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Sixteen Pairs of 4-Component Spinors for SL(4,C) and Four Types of Transformations with a Conjugate Space Which Has No Counterpart in SL(2,C)

Mehmet Ali Kuntman *

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Sixteen Pairs of 4-Component Spinors for SL(4,C) and Four Types of Transformations with a Conjugate Space Which Has No Counterpart in SL(2,C)

M. A. Kuntman

Independent Researcher; makuntman@gmail.com

Abstract: We define a spinor-Minkowski metric for SL(4,C). It is not a trivial generalization of the SL(2,C) metric and it involves the Minkowskian one. We define 4x4 version of the Pauli matrices and eight 4-component associated generalized eigenvectors that can be regarded as undotted covariant spinors. The 4-component spinors can be grouped into four categories. Each category transforms in its own way. The outer products of pairwise combinations of 4-component spinors can be associated with 4-vectors. Including the dotted covariant, undotted and dotted contravariant forms totally we have sixteen pairs of spinors. Eight of them live in the conjugate space which has no countepart in SL(2,C).

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1. Introduction

Let L_A be an element of SL(2,C). In an exponential form with parameters θ and η:

$$L_A = exp(-\frac{i}{2}(\vec{\theta}\cdot\vec{\sigma} + i\vec{\rho}\cdot\vec{\sigma})) \tag{1}$$

$\vec{\sigma}$ is the Pauli vector with $\sigma_1 = \sigma_x$, $\sigma_2 = \sigma_y$, $\sigma_3 = \sigma_z$. The subscript $_A$ is introduced in order to distinguish the other forms of L that will be introduced subsequently.

We rewrite L_A and its complex conjugate in the following compact forms:

$$L_A = exp(-\frac{i}{2}\vec{\pi}_A\cdot\vec{\sigma}), \qquad L_A^* = exp(\frac{i}{2}\vec{\pi}_A^*\cdot\vec{\sigma}^*) \tag{2}$$

$(\pi_A)_i = \theta_i + i\rho_i$ and * denotes complex conjugation. L_A corresponds to the Lorentz transformation with θ_i and ρ_i being the rotation and boost parameters, respectively.

It is well known that a complex version of the 4×4 Lorentz transformation matrix can be written as a matrix direct product of L_A and L_A^* [1,2]:

$$\lambda_A = L_A \otimes L_A^* \tag{3}$$

Let us write L_A in σ basis

$$L_A = \sum_{\mu=0}^{3}\alpha_\mu\sigma_\mu \tag{4}$$

σ_0 is the 4×4 identity. Explicit definitions of α_μ can be found in the Appendix. With the expansion given in Equation (4)

$$L_A \otimes L_A^* = \sum_{\mu,\nu=0}^{3}\alpha_\mu{\alpha^*}_\nu(\sigma_\mu\otimes\sigma_\nu^*) \tag{5}$$

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We can write Equation (5) as a matrix product of two matrices by defining new basis

$$E_\mu = \sigma_\mu \otimes \sigma_0, \qquad \widetilde{E}_\mu = \sigma_0 \otimes \sigma_\mu^* \tag{6}$$

$$L_A \otimes L_A^* = W_A \widetilde{W}_A \tag{7}$$

where $W_A = \sum \alpha_\mu E_\mu$, $\widetilde{W}_A = \sum \alpha_\mu^* \widetilde{E}_\mu$.

E and $\widetilde{E}$ basis obey the following commutation relations [1]:

$$[E_i, E_j] = 2i\epsilon_{ijk}E_k, \qquad [\widetilde{E}_i, \widetilde{E}_j] = -2i\epsilon_{ijk}\widetilde{E}_k, \qquad [E_i, \widetilde{E}_j] = 0 \tag{8}$$

Note that in this representation $\widetilde{E}_i \neq E_i^*$.

In order to obtain the familiar real 4×4 matrix form of the Lorentz transformation it is enough to change the basis [3–5]:

$$\Lambda_A = A(L_A \otimes L_A^*)A^{-1} \tag{9}$$

where

$$A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & i & -i & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}, \qquad A^{-1} = A^\dagger = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -i & 0 \\ 0 & 1 & i & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} \tag{10}$$

Now, it is straightforward to show that the 4×4 real Lorentz transformation matrix can be written as a commutative product of two matrices one being the complex conjugate of the other [6–8]:

$$\Lambda_A = [A(L_A \otimes I)A^{-1}][A(I \otimes L_A^*)A^{-1}] = Z_A Z_A^* = Z_A^* Z_A, \tag{11}$$

$$Z_A = A(L_A \otimes I)A^{-1}, \qquad Z_A^* = A(I \otimes L_A^*)A^{-1}. \tag{12}$$

Z_A and Z_A^* are the 4×4 versions of L_A and L_A^* matrices. They can be expressed in terms of Σ_i matrices:

$$Z_A = exp(-\frac{i}{2}\vec{\pi}_A \cdot \vec{\Sigma}), \qquad Z_A^* = exp(\frac{i}{2}\vec{\pi}_A^* \cdot \vec{\Sigma}^*). \tag{13}$$

$\vec{\Sigma} = (\Sigma_1, \Sigma_2, \Sigma_3)$ and Σ_i are 4×4 versions of Pauli matrices [9]:

$$\Sigma_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \quad \Sigma_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \\ 1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad \Sigma_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{14}$$

By definition, $\Sigma_\mu = A(\sigma_\mu \otimes \sigma_0)A^{-1}$, $(\mu = 0,1,2,3)$, Σ_0 is the 4×4 identity. Similarly, we define $\widetilde{\Sigma}$ basis as $\widetilde{\Sigma}_\mu = A(\sigma_0 \otimes \sigma_\mu^*)A^{-1}$. But now we have an important property that $\widetilde{\Sigma}_\mu = \Sigma_\mu^*$.

Σ_μ basis do not form a complete set for 4×4 matrices, but the set of $\Sigma_\mu \Sigma_\nu^*$ does [2]. Σ_i and Σ_i^* matrices are traceless Hermitian and they satisfy the following commutation relations:

$$[\Sigma_i, \Sigma_j] = 2i\epsilon_{ijk}\Sigma_k, \quad [\Sigma_i^*, \Sigma_j^*] = -2i\epsilon_{ijk}\Sigma_k, \quad [\Sigma_i, \Sigma_j^*] = 0 \tag{15}$$

The 2×2 Pauli matrices satisfy the first two relations, but the analogy breaks down because $\sigma_1 = \sigma_1^*$, $\sigma_2 = -\sigma_2^*$ and $\sigma_3 = \sigma_3^*$, hence σ_i does not commute with σ_j^* if $i \neq j$. On the other hand Σ_i commutes with Σ_j^* for all i, j.

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1 This representation will be studied in detail in Section 8

2 The basis set $\Sigma_\mu \Sigma_\nu^* = A(\sigma_\mu \otimes \sigma_\nu^*)A^{-1}$ is equivalent to the Dirac basis set, $\sigma_\mu \otimes \sigma_\nu$.

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From the Equation (12), Z_A can be found in terms of the elements of L_A:

$$Z_A = \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_1 & \alpha_0 & -i\alpha_3 & i\alpha_2 \\ \alpha_2 & i\alpha_3 & \alpha_0 & -i\alpha_1 \\ \alpha_3 & -i\alpha_2 & i\alpha_1 & \alpha_0 \end{pmatrix} \tag{16}$$

where $\alpha_0 = \frac{1}{2}(L_{11} + L_{22})$, $\alpha_1 = \frac{1}{2}(L_{12} + L_{21})$, $\alpha_2 = \frac{i}{2}(L_{12} - L_{21})$, and $\alpha_3 = \frac{1}{2}(L_{11} - L_{22})$. Hence, L_A can be written in terms of α_μ as

$$L_A = \begin{pmatrix} \alpha_0 + \alpha_3 & \alpha_1 - i\alpha_2 \\ \alpha_1 + i\alpha_2 & \alpha_0 - \alpha_3 \end{pmatrix} \tag{17}$$

We can write L_A and Z_A in terms of σ_μ and Σ_μ matrices:

$$L_A = \alpha_0\sigma_0 + \alpha_1\sigma_1 + \alpha_2\sigma_2 + \alpha_3\sigma_3 \tag{18}$$

$$Z_A = \alpha_0\Sigma_0 + \alpha_1\Sigma_1 + \alpha_2\Sigma_2 + \alpha_3\Sigma_3 \tag{19}$$

Or, simply

$$L_A = (++++)_\sigma, \qquad Z_A = (++++)_\Sigma. \tag{20}$$

Similarly we write L_A^* and Z_A^* in terms of α_μ^*:

$$L_A^* = \alpha_0^*\sigma_0^* + \alpha_1^*\sigma_1^* + \alpha_2^*\sigma_2^* + \alpha_3^*\sigma_3^* = (++++)^*_{\sigma^*} = (++-+)^*_\sigma \tag{21}$$

$$Z_A^* = \alpha_0^*\Sigma_0^* + \alpha_1^*\Sigma_1^* + \alpha_2^*\Sigma_2^* + \alpha_3^*\Sigma_3^* = (++++)^*_{\Sigma^*} \neq (++-+)^*_\Sigma \tag{22}$$

It is straightforward to show that Z_A commutes with Z_A^*, but L_A does not commute with L_A^*, in general. Actually, this property of Z matrices is more general. Let Z and Y be two matrices defined in Σ_μ and Σ_μ^* bases, respectively, Z and Y commute, because Σ_μ commutes with Σ_ν^*.

We also define the spinor metric g for SL(4,C) that corresponds to the spinor metric ϵ of SL(2,C):

$$g = g^{\mu\nu} = i\eta\Sigma_2^* = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -1 \\ -i & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}; \quad g^{-1} = g_{\mu\nu} = g^\dagger. \tag{23}$$

η is the mostly minus Minkowski metric[3].

Z_A preserves the spinor metric g and the Minkowski metric η [4]:

$$Z_A^T g Z_A = g, \qquad Z_A^T \eta Z_A = \eta \tag{24}$$

Since η is real, $Z_A^T \eta Z_A = \eta$ directly entails $\Lambda^T \eta \Lambda = \eta$. In an analogy with $\epsilon\sigma_i\epsilon^{-1} = -\sigma_i^*$, we have the following very useful relation:

$$g\Sigma_i g^{-1} = -\Sigma_i^* \tag{25}$$

In this note it will be shown that there are eight generalized eigenvectors of Σ_3^* that can be interpreted as 4-component undotted covariant spinors. They can be grouped pairwise into four

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[3] We can define the spinor metric for SL(4,C) as $i\eta\Sigma_1^*$ or $i\eta\Sigma_3^*$ if we like. These metrics also have the same properties as g.

[4] Besides Z_A, we will subsequently define other types Z_B, Z_C and Z_D that preserve both g and η. Dotted versions of all four types of transformations also preserve both g and η. All types of Z^* and $\dot{Z}^*$ preserve g^* and η ($\dot{Z} = (Z^{-1})^\dagger$).

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categories. Each category transforms in its own way. The outer products of spinor pairs can be associated with 4-vectors. When we include the dotted contravariant forms we get eight spinor pairs.

There is also a conjugate space. The dotted covariant and undotted contravariant spinor pairs live in this space. They double the number of spinor pairs, hence we have totally sixteen pairs. Due to the structural reasons there is no counterpart of this conjugate space in SL(2,C).

In the following sections we will study the first and the second categories in detail, and we will introduce the remaining ones in the subsequent sections.

2. The first and the second pairs and their transformation properties

Let $L_A = exp(-\frac{i}{2}\vec{\pi}_A \cdot \vec{\sigma})$ be the $(\frac{1}{2}, 0)$ representation of the Lorentz group that acts on the 2-component left-chiral spinor ξ_L:

$$\xi_L \to \xi'_L = L_A \xi_L. \tag{26}$$

where

$$L_A = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \tag{27}$$

In terms of the components u, v of ξ_L:

$$u \to u' = L_{11}u + L_{12}v, \qquad v \to v' = L_{21}u + L_{22}v. \tag{28}$$

Let us call this transformation scheme T_A.

Let $\dot{L}_A = exp(-\frac{i}{2}\vec{\pi}^*_A \cdot \vec{\sigma})$ be the dotted version corresponding to the $(0, \frac{1}{2})$ representation of the Lorentz group. $\dot{L}_A = (L_A^{-1})^\dagger$. Let ξ_R be the 2-component right-chiral spinor, $\xi_R = \epsilon\xi^*_L$, where

$$\epsilon = \epsilon^{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \epsilon^{-1} = \epsilon_{ab} = \epsilon^\dagger. \tag{29}$$

ξ_R transforms as

$$\xi_R \to \xi'_R = \dot{L}_A \xi_R \tag{30}$$

In terms of the components u, v, Equation (30) is equivalent to the scheme T_A given in Equation (28).

What happens when L_A acts on $\epsilon\xi_L$? In this case, in terms of the components

$$u \to u' = L_{22}u - L_{21}v, \qquad v \to v' = -L_{12}u + L_{11}v \tag{31}$$

Let us call this transformation scheme T_B. We can write T_B in a matrix form:

$$\begin{pmatrix} u \\ v \end{pmatrix} \to \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} L_{22} & -L_{21} \\ -L_{12} & L_{11} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \tag{32}$$

Let us name this transformation matrix as L_B. Note that, $L_B = (\dot{L}_A)^*$, and Equation (32) is nothing but the transformation of ξ_L under the action of L_B, which is a type T_B transformation.

Now, let $Z_A = exp(-\frac{i}{2}\vec{\pi}_A \cdot \vec{\Sigma})$ be the $(\frac{1}{2}, 0)$ representation of SL(4,C) that acts on a pair of 4-component undotted covariant spinors:

$$\chi_{(1)} \to \chi'_{(1)} = Z_A \chi_{(1)}, \qquad \chi_{(2)} \to \chi'_{(2)} = Z_A \chi_{(2)} \tag{33}$$

where $\chi_{(1)}$ and $\chi_{(2)}$ are the generalized eigenvectors of Σ^*_3 [5]:

[5] We may use the generalized eigenvectors of Σ_1 or Σ_2 matrices as well, but, in that case, we have to employ the other forms of the spinor metric.

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$$\chi_{(1)} = \frac{1}{\sqrt{2}} \begin{pmatrix} u \\ v \\ -iv \\ u \end{pmatrix}, \qquad \chi_{(2)} = \frac{1}{\sqrt{2}} \begin{pmatrix} v \\ u \\ iu \\ -v \end{pmatrix} \tag{34}$$

Indices in the parentheses are simply labels for 4-component spinors.

We also have another pair of generalized eigenvectors for Σ_3^*:

$$\chi_{(3)} = \frac{1}{\sqrt{2}} \begin{pmatrix} -v \\ u \\ -iu \\ -v \end{pmatrix}, \qquad \chi_{(4)} = \frac{1}{\sqrt{2}} \begin{pmatrix} u \\ -v \\ -iv \\ -u \end{pmatrix} \tag{35}$$

Transformation scheme of $\chi_{(3)}$ and $\chi_{(4)}$ is different from that of $\chi_{(1)}$ and $\chi_{(2)}$. Under the action of Z_A, $\chi_{(1)}$ and $\chi_{(2)}$ transform according to the scheme T_A, but $\chi_{(3)}$ and $\chi_{(4)}$ transform according to the scheme T_B. However, we may think in an alternative way: Suppose that $\chi_{(3)}$ and $\chi_{(3)}$ are different kind of objects with different transformation properties, such that another transformation matrix, Z_B, acts on them and under the action of Z_B they transform according to the scheme T_A:

$$\chi_{(3)} \to \chi'_{(3)} = Z_B \chi_{(3)}, \qquad \chi_{(4)} \to \chi'_{(4)} = Z_B \chi_{(4)}. \tag{36}$$

By definition $Z_B = A(L_B \otimes I)A^{-1}$:

$$Z_B = \begin{pmatrix} \alpha_0 & -\alpha_1 & \alpha_2 & -\alpha_3 \\ -\alpha_1 & \alpha_0 & i\alpha_3 & i\alpha_2 \\ \alpha_2 & -i\alpha_3 & \alpha_0 & i\alpha_1 \\ -\alpha_3 & -i\alpha_2 & -i\alpha_1 & \alpha_0 \end{pmatrix} = \alpha_0\Sigma_0 - \alpha_1\Sigma_1 + \alpha_2\Sigma_2 - \alpha_3\Sigma_3. \tag{37}$$

Or, simply

$$Z_B = (+-+-)_\Sigma \tag{38}$$

Although $\chi_{(1)}$ and $\chi_{(2)}$ have different transformation properties than $\chi_{(3)}$ and $\chi_{(4)}$, they are not completely independent:

$$-i\Sigma_2\chi_{(1)} = \chi_{(3)}, \qquad -i\Sigma_2\chi_{(2)} = \chi_{(4)} \tag{39}$$

where $-i\Sigma_2$ corresponds to a 180° CCW rotation about the x_2 axis in SL(4,C) [6]. For more about rotations in the spinor space see Secction 9.

Now let $\dot{Z}_A = exp(-\frac{i}{2}\vec{\pi}_A^* \cdot \vec{\Sigma})$ be the $(0, \frac{1}{2})$ representation. $\dot{Z}_A = (Z_A^{-1})^\dagger$. The first pair of 4-component spinors for this representation is

$$\dot{\chi}^{(1)} = \frac{1}{\sqrt{2}} \begin{pmatrix} \dot{v} \\ -\dot{u} \\ i\dot{u} \\ \dot{v} \end{pmatrix}, \qquad \dot{\chi}^{(2)} = \frac{1}{\sqrt{2}} \begin{pmatrix} -\dot{u} \\ \dot{v} \\ i\dot{v} \\ \dot{u} \end{pmatrix} \tag{40}$$

These are also the generalized eigenvectors of Σ_3^*. Under the action of $\dot{Z}_A$, $\dot{\chi}^{(1)}$ and $\dot{\chi}^{(2)}$ transform according to the scheme T_A.

$$\dot{\chi}^{(1)} \to \dot{\chi}^{(1)'} = \dot{Z}_A \dot{\chi}^{(1)}, \qquad \dot{\chi}^{(2)} \to \dot{\chi}^{(2)'} = \dot{Z}_A \dot{\chi}^{(2)} \tag{41}$$

[6] $-i\sigma_2$ corresponds to a 180° CCW rotation about the x_2 axis in SL(2,C).

The second pair of the generalized eigenvectors of Σ_3^* is defined as

$$\dot{\chi}^{(3)} = \frac{1}{\sqrt{2}}\begin{pmatrix} \dot{u} \\ \dot{v} \\ -i\dot{v} \\ \dot{u} \end{pmatrix}, \qquad \dot{\chi}^{(4)} = \frac{1}{\sqrt{2}}\begin{pmatrix} \dot{v} \\ \dot{u} \\ i\dot{u} \\ -\dot{v} \end{pmatrix} \tag{42}$$

Under the action of $\dot{Z}_A$, $\dot{\chi}^{(3)}$ and $\dot{\chi}^{(4)}$ transform according to the scheme T_B. But, they transform according to the scheme T_A under the action of $\dot{Z}_B$:

$$\dot{\chi}^{(3)} \to \dot{\chi}^{(3)'} = \dot{Z}_B\dot{\chi}^{(3)}, \qquad \dot{\chi}^{(4)} \to \dot{\chi}^{(4)'} = \dot{Z}_B\dot{\chi}^{(4)} \tag{43}$$

where $\dot{Z}_B = (Z_B^{-1})^\dagger$ by definition. We also have the following relations between them:

$$-i\Sigma_2\dot{\chi}^{(1)} = \dot{\chi}^{(3)}, \qquad -i\Sigma_2\dot{\chi}^{(2)} = \dot{\chi}^{(4)} \tag{44}$$

Note that $\chi^{(a)}$ can be related to $\chi_{(a)}$ only by the SL(4,C) metric g that involves the Minkowski metric. $\chi^{(a)} = g\chi_{(a)}$, and its dotted version is defined as [7]

$$\dot{\chi}^{(a)} = (g\chi_{(a)})^*. \tag{45}$$

The scalar product of 4-component spinors is defined in a similar way as 2-component spinors and they are invariant under the Z transformations.

We write various forms of Z and L matrices in a compact notation to manifest the parallelism between them:

$$L_A = (++++)_\sigma \qquad L_B = (+-+-)_\sigma \qquad \dot{L}_A = (+---)^*_\sigma \qquad \dot{L}_B = (++-+)^*_\sigma \tag{46}$$

$$Z_A = (++++)_\Sigma \qquad Z_B = (+-+-)_\Sigma \qquad \dot{Z}_A = (+---)^*_\Sigma \qquad \dot{Z}_B = (++-+)^*_\Sigma \tag{47}$$

3. Outer products of 4-component spinors

Let us define the outer product $W_L = \xi_L\xi_L^\dagger$ which transforms as

$$W_L \to W_L' = (L_A\xi_L)(L_A\xi_L)^\dagger = L_A W_L L_A^\dagger \tag{48}$$

This is a type T_A transformation. Determinant of W_L is zero, hence W_L can be associated with a null 4-vector through the substitutions, $t = \frac{1}{2}(u\dot{u} + v\dot{v})$, $x = \frac{1}{2}(u\dot{v} + v\dot{u})$, $y = \frac{i}{2}(u\dot{v} - v\dot{u})$, $z = \frac{1}{2}(u\dot{u} - v\dot{v})$:

$$W_L = \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix} \tag{49}$$

We also define the outer product $W_R = \xi_R\xi_R^\dagger$ which transforms as

$$W_R \to W_R' = (\dot{L}_A\xi_R)(\dot{L}_A\xi_R)^\dagger = \dot{L}_A W_R \dot{L}_A^\dagger \tag{50}$$

This is also a type T_A transformation. Determinant of W_R is zero and W_R can be associated with a null 4-vector:

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[7] The upper dot on a spinorial object simply means complex conjugation: $\dot{\chi}^{(a)} = (\chi^{(a)})^*$. But, the upper dot on an element of SL(2,C) or SL(4,C) has a particular meaning. $\dot{L}_A = exp(-\frac{i}{2}\vec{\pi}_A^* \cdot \vec{\sigma}) \neq L_A^*$. Similarly, $\dot{Z}_A = exp(-\frac{i}{2}\vec{\pi}_A^* \cdot \vec{\Sigma}) \neq Z_A^*$.

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$$W_R = \begin{pmatrix} t-z & -x+iy \\ -x-iy & t+z \end{pmatrix} \tag{51}$$

W_R can be obtained from W_L by parity inversion.

We define the outer product of 4-component spinors, $\mathcal{W}_{(aa)} = \chi_{(a)}\chi^\dagger_{(a)}$, which transform in a similar way as W_L:

$$\mathcal{W}_{(aa)} \to \mathcal{W}'_{(aa)} = (Z_A\chi_{(a)})(Z_A\chi_{(a)})^\dagger = Z_A\mathcal{W}_{(aa)}Z_A^\dagger. \tag{52}$$

For $a=1$ and $a=2$, $\mathcal{W}_{(aa)}$ transforms according to the scheme T_A. Equation (52) is the main motivation behind the interpretation of $\chi_{(a)}$ as a 4-component spinor for SL(4, C).

$\dot{\mathcal{W}}^{(aa)} = \dot{\chi}^{(a)}\dot{\chi}^{(a)\dagger}$ transform in a similar way as W_R:

$$\dot{\mathcal{W}}^{(aa)} \to \dot{Z}_A\dot{\mathcal{W}}^{(aa)}\dot{Z}_A^\dagger \tag{53}$$

For $a=1$ and $a=2$, $\dot{\mathcal{W}}^{(aa)}$ transform according to the scheme T_A.

We also have outer products of 4-component spinors of the other kind. For $a=3$ and $a=4$, $\mathcal{W}_{(aa)}$ and $\dot{\mathcal{W}}^{(aa)}$ transform according to the scheme T_A under the action of Z_B and $\dot{Z}_B$:

$$\mathcal{W}_{(aa)} \to Z_B\mathcal{W}_{(aa)}Z_B^\dagger, \qquad \dot{\mathcal{W}}^{(aa)} \to \dot{Z}_B\dot{\mathcal{W}}^{(aa)}\dot{Z}_B^\dagger \tag{54}$$

$\mathcal{W}_{(aa)}$ and $\dot{\mathcal{W}}^{(aa)}$ are Hermitian and zero determinant matrices and they are the basic elements of the outer product forms. In the next section it will be shown that combinations of these basic elements can be associated with 4-vectors.

4. Quaternion forms and 4-vectors

In general, we can treat t, x, y and z as variables that do not depend on u and v. Then, we can associate the following matrices X_L and X_R with 4-vectors, which are not necessarily null:

$$W_L \to X_L = \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix} = t\sigma_0 + x\sigma_1 + y\sigma_2 + z\sigma_3 = (++++)_\sigma \tag{55}$$

$$W_R \to X_R = \begin{pmatrix} t-z & -x+iy \\ -x-iy & t+z \end{pmatrix} = t\sigma_0 - x\sigma_1 - y\sigma_2 - z\sigma_3 = (+---)_\sigma \tag{56}$$

$\det X_L = \det X_R = t^2 - x^2 - y^2 - z^2$ and in general not zero [8] . X_L and X_R transform as

$$X_L \to X'_L = L_A X_L L_A^\dagger, \qquad X_R \to X'_R = \dot{L}_A X_R \dot{L}_A^\dagger \tag{57}$$

In SL(4,C), in order to have a similar representation, we have to introduce the following two column objects that are pairwise combinations of 4-component spinors:

$$\chi_A = \frac{1}{\sqrt{2}}\begin{pmatrix} u & v \\ v & u \\ -iv & iu \\ u & -v \end{pmatrix}, \quad \chi_B = \frac{1}{\sqrt{2}}\begin{pmatrix} -v & u \\ u & -v \\ -iu & -iv \\ v & -u \end{pmatrix}, \quad \dot{\chi}^A = \frac{1}{\sqrt{2}}\begin{pmatrix} \dot{v} & -\dot{u} \\ -\dot{u} & \dot{v} \\ i\dot{u} & i\dot{v} \\ \dot{v} & \dot{u} \end{pmatrix}, \quad \dot{\chi}^B = \frac{1}{\sqrt{2}}\begin{pmatrix} \dot{u} & \dot{v} \\ \dot{v} & \dot{u} \\ -i\dot{v} & i\dot{u} \\ \dot{u} & -\dot{v} \end{pmatrix} \tag{58}$$

where $\chi_A = (\chi_{(1)}, \chi_{(2)})$, $\chi_B = (\chi_{(3)}, \chi_{(4)})$, $\dot{\chi}^A = (\dot{\chi}^{(1)}, \dot{\chi}^{(2)})$ and $\dot{\chi}^B = (\dot{\chi}^{(3)}, \dot{\chi}^{(4)})$.

Let us define an outer product of the form $\mathcal{W}_A = \chi_A\chi_A^\dagger$, which is formally a quaternion:

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8 These are matrix representations of quaternions, because $-i\sigma_1, -i\sigma_2, -i\sigma_3$ matrices have the same properties as the Hamilton's quaternion basis, $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$: $X_L = t\sigma_0 + ix(-i\sigma_1) + iy(-i\sigma_2) + iz(-i\sigma_3) = t\mathbb{1} + ix\boldsymbol{i} + iy\boldsymbol{j} + iz\boldsymbol{k}$, $X_R = t1 - ix\boldsymbol{i} - iy\boldsymbol{j} - iz\boldsymbol{k}$.

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$$\mathcal{W}_A = \frac{1}{2}\begin{pmatrix} u\dot{u}+v\dot{v} & u\dot{v}+v\dot{u} & iu\dot{v}-iv\dot{u} & u\dot{u}-v\dot{v} \\ v\dot{u}+u\dot{v} & v\dot{v}+u\dot{u} & iv\dot{v}-iu\dot{u} & v\dot{u}-u\dot{v} \\ -iv\dot{u}+iu\dot{v} & -iv\dot{v}+iu\dot{u} & v\dot{v}+u\dot{u} & -iv\dot{u}-iu\dot{v} \\ u\dot{u}-v\dot{v} & u\dot{v}-v\dot{u} & iu\dot{v}+iv\dot{u} & u\dot{u}+v\dot{v} \end{pmatrix} \tag{59}$$

$\mathcal{W}_A$ can be written as a sum of two basic forms: $\mathcal{W}_A = \mathcal{W}_{(11)} + \mathcal{W}_{(22)}$. In its present form $\det\mathcal{W}_A = 0$ and $\mathcal{W}_A$ corresponds to a null 4-vector, but we can associate $\mathcal{W}_A$ with an arbitrary 4-vector in terms of the variables t, x, y and z:

$$\mathcal{W}_A \to \mathcal{Q}_A = \begin{pmatrix} t & x & y & z \\ x & t & -iz & iy \\ y & iz & t & -ix \\ z & -iy & ix & t \end{pmatrix} = t\Sigma_0 + x\Sigma_1 + y\Sigma_2 + z\Sigma_3. \tag{60}$$

$\mathcal{Q}_A = (++++)_\Sigma$ and it is the 4×4 version of X_L:

$$\mathcal{Q}_A = A(X_L \otimes I)A^{-1} \tag{61}$$

Similarly, we define $\dot{\mathcal{W}}^A$:

$$\dot{\mathcal{W}}^A = \dot{\chi}^A\dot{\chi}^{A\dagger} = \dot{\mathcal{W}}^{(11)} + \dot{\mathcal{W}}^{(22)} = \frac{1}{2}\begin{pmatrix} \dot{v}v+\dot{u}u & -\dot{v}u-\dot{u}v & -i\dot{v}u+i\dot{u}v & \dot{v}v-\dot{u}u \\ -\dot{u}v-\dot{v}u & \dot{u}u+\dot{v}v & i\dot{u}u-i\dot{v}v & -\dot{u}v+\dot{v}u \\ i\dot{u}v-i\dot{v}u & -i\dot{u}u+i\dot{v}v & \dot{u}u+\dot{v}v & i\dot{u}v+i\dot{v}u \\ \dot{v}v-\dot{u}u & -\dot{v}u+\dot{u}v & -i\dot{v}u-i\dot{u}v & \dot{v}v+\dot{u}u \end{pmatrix} \tag{62}$$

In terms of the variables t, x, y and z

$$\dot{\mathcal{W}}^A \to \dot{\mathcal{Q}}^A = \begin{pmatrix} t & -x & -y & -z \\ -x & t & iz & -iy \\ -y & -iz & t & ix \\ -z & iy & -ix & t \end{pmatrix} = t\Sigma_0 - x\Sigma_1 - y\Sigma_2 - z\Sigma_3. \tag{63}$$

$\dot{\mathcal{Q}}^A = (+---)_\Sigma$ and it is the 4×4 version of X_R:

$$\dot{\mathcal{Q}}^A = A(X_R \otimes I)A^{-1}. \tag{64}$$

$\dot{\mathcal{Q}}^A$ can be obtained from $\mathcal{Q}_A$ by parity inversion and they transform as

$$\mathcal{Q}_A \to Z_A\mathcal{Q}_A Z_A^\dagger, \qquad \dot{\mathcal{Q}}^A \to \dot{Z}_A\dot{\mathcal{Q}}^A\dot{Z}_A^\dagger \tag{65}$$

These are type T_A transformations, hence these forms correspond to 4-vectors.

The outer product $\mathcal{W}_B = \chi_B\chi_B^\dagger$ is also a quaternion:

$$\mathcal{W}_B = \frac{1}{2}\begin{pmatrix} v\dot{v}+u\dot{u} & -v\dot{u}-u\dot{v} & -iv\dot{u}+iu\dot{v} & v\dot{v}-u\dot{u} \\ -u\dot{v}-v\dot{u} & u\dot{u}+v\dot{v} & iu\dot{u}-iv\dot{v} & -u\dot{v}+v\dot{u} \\ iu\dot{v}-iv\dot{u} & -iu\dot{u}+iv\dot{v} & u\dot{u}+v\dot{v} & iu\dot{v}+iv\dot{u} \\ v\dot{v}-u\dot{u} & -v\dot{u}+u\dot{v} & -iv\dot{u}-iu\dot{v} & v\dot{v}+u\dot{u} \end{pmatrix} \tag{66}$$

In terms of variables t, x, y and z

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$$\mathcal{W}_B \to \mathcal{Q}_B = \begin{pmatrix} t & -x & y & -z \\ -x & t & iz & iy \\ y & -iz & t & ix \\ -z & -iy & -ix & t \end{pmatrix} = t\Sigma_0 - x\Sigma_1 + y\Sigma_2 - z\Sigma_3 \tag{67}$$

In short, $\mathcal{Q}_B = (+-+-)_\Sigma$

We also write $\dot{\mathcal{W}}^B$:

$$\dot{\mathcal{W}}^B = \dot{\chi}^B \dot{\chi}^{B\dagger} = \frac{1}{2}\begin{pmatrix} \dot{u}u + \dot{v}v & \dot{u}v + \dot{v}u & i\dot{u}v - i\dot{v}u & \dot{u}u - \dot{v}v \\ \dot{v}u + \dot{u}v & \dot{v}v + \dot{u}u & i\dot{v}v - i\dot{u}u & \dot{v}u - \dot{u}v \\ -i\dot{v}u + i\dot{u}v & -i\dot{v}v + i\dot{u}u & \dot{v}v + \dot{u}u & -i\dot{v}u - i\dot{u}v \\ \dot{u}u - \dot{v}v & \dot{u}v - \dot{v}u & i\dot{u}v + i\dot{v}u & \dot{u}u + \dot{v}v \end{pmatrix} \tag{68}$$

$$\dot{\mathcal{W}}^B \to \dot{\mathcal{Q}}^B = \begin{pmatrix} t & x & -y & z \\ x & t & -iz & -iy \\ -y & iz & t & -ix \\ z & iy & ix & t \end{pmatrix} = t\Sigma_0 + x\Sigma_1 - y\Sigma_2 + z\Sigma_3. \tag{69}$$

$\dot{\mathcal{Q}}^B = (++-+)_\Sigma$ and it can be obtained from $\mathcal{Q}_B$ by parity inversion. $\mathcal{Q}_B$ and $\dot{\mathcal{Q}}^B$ are transformed by Z_B and $\dot{Z}_B$:

$$\mathcal{Q}_B \to Z_B \mathcal{Q}_B Z_B^\dagger, \qquad \dot{\mathcal{Q}}^B \to \dot{Z}_B \dot{\mathcal{Q}}^B \dot{Z}_B^\dagger \tag{70}$$

These transformations obey the scheme T_A also, hence $\mathcal{Q}_B$ and $\dot{\mathcal{Q}}^B$ can be associated with 4-vectors.

With our compact notation it can be shown that the form of the transformation matrix matches the form of the transformed object. For example, $Z_A = (++++)_\Sigma$ acts on the form $\mathcal{Q}_A = (++++)_\Sigma$, $Z_B = (+-+-)_\Sigma$ acts on the form $\mathcal{Q}_B = (+-+-)_\Sigma$, $\dot{Z}_A = (+---)^*_\Sigma$ acts on the form $\dot{\mathcal{Q}}^A = (+---)_\Sigma$, and $\dot{Z}_B = (++-+)^*_\Sigma$ acts on the form $\dot{\mathcal{Q}}^B = (++-+)_\Sigma$.

5. Two more pairs of spinors

There are four eigenvectors of Σ_3^* that constitute a complete orthonormal set of basis:

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 1 \\ -i \\ 0 \end{pmatrix}, \quad e_4 = \begin{pmatrix} 0 \\ 1 \\ i \\ 0 \end{pmatrix}. \tag{71}$$

e_1 and e_3 correspond to $+1$ eigenvalue and e_2 and e_4 correspond to -1 eigenvalue. We obtain eight generalized eigenvectors by combining the basis corresponding to the same eigenvalue. For example, we can obtain the four generalized undotted covariant eigenvectors that we have previously studied as follows:

$$\chi_{(1)} = ue_1 + ve_3, \qquad \chi_{(2)} = ve_2 + ue_4, \tag{72}$$

$$\chi_{(3)} = -ve_1 + ue_3, \qquad \chi_{(4)} = ue_2 - ve_4, \tag{73}$$

We can obtain four more generalized eigenvectors of Σ_3^* by changing the sign or swapping u and v:

$$\chi_{(5)} = -ue_1 + ve_3, \qquad \chi_{(6)} = ve_2 - ue_4, \tag{74}$$

$$\chi_{(7)} = ve_1 + ue_3, \qquad \chi_{(8)} = ue_2 + ve_4, \tag{75}$$

Totally we get eight undotted covariant spinors:

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$$\chi_{(1)} = \begin{pmatrix} u \\ v \\ -iv \\ u \end{pmatrix}, \quad \chi_{(2)} = \begin{pmatrix} v \\ u \\ iu \\ -v \end{pmatrix}, \quad \chi_{(3)} = \begin{pmatrix} -v \\ u \\ -iu \\ -v \end{pmatrix}, \quad \chi_{(4)} = \begin{pmatrix} u \\ -v \\ -iv \\ -u \end{pmatrix}, \tag{76}$$

$$\chi_{(5)} = \begin{pmatrix} -u \\ v \\ -iv \\ -u \end{pmatrix}, \quad \chi_{(6)} = \begin{pmatrix} v \\ -u \\ -iu \\ -v \end{pmatrix}, \quad \chi_{(7)} = \begin{pmatrix} v \\ u \\ -iu \\ v \end{pmatrix}, \quad \chi_{(8)} = \begin{pmatrix} u \\ v \\ iv \\ -u \end{pmatrix}. \tag{77}$$

We can group $\chi_{(a)}$ $(a = 1, 2, \cdots 8)$ pairwise:

$$P_A = \{\chi_{(1)}, \chi_{(2)}\}, \quad P_B = \{\chi_{(3)}, \chi_{(4)}\}, \quad P_C = \{\chi_{(5)}, \chi_{(6)}\}, \quad P_D = \{\chi_{(7)}, \chi_{(8)}\}, \tag{78}$$

We already know that P_A transforms with Z_A and P_B transforms with Z_B. Following the same procedure that we have applied in the previous sections we can show that P_C and P_D transform with Z_C and Z_D respectively:

$$Z_C = A(L_C \otimes I)A^{-1}, \qquad Z_D = A(L_D \otimes I)A^{-1}, \tag{79}$$

where

$$L_C = \begin{pmatrix} L_{11} & -L_{12} \\ -L_{21} & L_{22} \end{pmatrix} = \begin{pmatrix} \alpha_0 + \alpha_3 & -\alpha_1 + i\alpha_2 \\ -\alpha_1 - i\alpha_2 & \alpha_0 - \alpha_3 \end{pmatrix} = (+--+)_\sigma \tag{80}$$

$$L_D = \begin{pmatrix} L_{22} & L_{21} \\ L_{12} & L_{11} \end{pmatrix} = \begin{pmatrix} \alpha_0 - \alpha_3 & \alpha_1 + i\alpha_2 \\ \alpha_1 - i\alpha_2 & \alpha_0 + \alpha_3 \end{pmatrix} = (++--)_\sigma \tag{81}$$

$$\dot{L}_C = (+++-)^*_\sigma = L^*_D \tag{82}$$

$$\dot{L}_D = (+-++)^*_\sigma = L^*_C \tag{83}$$

The corresponding SL(4,C) matrices are

$$Z_C = \begin{pmatrix} \alpha_0 & -\alpha_1 & -\alpha_2 & \alpha_3 \\ -\alpha_1 & \alpha_0 & -i\alpha_3 & -i\alpha_2 \\ -\alpha_2 & i\alpha_3 & \alpha_0 & i\alpha_1 \\ \alpha_3 & i\alpha_2 & -i\alpha_1 & \alpha_0 \end{pmatrix} = (+--+)_\Sigma \tag{84}$$

$$Z_D = \begin{pmatrix} \alpha_0 & \alpha_1 & -\alpha_2 & -\alpha_3 \\ \alpha_1 & \alpha_0 & i\alpha_3 & -i\alpha_2 \\ -\alpha_2 & -i\alpha_3 & \alpha_0 & -i\alpha_1 \\ -\alpha_3 & i\alpha_2 & i\alpha_1 & \alpha_0 \end{pmatrix} = (++--)_\Sigma \tag{85}$$

There are also the dotted versions:

$$\dot{Z}_C = \begin{pmatrix} \alpha^*_0 & \alpha^*_1 & \alpha^*_2 & -\alpha^*_3 \\ \alpha^*_1 & \alpha^*_0 & i\alpha^*_3 & i\alpha^*_2 \\ \alpha^*_2 & -i\alpha^*_3 & \alpha^*_0 & -i\alpha^*_1 \\ -\alpha^*_3 & -i\alpha^*_2 & i\alpha^*_1 & \alpha^*_0 \end{pmatrix} = (+++-)^*_\Sigma \tag{86}$$

$$\dot{Z}_D = \begin{pmatrix} \alpha^*_0 & -\alpha^*_1 & \alpha^*_2 & \alpha^*_3 \\ -\alpha^*_1 & \alpha^*_0 & -i\alpha^*_3 & i\alpha^*_2 \\ \alpha^*_2 & i\alpha^*_3 & \alpha^*_0 & i\alpha^*_1 \\ \alpha^*_3 & -i\alpha^*_2 & -i\alpha^*_1 & \alpha^*_0 \end{pmatrix} = (+-++)^*_\Sigma \tag{87}$$

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We also define eight dotted contravariat spinors $\dot{\chi}^{(a)} = (g\chi_{(a)})^*$ $(a = 1, 2, \cdots 8)$ which are also the generalized eigenvectors of Σ_3^*:

$$\dot{\chi}^{(1)} = \begin{pmatrix} \dot{v} \\ -\dot{u} \\ i\dot{u} \\ \dot{v} \end{pmatrix}, \quad \dot{\chi}^{(2)} = \begin{pmatrix} -\dot{u} \\ \dot{v} \\ i\dot{v} \\ \dot{u} \end{pmatrix}, \quad \dot{\chi}^{(3)} = \begin{pmatrix} \dot{u} \\ \dot{v} \\ -i\dot{v} \\ \dot{u} \end{pmatrix}, \quad \dot{\chi}^{(4)} = \begin{pmatrix} \dot{v} \\ \dot{u} \\ i\dot{u} \\ -\dot{v} \end{pmatrix} \tag{88}$$

$$\dot{\chi}^{(5)} = \begin{pmatrix} \dot{v} \\ \dot{u} \\ -i\dot{u} \\ \dot{v} \end{pmatrix}, \quad \dot{\chi}^{(6)} = \begin{pmatrix} \dot{u} \\ \dot{v} \\ i\dot{v} \\ -\dot{u} \end{pmatrix}, \quad \dot{\chi}^{(7)} = \begin{pmatrix} \dot{u} \\ -\dot{v} \\ i\dot{v} \\ \dot{u} \end{pmatrix}, \quad \dot{\chi}^{(8)} = \begin{pmatrix} -\dot{v} \\ \dot{u} \\ i\dot{u} \\ \dot{v} \end{pmatrix}. \tag{89}$$

We group them pairwise:

$$P^A = \{\dot{\chi}^{(1)}, \dot{\chi}^{(2)}\}, \quad P^B = \{\dot{\chi}^{(3)}, \dot{\chi}^{(4)}\}, \quad P^C = \{\dot{\chi}^{(5)}, \dot{\chi}^{(6)}\}, \quad P^D = \{\dot{\chi}^{(7)}, \dot{\chi}^{(8)}\} \tag{90}$$

Each pair of the dotted contravariant spinors transform with the associated dotted Z matrix.

We define four two-column undotted covariant objects (spinor pairs):

$$\chi_A = (\chi_{(1)}, \chi_{(2)}), \quad \chi_B = (\chi_{(3)}, \chi_{(4)}), \quad \chi_C = (\chi_{(5)}, \chi_{(6)}), \quad \chi_D = (\chi_{(7)}, \chi_{(8)}) \tag{91}$$

And we define the corresponding two-column dotted contravariant spinor pairs

$$\dot{\chi}^A = (\dot{\chi}^{(1)}, \dot{\chi}^{(2)}), \quad \dot{\chi}^B = (\dot{\chi}^{(3)}, \dot{\chi}^{(4)}), \quad \dot{\chi}^C = (\dot{\chi}^{(5)}, \dot{\chi}^{(6)}), \quad \dot{\chi}^D = (\dot{\chi}^{(7)}, \dot{\chi}^{(8)}) \tag{92}$$

Finally, we construct eight outer products that can be associated with 4-vectors:

$$\chi_A\chi_A^\dagger \to \mathcal{Q}_A = (++++)_\Sigma \qquad \dot{\chi}^A\dot{\chi}^{A\dagger} \to \dot{\mathcal{Q}}^A = (+---)_\Sigma \tag{93}$$

$$\chi_B\chi_B^\dagger \to \mathcal{Q}_B = (+-+-)_\Sigma \qquad \dot{\chi}^B\dot{\chi}^{B\dagger} \to \dot{\mathcal{Q}}^B = (++-+)_\Sigma \tag{94}$$

$$\chi_C\chi_C^\dagger \to \mathcal{Q}_C = (+--+)_\Sigma \qquad \dot{\chi}^C\dot{\chi}^{C\dagger} \to \dot{\mathcal{Q}}^C = (+++-)_\Sigma \tag{95}$$

$$\chi_D\chi_D^\dagger \to \mathcal{Q}_D = (++--)_\Sigma \qquad \dot{\chi}^D\dot{\chi}^{D\dagger} \to \dot{\mathcal{Q}}^D = (+-++)_\Sigma \tag{96}$$

Each form transforms in its own way with the matching Z or $\dot{Z}$ matrix.

6. Complex conjugated forms

There are also dotted covariant and undotted contravariant 4-component spinors which have no counterparts in SL(2,C): $\dot{\chi}_{(a)} = (\chi_{(a)})^*$, $\chi^{(a)} = (\dot{\chi}^{(a)})^*$. They are the generalized eigenvectors of Σ_3. We group them pairwise to get eight more two-column objects (spinor pairs):

$$\dot{\chi}_A = (\dot{\chi}_{(1)}, \dot{\chi}_{(2)}), \quad \dot{\chi}_B = (\dot{\chi}_{(3)}, \dot{\chi}_{(4)}), \quad \dot{\chi}_C = (\dot{\chi}_{(5)}, \dot{\chi}_{(6)}), \quad \dot{\chi}_D = (\dot{\chi}_{(7)}, \dot{\chi}_{(8)}) \tag{97}$$

$$\chi^A = (\chi^{(1)}, \chi^{(2)}), \quad \chi^B = (\chi^{(3)}, \chi^{(4)}), \quad \chi^C = (\chi^{(5)}, \chi^{(6)}), \quad \chi^D = (\chi^{(7)}, \chi^{(8)}) \tag{98}$$

All conjugated pairs live in the conjugate space and their outer products, that can be associated with 4-vectors, also live in the conjugate space:

$$\mathcal{Q}_A = (++++)_\Sigma \xrightarrow{\text{c.c.}} \dot{\mathcal{Q}}_A = (++++)_{\Sigma^*} = \dot{\chi}_A\dot{\chi}_A^\dagger \tag{99}$$

$$\mathcal{Q}_B = (+-+-)_\Sigma \xrightarrow{\text{c.c.}} \dot{\mathcal{Q}}_B = (+-+-)_{\Sigma^*} = \dot{\chi}_B\dot{\chi}_B^\dagger \tag{100}$$

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$$\mathcal{Q}_C = (+--+)_\Sigma \xrightarrow{\text{c.c.}} \dot{\mathcal{Q}}_C = (+--+)_{\Sigma^*} = \dot{\chi}_C \dot{\chi}_C^\dagger \tag{101}$$

$$\mathcal{Q}_D = (++--)_\Sigma \xrightarrow{\text{c.c.}} \dot{\mathcal{Q}}_D = (++--)_{\Sigma^*} = \dot{\chi}_D \dot{\chi}_D^\dagger \tag{102}$$

$$\dot{\mathcal{Q}}^A = (+---)_\Sigma \xrightarrow{\text{c.c.}} \mathcal{Q}^A = (+---)_{\Sigma^*} = \chi^A \chi^{A\dagger} \tag{103}$$

$$\dot{\mathcal{Q}}^B = (++-+)_\Sigma \xrightarrow{\text{c.c.}} \mathcal{Q}^B = (++-+)_{\Sigma^*} = \chi^B \chi^{B\dagger} \tag{104}$$

$$\dot{\mathcal{Q}}^C = (+++-)_\Sigma \xrightarrow{\text{c.c.}} \mathcal{Q}^C = (+++-)_{\Sigma^*} = \chi^C \chi^{C\dagger} \tag{105}$$

$$\dot{\mathcal{Q}}^D = (+-++)_\Sigma \xrightarrow{\text{c.c.}} \mathcal{Q}^D = (+-++)_{\Sigma^*} = \chi^D \chi^{D\dagger} \tag{106}$$

The space of conjugate quaternions is spanned by Σ_μ^* and it has no counterpart in SL(2, C). Dotted lower indexed and undotted upper indexed forms transform with Z_X^* and with $(\dot{Z}_X)^*$ matrices respectively ($X = A, B, C, D$), and all transformations obey the scheme T_A.

4-vector scalar product can be defined by using the conjugate forms in two equivalent ways. Let Q and P be two 4-vectors and let $\mathcal{Q}_X$ and $\mathcal{P}_X$ be the corresponding quaternions :

$$\mathsf{Q}\cdot\mathsf{P} = \frac{1}{4}\mathrm{Tr}(\mathcal{Q}_X^T \mathcal{P}^X) \tag{107}$$

Or, noting that $\mathcal{P}^X = g\mathcal{P}_X g^{-1}$, we can also write

$$\mathsf{Q}\cdot\mathsf{P} = \frac{1}{4}(\mathcal{Q}_{\mu\dot{\nu}}\mathcal{P}^{\mu\dot{\nu}}) \tag{108}$$

where now indices refer to the components and the summation convention is implied [9] [10].

7. Four types of transformations for SL(2,C)

We can suggest a similar formalism for SL(2,C). Let ξ_A, ξ_B, ξ_C and ξ_D be covariant spinors:

$$\xi_A = \begin{pmatrix} u \\ v \end{pmatrix}, \quad \xi_B = \begin{pmatrix} v \\ -u \end{pmatrix}, \quad \xi_C = \begin{pmatrix} u \\ -v \end{pmatrix}, \quad \xi_D = \begin{pmatrix} v \\ u \end{pmatrix} \tag{109}$$

and let ξ^A, ξ^B, ξ^C and ξ^D be contravariant spinors:

$$\xi^A = \begin{pmatrix} v \\ -u \end{pmatrix}, \quad \xi^B = \begin{pmatrix} -u \\ -v \end{pmatrix}, \quad \xi^C = \begin{pmatrix} -v \\ -u \end{pmatrix}, \quad \xi_D = \begin{pmatrix} u \\ -v \end{pmatrix} \tag{110}$$

where

$$\xi^A = \xi_B, \quad \xi^B = -\xi_A, \quad \xi^C = -\xi_D, \quad \xi^D = \xi_C \tag{111}$$

This proliferation is necessary for the symmetry in Figure 1.

We have the following transformation properties:

$$\xi_A \to \xi'_A = L_A\xi_A, \quad \xi_B \to \xi'_B = L_B\xi_B, \quad \xi_C \to \xi'_C = L_C\xi_C, \quad \xi_D \to \xi'_D = L_D\xi_D \tag{112}$$

$$\dot{\xi}^A \to \dot{\xi}'^A = \dot{L}_A\dot{\xi}^A, \quad \dot{\xi}^B \to \dot{\xi}'^B = \dot{L}_B\dot{\xi}^B, \quad \dot{\xi}^C \to \dot{\xi}'^C = \dot{L}_C\dot{\xi}^C, \quad \dot{\xi}^D \to \dot{\xi}'^D = \dot{L}_D\dot{\xi}^D \tag{113}$$

$$\dot{L}_A = L_B^*, \quad \dot{L}_B = L_A^*, \quad \dot{L}_C = L_D^*, \quad \dot{L}_D = L_C^* \tag{114}$$

All transformations obey the scheme T_A.

9 In these expressions complex conjugation is not explicit, but $\mathcal{P}^X = (\dot{\mathcal{P}}^X)^* = (....)_{\Sigma^*}$, i.e., $\mathcal{P}^X$ is based on Σ_μ^* matrices.
10 Actually, there is one more way: $\mathsf{Q}\cdot\mathsf{P} = \frac{1}{8}\mathrm{Tr}(\mathcal{Q}\dot{\mathcal{P}} + \mathcal{P}\dot{\mathcal{Q}})$

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We have the following outer products:

$$\xi_A \xi_A^\dagger \to X_A = (+ + ++)_\sigma \qquad \xi_B \xi_B^\dagger \to X_B = (+ - +-)_\sigma \tag{115}$$

$$\xi_C \xi_C^\dagger \to X_C = (+ - -+)_\sigma \qquad \xi_D \xi_D^\dagger \to X_D = (+ + --)_\sigma \tag{116}$$

$$\dot{\xi}^A \dot{\xi}^{A\dagger} \to \dot{X}^A = (+ - --)_\sigma \qquad \dot{\xi}^B \dot{\xi}^{B\dagger} \to \dot{X}^B = (+ + -+)_\sigma \tag{117}$$

$$\dot{\xi}^C \dot{\xi}^{C\dagger} \to \dot{X}^C = (+ + +-)_\sigma \qquad \dot{\xi}^D \dot{\xi}^{D\dagger} \to \dot{X}^D = (+ - ++)_\sigma \tag{118}$$

It is worth noting that complex conjugating these forms does not yield anything new. In order to emphasize the difference with 4-component spinor pairs let us define the following sets:

$$\xi = \{\xi_A, \xi_B, \xi_C, \xi_D, \dot{\xi}^A, \dot{\xi}^B, \dot{\xi}^C, \dot{\xi}^D\}, \quad \xi^* = \{\dot{\xi}_A, \dot{\xi}_B, \dot{\xi}_C, \dot{\xi}_D, \xi^A, \xi^B, \xi^C, \xi^D\}, \tag{119}$$

The outer products of the elements of the set ξ generate all forms of X matrices that are based on σ matrices, $X_\sigma = \{(+ + ++)_\sigma, (+ + +-)_\sigma, (+ + -+)_\sigma, (+ + --)_\sigma, (+ - ++)_\sigma, (+ - +-)_\sigma, (+ - -+)_\sigma, (+ - --)_\sigma\}$. These are the matrices given in Eqations (115)-(118). Similarly, the elements of the set ξ^* generate all forms of X matrices that are based on σ^* matrices, $X_{\sigma^*} = \{(+ + ++)_{\sigma^*}, (+ + +-)_{\sigma^*}, (+ + -+)_{\sigma^*}, (+ + --)_{\sigma^*}, (+ - ++)_{\sigma^*}, (+ - +-)_{\sigma^*}, (+ - -+)_{\sigma^*}, (+ - --)_{\sigma^*}\}$. But, the set X_σ is equal to the set X_{σ^*} [11].

Likewise, we define two sets for 4-component spinor pairs

$$\chi = \{\chi_A, \chi_B, \chi_C, \chi_D, \dot{\chi}^A, \dot{\chi}^B, \dot{\chi}^C, \dot{\chi}^D\}, \quad \chi^* = \{\dot{\chi}_A, \dot{\chi}_B, \dot{\chi}_C, \dot{\chi}_D, \chi^A, \chi^B, \chi^C, \chi^D\}, \tag{120}$$

The outer products of the elements of the set χ generate all forms of Q matrices that are based on Σ matrices, $Q_\Sigma = \{(+ + ++)_\Sigma, (+ + +-)_\Sigma, (+ + -+)_\Sigma, (+ + --)_\Sigma, (+ - ++)_\Sigma, (+ - +-)_\Sigma, (+ - -+)_\Sigma, (+ - --)_\Sigma\}$. These are the matrices given in Eqations (93)-(96). Similarly, the elements of the set χ^* generate all forms of Q matrices that are based on Σ^* matrices, $Q_{\Sigma^*} = \{(+ + ++)_{\Sigma^*}, (+ + +-)_{\Sigma^*}, (+ + -+)_{\Sigma^*}, (+ + --)_{\Sigma^*}, (+ - ++)_{\Sigma^*}, (+ - +-)_{\Sigma^*}, (+ - -+)_{\Sigma^*}, (+ - --)_{\Sigma^*}\}$. Now, the set Q_Σ is completely distinct from the set Q_{Σ^*}.

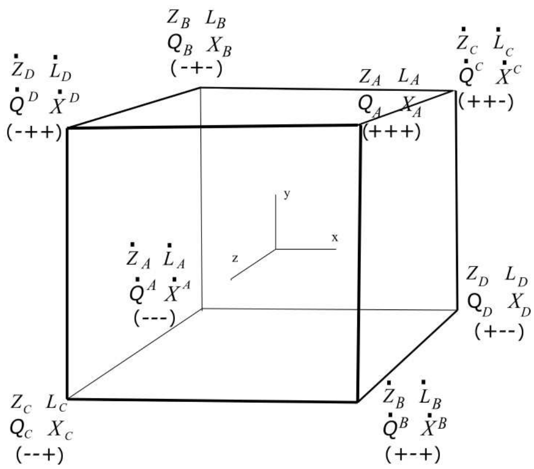

Figure 1. Reflections and inversions. Σ^* space is not shown.

[11] The set ξ is equal to the set ξ^* due to the relations given in Equation (111), and due to the relations $\sigma_1^* = \sigma_1$, $\sigma_2^* = -\sigma_2$, $\sigma_3^* = \sigma_3$, the set X_σ is equal to the set X_{σ^*}.

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8. Conjugate spaces in diagonal basis

It is possible to diagonalize Σ_3 and Σ_3^* simultaneously. Actually, in E and $\widetilde{E}$ basis both E_3 and $\widetilde{E}_3$ were already diagonal. Hence we can construct a similar 4-component spinor formalism by simply changing bases back to E and $\widetilde{E}$:

$$E_i = A^{-1}\Sigma_i A, \qquad \widetilde{E}_i = A^{-1}\Sigma_i^* A \tag{121}$$

where

$$A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & i & -i & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}, \qquad A^{-1} = A^\dagger \tag{122}$$

Then we have

$$E_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad E_2 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \qquad E_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{123}$$

$$\widetilde{E}_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad \widetilde{E}_2 = \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}, \qquad \widetilde{E}_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{124}$$

These are the definitions that were given in Equation (6). Note that $\widetilde{E}_i \neq E_i^*$.

The new bases obey the same commutation relations as the old bases. 4-component spinor pairs in the new basis φ can be obtained by the transformation $\varphi = A^{-1}\chi$

$$\varphi_A = \begin{pmatrix} u & 0 \\ 0 & u \\ v & 0 \\ 0 & v \end{pmatrix}, \qquad \varphi_B = \begin{pmatrix} -v & 0 \\ 0 & -v \\ u & 0 \\ 0 & u \end{pmatrix}, \qquad \varphi_C = \begin{pmatrix} -u & 0 \\ 0 & -u \\ v & 0 \\ 0 & v \end{pmatrix}, \qquad \varphi_D = \begin{pmatrix} v & 0 \\ 0 & v \\ u & 0 \\ 0 & u \end{pmatrix} \tag{125}$$

New spinor metric for this subspace is

$$g_E = A^T g_\Sigma A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}; \qquad g_\Sigma = g^{ab} = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -1 \\ -i & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{126}$$

Hence we obtain the following dotted contravariant spinor pairs:

$$\dot{\varphi}^A = \begin{pmatrix} \dot{v} & 0 \\ 0 & \dot{v} \\ -\dot{u} & 0 \\ 0 & -\dot{u} \end{pmatrix}, \qquad \dot{\varphi}^B = \begin{pmatrix} \dot{u} & 0 \\ 0 & \dot{u} \\ \dot{v} & 0 \\ 0 & \dot{v} \end{pmatrix}, \qquad \dot{\varphi}^C = \begin{pmatrix} \dot{v} & 0 \\ 0 & \dot{v} \\ \dot{u} & 0 \\ 0 & \dot{u} \end{pmatrix}, \qquad \dot{\varphi}^D = \begin{pmatrix} \dot{u} & 0 \\ 0 & \dot{u} \\ -\dot{v} & 0 \\ 0 & -\dot{v} \end{pmatrix} \tag{127}$$

Here we have $\varphi^A = -\varphi_B$, $\varphi^B = \varphi_A$, $\varphi^C = \varphi_D$ and $\varphi^D = -\varphi_C$. These relations are equivalent to the ones that given Equation (111). If this were the whole story we would have to conclude that the new representation is equivalent to SL(2,C), but there is another subspace with its own spinor metric that is based on $\widetilde{E}$. The spinor pairs associated with this second subspace can be obtained as $\dot{\widetilde{\varphi}} = A^{-1}\dot{\chi}$:

$$\dot{\tilde{\varphi}}_A = \begin{pmatrix} \dot{u} & 0 \\ \dot{v} & 0 \\ 0 & \dot{u} \\ 0 & \dot{v} \end{pmatrix}, \quad \dot{\tilde{\varphi}}_B = \begin{pmatrix} -\dot{v} & 0 \\ \dot{u} & 0 \\ 0 & -\dot{v} \\ 0 & \dot{u} \end{pmatrix}, \quad \dot{\tilde{\varphi}}_C = \begin{pmatrix} -\dot{u} & 0 \\ \dot{v} & 0 \\ 0 & -\dot{u} \\ 0 & \dot{v} \end{pmatrix}, \quad \dot{\tilde{\varphi}}_D = \begin{pmatrix} \dot{v} & 0 \\ \dot{u} & 0 \\ 0 & \dot{v} \\ 0 & \dot{u} \end{pmatrix} \tag{128}$$

The spinor metric for this subspace is [12]

$$g_{\tilde{E}} = A^T g_{\Sigma^*} A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{129}$$

Hence we obtain the associated contravariant forms as follows: [13]

$$\tilde{\varphi}^A = \begin{pmatrix} v & 0 \\ -u & 0 \\ 0 & v \\ 0 & -u \end{pmatrix}, \quad \tilde{\varphi}^B = \begin{pmatrix} u & 0 \\ v & 0 \\ 0 & u \\ 0 & v \end{pmatrix}, \quad \tilde{\varphi}^C = \begin{pmatrix} v & 0 \\ u & 0 \\ 0 & v \\ 0 & u \end{pmatrix}, \quad \tilde{\varphi}^D = \begin{pmatrix} u & 0 \\ -v & 0 \\ 0 & u \\ 0 & -v \end{pmatrix} \tag{130}$$

It may be useful to write the bases for the spinor pairs in both representations. Let $\mathcal{S}$ be the space of 4-component two column spinor pairs. The subspaces $\mathcal{S}_\Sigma$ and $\mathcal{S}_{\tilde{\Sigma}}$ associated with Σ and $\tilde{\Sigma}$ are spanned by

$$e^1_\Sigma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & i \\ 1 & 0 \end{pmatrix}, \quad e^2_\Sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ -i & 0 \\ 0 & -1 \end{pmatrix}; \quad e^1_{\tilde{\Sigma}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -i \\ 1 & 0 \end{pmatrix}, \quad e^2_{\tilde{\Sigma}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ i & 0 \\ 0 & -1 \end{pmatrix} \tag{131}$$

Here we use the property $\tilde{\Sigma} = \Sigma^*$. On the other hand, The subspaces $\mathcal{S}_E$ and $\mathcal{S}_{\tilde{E}}$ associated with E and $\tilde{E}$ are spanned by

$$e^1_E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad e^2_E = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad e^1_{\tilde{E}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e^2_{\tilde{E}} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{132}$$

Note that none of the bases for spinor pairs that given in Eqations (131) and (132) can be written as a linear combination of the others.

In the new basis the transformation matrix W_A that corresponds to Z_A takes the form

$$W_A = \begin{pmatrix} \alpha_0 + \alpha_3 & 0 & \alpha_1 - i\alpha_2 & 0 \\ 0 & \alpha_0 + \alpha_3 & 0 & \alpha_1 - i\alpha_2 \\ \alpha_1 + i\alpha_2 & 0 & \alpha_0 - \alpha_3 & 0 \\ 0 & \alpha_1 + i\alpha_2 & 0 & \alpha_0 - \alpha_3 \end{pmatrix} = \alpha_0 E_0 + \alpha_1 E_1 + \alpha_2 E_2 + \alpha_3 E_3 \tag{133}$$

In short, $W_A = (+ + ++)_E$ and W_A preserves the spinor metric g_E

$$W_A^T g_E W_A = g_E \tag{134}$$

12 $g_{\Sigma^*} = g^*_\Sigma = g^*$

13 We can also obtain 16 independent spinor pairs for the diagonal basis from the eigenvectors of E_3 and $\tilde{E}_3$ by applying the procedure described in Section 5

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Similarly, $\widetilde{W}_A$ corresponds to Z_A^*

$$\widetilde{W}_A = \begin{pmatrix} \alpha_0^*+\alpha_3^* & \alpha_1^*+i\alpha_2^* & 0 & 0 \\ \alpha_1^*-i\alpha_2^* & \alpha_0^*-\alpha_3^* & 0 & 0 \\ 0 & 0 & \alpha_0^*+\alpha_3^* & \alpha_1^*+i\alpha_2^* \\ 0 & 0 & \alpha_1^*-i\alpha_2^* & \alpha_0^*-\alpha_3^* \end{pmatrix} = \alpha_0^*\widetilde{E}_0+\alpha_1^*\widetilde{E}_1+\alpha_2^*\widetilde{E}_2+\alpha_3^*\widetilde{E}_3 \tag{135}$$

In short, $\widetilde{W}_A = (++++)^*_{\widetilde{E}}$ and $\widetilde{W}_A$ preserves the metric $g_{\widetilde{E}}$

$$\widetilde{W}_A^T g_{\widetilde{E}} \widetilde{W}_A = g_{\widetilde{E}} \tag{136}$$

W_A transforms the associated outer product $P_A = \varphi_A \varphi_A^\dagger$

$$P_A = \begin{pmatrix} t+z & 0 & x-iy & 0 \\ 0 & t+z & 0 & x-iy \\ x+iy & 0 & t-z & 0 \\ 0 & x+iy & 0 & t-z \end{pmatrix} = tE_0+xE_1+yE_2+zE_3 = (++++)_E \tag{137}$$

$\widetilde{W}_A$ transforms the associated outer product $\dot{\widetilde{P}}_A = \dot{\widetilde{\varphi}}_A \dot{\widetilde{\varphi}}_A^\dagger$

$$\dot{\widetilde{P}}_A = \begin{pmatrix} t+z & x+iy & 0 & 0 \\ x-iy & t-z & & \\ 0 & 0 & t+z & x+iy \\ 0 & 0 & x-iy & t-z \end{pmatrix} = t\widetilde{E}_0+x\widetilde{E}_1+y\widetilde{E}_2+z\widetilde{E}_3 = (++++)_{\widetilde{E}} \tag{138}$$

We can also write the 4×4 Lorentz transformation matrix λ_A in terms of L_{ij} by using the relations $\alpha_0+\alpha_3 = L_{11}$, $\alpha_1 - i\alpha_2 = L_{12}$, $\alpha_1 + i\alpha_2 = L_{21}$ and $\alpha_0 - \alpha_3 = L_{22}$:

$$\lambda_A = W_A\widetilde{W}_A = \begin{pmatrix} L_{11}L_{11}^* & L_{11}L_{12}^* & L_{12}L_{11}^* & L_{12}L_{12}^* \\ L_{11}L_{21}^* & L_{11}L_{22}^* & L_{12}L_{21}^* & L_{12}L_{22}^* \\ L_{21}L_{11}^* & L_{21}L_{12}^* & L_{22}L_{11}^* & L_{22}L_{12}^* \\ L_{21}L_{21}^* & L_{21}L_{22}^* & L_{22}L_{21}^* & L_{22}L_{22}^* \end{pmatrix} = L_A \otimes L_A^* \tag{139}$$

Everything looks nice and simple in the diagonal basis. But now λ_A does not transform the simple and familiar form of the contravariant real 4-vector $V_A^\mu = (t,\ x,\ y,\ z)^T$, instead, it acts on the complex 4-vector $\mathcal{V}_A^\mu = A^{-1}V_A^\mu$

$$\mathcal{V}_A^\mu = \begin{pmatrix} t+z \\ x-iy \\ x+iy \\ t-z \end{pmatrix} \tag{140}$$

Hence the Lorentz transformation of a 4-vector in the diagonal basis reads as follows:

$$\mathcal{V}_A^\mu \to \mathcal{V}'^\mu_A = \lambda_A \mathcal{V}_A^\mu \tag{141}$$

Furthermore, the Minkowski metric in the diagonal representation takes the form

$$\eta_E = \eta_{\widetilde{E}} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{142}$$

It can be shown that both W_A and $\widetilde{W}_A$ preserve η_E.

9. Rotations in spinor space

$U(\alpha) = \cos(\alpha/2) - i\hat{n}\cdot\vec{\sigma}\sin(\alpha/2)$ represents a CCW rotation about $\hat{n} = (n_x, n_y, n_z)$ by an angle α. If $\alpha = 180°$, $U(180°) = -i\hat{n}\cdot\vec{\sigma}$. For example, $-i\sigma_x$ represents a 180° CCW rotation about the x axis. When we apply this rotation on ξ_A we get $-i\sigma_x\xi_A = -i\xi_D$, i.e. a 180° rotation about the x axis transforms ξ_A into ξ_D apart from a global phase $-i$. Some of these transformations are shown in Figure 2. The relations that are not shown can be found by successive application of rotations, such as $\xi_A \to \xi_C = \xi_A \to \xi_B \to \xi_C$

$$(-\sigma_x)(i\sigma_y)\xi_A = \sigma_z\xi_A = \xi_C, \quad \text{or} \quad -i\sigma_z\xi_A = -i\xi_c \tag{143}$$

Likewise, $\xi_B \to \xi_D = \xi_B \to \xi_C \to \xi_D$

$$(-i\sigma_y)(-\sigma_x)\xi_B = \sigma_z\xi_B = \xi_D, \quad \text{or} \quad -i\sigma_z\xi_B = -i\xi_D \tag{144}$$

Here we used the property that a rotation in the opposite direction is induced by the Hermitian adjoint of the forward one.

In order to visualize the orientations of different types of spinors in the spinor space the flagpole picture may be helpful [23]. Space components of a 4-vector associated with a spinor is given by

$$x_i = \xi^\dagger\sigma_i\,\xi \tag{145}$$

For $\xi = \xi_A$, $\xi_A^\dagger\sigma_x\,\xi_A = x$, $\xi_A^\dagger\sigma_y\,\xi_A = y$, $\xi_A^\dagger\sigma_z\,\xi_A = z$, therefore the orientation of the flagpole associated with ξ_A can be marked as $(+++)$. On the other hand, For $\xi = \xi_B$, $\xi_B^\dagger\sigma_x\,\xi_B = -x$, $\xi_B^\dagger\sigma_y\,\xi_B = y$, $\xi_B^\dagger\sigma_z\,\xi_B = -z$, therefore the orientation of the flagpole associated with ξ_B is $(-+-)$, likewise for ξ_C, $(--+)$ and for ξ_D, $(+--)$. Hence, in Figure 1, a vector pointing from the origin to the corner of the cube can be used to describe the orientation of the spinor that carries same mark.

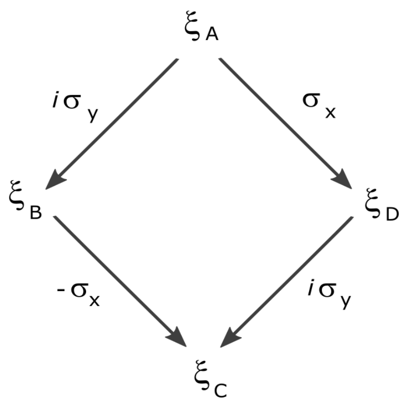

Figure 2. 180° rotations of spinors

Similar rotation schemes apply to the dotted contravariant 2-component spinors as well.

Rotations for 4-component spinor pairs is not much different. Now we have to replace σ_i by Σ_i. Here are some rotations for undotted covariant and dotted contravariant 4-component spinor pairs

$$-i\Sigma_y\chi_A = \chi_B, \quad -i\Sigma_x\chi_A = -i\chi_D, \quad -i\Sigma_x\chi_B = i\chi_C, \quad -i\Sigma_y\chi_D = \chi_C \tag{146}$$

$$-i\Sigma_y\dot{\chi}^A = \dot{\chi}^B, \quad -i\Sigma_y\dot{\chi}^A = i\dot{\chi}^D, \quad -i\Sigma_x\dot{\chi}^B = -i\dot{\chi}^C, \quad -i\Sigma_y\dot{\chi}^D = \dot{\chi}^C \tag{147}$$

The complex conjugated forms are rotated in a slightly different way. Now we have to replace Σ_i with Σ_i^*. For example, for dotted covariant spinor pairs

$$-i\Sigma_y^*\dot{\chi}_A = -\dot{\chi}_B, \quad -i\Sigma_x^*\dot{\chi}_A = -i\dot{\chi}_D, \quad -i\Sigma_x^*\dot{\chi}_B = i\dot{\chi}_C, \quad -i\Sigma_y^*\dot{\chi}_D = -\dot{\chi}_C \tag{148}$$

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Similar relations hold for undotted contravariant spinor pairs.

It is also possible to rotate the spinors about an axis perpendicular to their flagpole. For example, when we rotate ξ_A by 180° about an axis perpendicular to the flagpole of ξ_A, apart from a global phase, we get $\dot{\xi}^A$. But, the perpendicular axis is not unique and the global phase depends on its choice.

10. State space and inner product of states

When a Z matrix acts on a spinor pair the result is again a spinor pair. We may regard a spinor pair as a two column state vector which fits a certain pattern [14]:

$$|\alpha.\beta\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \\ -i\beta & i\alpha \\ \alpha & -\beta \end{pmatrix} \tag{149}$$

where α and β are two complex parameters. Previously defined pairs that given in Equation (58) and their dotted contravariant versions obey the same pattern [15]. Under the action of Z, $|\alpha, \beta\rangle$ transforms in such a way that the pattern is preserved

$$Z|\alpha.\beta\rangle \rightarrow |\alpha', \beta'\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} \alpha' & \beta' \\ \beta' & \alpha' \\ -i\beta' & i\alpha' \\ \alpha' & -\beta' \end{pmatrix} \tag{150}$$

Here Z can be type A, B, C or D. [16]

Now let us consider the inner product of two state vectors

$$\langle \alpha, \beta | \gamma, \delta \rangle = \begin{pmatrix} \dot{\alpha}\gamma + \dot{\beta}\delta & 0 \\ 0 & \dot{\alpha}\gamma + \dot{\beta}\delta \end{pmatrix} \tag{151}$$

Due to the specific pattern of the states the inner product always results in this form, therefore, for later convenience, we will define the inner product of two states as one half the trace of the 2×2 matrix and we will simply write:

$$\langle \alpha, \beta | \gamma, \delta \rangle = \dot{\alpha}\gamma + \dot{\beta}\delta \tag{152}$$

This definition is equivalent to the inner product of two 2-component spinors with components α, β and γ, δ.

11. Observables and states

We associate Σ_i with the components of an observable $\mathbf{\Sigma}$ along the axes x, y and z:

$$\Sigma_1 \rightarrow \mathbf{\Sigma}_x, \qquad \Sigma_2 \rightarrow \mathbf{\Sigma}_y, \qquad \Sigma_3 \rightarrow \mathbf{\Sigma}_z \tag{153}$$

---

14 $|\alpha.\beta\rangle$ is an eigenvector of Σ^*

15 The pairs that obey this pattern are the generalized eigenvectors of Σ_3^*. First column corresponds to the eigenvalue $+1$ and the second column corresponds to the eigenvalue -1.

16 Let $|\chi_{(x)}\rangle$ denote a state vector of any kind and let $|\dot{\chi}^{(x)}\rangle$ be its dotted contravariant version. These objects live in the state space and they all fit the pattern that given in Equation (149). All undotted covariant objects transform with $Z_{(x)}$ and all dotted contravariant objects transform with $\dot{Z}_{(x)}$ which are based on Σ_μ. Under these transformations the pattern of the transformed object remains the same. Similar considerations also apply to the conjugated counterparts that live in the conjugate space. The only difference is that, in this case, the sign of the imaginary unit in Equation (149) is flipped and all matrix forms are now based on Σ_μ^*.

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Let $\hat{\mathbf{n}}$ be a unit vector: $\hat{\mathbf{n}} = (n_x, n_y, n_z) = (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)$, and let $\vec{\mathbf{\Sigma}}^A = (\mathbf{\Sigma}_x, \mathbf{\Sigma}_y, \mathbf{\Sigma}_z)$. $\vec{\mathbf{\Sigma}}^A$ is the observable for Type-A states. For types B, C and D, $\vec{\mathbf{\Sigma}}^B = (-\mathbf{\Sigma}_x, \mathbf{\Sigma}_y, -\mathbf{\Sigma}_z)$, $\vec{\mathbf{\Sigma}}^C = (-\mathbf{\Sigma}_x, -\mathbf{\Sigma}_y, \mathbf{\Sigma}_z)$, $\vec{\mathbf{\Sigma}}^D = (\mathbf{\Sigma}_x, -\mathbf{\Sigma}_y, -\mathbf{\Sigma}_z)$.

The component of the observable $\vec{\mathbf{\Sigma}}^A$ along $\hat{\mathbf{n}}$ can be written as

$$\hat{\mathbf{n}} \cdot \vec{\mathbf{\Sigma}}^A = \begin{pmatrix} 0 & n_x & n_y & n_z \\ n_x & 0 & -in_z & in_y \\ n_y & in_z & 0 & -in_x \\ n_z & -in_y & in_x & 0 \end{pmatrix}. \tag{154}$$

This is the 4×4 analog of $\hat{\mathbf{n}} \cdot \vec{\sigma}$:

$$\hat{\mathbf{n}} \cdot \vec{\sigma} = \begin{pmatrix} n_z & n_x - in_y \\ n_x + in_y & -n_z \end{pmatrix} \tag{155}$$

In terms of θ and ϕ the eigenstates of $\hat{\mathbf{n}} \cdot \vec{\sigma}$ are

$$|\uparrow_{\theta,\phi}\rangle = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right)e^{i\phi} \end{pmatrix}, \qquad |\downarrow_{\theta,\phi}\rangle = \begin{pmatrix} -\sin\left(\frac{\theta}{2}\right)e^{-i\phi} \\ \cos\left(\frac{\theta}{2}\right) \end{pmatrix} \tag{156}$$

$|\uparrow_{\theta,\phi}\rangle$ and $|\downarrow_{\theta,\phi}\rangle$ states correspond to $+1$ and -1 eigenvalues respectively and they constitute a complete set of basis, hence

$$\hat{\mathbf{n}} \cdot \vec{\sigma} = |\uparrow_{\theta,\phi}\rangle\langle\uparrow_{\theta,\phi}| - |\downarrow_{\theta,\phi}\rangle\langle\downarrow_{\theta,\phi}|. \tag{157}$$

In SL(4,C), the eigenstates of $\hat{\mathbf{n}} \cdot \vec{\mathbf{\Sigma}}^A$ turn out to be the following two column orthogonal state vectors of Type-A that correspond to the eigenvalues $+1$ and -1, respectively:

$$|A^{\uparrow}_{\theta,\phi}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) & \sin\left(\frac{\theta}{2}\right)e^{i\phi} \\ \sin\left(\frac{\theta}{2}\right)e^{i\phi} & \cos\left(\frac{\theta}{2}\right) \\ -i\sin\left(\frac{\theta}{2}\right)e^{i\phi} & i\cos\left(\frac{\theta}{2}\right) \\ \cos\left(\frac{\theta}{2}\right) & -\sin\left(\frac{\theta}{2}\right)e^{i\phi} \end{pmatrix}, \quad |A^{\downarrow}_{\theta,\phi}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} -\sin\left(\frac{\theta}{2}\right)e^{-i\phi} & \cos\left(\frac{\theta}{2}\right) \\ \cos\left(\frac{\theta}{2}\right) & -\sin\left(\frac{\theta}{2}\right)e^{-i\phi} \\ -i\cos\left(\frac{\theta}{2}\right) & -i\sin\left(\frac{\theta}{2}\right)e^{-i\phi} \\ -\sin\left(\frac{\theta}{2}\right)e^{-i\phi} & -\cos\left(\frac{\theta}{2}\right) \end{pmatrix} \tag{158}$$

$|A^{\uparrow}_{\theta,\phi}\rangle$ and $|A^{\downarrow}_{\theta,\phi}\rangle$ constitute a complete set of basis for the state space, therefore

$$\hat{\mathbf{n}} \cdot \vec{\mathbf{\Sigma}}^A = |A^{\uparrow}_{\theta,\phi}\rangle\langle A^{\uparrow}_{\theta,\phi}| - |A^{\downarrow}_{\theta,\phi}\rangle\langle A^{\downarrow}_{\theta,\phi}| \tag{159}$$

In a similar way we can write $\hat{\mathbf{n}} \cdot \vec{\mathbf{\Sigma}}^B$, $\hat{\mathbf{n}} \cdot \vec{\mathbf{\Sigma}}^C$ and $\hat{\mathbf{n}} \cdot \vec{\mathbf{\Sigma}}^D$ in terms of the orthogonal states vectors associated with them.

According to the definition of the inner product that given in Equation (152), the expectation value of $\hat{\mathbf{n}} \cdot \vec{\mathbf{\Sigma}}^A$ in a state with a flagpole in the direction of a unit vector $\hat{\mathbf{m}} = (\sin\alpha\cos\gamma,\ \sin\alpha\sin\gamma,\ \cos\alpha)$ is

$$\langle \alpha, \gamma | \hat{\mathbf{n}} \cdot \vec{\mathbf{\Sigma}}^A | \alpha, \gamma \rangle = \hat{\mathbf{n}} \cdot \hat{\mathbf{m}} \tag{160}$$

Although, the state vector has four components the state space is only two dimensional [17]. It may be convenient to use $|\uparrow\rangle$ and $|\downarrow\rangle$ as basis states by letting $\theta = \phi = 0$ in Equation (158)

[17] The conjugate space is also two dimensional. Basis for the conjugate space can be obtained by simply changing the sign of the imaginary unit in Equation (161). Comjugate basis cannot be expressed as linear combinations of $|\uparrow\rangle$ and $|\downarrow\rangle$ basis.

$$|\uparrow\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & 1\\ 0 & i\\ 1 & 0\end{pmatrix}, \qquad |\downarrow\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1\\ 1 & 0\\ -i & 0\\ 0 & -1\end{pmatrix} \tag{161}$$

In these basis basic spinor pairs can be written as

$$|\chi_A\rangle = u|\uparrow\rangle + v|\downarrow\rangle, \quad |\chi_B\rangle = -v|\uparrow\rangle + u|\downarrow\rangle, \quad |\chi_C\rangle = -u|\uparrow\rangle + v|\downarrow\rangle, \quad |\chi_D\rangle = v|\uparrow\rangle + u|\downarrow\rangle \tag{162}$$

$$|\dot{\chi}^A\rangle = \dot{v}|\uparrow\rangle - \dot{u}|\downarrow\rangle, \quad |\dot{\chi}^B\rangle = \dot{u}|\uparrow\rangle + \dot{v}|\downarrow\rangle, \quad |\dot{\chi}^C\rangle = \dot{v}|\uparrow\rangle + \dot{u}|\downarrow\rangle, \quad |\dot{\chi}^D\rangle = \dot{u}|\uparrow\rangle - \dot{v}|\downarrow\rangle \tag{163}$$

We can write raising and lowering operators and their effect on the $|\uparrow\rangle$ and $|\downarrow\rangle$ states:

$$\mathbf{\Sigma}_+^A = \mathbf{\Sigma}_x + i\mathbf{\Sigma}_y = \begin{pmatrix}0 & 1 & i & 0\\ 1 & 0 & 0 & -1\\ i & 0 & 0 & -i\\ 0 & 1 & i & 0\end{pmatrix}, \qquad \mathbf{\Sigma}_-^A = \mathbf{\Sigma}_x - i\mathbf{\Sigma}_y = \begin{pmatrix}0 & 1 & -i & 0\\ 1 & 0 & 0 & 1\\ -i & 0 & 0 & -i\\ 0 & -1 & i & 0\end{pmatrix} \tag{164}$$

$$\mathbf{\Sigma}_+^A|\uparrow\rangle = 0, \qquad \mathbf{\Sigma}_+^A|\downarrow\rangle = |\uparrow\rangle, \qquad \mathbf{\Sigma}_-^A|\uparrow\rangle = |\downarrow\rangle, \qquad \mathbf{\Sigma}_-^A|\downarrow\rangle = 0 \tag{165}$$

Similar relations can be written for Type B, C and D states with the associated ladder operators:

$$\mathbf{\Sigma}_+^B = -\mathbf{\Sigma}_x + i\mathbf{\Sigma}_y \qquad \mathbf{\Sigma}_-^B = -\mathbf{\Sigma}_x - i\mathbf{\Sigma}_y \tag{166}$$

$$\mathbf{\Sigma}_+^C = -\mathbf{\Sigma}_x - i\mathbf{\Sigma}_y \qquad \mathbf{\Sigma}_-^C = -\mathbf{\Sigma}_x + i\mathbf{\Sigma}_y \tag{167}$$

$$\mathbf{\Sigma}_+^D = \mathbf{\Sigma}_x - i\mathbf{\Sigma}_y \qquad \mathbf{\Sigma}_-^D = \mathbf{\Sigma}_x + i\mathbf{\Sigma}_y \tag{168}$$

12. 8-component Dirac spinor

$$(i\gamma^\mu\partial_\mu - m)\Psi = 0 \tag{169}$$

The 4-component Dirac spinor Ψ is a solution to the Dirac equation given in Equation (169). But, we have already committed ourselves to the 4-component state vectors that have the specific form given in Equation (149). Therefore we seek a solution to the Dirac equation within our state space, but there does not exist a nontrivial one. In order to write a Dirac-like equation in our state space we have to go to higher dimensions [18]. In the following it will be shown that this is possible with 8-component spinors.

Let us rewrite Equation (169) with $8 \times 8 \quad G^\mu$ matrices

$$(iG^\mu\partial_\mu - m)\Phi = 0 \tag{170}$$

where

$$G^0 = \begin{pmatrix}I_{4\times4} & 0\\ 0 & -I_{4\times4}\end{pmatrix}, \qquad G^i = \begin{pmatrix}0 & \Sigma_i\\ -\Sigma_i & 0\end{pmatrix} \tag{171}$$

Clifford algebra of G^μ is the same as γ^μ

$$\{G^\mu, G^\nu\} = 2\eta^{\mu\nu}\mathbb{K}_{8\times8} \tag{172}$$

Let us write Φ in terms of spinor pairs

[18] There does not exists a fourth matrix that anticommutes with Σ_i matrices.

$$\Phi = \begin{pmatrix} \Omega \\ \Delta \end{pmatrix}, \qquad \Omega = \begin{pmatrix} \omega_1 & \omega_2 \\ \omega_2 & \omega_1 \\ -i\omega_2 & i\omega_1 \\ \omega_1 & -\omega_2 \end{pmatrix}, \qquad \Delta = \begin{pmatrix} \delta_1 & \delta_2 \\ \delta_2 & \delta_1 \\ -i\delta_2 & i\delta_1 \\ \delta_1 & -\delta_2 \end{pmatrix} \tag{173}$$

Then Equation (170) reads

$$\begin{pmatrix} i\partial_0 - m & 0 & 0 & 0 & 0 & i\partial_1 & i\partial_2 & i\partial_3 \\ 0 & i\partial_0 - m & 0 & 0 & i\partial_1 & 0 & \partial_3 & -\partial_2 \\ 0 & 0 & i\partial_0 - m & 0 & i\partial_2 & -\partial_3 & 0 & \partial_1 \\ 0 & 0 & 0 & i\partial_0 - m & i\partial_3 & \partial_2 & -\partial_1 & 0 \\ 0 & -i\partial_1 & -i\partial_2 & -i\partial_3 & -(i\partial_0 + m) & 0 & 0 & 0 \\ -i\partial_1 & 0 & -\partial_3 & \partial_2 & 0 & -(i\partial_0 + m) & 0 & 0 \\ -i\partial_2 & \partial_3 & 0 & -\partial_1 & 0 & 0 & -(i\partial_0 + m) & 0 \\ -i\partial_3 & -\partial_2 & \partial_1 & 0 & 0 & 0 & 0 & -(i\partial_0 + m) \end{pmatrix} \begin{pmatrix} \omega_1 & \omega_2 \\ \omega_2 & \omega_1 \\ -i\omega_2 & i\omega_1 \\ \omega_1 & -\omega_2 \\ \delta_1 & \delta_2 \\ \delta_2 & \delta_1 \\ -i\delta_2 & i\delta_1 \\ \delta_1 & -\delta_2 \end{pmatrix} = 0 \tag{174}$$

Let us try a plane wave solution

$$\omega_i = A_i e^{-i(Et - \vec{p}\cdot\vec{x})}, \qquad \delta_i = B_i e^{-i(Et - \vec{p}\cdot\vec{x})} \tag{175}$$

$$\begin{pmatrix} E-m & 0 & 0 & 0 & 0 & -p_1 & -p_2 & -p_3 \\ 0 & E-m & 0 & 0 & -p_1 & 0 & ip_3 & -ip_2 \\ 0 & 0 & E-m & 0 & -p_2 & -ip_3 & 0 & ip_1 \\ 0 & 0 & 0 & E-m & -p_3 & ip_2 & -ip_1 & 0 \\ 0 & p_1 & p_2 & p_3 & -(E+m) & 0 & 0 & 0 \\ p_1 & 0 & -ip_3 & ip_2 & 0 & -(E+m) & 0 & 0 \\ p_2 & ip_3 & 0 & -ip_1 & 0 & 0 & -(E+m) & 0 \\ p_3 & -ip_2 & ip_1 & 0 & 0 & 0 & 0 & -(E+m) \end{pmatrix} \begin{pmatrix} A_1 & A_2 \\ A_2 & A_1 \\ -iA_2 & iA_1 \\ A_1 & -A_2 \\ B_1 & B_2 \\ B_2 & B_1 \\ -iB_2 & iB_1 \\ B_1 & -B_2 \end{pmatrix} = 0 \tag{176}$$

A nontrivial solution exists only if the determinant of the coefficients vanishes

$$\det = (E^2 - \vec{p}\cdot\vec{p} - m^2)^4 = 0 \tag{177}$$

We have a nontrivial solution if $E^2 = \vec{p}\cdot\vec{p} + m^2$.

Equation (176) can be written in a compact form

$$\begin{pmatrix} E-m & -\vec{\mathbf{\Sigma}}\cdot\vec{p} \\ \vec{\mathbf{\Sigma}}\cdot\vec{p} & -(E+m) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0 \tag{178}$$

This is equivalent to Dirac's formulation. Hence, we expect positive and negative energy solutions for a system which has a two level intrinsic property.

In general, the 8-component spinor pair Φ transforms as

$$\Phi \rightarrow \Phi' = S\Phi \tag{179}$$

Since the Clifford algebra of G^μ and γ^μ is the same, covariance of the Dirac-like equation that given in Equation (170) can be verified by showing the existence of of a matrix S that satisfies the property [24]

$$S^{-1} G^\mu S = \Lambda^\mu{}_\nu G^\nu \tag{180}$$

As a special case, it is straightforward to show the existence of such a matrix S in the 8×8 version of the Weyl basis

$$G^0 = \begin{pmatrix} 0 & -\Sigma_0 \\ -\Sigma_0 & 0 \end{pmatrix}, \qquad G^i = \begin{pmatrix} 0 & \Sigma_i \\ -\Sigma_i & 0 \end{pmatrix} \tag{181}$$

In this basis we can write Equation (179) as

$$\begin{pmatrix} \Omega_L \\ \Delta_R \end{pmatrix} \rightarrow \begin{pmatrix} \Omega'_L \\ \Delta'_R \end{pmatrix} = \begin{pmatrix} Z & 0 \\ 0 & \dot{Z} \end{pmatrix} \begin{pmatrix} \Omega_L \\ \Delta_R \end{pmatrix} \tag{182}$$

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Appendix A

Appendix A.1. Various forms of Z and L matrices

Let us begin with the exponential form $Z_A = e^R$, where $R = -\frac{i}{2}\vec{\pi}\cdot\vec{\Sigma}$, $\pi_i = \theta_i + i\rho_i$. Let ϕ be the complex angle defined as $\phi = \frac{1}{2}\sqrt{\pi_1^2 + \pi_2^2 + \pi_3^2}$. Using the property $R^2 = -\phi^2 I$:

$$Z_A = \cos\phi I - \frac{i\sin\phi}{2\phi}\vec{\pi}\cdot\vec{\Sigma} = \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_1 & \alpha_0 & -i\alpha_3 & i\alpha_2 \\ \alpha_2 & i\alpha_3 & \alpha_0 & -i\alpha_1 \\ \alpha_3 & -i\alpha_2 & i\alpha_1 & \alpha_0 \end{pmatrix} \tag{183}$$

where

$$\alpha_0 = \cos\phi, \qquad \alpha_1 = -\frac{i\sin\phi}{2\phi}\pi_1, \qquad \alpha_2 = -\frac{i\sin\phi}{2\phi}\pi_2, \qquad \alpha_3 = -\frac{i\sin\phi}{2\phi}\pi_3. \tag{184}$$

Or in a compact form

$$Z_A = \alpha_0\Sigma_0 + \alpha_1\Sigma_1 + \alpha_2\Sigma_2 + \alpha_3\Sigma_3 = (++++)_\Sigma. \tag{185}$$

It is easy to show that

$$Z_A^{-1} = \alpha_0\Sigma_0 - \alpha_1\Sigma_1 - \alpha_2\Sigma_2 - \alpha_3\Sigma_3 = (+---)_\Sigma. \tag{186}$$

and

$$Z_A^\dagger = \alpha_0^*\Sigma_0 + \alpha_1^*\Sigma_1 + \alpha_2^*\Sigma_2 + \alpha_3^*\Sigma_3 = (++++)_\Sigma^* \tag{187}$$

where complex conjugation is applied only to α_μ.

The corresponding L_A is

$$L_A = \begin{pmatrix} \alpha_0 + \alpha_3 & \alpha_1 - i\alpha_2 \\ \alpha_1 + i\alpha_2 & \alpha_0 - \alpha_3 \end{pmatrix} \tag{188}$$

In terms of the Pauli matrices:

$$L_A = \alpha_0\sigma_0 + \alpha_1\sigma_1 + \alpha_2\sigma_2 + \alpha_3\sigma_3 = (++++)_\sigma. \tag{189}$$

In order to write Z_B we first find $L_B = (\dot{L}_A)^*$, where

$$\dot{L}_A = (L_A^{-1})^\dagger = \begin{pmatrix} \alpha_0^* - \alpha_3^* & -\alpha_1^* + i\alpha_2^* \\ -\alpha_1^* - i\alpha_2^* & \alpha_0^* + \alpha_3^* \end{pmatrix} = (+---)_\sigma^*. \tag{190}$$

From the definition $Z_B = A(L_B \otimes I)A^{-1}$:

$$Z_B = \begin{pmatrix} \alpha_0 & -\alpha_1 & \alpha_2 & -\alpha_3 \\ -\alpha_1 & \alpha_0 & i\alpha_3 & i\alpha_2 \\ \alpha_2 & -i\alpha_3 & \alpha_0 & i\alpha_1 \\ -\alpha_3 & -i\alpha_2 & -i\alpha_1 & \alpha_0 \end{pmatrix} = \alpha_0\Sigma_0 - \alpha_1\Sigma_1 + \alpha_2\Sigma_2 - \alpha_3\Sigma_3. \tag{191}$$

Or, simply

$$Z_B = (+-+-)_\Sigma \tag{192}$$

We write various forms of Z and L matrices in compact forms:

$$L_A = (++++)_\sigma, \quad L_B = (+-+-)_\sigma, \quad \dot{L}_A = (+---)_\sigma^*, \quad \dot{L}_B = (++-+)_\sigma^*. \tag{193}$$

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$$Z_A = (++++)_\Sigma, \quad Z_B = (+-+-)_\Sigma, \quad \dot{Z}_A = (+---)^*_\Sigma, \quad \dot{Z}_B = (++-+)^*_\Sigma. \tag{194}$$

Although, all types of Z and L matrices are in the same form, there is a very important difference between them. Because of the particular property of the Pauli matrices, $\sigma_1^* = \sigma_1$, $\sigma_3^* = \sigma_3$, but $\sigma_2^* = -\sigma_2$, we have the following relations:

$$L_B = \dot{L}_A^*, \qquad \dot{L}_B = L_A^*. \tag{195}$$

For example,

$$\dot{L}_A = (+---)^*_\sigma \to \dot{L}_A^* = (+---)_{\sigma^*} = (+-+-)_\sigma = L_B. \tag{196}$$

On the other hand we do not have a similar property with Σ matrices, hence

$$Z_B \neq \dot{Z}_A^*, \qquad \dot{Z}_B \neq Z_A^*. \tag{197}$$

.

The structural difference between SL(2,C) and SL(4,C) becomes more apparent when we write the matrices in exponential forms. In order to do this we have to define two types of $\vec{\pi}$: $\vec{\pi}_A = (\pi_1, \pi_2, \pi_3)$ and $\vec{\pi}_B = (-\pi_1, \pi_2, -\pi_3)$.

$$L_A = exp(-\frac{i}{2}\vec{\pi}_A \cdot \vec{\sigma}), \quad L_B = exp(-\frac{i}{2}\vec{\pi}_B \cdot \vec{\sigma}), \quad \dot{L}_A = exp(-\frac{i}{2}\vec{\pi}_A^* \cdot \vec{\sigma}), \quad \dot{L}_B = exp(-\frac{i}{2}\vec{\pi}_B^* \cdot \vec{\sigma}). \tag{198}$$

$$Z_A = exp(-\frac{i}{2}\vec{\pi}_A \cdot \vec{\Sigma}), \quad Z_B = exp(-\frac{i}{2}\vec{\pi}_B \cdot \vec{\Sigma}), \quad \dot{Z}_A = exp(-\frac{i}{2}\vec{\pi}_A^* \cdot \vec{\Sigma}), \quad \dot{Z}_B = exp(-\frac{i}{2}\vec{\pi}_B^* \cdot \vec{\Sigma}). \tag{199}$$

Due to the properties, $-\vec{\pi}_B \cdot \vec{\sigma} = \vec{\pi}_A \cdot \vec{\sigma}^*$ and $-\vec{\pi}_B \cdot \vec{\sigma}^* = \vec{\pi}_A \cdot \vec{\sigma}$, the relations in Equation (195) hold. But we do not have similar relations with the Σ matrices.

Appendix A.2. The Lorentz matrix

By definition

$$\Lambda = ZZ^* = Z^*Z \tag{200}$$

where $Z = Z_A$

$$Z_A = \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_1 & \alpha_0 & -i\alpha_3 & i\alpha_2 \\ \alpha_2 & i\alpha_3 & \alpha_0 & -i\alpha_1 \\ \alpha_3 & -i\alpha_2 & i\alpha_1 & \alpha_0 \end{pmatrix} \tag{201}$$

$$\Lambda = \left(\begin{array}{c|c|c|c} \begin{matrix}\alpha_0\alpha_0^* + \alpha_1\alpha_1^* \\ \alpha_2\alpha_2^* + \alpha_3\alpha_3^*\end{matrix} & \begin{matrix}\alpha_0\alpha_1^* + \alpha_1\alpha_0^* \\ +i(\alpha_3\alpha_2^* - \alpha_2\alpha_3^*)\end{matrix} & \begin{matrix}\alpha_0\alpha_2^* + \alpha_2\alpha_0^* \\ +i(\alpha_1\alpha_3^* - \alpha_3\alpha_1^*)\end{matrix} & \begin{matrix}\alpha_0\alpha_3^* + \alpha_3\alpha_0^* \\ +i(\alpha_2\alpha_1^* - \alpha_1\alpha_2^*)\end{matrix} \\ \hline \begin{matrix}\alpha_0\alpha_1^* + \alpha_1\alpha_0^* \\ -i(\alpha_3\alpha_2^* - \alpha_2\alpha_3^*)\end{matrix} & \begin{matrix}\alpha_0\alpha_0^* + \alpha_1\alpha_1^* \\ -\alpha_2\alpha_2^* - \alpha_3\alpha_3^*\end{matrix} & \begin{matrix}\alpha_1\alpha_2^* + \alpha_2\alpha_1^* \\ +i(\alpha_0\alpha_3^* - \alpha_3\alpha_0^*)\end{matrix} & \begin{matrix}\alpha_1\alpha_3^* + \alpha_3\alpha_1^* \\ +i(\alpha_2\alpha_0^* - \alpha_0\alpha_2^*)\end{matrix} \\ \hline \begin{matrix}\alpha_0\alpha_2^* + \alpha_2\alpha_0^* \\ -i(\alpha_1\alpha_3^* - \alpha_3\alpha_1^*)\end{matrix} & \begin{matrix}\alpha_1\alpha_2^* + \alpha_2\alpha_1^* \\ -i(\alpha_0\alpha_3^* - \alpha_3\alpha_0^*)\end{matrix} & \begin{matrix}\alpha_0\alpha_0^* - \alpha_1\alpha_1^* \\ +\alpha_2\alpha_2^* - \alpha_3\alpha_3^*\end{matrix} & \begin{matrix}\alpha_2\alpha_3^* + \alpha_3\alpha_2^* \\ +i(\alpha_0\alpha_1^* - \alpha_1\alpha_0^*)\end{matrix} \\ \hline \begin{matrix}\alpha_0\alpha_3^* + \alpha_3\alpha_0^* \\ -i(\alpha_2\alpha_1^* - \alpha_1\alpha_2^*)\end{matrix} & \begin{matrix}\alpha_1\alpha_3^* + \alpha_3\alpha_1^* \\ -i(\alpha_2\alpha_0^* - \alpha_0\alpha_2^*)\end{matrix} & \begin{matrix}\alpha_2\alpha_3^* + \alpha_3\alpha_2^* \\ -i(\alpha_0\alpha_1^* - \alpha_1\alpha_0^*)\end{matrix} & \begin{matrix}\alpha_0\alpha_0^* - \alpha_1\alpha_1^* \\ -\alpha_2\alpha_2^* + \alpha_3\alpha_3^*\end{matrix} \end{array}\right) \tag{202}$$

Appendix A.3. Geometric phase with 4-component spinors

When the polarization state of light undergoes a series of operations and returns to its original state, the final state differs in an additional overall phase factor from the original one. It was pointed out by Pancharatnam that the phase difference is not only due to the dynamical phase from the accumulated path lengths but also involves a geometric phase [32]. Berry introduced the corresponding theory for quantum mechanical state vector and re-derived the Pancharatnam's geometric phase [33,34]. If the cyclic path consists of only great circles, additional dynamical phase will not develop, and the

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geometric part increases by $\pm\Omega/2$ [19], where Ω is the solid angle that the geodesic path of cyclic operations subtends on the Poincare sphere [20].

The emergence of geometric phase can be demonstrated in SL(2,C) by considering three successive Hermitian operations on a spinor. For example, let $\hat{\boldsymbol{n}}_1$, $\hat{\boldsymbol{n}}_2$ and $\hat{\boldsymbol{n}}_3$ be three radial unit vectors at points 1, 2 and 3 on the Poincare sphere:

$$\hat{\boldsymbol{n}}_1 = \begin{pmatrix} \sin\theta_1\cos\phi_1 \\ \sin\theta_1\sin\phi_1 \\ \cos\theta_1 \end{pmatrix}, \quad \hat{\boldsymbol{n}}_2 = \begin{pmatrix} \sin\theta_2\cos\phi_2 \\ \sin\theta_2\sin\phi_2 \\ \cos\theta_2 \end{pmatrix}, \quad \hat{\boldsymbol{n}}_3 = \begin{pmatrix} \sin\theta_3\cos\phi_3 \\ \sin\theta_3\sin\phi_3 \\ \cos\theta_3 \end{pmatrix} \tag{203}$$

And let $|\xi_1\rangle$, $|\xi_2\rangle$ and $|\xi_3\rangle$ be the spinors that correspond to these radial unit vectors:

$$|\xi_1\rangle = \begin{pmatrix} \cos(\frac{\theta_1}{2}) \\ \sin(\frac{\theta_1}{2})e^{i\phi_1} \end{pmatrix}, \quad |\xi_2\rangle = \begin{pmatrix} \cos(\frac{\theta_2}{2}) \\ \sin(\frac{\theta_2}{2})e^{i\phi_2} \end{pmatrix}, \quad |\xi_3\rangle = \begin{pmatrix} \cos(\frac{\theta_3}{2}) \\ \sin(\frac{\theta_3}{2})e^{i\phi_3} \end{pmatrix} \tag{204}$$

We consider a closed loop $|\xi_1\rangle \to |\xi_2\rangle \to |\xi_3\rangle \to |\xi_1\rangle$ projection operations that starts by acting on $|\xi_1\rangle$ and finally brings the state back to $|\xi_1\rangle$ which may differ from the first one only by a phase angle γ:

$$\gamma = arg(\langle\xi_1|\Big[|\xi_1\rangle\langle\xi_1|\Big]\Big[|\xi_3\rangle\langle\xi_3|\Big]\Big[|\xi_2\rangle\langle\xi_2|\Big]|\xi_1\rangle) \tag{205}$$

For simplicity and without loss of generality when we choose $\theta_1 = \phi_1 = 0$

$$\gamma = \arctan\left(\frac{\sin\theta_2\sin\theta_3\sin(\phi_2-\phi_3)}{4\cos^2(\theta_2/2)\cos^2(\theta_3/2)+\sin\theta_2\sin\theta_3\cos(\phi_2-\phi_3)}\right) \tag{206}$$

This is the geometric phase and it's magnitude is equal to $\Omega/2$. The result can be checked by calculating the area subtended by the cyclic path in terms of the angular variables. In this example, the corners of the spherical triangle are defined by the unit vectors, $\hat{\boldsymbol{n}}_1$, $\hat{\boldsymbol{n}}_2$ and $\hat{\boldsymbol{n}}_3$, hence, it is convenient to use the following formula in order to calculate the area of the spherical triangle [35]:

$$\frac{\Omega}{2} = \arctan\left[\frac{\hat{\boldsymbol{n}}_1\cdot(\hat{\boldsymbol{n}}_2\times\hat{\boldsymbol{n}}_3)}{1+\hat{\boldsymbol{n}}_1\cdot\hat{\boldsymbol{n}}_2+\hat{\boldsymbol{n}}_2\cdot\hat{\boldsymbol{n}}_3+\hat{\boldsymbol{n}}_3\cdot\hat{\boldsymbol{n}}_1}\right] \tag{207}$$

As an example let $\theta_1 = \phi_1 = 0$, $\theta_2 = \pi/2$, $\phi_2 = 0$, $\theta_3 = \pi/2$, $\phi_3 = \pi/2$. From Equation (205), $\gamma = -\pi/2$ and from Equation (207) $\Omega/2 = \pi/2$. Hence, $\gamma = -\Omega/2$ [21].

Similar relations can be derived with Z matrices and 4-component spinor pairs. But, now the type of transformation is important, Z_X must act on the pair $|\chi_X\rangle$, ($X = A, B, C$ or D). As an example, this time, we demonstrate the emergence of the geometric phase with unitary operations (rotations) on the 4-component spinor pairs. In order to avoid the dynamical phase we consider trajectories on the great circles. Let us start with $|\chi_A\rangle$ at the point $\theta_1 = \phi_1 = 0$:

$$|\chi_A\rangle = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & i \\ 1 & 0 \end{pmatrix} \tag{208}$$

---

19 Sign depends on the sense of the cyclic path.
20 Geometric phase can also be defined and calculated for an open loop operations on the Poincare sphere.
21 Negative sign is due to the CCW cyclic path.

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First we rotate $|\chi_A\rangle$ about the y-axis CCW by 90°. A Z_A matrix with $\alpha_0 = 1/\sqrt{2}$, $\alpha_1 = 0$, $\alpha_2 = -i/\sqrt{2}$, $\alpha_3 = 0$ does this job:

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & -i & 0 \\ 0 & 1 & 0 & 1 \\ -i & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & i \\ 1 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ -i & i \\ 1 & -1 \end{pmatrix} \tag{209}$$

Then we apply a 90° CCW rotation about the z-axis

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & -i \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ -i & 0 & 0 & 1 \end{pmatrix} \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ -i & i \\ 1 & -1 \end{pmatrix} = \frac{1}{2\sqrt{2}}\begin{pmatrix} 1-i & 1+i \\ 1+i & 1-i \\ 1-i & 1+i \\ 1-i & -1-i \end{pmatrix} \tag{210}$$

Finally we rotate the state CCW about the x-axis by 90°:

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i & 0 & 0 \\ -i & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \frac{1}{2\sqrt{2}}\begin{pmatrix} 1-i & 1+i \\ 1+i & 1-i \\ 1-i & 1+i \\ 1-i & -1-i \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1-i & 0 \\ 0 & 1-i \\ 0 & 1+i \\ 1-i & 0 \end{pmatrix} = \frac{e^{-i\pi/4}}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & i \\ 1 & 0 \end{pmatrix} \tag{211}$$

The final state differs from the initial one by a phase angle $\gamma = -\pi/4$, and this is the expected geometric phase for this cyclic unitary operations on the Poincare sphere.

If we work with $|\chi_B\rangle$, $|\chi_C\rangle$ or $|\chi_D\rangle$ we have to use the associated form of the Z matrix. Similar operations are also possible with dotted and undotted contravariant states by using $\dot{Z}$ and $\dot{Z}^*$ matrices respectively.

Appendix A.4. Coherent parallel combination of transformations and interpretation of Z as a state of the transforming medium

Up to now Z matrices were abstract mathematical operators that transform abstract mathematical objects (spinors). In a real experiment the transformation of the physical state of the system is carried out by an apparatus. For example, the polarization state of a beam of light (or a single photon) is modified as it passes through an optical medium.

The overall effect of the interaction of light with a deterministic, i.e., non-depolarizing, medium or optical element can be described by a 2×2 complex matrix J, referred to as the Jones matrix [28]. In order to obtain the Jones matrix boost and rotation parameters of the Lorentz group should be replaced by spectroscopic parameters (diattenuation and retardation) associated with various anisotropy properties of the optical medium [22]. Jones matrix J differs from L of SL(2,C) by a complex constant [25], hence it is an element of GL(2,C). This complex overall factor k is due to the isotropic phase retardation (η) and isotropic amplitude absorption (κ): $k = e^{-i\chi/2}$, where $\chi = \eta - i\kappa$. Hence, in polarization optics, $L \to kL = J$. Similarly $Z \to kZ = N$. Accordingly, $\alpha_\mu \to k\alpha_\mu$. It may be appropriate to define new parameters, $\tau = k\alpha_0$, $\alpha = k\alpha_1$, $\beta = k\alpha_2$, $\gamma = k\alpha_3$ [18]:

$$N = kZ = \begin{pmatrix} \tau & \alpha & \beta & \gamma \\ \alpha & \tau & -i\gamma & i\beta \\ \beta & i\gamma & \tau & -i\alpha \\ \gamma & -i\beta & i\alpha & \gamma \end{pmatrix} \tag{212}$$

[22] In polarization optics, $\sigma_1 = \sigma_z$, $\sigma_2 = \sigma_x$, $\sigma_3 = \sigma_y$ convention is usual.

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In order to obtain the optical version of the Z matrix we have to modify Equation (184). After multiplying α_μ by k, we replace θ_i and ρ_i in $\pi_i = \theta_i + \rho_i$ by birefringence and dichroism parameters: LB and LB' for linear birefringence, CB for circular birefringence; LD and LD' for linear dichroism and CD for circular dichroism [23]. For example, the state of the medium that given by $\tau = 1/\sqrt{2},\ \alpha = 1/\sqrt{2},\ \beta = \gamma = 0$ is a horizontal linear polarizer, $\tau = 1/\sqrt{2},\ \alpha = 0,\ \beta = -1/\sqrt{2},\ \gamma = 0$ is a linear polarizer at 135°, $\tau = 1/\sqrt{2},\ \alpha = -i/\sqrt{2},\ \beta = \gamma = 0$ is a quarter wave plate (vertical fast axis), $\tau = 1/\sqrt{2},\ \alpha = 0,\ \beta = 0,\ \gamma = i/\sqrt{2}$ is a circular retarder ($\delta = \pi/2$). List of basic optical elements and their states can be found in the Appendix.

There are basically two types of light-medium interaction: serial and parallel [24]. In a serial combination Jones matrices act on a 2-component spinor in succession

$$J_n \cdots J_2 J_1 |\xi\rangle = J|\xi\rangle \tag{213}$$

In a parallel process, an optical recombination takes place during the light-medium interaction. When the light beam simultaneously illuminates different parts of the medium, each part having different optical properties, the light emerging from different parts, in general with different polarizations, may recombine into a single beam. If the medium is composed of several non-depolarizing (deterministic) components, each component with a well defined Jones matrix, then the matrix associated with the coherently combined overall optical system is simply given by a linear combination of the individual matrices of the components [26,27]:

$$aJ_1|\xi\rangle + bJ_2|\xi\rangle + cJ_3|\xi\rangle + \cdots = J|\xi\rangle \tag{214}$$

where

$$J = aJ_1 + bJ_2 + cJ_3 + \cdots \tag{215}$$

Complex coefficients $a, b, c, \ldots$ are generally functions of space, time and frequency and they play the role of probability amplitudes of quantum mechanics [25] [26]. Similar relations can be written in terms of N matrices:

$$aN_1|\chi\rangle + bN_2|\chi\rangle + cN_3|\chi\rangle + \cdots = N|\chi\rangle \tag{216}$$

$$N = aN_1 + bN_2 + cN_3 + \cdots \tag{217}$$

The 4×4 real matrix for transforming the Stokes vector of the light is the Mueller matrix M that is directly connected with the experimental work. If the medium is deterministic, M can be obtained from N matrix as, $M = NN^* = N^*N$. As opposed to J and N, M does not contain any information about the overall phase introduced by the material medium. M differs from the Lorentz transformation matrix Λ by a positive real constant [27].

The resultant matrix state N in Equation (217) corresponds to the nondepolarizing Mueller matrix of the coherently combined system. Without loss of generality we may restrict our presentation to a two-term coherent parallel combination, then M can be written in terms of N matrices as follows:

$$M = NN^* = aa^* N_1 N_1^* + bb^* N_2 N_2^* + ab^* N_1 N_2^* + ba^* N_2 N_1^*. \tag{218}$$

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[23] Birefringence parameters correspond to rotations and dichroism parameters correspond to boosts.
[24] We may think of more complicated combinations involving many branches with serial, parallel or mixed operations.
[25] Time, space and frequency dependencies may entail depolarization effects, which was disscussed in [18,20].
[26] When we interpret J_i as the *state* of the component system (medium), the linear combination given in Eqations (215) may look like a *superposition* of states, but the elements J_i in the sum are not representing different states of the same system, they are the states corresponding to different systems.
[27] $\Lambda = ZZ^* \rightarrow M = cc^* ZZ^* = NN^*$. Only the isotropic amplitude absorption survives in cc^*.

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In this expansion, $N_1N_1^*$ and $N_2N_2^*$ are the Mueller matrices of the nondepolarizing component systems, whereas, $N_1N_2^*$ and $N_2N_1^*$ are the matrices resulting from coherence that cannot be interpreted as Mueller matrices in the usual sense. Although, the combined term $ab^*N_1N_2^* + ba^*N_2N_1^*$ turns out to be a real matrix, it is still not a Mueller matrix [18].

It may be more convenient to work with vectors rather than matrices to represent optical media states. The *state* interpretation of the transformation matrices becomes more clear in the vector representation. The vector state can be defined as the first column of the N matrix [8]:

$$|h\rangle = \begin{pmatrix} \tau \\ \alpha \\ \beta \\ \gamma \end{pmatrix} \tag{219}$$

It is possible to decompose a given vector state $|h\rangle$ with respect to a complete basis set of component systems:

$$|h\rangle = a|h_1\rangle + b|h_2\rangle + c|h_3\rangle + d|h_4\rangle \tag{220}$$

Here, we simply apply the ordinary vector decomposition procedure. The natural basis are

$$|h_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad |h_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad |h_3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad |h_4\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{221}$$

These basis correspond respectively to free space (identity), half-wave plate (0° fast axis), half-wave plate (45° fast axis) and a circular retarder ($\delta = \pi$). We may use other states as basis if we like. For example, let $|h_1\rangle$ and $|h_2\rangle$ correspond to orthonormal vector states of a linear horizontal polarizer and a linear vertical polarizer, then the following expansion of $|h\rangle$ will correspond to a horizontal quarter-wave plate state [18]:

$$|h\rangle = \frac{1+i}{2}|h_1\rangle + \frac{1-i}{2}|h_2\rangle. \tag{222}$$

where $c = d = 0$ [28] and

$$|h_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad |h_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \qquad |h\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \\ 0 \\ 0 \end{pmatrix}, \tag{223}$$

Therefore, at least mathematically, we can consider an ideal quarter-wave plate state as a coherent linear combination of two orthogonal linear polarizer states. In practice, this means that, if it could be possible to combine two orthogonal polarizers coherently with the associated complex coefficients as given in Equation (222), we would obtain an artificial quarter wave plate that effectively responds to the incident light just like a genuine one.

In general, we can use non-orthogonal basis to decompose a given covariance vector $|h\rangle$. However, decomposition with respect to non-orthogonal basis is more involved: we have to take into account covariant and contravariant types of bases and expansion coefficients. As an example, the covariance vector of an ideal partial polarizer can be decomposed into two non-orthogonal states, one of them

[28] In this simple example, $c = d = 0$ and the state space is two dimensional, hence we can truncate the third and fourth components of the vector states.

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being the direct beam state which corresponds to the identity Mueller matrix, and the other component being a horizontal linear polarizer state, with a suitable coefficient.

There are coherent parallel combination experiments that demonstrate the state interpretation of J, N and $|h\rangle$ in light-nanoparticle interactions [18–21]. In a certain interval of wavelength, a nanorod oriented at an angle θ in the the x-y plane responds to the incident light propagating along the z axis as a linear polaizer and the associated vector state is

$$|h\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \cos(2\theta) \\ \sin(2\theta) \\ 0 \end{pmatrix} \tag{224}$$

It can be mathematically shown and experimentally observed that two crossed orthogonal identical nanorods respond to the incident light as an identity (free space)[18,19]:

$$|h\rangle = \frac{1}{\sqrt{2}}|h_1\rangle + \frac{1}{\sqrt{2}}|h_2\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{225}$$

In this expression, for simplicity, we let the Lorentzian polarizabilities of the nanorods equal to one.

In a three dimensional arrangement, if there is a spacing between the two nanorods as depicted in Figure A1, coherently combined system manifests optical activity along the z axis due to the relative phase and mutual interaction between the nanorods. It is worth noting that, since the nonorod vector states are in the form given in Equation (224), the fourth component that corresponds to the γ anisotropy, which is related to the optical activity, is always zero, i.e., for non-interacting nanorods no coherent linear combination can result in a vector state with $\gamma \neq 0$ [29]. For the system given in Figure A1, the emergence of optical activity can be described as a three term coherent combination of vector states, two of them associated with non-interacting nanorods and the third one being the state due the interaction:

$$|h\rangle = g\left[a|h_1\rangle + b|h_2\rangle + c_{int}|h_{int}\rangle\right] = g\left[a\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} + c_{int}\begin{pmatrix} 0 \\ 0 \\ 1+e^2 \\ i(1-e^2) \end{pmatrix}\right] \tag{226}$$

where $e = e^{-i2\pi Z/\lambda}$ is the phase difference due to the spacing between the nanorods along the z axis, c_{int} contains the Lorentzian polarizability and the interaction coefficient, the overall factor $g = G/(1-\Delta^2)$, G is a function of the Lorentzian polarizability and the far field factor, Δ is a compound factor involving phase, polarizability and interaction coefficient [21] [30] [31].

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[29] This is also true for crossed orthogonal nanorods.

[30] Lorentzian polarizability and the interaction coefficient are functions of the wavelength of the incident light, hence the phenomenon of plasmonic hybridization occurs at wavelengths that make the denominator of g zero.

[31] Optical activity can be observed even in a planar geometry in sideways scattering directions [21].

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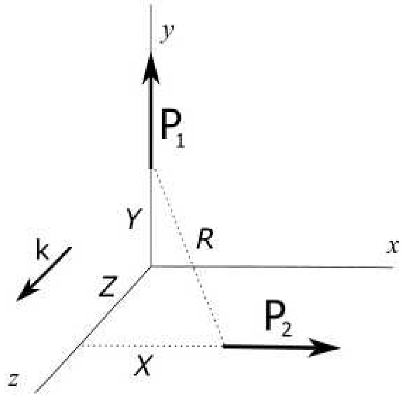

Figure A1. Optical activity in a coupled dimer. $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ are the dipoles associated with the nanorods, Light propagates along $\boldsymbol{k}$.

Appendix A.4.1. Unitary formulation of the rotation of the medium in space

J, N matrices and $|h\rangle$ vector are the states of the transforming medium or the optical element, therefore they are themselves subjected to transformations. Particularly, if the optical element is rotated CCW by an angle θ about an axis parallel to the direction of propagation of light, the state of the optical element is also rotated:

$$N(\theta) = U(2\theta)N(0)U(2\theta)^{\dagger} \tag{227}$$

$U(2\theta)$ is unitary 4×4 matrix, element of SL(4,C):

$$U(2\theta) = \cos(\theta) - i\hat{n} \cdot \vec{\Sigma} \sin(\theta) \tag{228}$$

$\hat{n}$ indicates the direction of propagation of light [32]. The corresponding Mueller matrix is rotated as [33]

$$M(\theta) = R(2\theta)M(0)R(-2\theta) \tag{229}$$

If we choose the direction of the light beam along the z axis

$$U(2\theta) = \begin{pmatrix} \cos(\theta) & 0 & 0 & -i\sin(\theta) \\ 0 & \cos(\theta) & -\sin(\theta) & 0 \\ 0 & \sin(\theta) & \cos(\theta) & 0 \\ -i\sin(\theta) & 0 & 0 & \cos(\theta) \end{pmatrix} \tag{230}$$

and

$$R(2\theta) = U(2\theta)U(2\theta)^* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(2\theta) & -\sin(2\theta) & 0 \\ 0 & \sin(2\theta) & \cos(2\theta) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{231}$$

32 The vector state $|h\rangle$ is rotated as $|h(\theta)\rangle = R(2\theta)|h(0)\rangle$.

33 $M(\theta) = N(\theta)N(\theta)^* = U(2\theta)N(0)U(2\theta)^{\dagger}(U(2\theta)N(0)U(2\theta)^{\dagger})^*$, because any complex conjugate form commutes with any Hermitian adjoint form (Σ_i^* commutes with Σ_j for all i, j), hence $M(\theta) = U(2\theta)U(2\theta)^*Z(0)Z(0)^*(U(2\theta)U(2\theta)^*)^{\dagger} = R(2\theta)M(0)R(-2\theta)$.

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Appendix A.5. Table for vector and matrix states of optical elements

This appendix contains a tabulated list of $|h\rangle$ vector states, N matrix states and their corresponding Mueller matrices M .

Optical element	$\|h\rangle$	N	M
Free space	$\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$
Half-wave plate (Ideal mirror)	$\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$	$\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&-i\\0&0&i&0\end{pmatrix}$	$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}$
Half-wave plate 45° fast axis	$\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$	$\begin{pmatrix}0&0&1&0\\0&0&0&i\\1&0&0&0\\0&-i&0&0\end{pmatrix}$	$\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}$
Circular retarder ($\delta=\pi$)	$\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$	$\begin{pmatrix}0&0&0&1\\0&0&-i&0\\0&i&0&0\\1&0&0&0\end{pmatrix}$	$\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix}$
Horizontal Linear Polarizer	$\begin{pmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\\0\\0\end{pmatrix}$	$\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}&0&0\\\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}&0&0\\0&0&\frac{1}{\sqrt{2}}&\frac{-i}{\sqrt{2}}\\0&0&\frac{i}{\sqrt{2}}&\frac{1}{\sqrt{2}}\end{pmatrix}$	$\begin{pmatrix}1&1&0&0\\1&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$
Vertical Linear Polarizer	$\begin{pmatrix}\frac{1}{\sqrt{2}}\\\frac{-1}{\sqrt{2}}\\0\\0\end{pmatrix}$	$\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{-1}{\sqrt{2}}&0&0\\\frac{-1}{\sqrt{2}}&\frac{1}{\sqrt{2}}&0&0\\0&0&\frac{1}{\sqrt{2}}&\frac{i}{\sqrt{2}}\\0&0&\frac{-i}{\sqrt{2}}&\frac{1}{\sqrt{2}}\end{pmatrix}$	$\begin{pmatrix}1&-1&0&0\\-1&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$
Linear Polarizer at 45°	$\begin{pmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\\0\end{pmatrix}$	$\begin{pmatrix}\frac{1}{\sqrt{2}}&0&\frac{1}{\sqrt{2}}&0\\0&\frac{1}{\sqrt{2}}&0&\frac{i}{\sqrt{2}}\\\frac{1}{\sqrt{2}}&0&\frac{1}{\sqrt{2}}&0\\0&\frac{-i}{\sqrt{2}}&0&\frac{1}{\sqrt{2}}\end{pmatrix}$	$\begin{pmatrix}1&0&1&0\\0&0&0&0\\1&0&1&0\\0&0&0&0\end{pmatrix}$
Linear Polarizer at 135°	$\begin{pmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{-1}{\sqrt{2}}\\0\end{pmatrix}$	$\begin{pmatrix}\frac{1}{\sqrt{2}}&0&\frac{-1}{\sqrt{2}}&0\\0&\frac{1}{\sqrt{2}}&0&\frac{-i}{\sqrt{2}}\\\frac{-1}{\sqrt{2}}&0&\frac{1}{\sqrt{2}}&0\\0&\frac{i}{\sqrt{2}}&0&\frac{1}{\sqrt{2}}\end{pmatrix}$	$\begin{pmatrix}1&0&-1&0\\0&0&0&0\\-1&0&1&0\\0&0&0&0\end{pmatrix}$
Circular Polarizer (right handed)	$\begin{pmatrix}\frac{1}{\sqrt{2}}\\0\\0\\\frac{1}{\sqrt{2}}\end{pmatrix}$	$\begin{pmatrix}\frac{1}{\sqrt{2}}&0&0&\frac{1}{\sqrt{2}}\\0&\frac{1}{\sqrt{2}}&\frac{-i}{\sqrt{2}}&0\\0&\frac{i}{\sqrt{2}}&\frac{1}{\sqrt{2}}&0\\\frac{1}{\sqrt{2}}&0&0&\frac{1}{\sqrt{2}}\end{pmatrix}$	$\begin{pmatrix}1&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&1\end{pmatrix}$
Circular Polarizer (left handed)	$\begin{pmatrix}\frac{1}{\sqrt{2}}\\0\\0\\\frac{-1}{\sqrt{2}}\end{pmatrix}$	$\begin{pmatrix}\frac{1}{\sqrt{2}}&0&0&\frac{-1}{\sqrt{2}}\\0&\frac{1}{\sqrt{2}}&\frac{i}{\sqrt{2}}&0\\0&\frac{-i}{\sqrt{2}}&\frac{1}{\sqrt{2}}&0\\\frac{-1}{\sqrt{2}}&0&0&\frac{1}{\sqrt{2}}\end{pmatrix}$	$\begin{pmatrix}1&0&0&-1\\0&0&0&0\\0&0&0&0\\-1&0&0&1\end{pmatrix}$

Optical element	$\lvert h\rangle$	N	M
QWP horizontal fast axis	$\begin{pmatrix}\frac{1}{\sqrt{2}}\\\frac{i}{\sqrt{2}}\\0\\0\end{pmatrix}$	$\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{i}{\sqrt{2}}&0&0\\\frac{i}{\sqrt{2}}&\frac{1}{\sqrt{2}}&0&0\\0&0&\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\\0&0&\frac{-1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\end{pmatrix}$	$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&-1&0\end{pmatrix}$
QWP vertical fast axis	$\begin{pmatrix}\frac{1}{\sqrt{2}}\\\frac{-i}{\sqrt{2}}\\0\\0\end{pmatrix}$	$\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{-i}{\sqrt{2}}&0&0\\\frac{-i}{\sqrt{2}}&\frac{1}{\sqrt{2}}&0&0\\0&0&\frac{1}{\sqrt{2}}&\frac{-1}{\sqrt{2}}\\0&0&\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\end{pmatrix}$	$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&-1\\0&0&1&0\end{pmatrix}$
QWP fast axis 135º	$\begin{pmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{i}{\sqrt{2}}\\0\end{pmatrix}$	$\begin{pmatrix}\frac{1}{\sqrt{2}}&0&\frac{i}{\sqrt{2}}&0\\0&\frac{1}{\sqrt{2}}&0&\frac{-1}{\sqrt{2}}\\\frac{i}{\sqrt{2}}&0&\frac{1}{\sqrt{2}}&0\\0&\frac{1}{\sqrt{2}}&0&\frac{1}{\sqrt{2}}\end{pmatrix}$	$\begin{pmatrix}1&0&0&0\\0&0&0&-1\\0&0&1&0\\0&1&0&0\end{pmatrix}$
QWP fast axis 45º	$\begin{pmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{-i}{\sqrt{2}}\\0\end{pmatrix}$	$\begin{pmatrix}\frac{1}{\sqrt{2}}&0&\frac{-i}{\sqrt{2}}&0\\0&\frac{1}{\sqrt{2}}&0&\frac{1}{\sqrt{2}}\\\frac{-i}{\sqrt{2}}&0&\frac{1}{\sqrt{2}}&0\\0&\frac{-1}{\sqrt{2}}&0&\frac{1}{\sqrt{2}}\end{pmatrix}$	$\begin{pmatrix}1&0&0&0\\0&0&0&1\\0&0&1&0\\0&-1&0&0\end{pmatrix}$
Circular retarder ($\delta=\frac{\pi}{2}$)	$\begin{pmatrix}\frac{1}{\sqrt{2}}\\0\\0\\\frac{i}{\sqrt{2}}\end{pmatrix}$	$\begin{pmatrix}\frac{1}{\sqrt{2}}&0&0&\frac{i}{\sqrt{2}}\\0&\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}&0\\0&\frac{-1}{\sqrt{2}}&\frac{1}{\sqrt{2}}&0\\\frac{i}{\sqrt{2}}&0&0&\frac{1}{\sqrt{2}}\end{pmatrix}$	$\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&-1&0&0\\0&0&0&1\end{pmatrix}$
Circular retarder ($\delta=-\frac{\pi}{2}$)	$\begin{pmatrix}\frac{1}{\sqrt{2}}\\0\\0\\\frac{-i}{\sqrt{2}}\end{pmatrix}$	$\begin{pmatrix}\frac{1}{\sqrt{2}}&0&0&\frac{-i}{\sqrt{2}}\\0&\frac{1}{\sqrt{2}}&\frac{-1}{\sqrt{2}}&0\\0&\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}&0\\\frac{-i}{\sqrt{2}}&0&0&\frac{1}{\sqrt{2}}\end{pmatrix}$	$\begin{pmatrix}1&0&0&0\\0&0&-1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}$

QWP stands for Quarter Wave Plate.

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