_u\right) = \sum_{u\in G}\zeta_u(A\delta_{h^{-1}})\rho_g J\delta_u$$

$$= \sum_{u\in G}\zeta_u(A\delta_{h^{-1}})\rho_g\delta_{u^{-1}} = \sum_{u\in G}\zeta_u(A\delta_{h^{-1}})\delta_{u^{-1}g^{-1}}.$$

Clearly Φ is linear. Since $J^2 = I_{\ell^p(G)}$, Φ is multiplicative. Through a direct calculation we see that inverse of Φ is the map $\lambda(G)'' \ni B \mapsto JBJ \in \rho(G)''$. $\square$

Following is the most important result for generators of group-frames for Banach spaces.

Theorem 2.17. *Let $\pi : G \to \mathcal{II}(\mathcal{X})$ be a invertible isometric representation of a group G which admits a functional-vector (f,τ). Then*

$$f[\mathbb{I}^*(\pi(G)'')^{-1}, \mathbb{I}^*(\pi(G)'')]\tau \subseteq \mathcal{F}_G(\pi,\mathcal{X}). \tag{14}$$

Proof. Let $g \in G$ and define $\Delta_g := \lambda_g|_{\theta_f(\mathcal{X})}$. Theorem 2.8 says that π is invertible isometrically equivalent to the representation Δ with functional-vector $(\zeta_e(\theta_f\theta_\tau), \theta_f\theta_\tau\delta_e)$. Therefore, without loss of generality we may assume that

$$\mathcal{X} = \theta_f(\mathcal{X}), \quad \pi = \Delta, \quad G = \{\lambda_g|_{\theta_f(\mathcal{X})} : g \in G\}, \quad \tau = \theta_f\theta_\tau\delta_e, \quad f = \zeta_e(\theta_f\theta_\tau).$$

Let $U : \theta_f(\mathcal{X}) \to \theta_f(\mathcal{X})$ be an invertible isometry such that $U \in \mathbb{I}^*(\pi(G)'')$. We need to show that $(fU^{-1}, U\tau) \in \mathcal{F}_G(\pi,\mathcal{X})$. Define $\tilde{\tau} := U\tau$, $\tilde{\tau}_g := \pi_g\tilde{\tau}$ and $\tilde{f} := fU^{-1}$, $\tilde{f}_g := \tilde{f}\pi_{g^{-1}}$ for all $g \in G$. To prove the theorem, now it suffices to show that $(\{\tilde{f}_g\}_{g\in G}, \{\tilde{\tau}_g\}_{g\in G})$ is an group-p-USF for $\mathcal{X}$. Let $x = \theta_f y \in \theta_f(\mathcal{X})$. Using Theorem 2.15 and Theorem 2.16, we may assume that $U \in \{\lambda_g|_{\theta_f(\mathcal{X})} : g \in G\}'$. Since $\theta_f\theta_\tau z = z$ for all $z \in \theta_f(\mathcal{X})$ and $U : \theta_f(\mathcal{X}) \to \theta_f(\mathcal{X})$, we then have $\theta_f\theta_\tau U = U$. Therefore

$$\sum_{g\in G}\tilde{f}_g(x)\tilde{\tau}_g = \sum_{g\in G}(\tilde{f}\pi_{g^{-1}})(x)\pi_g\tilde{\tau} = \sum_{g\in G}((fU^{-1})\pi_{g^{-1}})(x)\pi_g U\tau$$

$$= \sum_{g\in G}f(U^{-1}\pi_{g^{-1}}x)\pi_g U\tau = \sum_{g\in G}(\zeta_e\theta_f\theta_\tau)(U^{-1}\pi_{g^{-1}}x)\pi_g U\theta_f\theta_\tau\delta_e$$

$$= \sum_{g\in G}\zeta_e(\theta_f\theta_\tau U^{-1}\pi_{g^{-1}}x)\pi_g U\theta_f\theta_\tau\delta_e = \sum_{g\in G}\zeta_e(\theta_f\theta_\tau U^{-1}\lambda_{g^{-1}}x)\lambda_g U\theta_f\theta_\tau\delta_e$$

$$= \sum_{g\in G}\zeta_e(\theta_f\theta_\tau\lambda_{g^{-1}}U^{-1}x)U\lambda_g\theta_f\theta_\tau\delta_e = \sum_{g\in G}\zeta_e(\lambda_{g^{-1}}\theta_f\theta_\tau U^{-1}x)U\theta_f\theta_\tau\lambda_g\delta_e$$

$$= \sum_{g\in G}\zeta_g(\theta_f\theta_\tau U^{-1}x)U\theta_f\theta_\tau\delta_g = U\theta_f\theta_\tau\theta_f\theta_\tau U^{-1}x$$

$$= U\theta_f\theta_\tau U^{-1}x = UU^{-1}x = x.$$

$\square$

Han and Larson characterized (with an equality) frame vectors for Hilbert spaces (see Theorem 6.17 in [1]). Later Kaftal, Larson and Zhang noticed that the set of all frame vectors is path-connected in norm topology (see Theorem 8.1 in [28]). Based on these, we raise the following questions.

Problem 2.18. *Given an invertible isometric representation* $\pi : G \to \mathbb{I}^*(\mathcal{X})$ *which admits a functional-vector* (f, τ), *characterize* $\mathcal{F}_G(\pi, \mathcal{X})$. *In particular, classify Banach spaces, groups and representations such that*

(i) $f[\mathbb{I}^*(\pi(G)')^{-1}, \mathbb{I}^*(\pi(G)')]\tau = \mathcal{F}_G(\pi, \mathcal{X})$.
(ii) $f[\mathbb{I}^*(\pi(G)'')^{-1}, \mathbb{I}^*(\pi(G)'')]\tau = \mathcal{F}_G(\pi, \mathcal{X})$.

Problem 2.19. *Is* $\mathcal{F}_G(\pi, \mathcal{X}) \subseteq \mathcal{X}^* \times \mathcal{X}$ *path connected in the norm-topology?*

3. Finite Gabor-Schauder frames

In this section we study Schauder frame generated by time-frequency shifts on finite abelian groups for finite dimensional Banach space $\mathbb{C}^{o(G)}$. Our main motivation comes from the Gabor analysis on finite abelian groups presented by Pfander in [30] and Feichtinger, Kozek and Luef in [31]. Let G be a finite abelian group with identity e, $o(G)$ be the order of G and $\widehat{G}$ be the set of all characters of G. We denote the circle group by $\mathbb{T}$. Then $\widehat{G}$ becomes a group with respect to pointwise multiplication of characters. The character which sends every element of G to 1 is called as identity character and is denoted by 1_G. Let $\{\delta_g\}_{g \in G}$ be the standard basis for $\mathbb{C}^{o(G)}$ and $\{\zeta_g\}_{g \in G}$ be the coordinate functionals associated with $\{\delta_g\}_{g \in G}$. For $x = (x_g)_{g \in G}, y = (y_g)_{g \in G} \in \mathbb{C}^{o(G)}$, we set

$$x \cdot y := (x_g y_g)_{g \in G}$$

and

$$x^* := (\overline{x_g})_{g \in G}.$$

From the classical Fourier analysis on finite abelian groups, we note that we have the following properties (see [32]).

(1) $o(G) = o(\widehat{G})$.
(2) If $\xi, \chi \in \widehat{G}$, then $\langle \xi, \chi \rangle = \delta_{\xi, \chi}$, where

$$\langle \xi, \chi \rangle := \frac{1}{o(G)} \sum_{g \in G} \xi(g) \overline{\chi(g)}. \tag{15}$$

(3) If $g, h \in G$, then $\langle g, h \rangle = \delta_{g,h}$, where

$$\langle g, h \rangle := \frac{1}{o(\widehat{G})} \sum_{\xi \in \widehat{G}} \xi(g) \overline{\xi(h)}. \tag{16}$$

Given $\xi \in \widehat{G}$, the **modulation operator** $M_\xi : \mathbb{C}^{o(G)} \to \mathbb{C}^{o(G)}$ is defined by

$$M_\xi(x_g)_{g \in G} := (\xi(g) x_g)_{g \in G}, \quad \forall (x_g)_{g \in G} \in \mathbb{C}^{o(G)}.$$

Given $k \in G$, the **translation operator** $T_k : \mathbb{C}^{o(G)} \to \mathbb{C}^{o(G)}$ by

$$T_k(x_g)_{g \in G} := (x_{g-k})_{g \in G}, \quad \forall (x_g)_{g \in G} \in \mathbb{C}^{o(G)}.$$

Let $\lambda := (k, \xi) \in G \times \widehat{G}$. By composing modulation operator and translation operator we get the **time-frequency shift operator** $\pi(k, \xi) : \mathbb{C}^{o(G)} \to \mathbb{C}^{o(G)}$ defined as

$$\pi(\lambda) = \pi(k, \xi) := M_\xi T_k.$$

Thus the action of $\pi(\lambda)$ is given by

$$\pi(\lambda)(x_g)_{g \in G} = (\xi(g)x_{g-k})_{g \in G}, \quad \forall (x_g)_{g \in G} \in \mathbb{C}^{o(G)}.$$

Following properties of time-frequency shift operators will be used and are well-known.

Theorem 3.1. *[31] Let $\lambda = (k, \xi), \mu = (l, \chi) \in G \times \widehat{G}$. Then*

(i) $\pi(\lambda + \mu) = \overline{\chi(k)}\pi(\lambda)\pi(\mu)$.
(ii) $\pi(\lambda)\pi(\mu) = \overline{\chi(k)}\xi(l)\pi(\mu)\pi(\lambda)$.
(iii) $\pi(\lambda)^{-1} = \overline{\xi(k)}\pi(-\lambda)$.

Motivated from discrete Gabor analysis over finite abelian groups (see [30,31]) we set the following notion. We emphasis here that, even though it is true that $\mathbb{C}^{o(G)}$ is a Hilbert space, given a Banach space structure on it, thinking of $\mathbb{C}^{o(G)}$ as a Hilbert space in frame theory will not work. A recent influential instance is in defining the notion of '**Frame Potential**' for Banach spaces where usual direct generalization of Hilbert space frame potential failed (see Proposition 2.5 in [33]).

Definition 3.2. *Given a subgroup $\Lambda \subseteq G \times \widehat{G}$, a nonzero $f \in (\mathbb{C}^{o(G)})^*$ and a nonzero $\tau \in \mathbb{C}^{o(G)}$, the pair $(\{f(\pi(\lambda)^{-1})\}_{\lambda \in \Lambda}, \{\pi(\lambda)\tau\}_{\lambda \in \Lambda})$ is called as a **Gabor-Schauder system**. If the operator*

$$S_{f,\tau,\Lambda} : \mathbb{C}^{o(G)} \ni x \mapsto \sum_{\lambda \in \Lambda} f(\pi(\lambda)^{-1}x)\pi(\lambda)\tau \in \mathbb{C}^{o(G)}$$

*is invertible, then $(\{f(\pi(\lambda)^{-1})\}_{\lambda \in \Lambda}, \{\pi(\lambda)\tau\}_{\lambda \in \Lambda})$ is called as a **Gabor-Schauder frame**.*

We are interested in subgroups Λ of $G \times \widehat{G}$, which will give Gabor-Schauder frames. First we show that the full lattice $G \times \widehat{G}$ will give a Gabor-Schauder frame. Given a nonzero $f \in (\mathbb{C}^{o(G)})^*$, we define analysis operator

$$W_f : \mathbb{C}^{o(G)} \ni x \mapsto W_f x := (f(\pi(\lambda)^{-1}x)_{\lambda \in G \times \widehat{G}} \in \mathbb{C}^{o(G \times \widehat{G})}$$

and given a nonzero $\tau \in \mathbb{C}^{o(G)}$, we define synthesis operator

$$V_\tau : \mathbb{C}^{o(G \times \widehat{G})} \ni (a_\lambda)_{\lambda \in G \times \widehat{G}} \mapsto \sum_{\lambda \in G \times \widehat{G}} a_\lambda \pi(\lambda)\tau \in \mathbb{C}^{o(G)}.$$

Our first result is that composition of previous two operators give scalar times identity.

Theorem 3.3. *For $f \in (\mathbb{C}^{o(G)})^*$ and $\tau \in \mathbb{C}^{o(G)}$,*

$$V_\tau W_f = o(G)f(\tau)I_{\mathbb{C}^{o(G)}}. \tag{17}$$

Proof. Let $x = (x_g)_{g \in G} \in \mathbb{C}^{o(G)}$ and $\tau = (\tau_g)_{g \in G}$. Then

$$V_\tau W_f x = V_\tau(f[\pi(\lambda)^{-1}x])_{\lambda \in G \times \widehat{G}} = \sum_{\lambda \in G \times \widehat{G}} f[\pi(\lambda)^{-1}x]\pi(\lambda)\tau$$

$$= \sum_{(k,\xi) \in G \times \widehat{G}} f[\overline{\xi(k)}\pi(-k,-\xi)(x_g)_{g \in G}]\pi(k,\xi)(\tau_g)_{g \in G}$$

$$= \sum_{(k,\xi) \in G \times \widehat{G}} f[\overline{\xi(k)}((-\xi)(h)x_{h+k})_{h \in G}](\xi(g)\tau_{g-k})_{g \in G} = \sum_{(k,\xi) \in G \times \widehat{G}} f[(\overline{\xi(k+h)}x_{h+k})_{h \in G}](\xi(g)\tau_{g-k})_{g \in G}$$

$$= \left(\sum_{(k,\xi) \in G \times \widehat{G}} f[(\overline{\xi(k+h)}x_{h+k})_{h \in G}]\xi(g)\tau_{g-k} \right)_{g \in G} = \left(\sum_{(k,\xi) \in G \times \widehat{G}} f[(\overline{\xi(k+h)}\xi(g)x_{h+k})_{h \in G}]\tau_{g-k} \right)_{g \in G}$$

$$= \left(\sum_{k \in G} \sum_{\xi \in \widehat{G}} f[(\overline{\xi(k+h)}\xi(g)x_{h+k})_{h \in G}]\tau_{g-k} \right)_{g \in G} = \left(\sum_{k \in G} f\left[\sum_{\xi \in \widehat{G}} (\overline{\xi(k+h)}\xi(g)x_{h+k})_{h \in G}\right]\tau_{g-k} \right)_{g \in G}$$

$$= \left(\sum_{k \in G} f\left[\sum_{\xi \in \widehat{G}} \overline{\xi(k+h)}\xi(g)x_{h+k}\right]_{h \in G} \tau_{g-k} \right)_{g \in G} = o(G)\left(\sum_{k \in G} f\left[\delta_{k+h,g}x_{h+k}\right]_{h \in G} \tau_{g-k} \right)_{g \in G}$$

$$= o(G) \sum_{g \in G} \sum_{k \in G} f\left[\delta_{k+h,g}x_{h+k}\right]_{h \in G} \tau_{g-k}\delta_g = o(G) \sum_{g \in G} \sum_{k \in G} f\left[\sum_{h \in G}\delta_{k+h,g}x_{h+k}\delta_h\right]\tau_{g-k}\delta_g$$

$$= o(G) \sum_{g \in G} \sum_{k \in G} \sum_{h \in G}\delta_{k+h,g}x_{h+k}f[\delta_h]\tau_{g-k}\delta_g = o(G) \sum_{g \in G} \sum_{k \in G} x_g f[\delta_{g-k}]\tau_{g-k}\delta_g$$

$$= o(G) \sum_{g \in G} x_g f\left[\sum_{k \in G}\tau_{g-k}\delta_{g-k}\right]\delta_g = o(G) \sum_{g \in G} x_g f[(\tau_k)_{k \in G}]\delta_g = o(G)f(\tau)x.$$

□

We call Equation (17) as **Schauder-Moyal formula** for Banach space. It is easy to see that for Hilbert spaces, whenever f is determined by τ, Schauder-Moyal formula reduces to familiar Moyal formula. Schauder-Moyal formula immediately gives the following corollary.

Corollary 3.4. *If $f \in (\mathbb{C}^{o(G)})^*$ and $\tau \in \mathbb{C}^{o(G)}$ are such that $f(\tau) \neq 0$, then*

$$x = \frac{1}{o(G)f(\tau)} \sum_{\lambda \in G \times \widehat{G}} f(\pi(\lambda)^{-1}x)\pi(\lambda)\tau, \quad \forall x \in \mathbb{C}^{o(G)}. \tag{18}$$

In other words,

$$\left(\{f(\pi(\lambda)^{-1})\}_{\lambda \in G \times \widehat{G}}, \{\pi(\lambda)\tau\}_{\lambda \in G \times \widehat{G}} \right)$$

is a Gabor-Schauder frame for $\mathbb{C}^{o(G)}$.

We call Equation (18) as the **inversion formula for Banach space**. It reduces to inversion formula for the short-time Fourier transform for Hilbert spaces, whenever f is determined by τ.

Recall that given linear operators $T, S : \mathbb{C}^{o(G)} \to \mathbb{C}^{o(G)}$, if we define

$$\langle T, S \rangle_{\text{HS}} := \sum_{g \in G} \langle T\delta_g, S\delta_g \rangle,$$

then the space $\mathcal{L}(\mathbb{C}^{o(G)})$ of all linear operators from $\mathbb{C}^{o(G)}$ to itself is a Hilbert space w.r.t. inner product $\langle T, S \rangle_{\text{HS}}$. We denote this Hilbert space by $\text{HS}(\mathbb{C}^{o(G)})$. We need the following result in continuation.

Theorem 3.5. *[31] The family*

$$\left\{ \frac{1}{\sqrt{o(G)}} \pi(\lambda) \right\}_{\lambda \in G \times \widehat{G}} \tag{19}$$

is an orthonormal basis for $HS(\mathbb{C}^{o(G)})$.

Recall that given a subgroup (also known as lattice) $\Lambda \subseteq G \times \widehat{G}$, we define the adjoint subgroup of Λ, denoted by Λ^0 as

$$\Lambda^0 := \{ \mu \in G \times \widehat{G} : \pi(\lambda)\pi(\mu) = \pi(\mu)\pi(\lambda), \forall \lambda \in \Lambda \}.$$

Theorem 3.1 says that Λ^0 is a subgroup of $G \times \widehat{G}$. Now given a nonzero vector $\tau \in \mathbb{C}^{o(G)}$ and a nonzero functional $f \in (\mathbb{C}^{o(G)})^*$, we define the **Gabor-Schauder frame operator** $S_{f,\tau,\Lambda} : \mathbb{C}^{o(G)} \to \mathbb{C}^{o(G)}$ as

$$S_{f,\tau,\Lambda} x := \sum_{\lambda \in \Lambda} f(\pi(\lambda)^{-1} x) \pi(\lambda) \tau, \quad \forall x \in \mathbb{C}^{o(G)}.$$

Following key result will be used repeatedly. It mainly uses group properties of Λ.

Theorem 3.6. *Let* Λ *be a subgroup of* $G \times \widehat{G}$, $\tau \in \mathbb{C}^{o(G)}$ *and* $f \in (\mathbb{C}^{o(G)})^*$. *Then for each* $\mu \in \Lambda$, *the time-frequency shifts* $\pi(\mu)$ *commute with the Gabor-Schauder frame operator* $S_{f,\tau,\Lambda}$.

Proof. Since $\pi(\mu)$ is invertible, to show $\pi(\mu) S_{f,\tau,\Lambda} = S_{f,\tau,\Lambda} \pi(\mu)$ it suffices to show that $\pi(\mu)^{-1} S_{f,\tau,\Lambda} \pi(\mu) = S_{f,\tau,\Lambda}$. Let $x \in \mathbb{C}^{o(G)}$ and $\mu = (l, \chi)$. Now using the fact that G is a group and using Theorem 3.1,

$$
\begin{aligned}
\pi(l,\chi)^{-1} S_{f,\tau,\Lambda} \pi(l,\chi) x &= \pi(l,\chi)^{-1} \left(\sum_{(k,\xi) \in \Lambda} f[\pi(k,\xi)^{-1} \pi(l,\chi) x] \pi(k,\xi) \tau \right) \\
&= \sum_{(k,\xi) \in \Lambda} f[\pi(k,\xi)^{-1} \pi(l,\chi) x] \pi(l,\chi)^{-1} \pi(k,\xi) \tau \\
&= \sum_{(k,\xi) \in \Lambda} f[(\pi(l,\chi)^{-1} \pi(k,\xi))^{-1} x] \pi(l,\chi)^{-1} \pi(k,\xi) \tau \\
&= \sum_{(k,\xi) \in \Lambda} f[(\overline{\chi(l)} \pi(-l,-\chi) \pi(k,\xi))^{-1} x] \overline{\chi(l)} \pi(-l,-\chi) \pi(k,\xi) \tau \\
&= \sum_{(k,\xi) \in \Lambda} f[(\pi(-l,-\chi) \pi(k,\xi))^{-1} x] \pi(-l,-\chi) \pi(k,\xi) \tau \\
&= \sum_{(k,\xi) \in \Lambda} f[(\overline{\xi(-l)} \pi(-l+k,-\chi+\xi))^{-1} x] \overline{\xi(-l)} \pi(-l+k,-\chi+\xi) \tau \\
&= \sum_{(k,\xi) \in \Lambda} f[\pi(-l+k,-\chi+\xi)^{-1} x] \pi(-l+k,-\chi+\xi) \tau = S_{f,\tau,\Lambda} x.
\end{aligned}
$$

□

Recall that (see [34]) a pair $(\{f_j\}_{j=1}^n, \{\tau_j\}_{j=1}^n)$ is said to be an approximate Schauder frame (written as ASF) for $\mathbb{C}^{o(G)}$ if the operator

$$S_{f,\tau} : \mathbb{C}^{o(G)} \ni x \mapsto S_{f,\tau} x := \sum_{j=1}^n f_j(x) \tau_j \in \mathbb{C}^{o(G)}$$

is invertible. Also we recall that an ASF $(\{g_j\}_{j=1}^n, \{\omega_j\}_{j=1}^n)$ for $\mathbb{C}^{o(G)}$ is said to be a dual for $(\{f_j\}_{j=1}^n, \{\tau_j\}_{j=1}^n)$ if

$$x = \sum_{j=1}^n f_j(x)\omega_j = \sum_{j=1}^n g_j(x)\tau_j, \quad \forall x \in \mathbb{C}^{o(G)}.$$

It is an easy observation that the ASF $(\{f_j S_{f,\tau}^{-1}\}_{j=1}^n, \{S_{f,\tau}^{-1}\tau_j\}_{j=1}^n)$ is always a dual for $(\{f_j\}_{j=1}^n, \{\tau_j\}_{j=1}^n)$. This is called as canonical dual of $(\{f_j\}_{j=1}^n, \{\tau_j\}_{j=1}^n)$. Following corollary says that canonical dual of Gabor-Schauder frame is again a Gabor-Schauder frame.

Corollary 3.7. *Let Λ be a subgroup of $G \times \widehat{G}$, $\tau \in \mathbb{C}^{o(G)}$ and $f \in (\mathbb{C}^{o(G)})^*$. If $(\{f(\pi(\lambda)^{-1})\}_{\lambda \in \Lambda}, \{\pi(\lambda)\tau\}_{\lambda \in \Lambda})$ is a Gabor-Schauder frame for $\mathbb{C}^{o(G)}$, then its canonical dual can be written as $(\{\phi(\pi(\lambda)^{-1})\}_{\lambda \in \Lambda}, \{\pi(\lambda)\omega\}_{\lambda \in \Lambda})$ for some $\omega \in \mathbb{C}^{o(G)}$ and $\phi \in (\mathbb{C}^{o(G)})^*$. In other words, there exist $\omega \in \mathbb{C}^{o(G)}$ and $\phi \in (\mathbb{C}^{o(G)})^*$ such that $(\{\phi(\pi(\lambda)^{-1})\}_{\lambda \in \Lambda}, \{\pi(\lambda)\omega\}_{\lambda \in \Lambda})$ is a Gabor-Schauder frame for $\mathbb{C}^{o(G)}$ and*

$$x = \sum_{\lambda \in \Lambda} \phi(\pi(\lambda)^{-1}x)\pi(\lambda)\tau = \sum_{\lambda \in \Lambda} f(\pi(\lambda)^{-1}x)\pi(\lambda)\omega, \quad \forall x \in \mathbb{C}^{o(G)}.$$

Proof. Canonical dual for $(\{f(\pi(\lambda)^{-1})\}_{\lambda \in \Lambda}, \{\pi(\lambda)\tau\}_{\lambda \in \Lambda})$ is given by $(\{f(\pi(\lambda)^{-1}S_{f,\tau,\Lambda}^{-1})\}_{\lambda \in \Lambda}, \{S_{f,\tau,\Lambda}^{-1}\pi(\lambda)\tau\}_{\lambda \in \Lambda})$. Define $\omega := S_{f,\tau,\Lambda}^{-1}\tau$ and $\phi := fS_{f,\tau,\Lambda}^{-1}$. Then using Theorem 3.6,

$$\{S_{f,\tau,\Lambda}^{-1}\pi(\lambda)\tau\}_{\lambda \in \Lambda} = \{\pi(\lambda)S_{f,\tau,\Lambda}^{-1}\tau\}_{\lambda \in \Lambda} = \{\pi(\lambda)\omega\}_{\lambda \in \Lambda},$$

$$\{f(\pi(\lambda)^{-1}S_{f,\tau,\Lambda}^{-1})\}_{\lambda \in \Lambda} = \{f(S_{f,\tau,\Lambda}^{-1}\pi(\lambda)^{-1})\}_{\lambda \in \Lambda} = \{\phi(\pi(\lambda)^{-1})\}_{\lambda \in \Lambda}.$$

□

Next result is the key result of this section.

Theorem 3.8. *Let Λ be a subgroup of $G \times \widehat{G}$ and $\tau \in \mathbb{C}^{o(G)}$, $f \in (\mathbb{C}^{o(G)})^*$. Then*

$$S_{f,\tau,\Lambda}x = \sum_{\lambda \in \Lambda} f(\pi(\lambda)^{-1}x)\pi(\lambda)\tau = \frac{o(\Lambda)}{o(G)} \sum_{\mu \in \Lambda^0} f(\pi(\mu)^{-1}\tau)\pi(\mu)x = \frac{o(\Lambda)}{o(G)}S_{f,x,\Lambda^0}\tau, \quad \forall x \in \mathbb{C}^{o(G)}.$$

(20)

Proof. Theorem 3.5 says that

$$S_{f,\tau,\Lambda} = \sum_{\mu \in G \times \widehat{G}} \left\langle S_{f,\tau,\Lambda}, \frac{1}{\sqrt{o(G)}}\pi(\mu) \right\rangle_{\mathrm{HS}} \frac{1}{\sqrt{o(G)}}\pi(\mu)$$

$$= \sum_{\mu \in G \times \widehat{G}} c_\mu \pi(\mu), \quad \text{where } c_\mu := \frac{1}{o(G)} \left\langle S_{f,\tau,\Lambda}, \pi(\mu) \right\rangle_{\mathrm{HS}}, \forall \mu \in G \times \widehat{G}.$$

Let $\mu \in G \times \widehat{G}$ and $\lambda \in \Lambda$. Theorem 3.1 then gives a $d_{\mu,\lambda} \in \mathbb{T}$ such that $\pi(\lambda)^{-1}\pi(\mu)\pi(\lambda) = d_{\mu,\lambda}\pi(\mu)$. Using Theorem 3.6,

$$\sum_{\mu \in G \times \widehat{G}} c_\mu \pi(\mu) = \pi(\lambda)^{-1} S_{f,\tau,\Lambda} \pi(\lambda) = \pi(\lambda)^{-1} \left(\sum_{\mu \in G \times \widehat{G}} c_\mu \pi(\mu) \right) \pi(\lambda)$$

$$= \sum_{\mu \in G \times \widehat{G}} c_\mu \pi(\lambda)^{-1} \pi(\mu) \pi(\lambda) = \sum_{\mu \in G \times \widehat{G}} c_\mu d_{\mu,\lambda} \pi(\mu).$$

Therefore

$$\sum_{\mu \in G \times \widehat{G}} c_\mu (1 - d_{\mu,\lambda}) \pi(\mu) = 0, \quad \forall \lambda \in \Lambda.$$

Since $\{\pi(\lambda)\}_{\lambda \in G \times \widehat{G}}$ is linearly independent (Theorem 3.5), we then have $c_\mu(1 - d_{\mu,\lambda}) = 0$, for all $\lambda \in \Lambda$, for all $\mu \in G \times \widehat{G}$. Let $\mu \notin \Lambda^0$. We claim that $c_\mu = 0$. If this is not true, then $1 - d_{\mu,\lambda} = 0$ for all $\lambda \in \Lambda$. But then we have $\pi(\mu)\pi(\lambda) = \pi(\lambda)\pi(\mu)$ for all $\lambda \in \Lambda$ which says $\mu \in \Lambda^0$ which is impossible. So claim holds. Hence the formula for the Gabor-Schauder frame operator reduces to

$$S_{f,\tau,\Lambda} = \sum_{\mu \in \Lambda^0} c_\mu \pi(\mu). \tag{21}$$

Now let $\mu \in \Lambda^0$. Then using Equation (19),

$$c_\mu = \frac{1}{o(G)} \left\langle S_{f,\tau,\Lambda}, \pi(\mu) \right\rangle_{\text{HS}} = \frac{1}{o(G)} \sum_{g \in G} \langle S_{f,\tau,\Lambda} \delta_g, \pi(\mu) \delta_g \rangle$$

$$= \frac{1}{o(G)} \sum_{g \in G} \left\langle \sum_{\lambda \in \Lambda} f(\pi(\lambda)^{-1}\delta_g)\pi(\lambda)\tau, \pi(\mu)\delta_g \right\rangle = \frac{1}{o(G)} \sum_{\lambda \in \Lambda} \sum_{g \in G} f(\pi(\lambda)^{-1}\delta_g)\langle \pi(\lambda)\tau, \pi(\mu)\delta_g \rangle$$

$$= \frac{1}{o(G)} \sum_{\lambda \in \Lambda} \sum_{g \in G} f(\pi(\lambda)^{-1}\delta_g)\langle \pi(\mu)^{-1}\pi(\lambda)\tau, \delta_g \rangle = \frac{1}{o(G)} \sum_{\lambda \in \Lambda} \sum_{g \in G} f \left[\langle \pi(\mu)^{-1}\pi(\lambda)\tau, \delta_g \rangle \pi(\lambda)^{-1}\delta_g \right]$$

$$= \frac{1}{o(G)} \sum_{\lambda \in \Lambda} f \left[\sum_{g \in G} \langle \pi(\mu)^{-1}\pi(\lambda)\tau, \delta_g \rangle \pi(\lambda)^{-1}\delta_g \right] = \frac{1}{o(G)} \sum_{\lambda \in \Lambda} f \left[\pi(\lambda)^{-1} \sum_{g \in G} \langle \pi(\mu)^{-1}\pi(\lambda)\tau, \delta_g \rangle \delta_g \right]$$

$$= \frac{1}{o(G)} \sum_{\lambda \in \Lambda} f(\pi(\lambda)^{-1}\pi(\mu)^{-1}\pi(\lambda)\tau) = \frac{1}{o(G)} \sum_{\lambda \in \Lambda} f(\pi(\mu)^{-1}\pi(\lambda)^{-1}\pi(\lambda)\tau)$$

$$= \frac{1}{o(G)} \sum_{\lambda \in \Lambda} f(\pi(\mu)^{-1}\tau) = \frac{o(\Lambda)}{o(G)} f(\pi(\mu)^{-1}\tau).$$

By substituting the value of c_μ in Equation (21) finally gives

$$S_{f,\tau,\Lambda} = \frac{o(\Lambda)}{o(G)} \sum_{\mu \in \Lambda^0} f(\pi(\mu)^{-1}\tau)\pi(\mu).$$

□

We call Equation (20) as **Fundamental Identity of Gabor-Schauder Frames (FIGSF)** or **Schauder-Wexler-Raz identity** or **Schauder-Janssen representation** of Gabor-Schauder frame operator $S_{f,\tau,\Lambda}$. It represents the frame operator corresponding to Λ in terms of frame operator corresponding to Λ^0. For Hilbert spaces it reduces to the usual Fundamental Identity of Gabor Analysis (famously known as FIGA) derived in [30] and [31]. First major consequence of FIGSF is the following criterion.

Theorem 3.9. *Let* Λ *be a subgroup of* $G \times \widehat{G}$ *and* $\tau \in \mathbb{C}^{o(G)}$, $f \in (\mathbb{C}^{o(G)})^*$. *Then*

$$x = \sum_{\lambda \in \Lambda} f(\pi(\lambda)^{-1}x)\pi(\lambda)\tau, \quad \forall x \in \mathbb{C}^{o(G)} \iff f(\pi(\mu)^{-1}\tau) = \frac{o(G)}{o(\Lambda)}\delta_{\mu,0}, \quad \forall \mu \in \Lambda^0. \tag{22}$$

Proof. ($\Rightarrow$) We have that the Gabor-Schauder frame operator $S_{f,\tau,\Lambda}$ is identity operator. Theorem 3.8 then says that

$$I_{\mathbb{C}^{o(G)}} = \frac{o(\Lambda)}{o(G)} \sum_{\mu \in \Lambda^0} f(\pi(\mu)^{-1}\tau)\pi(\mu) = \frac{o(\Lambda)}{o(G)} f(\tau) I_{\mathbb{C}^{o(G)}} + \frac{o(\Lambda)}{o(G)} \sum_{\mu \in \Lambda^0 \setminus \{(0,0)\}} f(\pi(\mu)^{-1}\tau)\pi(\mu).$$

By Theorem 3.5, the collection $\{\pi(\lambda)\}_{\lambda \in G \times \widehat{G}}$ is linearly independent. Thus the validity of previous equation gives

$$f(\tau) = \frac{o(G)}{o(\Lambda)}, \quad f(\pi(\mu)^{-1}\tau) = 0, \forall \mu \in \Lambda^0 \setminus \{(0,0)\}.$$

($\Leftarrow$) Using Equation (20),

$$S_{f,\tau,\Lambda}x = \frac{o(\Lambda)}{o(G)} \sum_{\mu \in \Lambda^0} f(\pi(\mu)^{-1}\tau)\pi(\mu)x = \frac{o(\Lambda)}{o(G)} f(\tau)x + \frac{o(\Lambda)}{o(G)} \sum_{\mu \in \Lambda^0 \setminus \{(0,0)\}} f(\pi(\mu)^{-1}\tau)\pi(\mu)x = x, \quad \forall x \in \mathbb{C}^{o(G)}.$$

□

We call Equation (22) as **Schauder-Wexler-Raz criterion**. This is a generalization of Wexler-Raz criterion derived in [30] and [31]. The criterion says when we can say that certain pairs give Gabor-Schauder frames by checking an algebraic equation on the adjoint lattice. We conclude by deriving the following result, which we call **partial Schauder Ron-Shen duality**. Note that for Hilbert spaces, the theorem is if and only if' (see [30] and [31]).

Theorem 3.10. *Let* Λ *be a subgroup of* $G \times \widehat{G}$ *and* $\tau \in \mathbb{C}^{o(G)}$, $f \in (\mathbb{C}^{o(G)})^*$. *If* $(\{f(\pi(\lambda)^{-1})\}_{\lambda \in \Lambda}, \{\pi(\lambda)\tau\}_{\lambda \in \Lambda})$ *is a Gabor-Schauder frame for* $\mathbb{C}^{o(G)}$, *then both sets* $\{\pi(\mu)^{-1}\tau\}_{\mu \in \Lambda^0}$ *and* $\{f(\pi(\mu)^{-1})\}_{\mu \in \Lambda^0}$ *are linearly independent.*

Proof. Without loss of generality we may assume that $S_{f,\tau,\Lambda} = I_{\mathbb{C}^{o(G)}}$. Now Theorem 3.9 says that

$$f(\pi(\mu)^{-1}\tau) = \frac{o(G)}{o(\Lambda)}\delta_{\mu,0}, \quad \forall \mu \in \Lambda^0.$$

Let $\lambda, \mu \in \Lambda^0$ be such that $\lambda \neq \mu$. Then, since Λ^0 is a group, we get that $f(\pi(\lambda)^{-1}\pi(\mu)^{-1}\tau) = 0$. We now suppose that $\sum_{\mu \in \Lambda^0} c_\mu \pi(\mu)^{-1}\tau = 0$ for some $c_\mu \in \mathbb{C}$. Then for each $\lambda \in \Lambda^0$, we have

$$0 = f\left(\pi(\lambda)^{-1}\left(\sum_{\mu \in \Lambda^0} c_\mu \pi(\mu)^{-1}\tau\right)\right) = \sum_{\mu \in \Lambda^0} c_\mu f(\pi(\lambda)^{-1}\pi(\mu)^{-1}\tau) = c_\lambda \frac{o(G)}{o(\Lambda)}.$$

Therefore $\{\pi(\lambda)\tau\}_{\lambda \in \Lambda^0}$ is linearly independent. On the other hand, let $\sum_{\mu \in \Lambda^0} d_\mu f(\pi(\mu)^{-1}) = 0$ for some $d_\mu \in \mathbb{C}$. Then for each $\lambda \in \Lambda^0$, we have

$$0 = \sum_{\mu \in \Lambda^0} d_\mu f(\pi(\mu)^{-1}\pi(\lambda)^{-1}\tau) = d_\lambda \frac{o(G)}{o(\Lambda)}.$$

Therefore $\{f(\pi(\mu)^{-1})\}_{\mu \in \Lambda^0}$ is linearly independent. □

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