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Article

Evaluation of Various Generalized Pareto Probability Distributions for Flood Frequency Analysis

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Abstract: This article analyzes 6 probability distributions from the Generalized Pareto family, with 3, 4 and 5 parameters, with main purpose to identify other distributions from this family with applicability in flood frequency analysis compared to the distribution already used in the literature from this family such as Generalized Pareto Type II and Wakeby. This analysis is part of a larger and more complex research carried out in the Faculty of Hydrotechnics regarding the elaboration of a norm for flood frequency analysis using the linear moments method. In Romania, the standard method of parameter estimation is the method of ordinary moments, thus the transition from this method to the method of linear moments is desired. All the necessary elements for the distributions use are presented like, the probability density functions, the complementary cumulative distribution functions, the quantile functions, the exact and approximate relations for estimating parameters, for both methods of parameters estimation. All these elements are necessary for a proper transition between the two methods, especially since the use of the method of ordinary moments is done by choosing the skewness of the observed data depending on the origin of the maximum flows. A flood frequency analysis case study, using annual maximum and annual exceedance series, was carried out for the Prigor river, to numerically present the analyzed distributions. The performance of this distributions is evaluated using relative mean error, relative absolute error and linear moments diagram.

Keywords: floods; frequency analysis; extreme value statistics; Pareto; Wakeby; estimation parameters; approximate form; method of ordinary moments; method of linear moments

1. Introduction

The frequency analysis of extreme events in hydrology is of particular importance, with the aim of determining the probability of occurrence of extreme events of a given magnitude.

Flood frequency analysis is important because it determines the maximum flow with certain exceeding probabilities, they have a defining role in the design of dams [1] and in water management [2], and can have significant impacts on human lives, infrastructure, and the environment.

Together with the distributions from the Gamma family and Generalized Extreme Values, the Generalized Pareto Type 2 distribution (PGII) represents one of the most used distributions in flood frequency analysis [3–5], especially in the analysis of partial series using the Annual Exceedance Series (AES) or Peak Over Thresholds (POT).

Among the Generalized Pareto distributions analyzed in this article, the ones that received considerable attention in flood frequency analysis are the Generalized Pareto distribution Type II (PGII) using AES or POT [3–5], respectively the Generalized Pareto distribution Type III (PGIII) and the Wakeby (WK5) distribution [4,8,9], in the analysis with the Annual Maximum Series (AMS). The PGIII is also known in literature as the Log-Logistic distribution or Generalized Logistic distribution [4,6,7].

The PGII distribution has a broad use in the analysis of extreme events such as, precipitation frequency analysis [10–15], in low flow frequency analysis [16] and in flood frequency analysis [4,5,17–20].

Based on Rao and Hamed [4], respectively Hosking and Wallis [21] the Generalized Pareto distribution “ *is the logical choice for modeling flood magnitudes that exceed a fixed threshold when it is reasonable to assume that successive floods follow a Poisson process and have independent magnitudes*”.

In the case of Wakeby distribution, it is a quantile distribution (a distribution expressible in inverse form) which in certain situations [4,9,22] turns into the Generalized Pareto Type II distribution. Its application, using the close form equations for the first four central moments, was realized for the first time by Anghel and Ilinca [22,23], both for the four and five parameters Wakeby distributions.

Other distributions from the Generalized Pareto family have received little to none attention, such as the four parameters Generalized Pareto Type IV (PGIV4), the three parameters Generalized Pareto Type IV (PGIV3) and the Generalized Pareto Type I (PGI) also known as Pearson XI distribution.

Taking into account this, one of the objectives of the article consists in the analysis of the applicability of other distributions belonging to the same family of Generalized Pareto distribution in flood frequency analysis. For a comprehensive analysis, in this article all the distributions belonging to this family are comparatively analyzed. Although some of these distributions have been used in the frequency analysis of extreme events in hydrology, this article brings new elements for these distributions that help to better understand and apply them in hydrology and beyond.

The research from this article is part of a larger and more complex research carried out in the Faculty of Hydrotechnics, to identify the distributions from different families of distributions, which have applicability in frequency analysis of extreme values, partial results presented in other materials [22–24].

In this article, the estimation methods of the parameters of these distributions are method of ordinary moments (MOM) and the method of linear moments (L-moments), for some of them being necessary to solve nonlinear systems of equations, which leads to some difficulties in using these distributions. Thus, for the ease of applications of these distributions, parameter approximation relations are presented, using polynomial, exponential or rational functions.

Only these two methods of estimating parameters are analyzed in this article, because MOM is the “parent” method in Romania, and the L-moments method is the method that is intended to be used in the new regulations regarding the analysis of extreme phenomena in hydrology, being a much more stable and less sensitive to short lengths of data, as is the general case of hydrometry in Romania.

All the mathematical elements necessary to use these distributions in the flood frequency analysis are presented.

New elements are presented, such as: the first six raw and central moments for PGII, PGI; relations for estimating the parameters with L-moments for the PGIV4, PGIV3 and PGI distributions; new approximate parameter estimation relations for the PGII, PGI and PGIII distributions; the frequency factors for all analyzed distributions, both for MOM and L-moments; approximation relations of the frequency factors for the PGI, PGII and PGIII distributions.

Thus, all these novelty elements for these distributions presented in Table 1 will help hydrology researchers to better understand and easily apply these distributions.

Table 1. Novelty elements.

New Elements	Distribution
Exact parameter estimation	PGIV4, PGI
Approximate estimation of parameters	PGIII, PGII, PGI
The frequency factor for MOM	PGIV4, PGIV3, PGIII, PGII, PGI, WK5
The frequency factor for L-moments	PGIV4, PGIV3, PGIII, PGII, PGI, WK5
Approximate estimation of the frequency factor	PGIII, PGII

Raw and central moments*	PGIV, PGIII, PGII
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*are presented in Supplementary file.

The raw and central moments of the analyzed distributions were determined using the methodology presented in the Supplementary file, based on the probability density functions. It is for the first time that, for the Generalized Pareto distributions Type II and Type III, the raw and central moments up to order 6 are presented, important, along with the frequency factor, in establishing the confidence interval (for MOM) using the Kite approximation [4]. Also for the first time, the frequency factor of these distributions is presented based on the L-moments method, an important aspect in determining the confidence interval using the Chow approximation [4], the latter was used in hydrology only based on the estimation of parameters with MOM.

In order to verify the performances of the proposed distributions, a flood frequency analysis is carried out, using the Annual Maximum Series (AMS) and the Annual Exceedance Series (AES) for the Prigor river, as a case study. All results are presented in comparison with the Pearson III distribution, which is the “parent” distribution in flood frequency analysis in Romania [22]. The purpose of the article was to identify other distributions from this family with applicability in flood frequency analysis compared to the distributions already used in the literature from these families such as PGIII and PGII. The article does not exclude the applicability of other distributions from other families (Gamma, GEV, Beta) in flood frequency analysis, especially since these families were also analyzed within the research carried out in the Faculty of Hydrotechnics and presented in other materials [22–24].

Comparing the results and choosing the best distribution is based on the performance indicators: relative mean error (RME), relative absolute error (RAE) and L-skewness (τ_3)-L-kurtosis (τ_4) diagram.

The article is organized as follows. The description of the statistical distributions by presenting the density function, the complementary cumulative function and the quantile function, in Section 2.1. The presentation of the relations for exact calculation and the approximate relations for determining the parameters of the distributions, in Section 2.2. Case studies by applying these distributions in flood frequency analysis for the Prigor river, in Section 3. Results, discussions and conclusions, in Sections 3, 4 and 5.

2. Methods

The frequency analysis consists in determining the flows with certain exceedance probabilities using AMS, respectively AES, with the Prigor river as a case study.

The series of maximum annual flows consists in choosing the maximum value corresponding to each year. In many cases, the lower maximum values of the annual data series do not always represent "floods". Thus, the use of frequency analysis using AES is required, which allows secondary events, which exceed certain annual maximum values, to be considered as "floods".

The AES was established by descending sorting of all independent maximum values and choosing the first "n" values corresponding to the "n" number of years of analysis [14,15].

Using this criterion, it is important to verify that two or more maximum flow values do not come from the same flood. The independence of flows was verified and established based on the Cunnane criteria, respectively USWRC 1976 [25].

The determination of the maximum flows was carried out in stages according to Figure 1. The verification of the character of outliers (Grubbs, Pilon, Quartile method), homogeneity and independence of flows, were carried out in the data curation phase. No outliers have been detected.

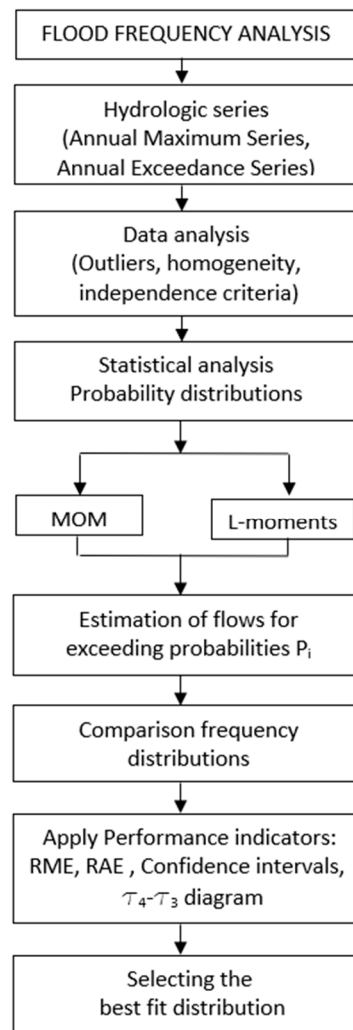


Figure 1. Methodological approach.

The estimation of the parameters of the analyzed distributions was done with MOM and L-moments. The MOM estimation has the disadvantage that for high-order moments and short data series in many cases it generates unrealistic values because the high-order moments require correction [4,5,26,27]. For skewness, the correction can be made using the Bobee relation [26] or, as is the practice in Romania [22,28–31], being established according to the origin of the maximum flows, by multiplying the coefficient of variation (C_v) by a coefficient reflecting this origin. In many cases the choice is subjective and without rigor [22,32].

The L-moments method is a more stable and robust method, having the advantage that it is less sensitive to short data lengths, compared to other parameter estimation methods such as Method of Ordinary Moments, Method of Maximum Likelihood Estimation or the Method of Least Squares [3,22,33].

Considering that in many cases to estimate the parameters of the analyzed distributions, it is necessary to solve some systems of non-linear equations, approximate relations for estimating the parameters were determined in the case of distributions where skewness and L-skewness depend on a single parameter. Also, for a simplified and fast calculation taking into account the fact that the inverse function can be expressed with the frequency factor for both MOM [4,23,34] and L moments [23], the approximation relations of the factor of frequency (and the coefficients of these relations) for the most frequent exceeding probabilities in the analysis of maximum flows. The estimation errors of both the parameters and the frequency factors are between 10^{-3} - 10^{-4} .

The quantile results are compared with those of the Pearson III distribution, which is the “parent” distribution in Romania in the analysis of extreme events in hydrology, especially in the flood frequency analysis [22,28,29].

2.1. Probability Distributions

In Table 2 are presented the probability density function, $f(x)$; the complementary cumulative distribution function, $F(x)$, and quantile function, $x(p)$, for analyzed distributions [4–6,14–17]. All $F(x)$ and $x(p)$ of the analysed distributions were determined using the methodology presented in the Supplementary file, using only $f(x)$

Table 2. The analyzed probability distributions.

Distr.	$f(x)$	$F(x)$	$x(p)$
PGIV4	$\frac{\lambda \cdot \left(\frac{x-\gamma}{\beta}\right)^{\frac{1}{\alpha}-1} \cdot \left(\left(\frac{x-\gamma}{\beta}\right)^{\frac{1}{\alpha}} + 1\right)^{-\lambda-1}}{\alpha \cdot \beta}$	$\left(\left(\frac{x-\gamma}{\beta}\right)^{\frac{1}{\alpha}} + 1\right)^{-\lambda}$	$\gamma + \beta \cdot \left(\frac{1}{p^{\frac{1}{\lambda}}} - 1\right)^{\alpha}$
PGIV3	$\frac{\lambda \cdot \left(\frac{x}{\beta}\right)^{\frac{1}{\alpha}-1} \cdot \left(\left(\frac{x}{\beta}\right)^{\frac{1}{\alpha}} + 1\right)^{-\lambda-1}}{\alpha \cdot \beta}$	$1 - \left(\left(\frac{x}{\beta}\right)^{\frac{1}{\alpha}} + 1\right)^{-\lambda}$	$\beta \cdot \left(\frac{1}{p^{\frac{1}{\lambda}}} - 1\right)^{\alpha}$
PGIII	$\frac{\alpha \cdot \left(\frac{x-\gamma}{\beta}\right)^{\alpha-1} \cdot \left(\left(\frac{x-\gamma}{\beta}\right)^{\alpha} + 1\right)^{-2}}{\beta}$	$\left(1 + \left(\frac{x-\gamma}{\beta}\right)^{\alpha}\right)^{-1}$	$\gamma + \beta \cdot \left(\frac{1}{p} - 1\right)^{\frac{1}{\alpha}}$
PGII	$\frac{1}{\beta} \cdot \left(1 - \frac{\alpha}{\beta} \cdot (x - \gamma)\right)^{\frac{1}{\alpha}-1}$	$\left(1 - \frac{\alpha}{\beta} \cdot (x - \gamma)\right)^{\frac{1}{\alpha}}$	$\gamma + \frac{\beta}{\alpha} \cdot (1 - p^{\alpha})$
PGI	$\frac{\alpha}{\beta} \cdot \left(\frac{x-\gamma}{\beta}\right)^{-\alpha-1}$	$\left(\frac{x-\gamma}{\beta}\right)^{-\alpha}$	$\gamma + \beta \cdot p^{-\frac{1}{\alpha}}$
WK5	No closed form	No closed form	$\xi + \frac{\alpha}{\beta} \cdot (1 - p^{\beta}) - \frac{\gamma}{\delta} \cdot (1 - p^{-\delta})$

2.2. Parameter Estimation

The parameters estimation of the analyzed statistical distributions is presented for MOM and L-moments, some of the most used methods in hydrology for parameter estimation [3–5,7,22].

2.2.1. Generalized Pareto Type IV (PGIV4)

The equations needed to estimate the parameters with MOM have the following expressions:

$$\mu = \gamma + \frac{\beta \cdot \alpha \cdot \Gamma(\alpha) \cdot \Gamma(\lambda - \alpha)}{\lambda \cdot \Gamma(\lambda)} \quad (1)$$

$$\sigma^2 = \frac{\beta^2}{\lambda \cdot \Gamma(\lambda)} \cdot \left(\Gamma(2 \cdot \alpha + 1) \cdot \Gamma(\lambda - 2 \cdot \alpha) - \frac{\Gamma(\alpha + 1)^2 \cdot \Gamma(\lambda - \alpha)^2}{\lambda \cdot \Gamma(\lambda)} \right) \quad (2)$$

Because they are too long, the relations for estimating skewness and kurtosis are presented in Appendix F.

The equations needed to estimate the parameters with L-moments have the following expressions:

$$L_1 = \gamma + \beta \cdot \frac{\Gamma(\lambda - \alpha) \cdot \Gamma(1 + \alpha)}{\Gamma(\lambda)} \quad (3)$$

$$L_2 = \beta \cdot \left(\frac{\Gamma(\lambda - \alpha) \cdot \Gamma(1 + \alpha)}{\Gamma(\lambda)} - \frac{\Gamma(2 \cdot \lambda - \alpha) \cdot \Gamma(1 + \alpha)}{\Gamma(2 \cdot \lambda)} \right) \quad (4)$$

$$L_3 = \beta \cdot \left(\frac{\Gamma(\lambda - \alpha) \cdot \Gamma(1 + \alpha)}{\Gamma(\lambda)} - 3 \cdot \frac{\Gamma(2 \cdot \lambda - \alpha) \cdot \Gamma(1 + \alpha)}{\Gamma(2 \cdot \lambda)} + 2 \cdot \frac{\Gamma(3 \cdot \lambda - \alpha) \cdot \Gamma(1 + \alpha)}{\Gamma(3 \cdot \lambda)} \right) \quad (5)$$

$$L_4 = \beta \cdot \left(\frac{\Gamma(\lambda - \alpha) \cdot \Gamma(1 + \alpha)}{\Gamma(\lambda)} - 6 \cdot \frac{\Gamma(2 \cdot \lambda - \alpha) \cdot \Gamma(1 + \alpha)}{\Gamma(2 \cdot \lambda)} + 10 \cdot \frac{\Gamma(3 \cdot \lambda - \alpha) \cdot \Gamma(1 + \alpha)}{\Gamma(3 \cdot \lambda)} - 5 \cdot \frac{\Gamma(4 \cdot \lambda - \alpha) \cdot \Gamma(1 + \alpha)}{\Gamma(4 \cdot \lambda)} \right) \quad (6)$$

where L_1, L_2, L_3, L_4 represent the first four linear moments.

2.2.2. Generalized Pareto Type IV (PGIV3)

The distribution represents a particular case of the Pareto IV distribution when the position parameter $\gamma = 0$. It is also known as the beta_p or Singh-Maddala distribution [6,33].

The equations needed to estimate the parameters with MOM have the following expressions:

$$\mu = \beta \cdot \frac{\Gamma(\alpha + 1) \cdot \Gamma(\lambda - \alpha)}{\Gamma(\lambda)} \quad (7)$$

$$\sigma^2 = \frac{\beta^2}{\Gamma(\lambda)} \cdot \left(\Gamma(2 \cdot \alpha + 1) \cdot \Gamma(\lambda - 2 \cdot \alpha) - \frac{\Gamma(\alpha + 1)^2 \cdot \Gamma(\lambda - \alpha)^2}{\Gamma(\lambda)} \right) \quad (8)$$

$$C_s = \frac{\Gamma(\lambda)^2 \cdot \Gamma(\lambda - 3 \cdot \alpha) \cdot \Gamma(1 + 3 \cdot \alpha) + 2 \cdot \Gamma(\lambda - \alpha)^3 \cdot \Gamma(1 + \alpha)^3 - 3 \cdot \Gamma(\lambda) \cdot \Gamma(\lambda - 2 \cdot \alpha) \cdot \Gamma(1 + 2 \cdot \alpha) \cdot \Gamma(\lambda - \alpha) \cdot \Gamma(1 + \alpha)}{\left(\Gamma(\lambda) \cdot \Gamma(\lambda - 2 \cdot \alpha) \cdot \Gamma(1 + 2 \cdot \alpha) - \Gamma(\lambda - \alpha)^2 \cdot \Gamma(1 + \alpha)^2 \right)^{1.5}} \quad (9)$$

The equations needed to estimate the parameters with L-moments have the following expressions [33]:

$$L_1 = \beta \cdot \frac{\Gamma(\lambda - \alpha) \cdot \Gamma(1 + \alpha)}{\Gamma(\lambda)} \quad (10)$$

$$L_2 = \beta \cdot \left(\frac{\Gamma(\lambda - \alpha) \cdot \Gamma(1 + \alpha)}{\Gamma(\lambda)} - \frac{\Gamma(2 \cdot \lambda - \alpha) \cdot \Gamma(1 + \alpha)}{\Gamma(2 \cdot \lambda)} \right) \quad (11)$$

$$L_3 = \beta \cdot \left(\frac{\Gamma(\lambda - \alpha) \cdot \Gamma(1 + \alpha)}{\Gamma(\lambda)} - 3 \cdot \frac{\Gamma(2 \cdot \lambda - \alpha) \cdot \Gamma(1 + \alpha)}{\Gamma(2 \cdot \lambda)} + 2 \cdot \frac{\Gamma(3 \cdot \lambda - \alpha) \cdot \Gamma(1 + \alpha)}{\Gamma(3 \cdot \lambda)} \right) \quad (12)$$

2.2.3. Generalized Pareto Type III (PGIII)

The equations needed to estimate the parameters with MOM have the following expressions [3,4]:

$$\mu = \gamma + \beta \cdot \Gamma\left(1 + \frac{1}{\alpha}\right) \cdot \Gamma\left(1 - \frac{1}{\alpha}\right) \quad (13)$$

$$\sigma^2 = \beta^2 \cdot \left(\Gamma\left(1 + \frac{2}{\alpha}\right) \cdot \Gamma\left(1 - \frac{2}{\alpha}\right) - \Gamma\left(1 + \frac{1}{\alpha}\right)^2 \cdot \Gamma\left(1 - \frac{1}{\alpha}\right)^2 \right) \quad (14)$$

$$C_s = \frac{\Gamma\left(1 + \frac{3}{\alpha}\right) \cdot \Gamma\left(1 - \frac{3}{\alpha}\right) + 2 \cdot \Gamma\left(1 + \frac{1}{\alpha}\right)^3 \cdot \Gamma\left(1 - \frac{1}{\alpha}\right)^3 - 3 \cdot \Gamma\left(1 + \frac{2}{\alpha}\right) \cdot \Gamma\left(1 - \frac{2}{\alpha}\right) \cdot \Gamma\left(1 + \frac{1}{\alpha}\right) \cdot \Gamma\left(1 - \frac{1}{\alpha}\right)}{\left(\Gamma\left(1 + \frac{2}{\alpha}\right) \cdot \Gamma\left(1 - \frac{2}{\alpha}\right) - \Gamma\left(1 + \frac{1}{\alpha}\right)^2 \cdot \Gamma\left(1 - \frac{1}{\alpha}\right)^2 \right)^{1.5}} \quad (15)$$

The shape parameter can be obtained approximately depending on the skewness coefficient, using the following function:

if $0.1 \leq C_s \leq 7$:

$$\alpha = \exp \left(\begin{aligned} &2.2463456 - 0.8505372 \cdot \ln(C_s) + 0.1230871 \cdot \ln(C_s)^2 + \\ &0.0511417 \cdot \ln(C_s)^3 + 0.0024655 \cdot \ln(C_s)^4 - 0.006948 \cdot \ln(C_s)^5 - \\ &0.0017015 \cdot \ln(C_s)^6 + 0.0004746 \cdot \ln(C_s)^7 + 0.0001676 \cdot \ln(C_s)^8 \end{aligned} \right) \quad (16)$$

$$\beta = \frac{\sigma}{\sqrt{\Gamma\left(1+\frac{2}{\alpha}\right) \cdot \Gamma\left(1-\frac{2}{\alpha}\right) - \Gamma\left(1+\frac{1}{\alpha}\right)^2 \cdot \Gamma\left(1-\frac{1}{\alpha}\right)^2}} \quad (17)$$

$$\gamma = \mu - \frac{\beta}{\alpha} \cdot \Gamma\left(\frac{1}{\alpha}\right) \cdot \Gamma\left(1-\frac{1}{\alpha}\right) \quad (18)$$

The equations needed to estimate the parameters with L-moments have the following expressions:

$$\alpha = \frac{1}{\tau_3} = \frac{L_2}{L_3} \quad (19)$$

$$\beta = \frac{\alpha^2 \cdot L_2}{\Gamma\left(\frac{1}{\alpha}\right) \cdot \Gamma\left(1-\frac{1}{\alpha}\right)} \quad (20)$$

$$\gamma = L_1 - \frac{\beta}{\alpha} \cdot \Gamma\left(\frac{1}{\alpha}\right) \cdot \Gamma\left(1-\frac{1}{\alpha}\right) \quad (21)$$

2.2.4. Generalized Pareto Type II (PGII)

The equations needed to estimate the parameters with MOM have the following expressions [3–5]:

$$\mu = \gamma + \frac{\beta}{\alpha + 1} \quad (22)$$

$$\sigma^2 = \frac{\beta^2}{(\alpha + 1)^2 \cdot (2 \cdot \alpha + 1)} \quad (23)$$

$$C_s = \frac{\alpha}{|\alpha|} \cdot \frac{2 \cdot (\alpha - 1) \cdot (2 \cdot \alpha + 1)^{0.5}}{3 \cdot \alpha + 1} \quad (24)$$

The shape parameter can be obtained approximately depending on the skewness coefficient, using the following functions:

if $C_s < 2$:

$$\alpha = 0.999946019 - 1.154837085 \cdot C_s + 0.559584297 \cdot C_s^2 - 0.152516332 \cdot C_s^3 - 0.018234805 \cdot C_s^4 \quad (25)$$

if $C_s \geq 2$:

$$\alpha = \frac{1.680766638 - 0.91819165 \cdot C_s + 0.043608198 \cdot C_s^2 - 0.002324901 \cdot C_s^3}{1 + 1.805341085 \cdot C_s} \quad (26)$$

$$\beta = \sigma \cdot (\alpha + 1) \cdot \sqrt{2 \cdot \alpha + 1} \quad (27)$$

$$\gamma = \mu - \frac{\beta}{\alpha + 1} \quad (28)$$

The frequency factor (for MOM), presented in Appendix B, can be obtained approximately, using a polynomial function of skewness and probability, whose coefficients are presented in table D1 from Appendix D.

The equations needed to estimate the parameters with L-moments have the following expressions:

$$L_1 = \gamma + \frac{\beta}{\alpha + 1} \quad (29)$$

$$L_2 = \frac{\beta}{(\alpha + 1) \cdot (\alpha + 2)} \quad (30)$$

$$L_3 = \frac{-\beta \cdot (\alpha - 1)}{(\alpha + 1) \cdot (\alpha + 2) \cdot (\alpha + 3)} \quad (31)$$

Based on these equations, the parameters have the following expressions:

$$\alpha = \frac{1 - 3 \cdot \tau_3}{\tau_3 + 1} \quad (32)$$

$$\beta = L_2 \cdot (\alpha + 1) \cdot (\alpha + 2) \quad (33)$$

$$\gamma = L_1 - L_2 \cdot (\alpha + 2) \quad (34)$$

The frequency factor (for L-moments) can be obtained approximately, using a polynomial function depending on skewness and probability, whose coefficients are presented in table D2 from Appendix D.

2.2.5. Generalized Pareto Type I (PGI)

The equations needed to estimate the parameters with MOM have the following expressions:

$$\mu = \gamma + \frac{\beta \cdot \alpha}{\alpha - 1} \quad (35)$$

$$\sigma^2 = \frac{\alpha \cdot \beta^2}{(\alpha - 1)^2 \cdot (\alpha - 2)} \quad (36)$$

$$C_s = \frac{\alpha}{|\alpha|} \cdot \frac{2 \cdot (\alpha + 1) \cdot (2 \cdot \alpha - 2)^{0.5}}{(\alpha - 3) \cdot \sqrt{\alpha}} \quad (37)$$

The shape parameter can be obtained approximately depending on the skewness coefficient, using the following functions:

if $0.1 \leq C_s \leq 1.5$:

$$\alpha = \frac{-1.00404609 - 2.127093299 \cdot C_s - 1.230376348 \cdot C_s^2 - 0.100951553 \cdot C_s^3}{1 + 1.163661138 \cdot C_s - 0.532187406 \cdot C_s^2} \quad (38)$$

if $1.5 < C_s < 2.8$:

$$\alpha = \frac{25953.828411268 - 70057.489315297 \cdot C_s + 66202.507287974 \cdot C_s^2 - 34489.86058943 \cdot C_s^3 + 8475.284356191 \cdot C_s^4 - 823.56972663 \cdot C_s^5}{1 + 2982.326081943 \cdot C_s - 1054.442301104 \cdot C_s^2} \quad (39)$$

$$\beta = \sigma \cdot (\alpha - 1) \cdot \sqrt{\frac{\alpha - 2}{\alpha}} \quad (40)$$

$$\gamma = \mu - \frac{\beta \cdot \alpha}{\alpha - 1} \quad (41)$$

The equations needed to estimate the parameters with MOM have the following expressions:

$$L_1 = \gamma + \frac{\beta \cdot \alpha}{\alpha - 1} \quad (42)$$

$$L_2 = \frac{\alpha \cdot \beta}{(\alpha - 1) \cdot (2 \cdot \alpha - 1)} \quad (43)$$

$$L_3 = \frac{\alpha \cdot \beta \cdot (\alpha + 1)}{(\alpha - 1) \cdot (2 \cdot \alpha - 1) \cdot (3 \cdot \alpha - 1)} \quad (44)$$

Based on these equations, the parameters have the following expressions:

$$\alpha = \frac{\tau_3 + 1}{3 \cdot \tau_3 - 1} \quad (45)$$

$$\beta = \frac{2 \cdot L_2}{3} - \frac{4 \cdot L_2 \cdot (7 \cdot \tau_3 - 5)}{3 \cdot (\tau_3 + 1) \cdot (3 \cdot \tau_3 - 1)} \quad (46)$$

$$\gamma = \frac{L_2 + 3 \cdot L_1}{3} - \frac{8 \cdot L_2}{9 \cdot \tau_3 - 3} \quad (47)$$

2.2.6. The Five-Parameter Wakeby Distribution (WK5)

The equations needed to estimate the parameters with MOM have the following expressions:

$$\mu = \xi + \frac{\alpha}{\beta+1} - \frac{\gamma}{\delta-1} \quad (48)$$

$$\sigma^2 = \frac{\alpha^2}{(\beta+1)^2 \cdot (2 \cdot \beta + 1)} - \frac{2 \cdot \alpha \cdot \gamma}{(\beta+1) \cdot (\beta+1-\delta) \cdot (\delta-1)} - \frac{\gamma^2}{(\delta-1)^2 \cdot (2 \cdot \delta - 1)} \quad (49)$$

The equations for skewness and kurtosis are presented in Appendix E.

The equations needed to estimate the parameters with L-moments have the following expressions [9]:

$$L_1 = \xi + \frac{\alpha}{\beta+1} + \frac{\gamma}{1-\delta} \quad (50)$$

$$L_2 = \frac{\alpha}{(\beta+1) \cdot (\beta+2)} + \frac{\gamma}{(2-\delta) \cdot (1-\delta)} \quad (51)$$

$$L_3 = \frac{\gamma \cdot (\delta+1)}{(1-\delta) \cdot (2-\delta) \cdot (3-\delta)} + \frac{\alpha \cdot (1-\beta)}{(\beta+1) \cdot (\beta+2) \cdot (\beta+3)} \quad (52)$$

$$L_4 = \frac{\gamma \cdot (\delta+1) \cdot (\delta+2)}{(4-\delta) \cdot (3-\delta) \cdot (2-\delta) \cdot (1-\delta)} + \frac{\alpha \cdot (2-\beta) \cdot (1-\beta)}{(\beta+1) \cdot (\beta+2) \cdot (\beta+3) \cdot (\beta+4)} \quad (53)$$

3. Case study

The presented case study consists in the determination of maximum annual flows, on the Prigor River, Romania, using the proposed probability distributions.

The Prigor River is the left tributary of the Nera River, and it is located in the south-western part of Romania, as shown in Figure 2. The geographical coordinates of the location are 44°55'25.5"N 22°07'21.7"E.

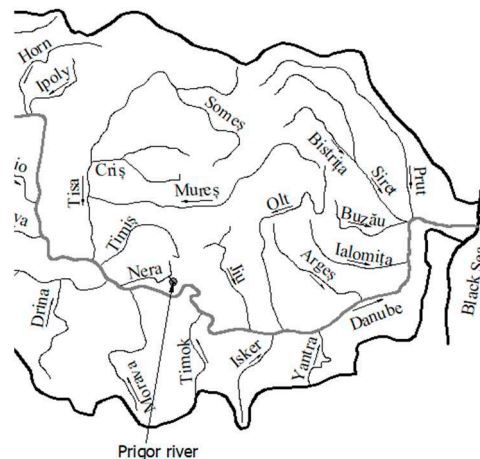


Figure 2. The Prigor River location—Prigor hydrometric station.

The main morphometric characteristics of the river are presented in Table 3 [35].

Table 3. The morphometric characteristics.

Length [km]	Average Stream Slope [‰]	Sinuosity Coefficient [-]	Average Altitude, [m]	Watershed Area, [km ²]
33	22	1.83	713	153

In the section of the hydrometric station, the watershed area is 153 km² and the average altitude is 713 m. The river has a length of 33 km, with an average slope of 22 ‰ and a sinuosity coefficient of 1.83.

There are 31 annual maximum flows, with the values presented in Table 4.

For the analysis with AES, the maximum flows resulting from the selection are presented in Table 5.

Table 4. The AMS from the Prigor hydrometric station.

		AMS										
		1990	1991	1992	1993	1994	1995	1996	1997	1998	1999	2000
Flow	[m ³ /s]	9.96	15	10.1	14.8	7.30	21.2	18.2	21.4	13.1	14.5	35
		2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011
Flow	[m ³ /s]	19.9	22.1	11.8	80.3	88	51.6	72.2	16.2	42.6	28.5	12.8
		2012	2013	2014	2015	2016	2017	2018	2019	2020		
Flow	[m ³ /s]	31.2	24.1	52.2	21.1	18.9	6.40	24.9	15.1	36.6		

Table 5. The AES from the Prigor hydrometric station.

		AES										
		1995	1996	1997	2000	2001	2002	2004	2004	2004	2005	2005
Flow	[m ³ /s]	21.2	18.2	21.4	35	19.9	22.1	80.3	22.2	19.2	88	38.9
		2005	2005	2006	2007	2007	2007	2008	2009	2009	2010	2012
Flow	[m ³ /s]	24	17.5	51.6	72.2	33.8	15.9	16.2	42.6	23.1	28.5	31.2
		2012	2012	2013	2014	2015	2016	2016	2018	2020		
Flow	[m ³ /s]	27.3	18.7	24.1	52.2	21.1	18.9	16.8	24.9	36.6		

The main statistical indicators of the data series, are presented in Table 6.

Table 6. The statistical indicators of the data series.

Prigor River	Statistical Indicators											
	μ	σ	C_v	C_s	C_k	L_1	L_2	L_3	L_4	τ_2	τ_3	τ_4
	[m ³ /s]	[m ³ /s]	[-]	[-]	[-]	[m ³ /s]	[m ³ /s]	[m ³ /s]	[m ³ /s]	[-]	[-]	[-]
AMS	27.6	21.1	0.762	1.66	5.16	27.6	10.7	4.26	2.43	0.386	0.399	0.228
AES	31.7	18.9	0.595	1.83	5.77	31.7	9.30	4.22	2.14	0.293	0.454	0.230

where $\mu, \sigma, C_v, C_s, C_k, L_1, L_2, L_3, L_4, \tau_2, \tau_3, \tau_4$ represent the mean, the standard deviation, the coefficient of variation, the skewness, the kurtosis, the four L-moments, the L-coefficient of variation, the L-skewness, and the L-kurtosis, respectively. For parameter estimation with L-moments, the data series must be in ascending order for the calculation of natural estimators, respectively L-moments.

4. Results

The proposed distributions from the Generalized Pareto family, were applied to perform a flood frequency analysis using the annual maximum series (AMS) and annual exceedance series (AES) analysis, on the Prigor river.

MOM and L-moments were used to estimate the parameters of the distributions. For the MOM, the skewness coefficient was chosen depending on the origin of the flows according to Romanian regulations [22,28], based on some multiplication coefficients for the coefficient of variation (C_v). For the Prigor river, the multiplication coefficient 3 applied to the coefficient of variation resulting in a skewness of 2.29 for AMS and 1.786 for AES.

In Table 7 and 8 are presented the results values of quantile distributions, for some of the most common exceedance probabilities in flood frequency analysis.

Table 7. Quantile results of the analyzed distributions for AMS.

Distributi on	Annual Maximum Series (AMS)															
	Exceedance probabilities [%]															
	MOM								L-moments							
	0.01	0.1	0.5	1	2	3	5	80	0.01	0.1	0.5	1	2	3	5	80
PE3	214	160	123	107	90.7	81.5	69.5	12.0	231	172	130	113	95.4	85.3	72.7	11.4
PGIV4	265	166	118	100	84.7	76.2	66.1	11.3	364	217	145	119	96.9	84.9	70.9	11.7
PGIV3	260	166	118	101	85.5	76.9	66.6	11.3	813	323	169	128	96.8	82.1	66.5	12.6
PGIII	279	169	117	98.7	82.7	74.2	64.3	12.0	800	320	168	128	95.7	82.1	66.6	12.5
PGII	225	162	122	106	89.6	80.4	69.1	11.8	329	207	142	118	96.8	85.1	71.4	11.7
PGI	171	138	112	100	87.6	80	70.2	10.5	329	207	142	118	86.8	85.1	71.4	11.7
WK5	227	163	122	106	89.4	80.3	69.0	11.8	358	216	145	120	97.0	85.0	71.0	11.7

Table 8. Quantile results of the analyzed distributions for AES.

Distributi on	Annual Exceedance Series (AES)															
	Exceedance probabilities [%]															
	MOM								L-moments							
	0.01	0.1	0.5	1	2	3	5	80	0.01	0.1	0.5	1	2	3	5	80
PE3	178	138	110	97.7	85.4	78.2	69.1	16.6	233	172	130	113	95.3	85.3	72.9	18.0
PGIV4	233	150	108	93.5	80.3	73.2	64.9	17.2	277	190	137	116	96.0	85.1	72.0	18.1
PGIV3	219	146	108	94.3	81.6	74.6	66.1	16.6	836	327	170	128	96.4	81.7	66.4	19.2
PGIII	232	150	109	93.7	80.4	73.2	64.7	17.4	940	338	168	126	94.4	80.1	65.3	18.8
PGII	170	136	110	98.0	86.0	78.9	69.7	16.6	452	240	150	121	96.4	83.9	69.9	18.3
PGI	134	117	101	92.6	83.5	77.7	70.0	15.6	452	240	150	121	96.4	83.9	69.9	18.3
WK5	150	128	108	98.1	87.3	80.5	71.4	17.0	202	164	130	114	96.7	86.6	73.7	18.2

Figures 3 and 4, show the fitting distributions for AMS and AES, for the Prigor river. For plotting positions, the Alexeev formula was used [2].

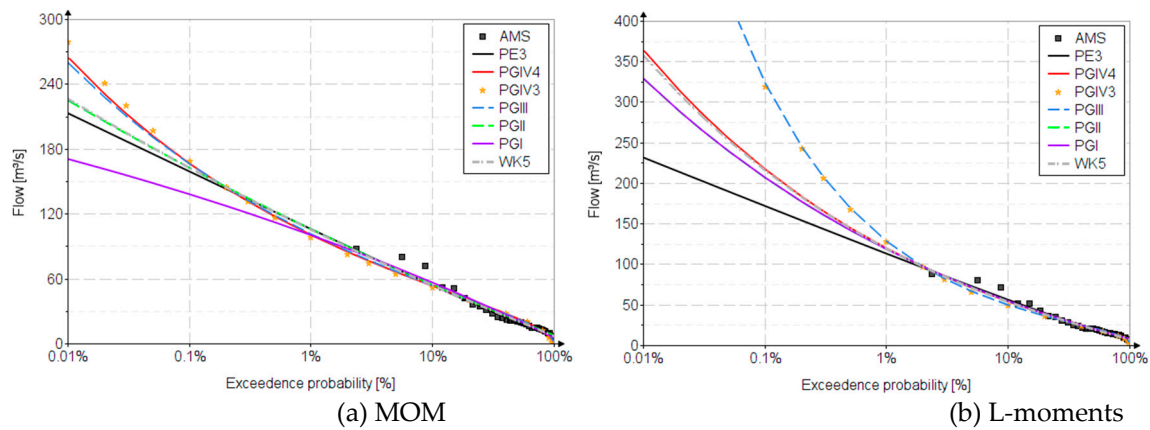


Figure 3. The fitting distributions for AMS.

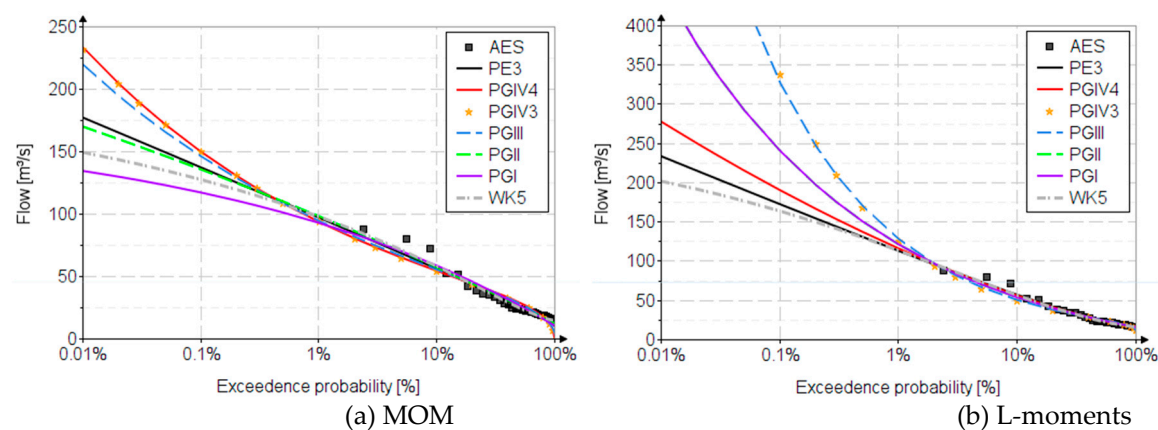


Figure 4. The fitting distributions for AES.

In Figure S1, from the supplementary material, the confidence interval for each analyzed distribution is presented, both for MOM and L-moments, using Chow's relation [4,34], for 95% confidence intervals.

Table 9 shows the values of the distributions parameters for the two methods of estimating and for both AMS and AES.

Table 9. Parameter values of each distribution for MOM and L-moments.

Distri	AMS												AES											
	MOM						L-moments						MOM						L-moments					
	α	β	γ	λ	δ	ξ	α	β	γ	λ	δ	ξ	α	β	γ	λ	δ	ξ	α	β	γ	λ	δ	ξ
PE3	0.766	24.1	9.2	-	-	-	0.69 4	26.9	9.0	-	-	-	1.254	16.9	10.6	-	-	-	0.52 6	28.2	16.9	-	-	-
PGIV ₄	0.505	47.4	-	2.66	-	-	0.94 0	80.07	4.75	1.9	-	-	0.222	49.3	-17	1.26	-	-	1.29	164 9	16.2	42.9	-	-
PGIV ₃	0.58	52.7	-	3.27	-	-	0.37	20.3	-	0.92	-	-	0.43	146.7	-	2.55	-	-	0.15	19.6	-	0.36	-	-
PGIII	5.23	52.9	-	-	-	-	2.51	20.30	9.0	-	-	-	6.07	557.1	-28	-	-	-	2.2	14.2	11.2	-	-	-
PGII	-	19.3	7.5	-	-	-	-	17.17	7.8	-	-	-	0.039	20.4	12.1	-	-	-	-0.25	12.2	15.4	-	-	-
PGI	-15.4	-367	373	-	-	-	7.14	122	-	-	-	-	-	-	176	-	-	-	4.02 3	49.2	-	-	-	-
WK5	3.317	4.01 4	19	-	0.04 7	-	7.057	2.822	0.9	16	-	0.177	56	-20.5	1.822	7.5	-	0.133	14.7	-508	0.28	515	-	16.3 0.25 3

The performance of the analyzed distribution was evaluated using the relative mean error (RME) criterion, the relative absolute error (RAE) criterion [36–38].

$$RME = \frac{1}{n} \cdot \sqrt{\sum_{i=1}^n \left(\frac{x_i - x(p)}{x_i} \right)^2}$$

(54)

$$RAE = \frac{1}{n} \cdot \sum_{i=1}^n \left| \frac{x_i - x(p)}{x_i} \right|$$

(55)

where $n, x_i, x(p)$ represent sample size, observed value, and estimated value for a given probability, respectively. For the RME and RAE performance indicators, the best model is the one with the minimum values. For the L-moments method, the variation diagram is additionally used for choosing the best fit model.

The distributions performance values are presented in Tables 10 and 11. The values for the best model are highlighted in bold.

Table 10. Distributions performance values for AMS.

Distr.	Statistical measures							
	Methods of parameters estimation						AES values	
	MOM		L-moments					
	RME	RAE	RME	RAE	τ_3	τ_4	τ_3	τ_4
PE3	0.0238	0.0879	0.0224	0.0902	0.399	0.192	0.399	0.228
PGIV4	0.0352	0.1574	0.0184	0.0772		0.228		
PGIV3	0.0300	0.1394	0.0183	0.0750		0.303		
PGIII	0.0533	0.2081	0.0181	0.0736		0.299		
PGII	0.0228	0.1047	0.0190	0.0787		0.221		
PGI	0.0470	0.2327	0.0646	0.2671		0.221		
WK5	0.0184	0.0807	0.0185	0.0775		0.228		

Table 11. Distributions performance values for AES.

Distr.	Statistical measures							
	Methods of parameters estimation						AES values	
	MOM		L-moments					
	RME	RAE	RME	RAE	τ_3	τ_4		
PE3	0.0219	0.1063	0.0093	0.0405	0.454	0.220	0.454	0.230
PGIV4	0.0395	0.1736	0.0094	0.0376		0.230		
PGIV3	0.0347	0.1556	0.0156	0.0680		0.348		
PGIII	0.0399	0.1747	0.0153	0.0656		0.338		
PGII	0.0195	0.0967	0.0108	0.0394		0.272		
PGI	0.0269	0.1313	0.0108	0.0394		0.272		
WK5	0.0137	0.0689	0.0086	0.0359		0.230		

5. Discussion

In this article, the applicability of the distributions from the Generalized Pareto family in flood frequency analysis was analyzed, with the Prigor river as a case study.

The analysis was performed using AMS and AES. As can be seen both graphically (Figure 4) and tabularly (Table 7), the analysis with AMS is more conservative than the analysis with AES.

The main advantage of the AMS analysis is the ease of data selection, these being chosen as the maximum flow corresponding to each year. The disadvantage of the analysis is the use of maximum flows characteristic of each year, which do not always represent floods. These values located in the right-hand (high probabilities), lead to a steeper graph with higher values of quantiles in the field of low probabilities (left-hand).

The advantage of the analysis with AES is that the flows of the data series always represent floods. The disadvantage is the greater effort in data selection, through additional analyzes regarding data independence, respecting the condition that these maximum flows do not come from the same flood.

The estimation methods of the analyzed distribution parameters were performed for MOM (standard method in Romania) and L-moments, two of the most used estimation methods in hydrology. In general, the L-moments method is a much more stable and robust method, being less influenced by the length of the observed data.

For the MOM analysis, the skewness was chosen depending on the origin of the flows, as is the hydrological practice in Romania.

All analyzed distributions represent particular cases of the Generalized Pareto distribution, being distributions of 3 and 4 parameters. The Wakeby distribution, which is a five-parameter distribution, was analyzed because it has as its particular case the PGII distribution, being a quantile function whose structure is made up of two quantile functions of the PGII distribution.

All the results obtained in the case study are presented compared to the Pearson III distribution, which is considered the “parent” distribution in Romania, for the most used exceedance probabilities in hydrology.

According to Tabel 10 and Tabel 11, the best fitted distributions with the smallest RME and RAE values are WK5 and PGII distributions. But based only on these indicators, the best fit distribution cannot always be selected, because these statistical indicators only properly evaluate the probability area of the observed values. Outside of this domain (left hand), they lose their relevance, because, in general, the data sets in Romania are not large enough ($n > 80$). Thus, considering that it is desired to implement the L-moments method in Romania, it is necessary to use an additional criterion for choosing the distributions in the analysis with L-moments, namely the $\tau_3 - \tau_4$ variation diagram. It is recommended to use distributions that have the values of these indicators very close to those of the observed data, especially in the case of 3-parameter distributions.

From the obtained results with L-moments, it can be seen that the best results are with the PGIV4 and WK5 distributions, which best approximate the statistical indicators of the data set, τ_3 and τ_4 . PGII and PGI distributions give satisfactory results because the natural values τ_3 and τ_4 of the distributions are close to those of the data sets.

For PE3, PGIV3 and PGIII, both for MOM and L-moments, the resulting values are characterized by a high degree of uncertainty, especially in the area of small exceedance probabilities (left-hand), due to the fact that a proper calibration of the higher moments cannot be done.

As observed in other materials [24], the apparent stability of the Pearson III distribution is due to the fact that the variation of the shape parameter for the two estimation methods does not differ much, except in the upper area of C_s and τ_3 . The same cannot be said about the PGI, PGII and PGIII distributions, in which the variation is extremely large.

Figure 5 shows the variation graph of the shape parameter for the PE3, PGIII, PGII and PGI distributions. As it could be observed in Section 2.1, both skewness and L-skewness depend only on the shape coefficient α .

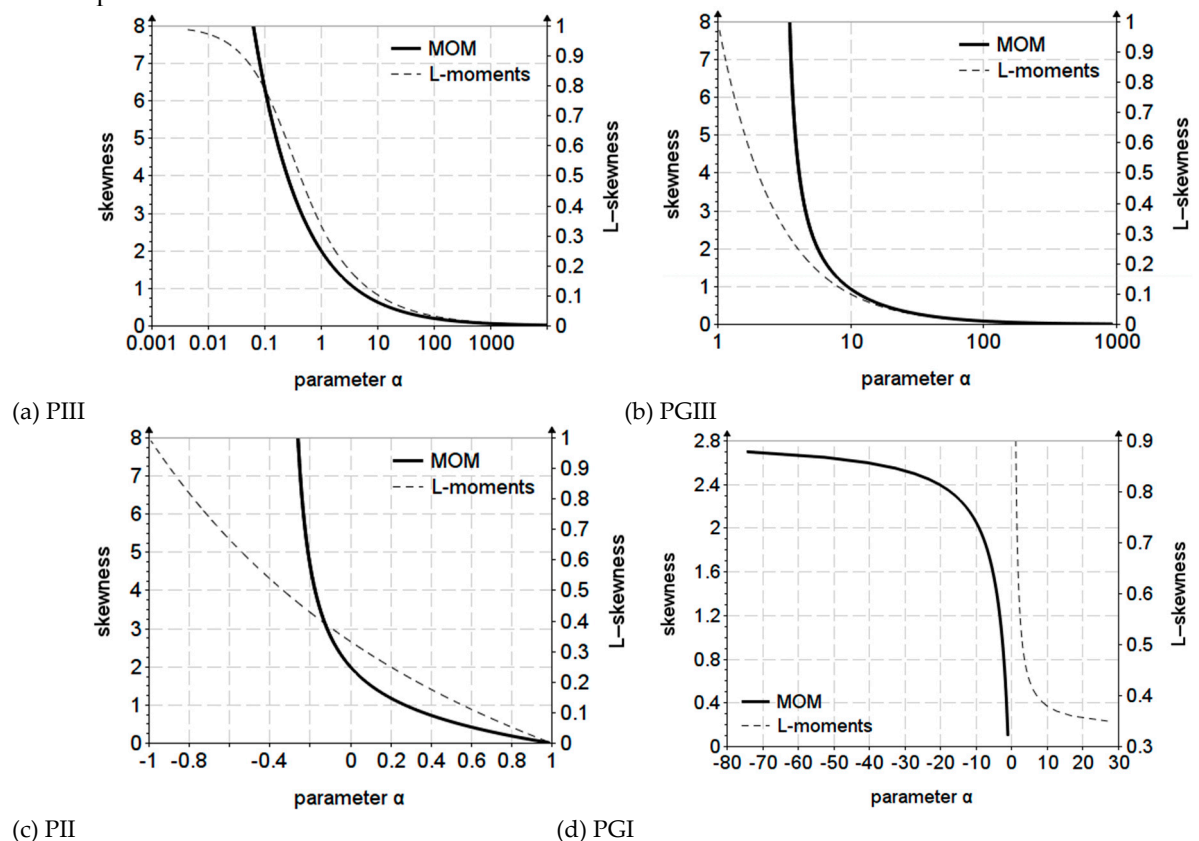


Figure 5. The variation of parameter α .

The results of the quantiles obtained with the L-moments method, for the PGII and PGI distributions, both for AES and for AMS, and presented in tables 4 and 5, are the same, the two distributions being mutual special cases. The distributions WK5 and PGIV4 best approximate, as expected, the values of the indicators obtained with L moments, being distributions of 5 and 4 parameters, respectively.

Figure 6 shows the $\tau_3 - \tau_4$ variation diagram of the distributions, as well as their relation to the values of the two indicators of the data sets.

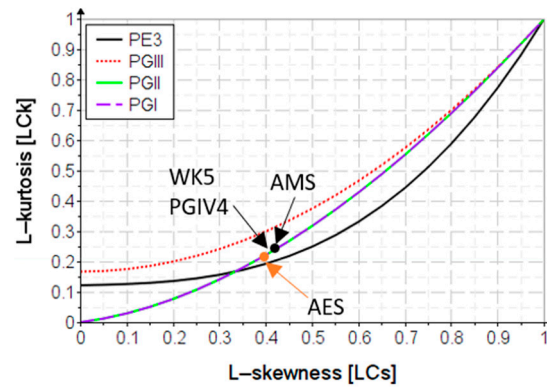


Figure 6. The $\tau_3 - \tau_4$ diagram for analyzed distributions.

Regarding the results obtained with MOM, for AES it can also be observed that the PGII and WK5 distributions have extremely close quantile values, due to the fact that for the same C_s they correspond to the same value of C_k , the WK5 distribution becoming the PGII distribution, case as it was highlighted in other materials [4,8,9,22].

The WK5 distribution, although it is a distribution that was introduced in flood frequency analysis to achieve the so-called "separation effect" described by Matalas [9,22], it can be seen that it is extremely sensitive depending on the analysis used (AMS or AES), and this is due to the particular cases in which it can take.

Figure 7 shows the skewness (C_s)-kurtosis (C_k) variation diagram of the distributions.

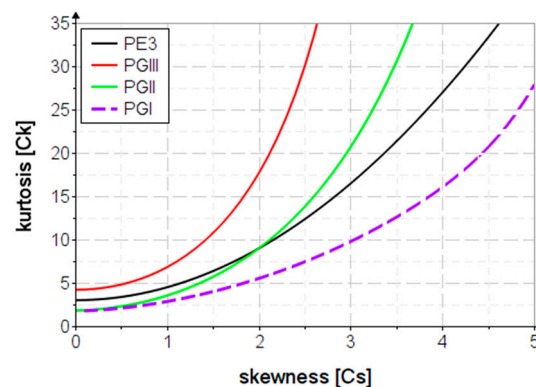


Figure 7. The $C_s - C_k$ diagram for the analyzed distributions.

6. Conclusions

Generalized Pareto distribution represents a usual distribution used in the analysis of extreme events in hydrology. In flood frequency analysis this is especially applied using the partial series of maximum flows.

This article presents 5 distributions, of 3, 4 and 5 parameters that represent different forms of the Generalized Pareto distribution, some of them received limited attention in flood frequency analysis.

These distributions were analyzed (besides other families of distributions) in the research carried out in the Faculty of Hydrotechnics regarding the elaboration of a norm in Romania for the frequency analysis using L-moments method.

The main purpose of the article was to identify other distributions from the same family that have applicability in flood frequency analysis using both AMS and AES.

For the transparency and ease of use of these distributions, all the necessary elements for their use are presented, such as the exact and approximate relations of parameters estimation, and of frequency factors, care eliminates the need for iterative numerical calculation, thus facilitating their applicability.

The performances of these distributions were verified in flood frequency analysis of the Prigor River, using the Annual Maximum Series and the Annual Exceedance Series.

The results were evaluated using relative mean error and relative absolute error, which uses the inverse function of the distribution. In general, performance indicators have the disadvantage that they are only valid for the range of recorded values. Thus, in the case of the analysis with L-moments, the additional selection criterion of the best fit distribution is represented by the values of τ_3 and τ_4 , based on the $\tau_3 - \tau_4$ diagram, compared to that of the data sets, being a more robust indicator over the entire range of excess probabilities.

Based on the study results, and also from the research carried out in the Faculty of Hydrotechnics for other sites, for flood frequency analysis and the L-moments estimation method, good candidates, from the Generalized Pareto family, are the PGIV4 and WK5 distributions, being distributions of four and five parameters, having the advantage that it can calibrate all linear moments.

Regarding the Wakeby distribution, this requires an additional analysis, because in some cases it turns into the PGII distribution, which is a three-parameter distribution, not achieving a satisfactory calibration of the linear moments.

In general, the three parameters distributions can be used in the analysis with L-moments, but the selection of their use must be made based on the $\tau_3 - \tau_4$ diagram, so that the natural values τ_3 and τ_4 of the distribution to be very close to those of the observed data. Based on the work of Anghel si Ilinca [23], in Appendix A is presented the $\tau_3 - \tau_4$ diagram for a wide range of distributions used in hydrology.

Mathematical support in statistical analysis is useful because the use of software (EasyFit, HEC-SSP, etc) without knowledge of mathematical foundations often leads to superficial analyzes with negative consequences. Another important aspect of the presentation of all the mathematical elements necessary for the application of these distributions is the fact that this software dedicated to statistical analysis are limited, and do not offer the possibility of choosing the skewness coefficient depending on the origin of the maximum flows, as is the practice in Romania.

The research in this article is part of a more complex research carried out within the Faculty of Hydrotechnics, with the main aim of establishing the necessary guidelines for a robust, clear and concise norm regarding the determination of the maximum flow using the L-moment estimation method.

Supplementary Materials: The following supporting information can be downloaded at the website of this paper posted on Preprints.org.

Author Contributions: Conceptualization, C.I. and C.G.A.; methodology, C.I. and C.G.A.; software, C.I. and C.G.A.; validation, C.I. and C.G.A.; formal analysis, C.I. and C.G.A.; investigation, C.I. and C.G.A.; resources, C.I. and C.G.A.; data curation, C.I. and C.G.A.; writing—original draft preparation, C.I. and C.G.A.; writing—review and editing, C.I. and C.G.A.; visualization, C.I. and C.G.A.; supervision, C.I. and C.G.A.; project administration, C.I. and C.G.A.; funding acquisition, C.I. and C.G.A. All authors have read and agreed to the published version of the manuscript.

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Abbreviations

MOM	the method of ordinary moments
L-moments	the method of linear moments
μ	expected value; arithmetic mean
σ	standard deviation
C_v	coefficient of variation
C_s	coefficient of skewness; skewness
C_k	coefficient of kurtosis; kurtosis
L_1, L_2, L_3	linear moments
τ_2, LC_v	coefficient of variation based on the L-moments method
τ_3, LC_s	coefficient of skewness based on the L-moments method
τ_4, LC_k	coefficient of kurtosis based on the L-moments method
Distr.	Distributions
RME	relative mean error
RAE	relative absolute error
x_i	observed values

Appendix A. The Variation of L-kurtosis - L-skewness

In the next section are presented the variation of L-kurtosis depending on the positive L-skewness, obtained with the L-moments method, for certain theoretical distributions often used in hydrology.

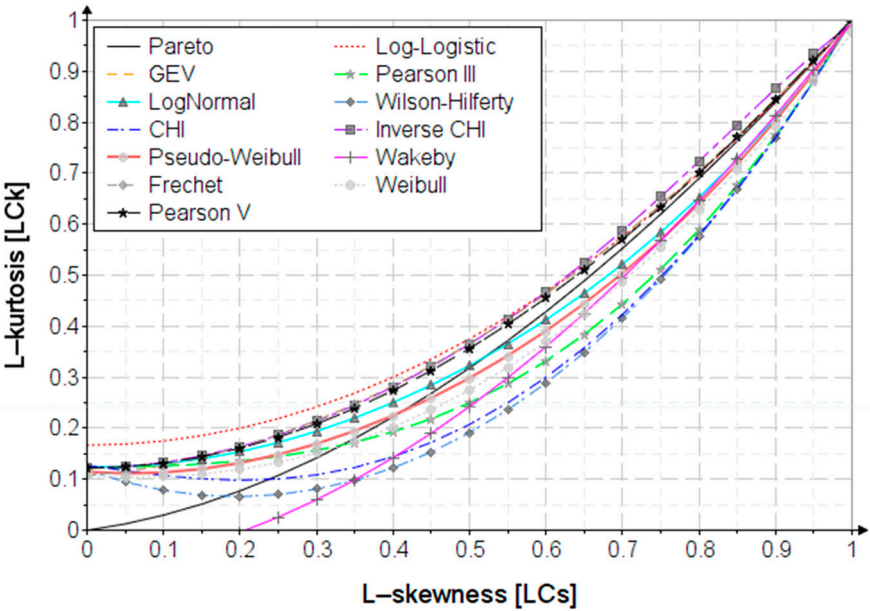


Figure A1. The variation diagram of $LC_s - LC_k$.

Appendix B. The Frequency Factors for the Analyzed Distributions

Table A1 shows the expressions of the frequency factors for MOM and L-moments.

Table A1. Frequency factors.

Frequency factor, $K_p(p)$

Distributi on	Quantile function (inverse function)	
	Method of ordinary moments (MOM)	L-moments
	$x(p) = \mu + \sigma \cdot K_p(p)$	$x(p) = L_1 + L_2 \cdot K_p(p)$
PGIV4	$\frac{\Gamma(\lambda) \cdot \left(p^{\frac{1}{\lambda}} - 1 \right) - \Gamma(\alpha+1) \cdot \Gamma(\lambda-\alpha)}{\sqrt{\Gamma(2 \cdot \alpha+1) \cdot \Gamma(\lambda-2 \cdot \alpha) \cdot \Gamma(\lambda) - \Gamma(\alpha+1)^2 \cdot \Gamma(\lambda)}}$	$\frac{\left(p^{\frac{1}{\lambda}} - 1 \right)^\alpha - \frac{\Gamma(\alpha+1) \cdot \Gamma(\lambda-\alpha)}{\Gamma(\lambda)}}{\Gamma(\alpha+1) \cdot \left(\frac{\Gamma(\lambda-\alpha)}{\Gamma(\lambda)} - \frac{\Gamma(2 \cdot \lambda-\alpha)}{\Gamma(2 \cdot \lambda)} \right)}$
PGIV3	$\frac{\Gamma(\lambda) \cdot \Gamma\left(p^{\frac{1}{\lambda}} - 1 \right)^\alpha - \Gamma(\alpha+1) \cdot \Gamma(\lambda-\alpha)}{\sqrt{\Gamma(2 \cdot \alpha+1) \cdot \Gamma(\lambda-2 \cdot \alpha) \cdot \Gamma(\lambda) - \Gamma(\alpha+1)^2 \cdot \Gamma(\lambda)}}$	$\frac{\Gamma\left(p^{\frac{1}{\lambda}} - 1 \right)^\alpha - \frac{\Gamma(\alpha+1) \cdot \Gamma(\lambda-\alpha)}{\Gamma(\lambda)}}{\Gamma(\alpha+1) \cdot \left(\frac{\Gamma(\lambda-\alpha)}{\Gamma(\lambda)} - \frac{\Gamma(2 \cdot \lambda-\alpha)}{\Gamma(2 \cdot \lambda)} \right)}$
PGIII	$\frac{\left(\frac{1}{p} - 1 \right)^\alpha - \frac{\pi}{\alpha \cdot \sin\left(\frac{\pi}{\alpha} \right)}}{\sqrt{\frac{2 \cdot \pi}{\alpha \cdot \sin\left(\frac{2 \cdot \pi}{\alpha} \right)} - \frac{\pi^2}{\alpha^2 \cdot \sin\left(\frac{\pi}{\alpha} \right)^2}}}$	$\frac{\left(\frac{1-p}{p} \right)^\alpha \cdot \alpha^2 \cdot \sin\left(\frac{\pi}{\alpha} \right)}{\pi} - \alpha$
PGII	$\frac{\sqrt{1-2 \cdot \alpha}}{\alpha} \cdot \left(1 - (1+\alpha) \cdot p^\alpha \right)$	$1 + \frac{2}{\alpha} \cdot \left(1 - p^\alpha \right) - (\alpha+3) \cdot p^\alpha$
PGI	$\frac{\alpha}{ \alpha } \cdot \frac{(\alpha-1) \cdot \sqrt{\alpha-2} \cdot \left((1-p)^{\frac{1}{\alpha}} - \frac{\alpha}{\alpha-1} \right)}{\sqrt{\alpha}}$	$\frac{\alpha}{ \alpha } \cdot \frac{(\alpha-1) \cdot (2 \cdot \alpha-1) \cdot \left((1-p)^{\frac{1}{\alpha}} - \frac{\alpha}{\alpha-1} \right)}{\alpha}$
WK5	$\frac{\frac{\alpha}{\beta+1} + \frac{\gamma}{\delta-1}}{\sqrt{\frac{\alpha^2}{(\beta+1)^2 \cdot (2 \cdot \beta+1)} - \frac{2 \cdot \alpha \cdot \gamma}{(\beta+1) \cdot (\beta+1-\delta) \cdot (\delta-1)} - \frac{\gamma^2}{(\delta-1)^2 \cdot (2 \cdot \delta-1)}}}$	$\frac{\frac{\alpha}{\beta} \cdot (1-p^\beta) - \frac{\gamma}{\delta} \cdot (1-p^{-\delta}) - \frac{\alpha}{\beta+1} - \frac{\gamma}{1-\delta}}{\frac{\alpha}{(\beta+1) \cdot (\beta+2)} + \frac{\gamma}{(2-\delta) \cdot (1-\delta)}}$

Appendix C. Estimation of the Frequency Factor for the PGIII Distribution

The frequency factor, for MOM, can be estimated using a polynomial function:

$$K_p(p) = a + b \cdot C_s + c \cdot C_s^2 + d \cdot C_s^3 + e \cdot C_s^4 + f \cdot C_s^5 + g \cdot C_s^6 + h \cdot C_s^7$$

Table A2. The frequency factor for estimation with MOM.

P [%]	a	b	c	d	e	f	g	h
0.01	5.111737	2.313409	1.34999E+00	-7.76028E-01	1.82704E-01	-2.32091E-02	1.5563E-03	-4.330E-05
0.1	3.808941	1.377403	3.45331E-01	-3.18752E-01	9.03167E-02	-1.30843E-02	9.7470E-04	-2.960E-05
0.5	2.912620	0.812083	1.70251E-02	-1.14635E-01	3.94326E-02	-6.29710E-03	4.9970E-04	-1.590E-05
1	2.527259	0.600436	-5.35487E-02	-5.65675E-02	2.33015E-02	3.98010E-03	3.2800E-04	1.070E-05
2	2.140031	0.411098	-9.11407E-02	-1.47256E-02	1.07835E-02	-2.10400E-03	1.8480E-04	-6.200E-06

3	1.911447	0.311395	-1.00372E-01	2.88960E-03	5.08580E-03	-1.21600E-03	1.1520E-04	-4.100E-06
5	1.619324	0.197967	-1.00797E-01	1.85338E-02	-4.64200E-04	-3.15600E-04	4.3000E-05	-1.700E-06
10	1.208950	0.066749	-8.47851E-02	2.91844E-02	-5.24780E-03	5.25900E-04	-2.7400E-05	6.000E-07
20	0.763538	-0.0362722	-5.31111E-02	2.86703E-02	-6.99220E-03	9.32800E-04	-6.5700E-05	1.900E-06
40	0.224364	-0.103222	-6.80020E-03	1.62353E-02	-5.34190E-03	8.40400E-04	-6.63000E-05	2.100E-06
50	0.001255	-0.111838	1.19116E-02	8.67100E-03	-3.76100E-03	6.53200E-04	-5.44000E-05	1.8000E-06
80	-0.762791	-0.0547579	5.93627E-02	-2.07334E-02	3.89270E-03	-4.16000E-04	2.37000E-05	-6.000E-07
90	-1.210684	0.0415455	6.68030E-02	-3.60124E-02	8.85020E-03	-1.19170E-03	8.48000E-05	-2.500E-06

Table A3. The frequency factor for estimation with L-moments.

P [%]	a	b	c	d	e	f	g	h
0.01	9.0865E+00	-9.0870E+00	-5.6824E+00	1.4136E+01	-1.9532E+01	1.5682E+01	-6.8872E+00	1.2851E+00
0.1	6.8300E+00	-6.8367E+00	-4.3709E+00	8.5137E+00	-9.3692E+00	6.1000E+00	-2.2178E+00	3.5120E-01
0.5	5.3161E+00	-5.3440E+00	-3.2817E+00	4.4099E+00	-2.2009E+00	-1.0766E+00	1.7928E+00	6.1653E-01
1	4.6040E+00	-4.6529E+00	-2.9168E+00	3.6177E+00	-1.7699E+00	-6.9899E-01	1.2546E+00	-4.3830E-01
2	3.8961E+00	-3.9826E+00	-2.5218E+00	2.8156E+00	-1.2339E+00	-6.2320E-01	9.9972E-01	-3.5026E-01
3	3.4784E+00	-3.5994E+00	-2.2840E+00	2.4127E+00	-1.0762E+00	-3.9996E-01	7.3125E-01	-2.6304E-01
5	2.9457E+00	-3.1313E+00	-1.9655E+00	1.9096E+00	-8.1060E-01	-2.9906E-01	5.5319E-01	-2.0219E-01
10	2.1976E+00	-2.5343E+00	-1.4975E+00	1.3114E+00	-5.3308E-01	-1.5403E-01	3.3379E-01	-1.2411E-01
20	1.3864E+00	-2.0186E+00	-9.5952E-01	8.1735E-01	-2.7803E-01	-7.4878E-02	1.8663E-01	-5.9390E-02
40	4.0582E-01	-4.0582E-01	-2.5879E-01	3.8534E-01	1.6327E-01	-4.2751E-02	-6.2078E-02	8.8377E-02
50	1.9341E-04	-1.6511E+00	3.3144E-02	3.7384E-01	1.9857E-01	1.1770E-02	-3.1091E-02	6.4863E-02
80	-1.3869E+00	-5.2845E-01	-9.5705E-02	1.2009E+00	-4.2838E-01	4.7547E-02	4.0683E-01	-2.1635E-01
90	-2.1977E+00	1.0582E+00	-1.2265E-01	7.0118E-01	-4.1310E-01	-1.1862E-01	1.7678E-01	-8.4297E-02

Appendix D. Estimation of the Frequency Factor for the PGII Distribution

The frequency factor, for MOM, can be estimated using a polynomial function:

$$K_p(p)=a+b\cdot C_s+c\cdot C_s^2+d\cdot C_s^3+e\cdot C_s^4+f\cdot C_s^5+g\cdot C_s^6+h\cdot C_s^7$$

Table A4. The frequency factor for estimation with MOM.

P [%]	a	b	c	d	e	f	g	h
0.01	1.932014	0.061904	2.87112E+00	-7.95586E-01	4.95418E-02	1.03747E-02	-1.7871E-03	7.910E-05
0.1	1.809609	0.670411	2.12699E+00	-1.15732E+00	2.84537E-01	-3.77979E-02	2.6299E-03	-7.530E-05
0.5	1.704417	1.213220	7.62347E-01	-6.51934E-01	1.98072E-01	-3.05909E-02	2.3988E-03	-7.580E-05
1	1.664419	1.314780	1.78438E-01	-3.55564E-01	1.25632E-01	-2.09008E-02	1.7176E-03	-5.610E-05
2	1.622183	1.265729	-2.90832E-01	-7.64258E-02	5.00117E-02	-9.92700E-03	8.9160E-04	-3.080E-05
3	1.590286	1.151049	-4.79305E-01	5.83412E-02	1.03575E-02	-3.85500E-03	4.1630E-04	-1.570E-05
5	1.530888	0.909001	-5.98849E-01	1.79061E-01	-2.88591E-02	2.47620E-03	-9.6700E-05	9.000E-07
10	1.377562	0.424384	-5.22494E-01	2.28357E-01	-5.36328E-02	7.14020E-03	-5.0690E-04	1.490E-05
20	1.047703	-0.1346753	-1.97916E-01	1.36073E-01	-3.93659E-02	5.98720E-03	-4.6780E-04	1.480E-05
40	0.355375	-0.4660863	1.82363E-01	-3.47085E-02	2.20900E-03	2.60000E-04	-4.8700E-05	2.100E-06
50	0.0051002	-0.428129	2.42055E-01	-7.58534E-02	1.42094E-02	-1.57970E-03	9.61000E-05	-2.5000E-06
80	-1.042946	0.1095022	7.44175E-02	-5.46187E-02	1.56946E-02	-2.36020E-03	1.82600E-04	-5.700E-06
90	-1.389758	0.3823067	-6.62687E-02	-9.67300E-03	6.70430E-03	-1.27020E-03	1.09700E-04	-3.700E-06

The frequency factor, for L-moments, can be estimated using a polynomial function:

$$K_p(p)=a+b\cdot \tau_3+c\cdot \tau_3^2+d\cdot \tau_3^3+e\cdot \tau_3^4+f\cdot \tau_3^5+g\cdot \tau_3^6+h\cdot \tau_3^7+i\cdot \tau_3^8+j\cdot \tau_3^9$$

Table A5. The frequency factor for estimation with L-moments.

P [%]	a	b	c	d	e	f	g	h	i	j
0.01	2.8976E+00	1.6049E+01	-1.6633E+02	2.1072E+03	-1.1711E+04	4.1260E+04	-8.4364E+04	1.0671E+05	-6.8465E+04	1.4604E+04
0.1	2.9921E+00	7.9122E+00	2.1160E+01	4.6340E+01	2.0703E+02	-3.9480E+02	1.4671E+03	-2.0265E+03	6.6772E+02	-
0.5	2.9751E+00	7.0733E+00	2.1320E+01	5.3452E+00	9.7924E+01	-9.9393E+01	-9.5564E+01	-5.9326E+01	-	-

1	2.9389E+00	6.8741E+00	1.4223E+01	1.0895E+01	3.3199E+01	-1.1624E+02	4.7113E+01	-	-	-
2	2.8667E+00	6.5598E+00	3.8790E+00	2.8740E+01	-6.6614E+01	2.3564E+01	-	-	-	-
3	2.8214E+00	5.3368E+00	6.0252E+00	1.9543E+00	-3.1305E+01	1.4174E+01	-	-	-	-
5	2.7060E+00	3.9951E+00	4.1471E+00	-1.3220E+01	-2.6956E+00	4.0738E+00	-	-	-	-
10	2.4017E+00	2.0082E+00	-1.6983E+00	-9.9252E+00	7.8868E+00	-1.6725E+00	-	-	-	-
20	1.7989E+00	-4.8676E-01	-4.2616E+00	1.0072E+00	1.7490E+00	-8.0770E-01	-	-	-	-
40	6.0025E-01	-2.4049E+00	-3.1267E-01	2.3727E+00	-1.6966E+00	4.4147E-01	-	-	-	-
50	3.8891E-04	-2.3323E+00	1.3905E+00	4.8092E-01	-8.1390E-01	2.7475E-01	-	-	-	-
80	-1.8003E+00	5.2569E-01	1.0120E+00	-1.3741E+00	8.6642E-01	-2.2985E-01	-	-	-	-
90	-2.4001E+00	2.1286E+00	-9.6924E-01	2.3252E-01	4.1754E-02	-3.3573E-02	-	-	-	-

Appendix E. The Skewness and Kurtosis for the WK5 Distribution

$$C_s = \frac{\frac{6 \cdot \alpha \cdot \gamma^2 \cdot (\delta^2 - \delta \cdot \beta + \delta - 1)}{(\beta + 1) \cdot (\beta + 1 - 2 \cdot \delta) \cdot (\beta + 1 - \delta) \cdot (\delta - 1)^2 \cdot (2 \cdot \delta - 1)} + \frac{6 \cdot \alpha^2 \cdot \gamma \cdot (\beta^2 - 1 - \beta \cdot (1 + \delta))}{(\beta + 1)^2 \cdot (2 \cdot \beta + 1) \cdot (\beta + 1 - \delta) \cdot (\delta - 1) \cdot (2 \cdot \beta + 1 - \delta)}}{(\delta - 1)^3 \cdot (1 + \delta \cdot (6 \cdot \delta - 5))} - \frac{2 \cdot \gamma^3 \cdot (1 + \delta)}{(\beta + 1)^3 \cdot (1 + \beta \cdot (5 + 6 \cdot \beta))}$$

$$C_k = \frac{\left(\frac{\alpha^4}{(\beta + 1)^2 \cdot (2 \cdot \beta + 1)} - \frac{\gamma^2}{(\delta - 1)^2 \cdot (2 \cdot \delta - 1)} + \frac{2 \cdot \alpha \cdot \lambda}{(\beta + 1) \cdot (\delta - 1) \cdot (\delta - \beta - 1)} \right)^{1.5}}{3 \cdot \left(\frac{\alpha^4 \cdot (2 \cdot \beta^2 - \beta + 3)}{(\beta + 1)^4 \cdot (24 \cdot \beta^3 + 26 \cdot \beta^2 + 9 \cdot \beta + 1)} - \frac{\gamma^4 \cdot (3 + \delta + 2 \cdot \delta^2)}{(\delta - 1)^4 \cdot (4 \cdot \delta - 1) \cdot (1 + \delta \cdot (6 \cdot \delta - 5))} + \right. \\ \left. 2 \cdot \alpha^2 \cdot \gamma^2 \cdot \left(\frac{-9 - \beta \cdot (30 + \beta \cdot (29 + 4 \cdot \beta \cdot (2 + \beta))) + \beta \cdot \delta \cdot (71 + \beta \cdot (43 + 2 \cdot \beta \cdot (7 + 4 \cdot \beta))) - \delta^2 \cdot (29 + \beta \cdot (43 + 4 \cdot (6 + 7 \cdot \beta))) + 2 \cdot \delta^3 \cdot (4 + 7 \cdot \beta \cdot (1 + 2 \cdot \beta)) - 4 \cdot \delta^4 \cdot (1 + 2 \cdot \beta) + 30 \cdot \delta}{(\beta + 1)^2 \cdot (2 \cdot \beta + 1) \cdot (1 + \beta - 2 \cdot \delta) \cdot (1 + 2 \cdot \beta - 2 \cdot \delta) \cdot (1 + \beta - \delta) \cdot (1 + 2 \cdot \beta - \delta) \cdot (\delta - 1)^2 \cdot (2 \cdot \delta - 1)} \right) + \right. \\ \left. 4 \cdot \alpha \cdot \gamma^4 \cdot \left(\frac{3 + 2 \cdot \beta + \beta^2 - \delta \cdot (12 + \beta \cdot (3 + \beta)) - \delta^2 (11 + \beta + 2 \cdot \beta^2) - 2 \cdot \delta^3 \cdot (1 + 3 \cdot \beta) - 4 \cdot \delta^4}{(1 + \beta) \cdot (1 + \beta - 3 \cdot \delta) \cdot (1 + \beta - 2 \cdot \delta) \cdot (1 + \beta - \delta) \cdot (\delta - 1)^3 \cdot (1 + \delta \cdot (6 \cdot \delta - 5))} \right) + \right. \\ \left. 4 \cdot \alpha^3 \cdot \gamma \cdot \left(\frac{3 + 4 \cdot \beta^4 + \beta^3 \cdot (2 - 6 \cdot \delta) + \delta \cdot (\delta - 2) + \beta \cdot (12 + \delta \cdot (\delta - 3)) + \beta^2 \cdot (11 + \delta \cdot (2 \cdot \delta - 1))}{(1 + \beta)^3 \cdot (1 + \beta \cdot (6 \cdot \beta + 5)) \cdot (1 + \beta - \delta) \cdot (1 + 2 \cdot \beta - \delta) \cdot (1 + 3 \cdot \beta - \delta) \cdot (\delta - 1)} \right) \right)^2}{\left(\frac{\gamma^2}{(\delta - 1)^2 \cdot (2 \cdot \delta - 1)} - \frac{\alpha^4}{(\beta + 1)^2 \cdot (2 \cdot \beta + 1)} + \frac{2 \cdot \alpha \cdot \lambda}{(\beta + 1) \cdot (\delta - 1) \cdot (\delta - \beta - 1)} \right)^2}$$

Appendix F. The Skewness and Kurtosis for the PGIIV Distribution

$$C_s = \frac{\frac{\Gamma(1 + 3 \cdot \alpha) \cdot \Gamma(\lambda - 3 \cdot \alpha)}{\lambda \cdot \Gamma(\lambda)} + \frac{2 \cdot \Gamma(1 + \alpha)^3 \cdot \Gamma(\lambda - \alpha)^3}{\Gamma(\lambda + 1)^3} - \frac{3 \cdot \alpha \cdot \Gamma(1 + 2 \cdot \alpha) \cdot \Gamma(\lambda - 2 \cdot \alpha) \cdot \Gamma(\alpha) \cdot \Gamma(\lambda - \alpha)}{\Gamma(\lambda + 1)^2}}{\left(\frac{\Gamma(1 + 2 \cdot \alpha) \cdot \Gamma(\lambda - 2 \cdot \alpha)}{\lambda \cdot \Gamma(\lambda)} - \frac{\Gamma(1 + \alpha)^2 \cdot \Gamma(\lambda - \alpha)^2}{\Gamma(\lambda + 1)^2} \right)^{3/2}}$$

$$C_k = \frac{\frac{\Gamma(1+4\cdot\alpha)\cdot\Gamma(\lambda-4\cdot\alpha)}{\Gamma(\lambda+1)} - 4\cdot\frac{\Gamma(1+3\cdot\alpha)\cdot\Gamma(\lambda-3\cdot\alpha)\cdot\Gamma(1+\alpha)\cdot\Gamma(\lambda-\alpha)}{\Gamma(\lambda+1)^2} + 6\cdot\frac{\Gamma(1+2\cdot\alpha)\cdot\Gamma(\lambda-2\cdot\alpha)\cdot\Gamma(1+\alpha)^2\cdot\Gamma(\lambda-\alpha)^2}{\Gamma(\lambda+1)^3} - 3\cdot\frac{\Gamma(1+\alpha)^4\cdot\Gamma(\lambda-\alpha)^4}{\Gamma(\lambda+1)^4}}{\left(\frac{\Gamma(1+2\cdot\alpha)\cdot\Gamma(\lambda-2\cdot\alpha)}{\Gamma(\lambda+1)} - \frac{\Gamma(1+\alpha)^2\cdot\Gamma(\lambda-\alpha)^2}{\Gamma(\lambda+1)^2}\right)^2}$$

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