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A Common Approach to Three Open Problems in Number Theory

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Article

A Common Approach to Three Open Problems in Number Theory

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Abstract: The following system of equations $\{x_1 \cdot x_1 = x_2,\ x_2 \cdot x_2 = x_3,\ 2^{2^{x_1}} = x_3,\ x_4 \cdot x_5 = x_2,\ x_6 \cdot x_7 = x_2\}$ has exactly one solution in $(\mathbb{N} \setminus \{0,1\})^7$, namely $(2,4,16,2,2,2,2)$. Hypothesis 1 states that if a system of equations $\mathcal{S} \subseteq \{x_i \cdot x_j = x_k : i,j,k \in \{1,\ldots,7\}\} \cup \{2^{2^{x_j}} = x_k : j,k \in \{1,\ldots,7\}\}$ has at most five equations and at most finitely many solutions in $(\mathbb{N} \setminus \{0,1\})^7$, then each such solution $(x_1,\ldots,x_7)$ satisfies $x_1,\ldots,x_7 \leqslant 16$. Hypothesis 1 implies that there are infinitely many composite numbers of the form $2^{2^n}+1$. Hypotheses 2 and 3 are of similar kind. Hypothesis 2 implies that if the equation $x!+1=y^2$ has at most finitely many solutions in positive integers x and y, then each such solution (x,y) belongs to the set $\{(4,5),(5,11),(7,71)\}$. Hypothesis 3 implies that if the equation $x(x+1)=y!$ has at most finitely many solutions in positive integers x and y, then each such solution (x,y) belongs to the set $\{(1,2),(2,3)\}$. We describe semi-algorithms $\texttt{sem}_j$ $(j=1,2,3)$ that never terminate. For every $j \in \{1,2,3\}$, if Hypothesis j is true, then $\texttt{sem}_j$ endlessly prints consecutive positive integers starting from 1. For every $j \in \{1,2,3\}$, if Hypothesis j is false, then $\texttt{sem}_j$ prints a finite number (including zero) of consecutive positive integers starting from 1.

Keywords: Brocard's problem; Brocard-Ramanujan equation $x!+1=y^2$; composite Fermat numbers; composite numbers of the form $2^{2^n}+1$; Erdös' equation $x(x+1)=y!$

MSC: 11D61; 11D85

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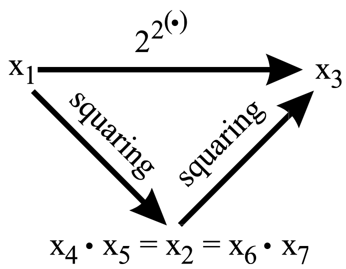
1. Composite numbers of the form $2^{2^n}+1$

Let $\mathcal{A}$ denote the following system of equations:

$$\left\{x_i \cdot x_j = x_k : \ i,j,k \in \{1,\ldots,7\}\right\} \cup \left\{2^{2^{x_j}} = x_k : \ j,k \in \{1,\ldots,7\}\right\}$$

The following subsystem of $\mathcal{A}$

$$\begin{cases} x_1 \cdot x_1 = x_2 \\ x_2 \cdot x_2 = x_3 \\ 2^{2^{x_1}} = x_3 \\ x_4 \cdot x_5 = x_2 \\ x_6 \cdot x_7 = x_2 \end{cases}$$



has exactly one solution in $(\mathbb{N} \setminus \{0,1\})^7$, namely $(2,4,16,2,2,2,2)$.

Hypothesis 1. *If a system of equations $\mathcal{S} \subseteq \mathcal{A}$ has at most five equations and at most finitely many solutions in $(\mathbb{N} \setminus \{0,1\})^7$, then each such solution $(x_1,\ldots,x_7)$ satisfies $x_1,\ldots,x_7 \leqslant 16$.*

Lemma 1. *([7, p. 109]). For every non-negative integers x and y, $x+1=y$ if and only if $2^{2^x} \cdot 2^{2^x} = 2^{2^y}$.*

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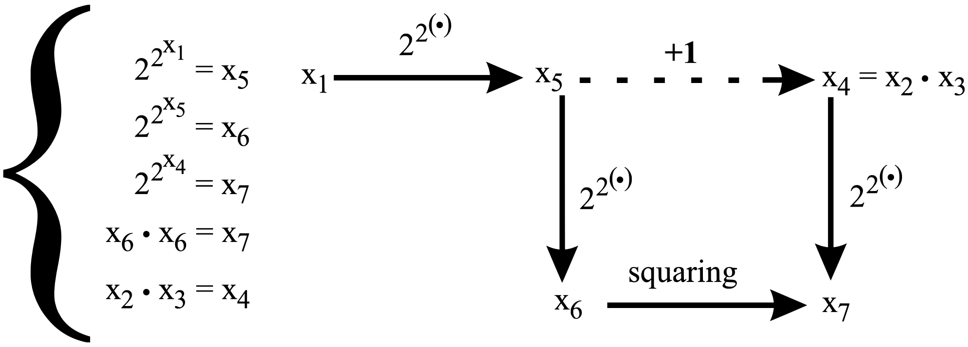
Theorem 1. *Hypothesis 1 implies that* $2^{2^{x_1}}+1$ *is composite for infinitely many integers* x_1 *greater than* 1.

Proof. Assume, on the contrary, that Hypothesis 1 holds and $2^{2^{x_1}}+1$ is composite for at most finitely many integers x_1 greater than 1. Then, the equation

$$x_2 \cdot x_3 = 2^{2^{x_1}}+1$$

has at most finitely many solutions in $(\mathbb{N} \setminus \{0,1\})^3$. By Lemma 1, in positive integers greater than 1, the following subsystem of $\mathcal{A}$



has at most finitely many solutions in $(\mathbb{N} \setminus \{0,1\})^7$ and expresses that

$$\begin{cases} x_2 \cdot x_3 &= 2^{2^{x_1}}+1 \\ x_4 &= 2^{2^{x_1}}+1 \\ x_5 &= 2^{2^{x_1}} \\ x_6 &= 2^{2^{2^{2^{x_1}}}} \\ x_7 &= 2^{2^{2^{2^{x_1}}}}+1 \end{cases}$$

Since $641 \cdot 6700417 = 2^{2^5}+1 > 16$, we get a contradiction. □

Most mathematicians believe that $2^{2^n}+1$ is composite for every integer $n \geqslant 5$, see [2, p. 23].

Open Problem 1. *([3, p. 159]). Are there infinitely many composite numbers of the form* $2^{2^n}+1$*?*

Primes of the form $2^{2^n}+1$ are called Fermat primes, as Fermat conjectured that every integer of the form $2^{2^n}+1$ is prime, see [3, p. 1]. Fermat remarked that $2^{2^0}+1=3$, $2^{2^1}+1=5$, $2^{2^2}+1=17$, $2^{2^3}+1=257$, and $2^{2^4}+1=65537$ are all prime, see [3, p. 1].

Open Problem 2. *([3, p. 158]). Are there infinitely many prime numbers of the form* $2^{2^n}+1$*?*

2. The Brocard-Ramanujan equation $x!+1=y^2$

Let $\mathcal{B}$ denote the following system of equations:

$$\{x_i \cdot x_j = x_k : i,j,k \in \{1,\ldots,6\}\} \cup \{x_j! = x_k : (j,k \in \{1,\ldots,6\}) \wedge (j \neq k)\}$$

The following subsystem of $\mathcal{B}$

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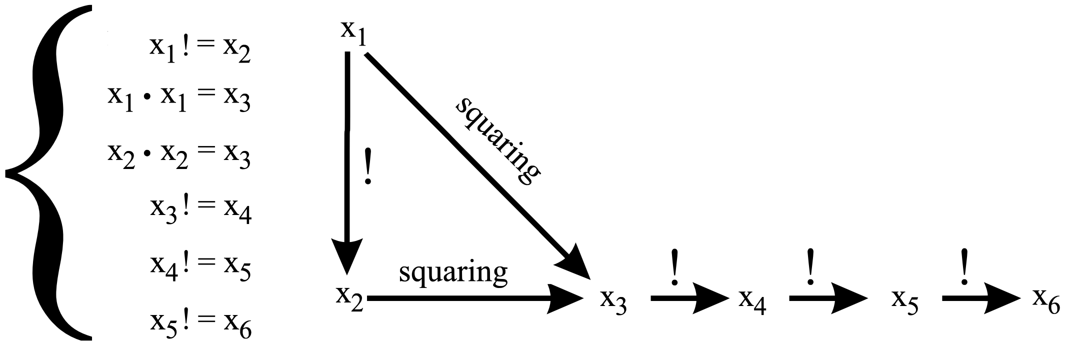


has exactly two solutions in positive integers, namely $(1, \ldots, 1)$ and $(2, 2, 4, 24, 24!, (24!)!)$.

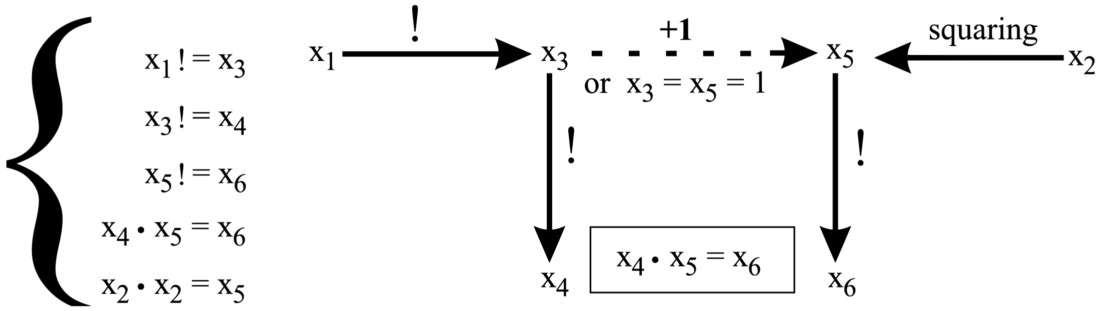
Hypothesis 2. *If a system of equations $\mathcal{S} \subseteq \mathcal{B}$ has at most finitely many solutions in positive integers $x_1, \ldots, x_6$, then each such solution $(x_1, \ldots, x_6)$ satisfies $x_1, \ldots, x_6 \leqslant (24!)!$.*

Lemma 2. *For every positive integers x and y, $x! \cdot y = y!$ if and only if*

$$(x + 1 = y) \vee (x = y = 1)$$

Theorem 2. *Hypothesis 2 implies that if the equation $x_1! + 1 = x_2^2$ has at most finitely many solutions in positive integers x_1 and x_2, then each such solution (x_1, x_2) belongs to the set $\{(4, 5), (5, 11), (7, 71)\}$.*

Proof. The following system of equations $\mathcal{B}_1$



is a subsystem of $\mathcal{B}$. By Lemma 2, in positive integers, the system $\mathcal{B}_1$ expresses that $x_1 = \ldots = x_6 = 1$ or

$$\begin{cases} x_1! + 1 &= x_2^2 \\ x_3 &= x_1! \\ x_4 &= (x_1!)! \\ x_5 &= x_1! + 1 \\ x_6 &= (x_1! + 1)! \end{cases}$$

If the equation $x_1! + 1 = x_2^2$ has at most finitely many solutions in positive integers x_1 and x_2, then $\mathcal{B}_1$ has at most finitely many solutions in positive integers $x_1, \ldots, x_6$ and Hypothesis 2 implies that every tuple $(x_1, \ldots, x_6)$ of positive integers that solves $\mathcal{B}_1$ satisfies $(x_1! + 1)! = x_6 \leqslant (24!)!$. Hence, $x_1 \in \{1, \ldots, 23\}$. If $x_1 \in \{1, \ldots, 23\}$, then $x_1! + 1$ is a square only for $x_1 \in \{4, 5, 7\}$. □

It is conjectured that $x! + 1$ is a square only for $x \in \{4, 5, 7\}$, see [8, p. 297]. A weak form of Szpiro's conjecture implies that the equation $x! + 1 = y^2$ has only finitely many solutions in positive integers, see [6].

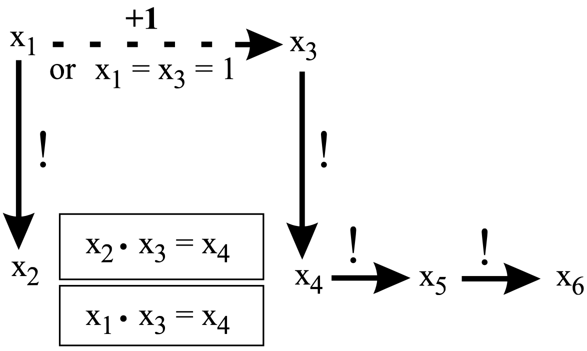
3. Erdös' equation $x(x+1) = y!$

Let C denote the following system of equations:

$$\{x_i \cdot x_j = x_k : (i,j,k \in \{1,\ldots,6\}) \wedge (i \neq j)\} \cup \{x_j! = x_k : (j,k \in \{1,\ldots,6\}) \wedge (j \neq k)\}$$

The following subsystem of C

$$\begin{cases} x_1! = x_2 \\ x_3! = x_4 \\ x_2 \cdot x_3 = x_4 \\ x_1 \cdot x_3 = x_4 \\ x_4! = x_5 \\ x_5! = x_6 \end{cases}$$



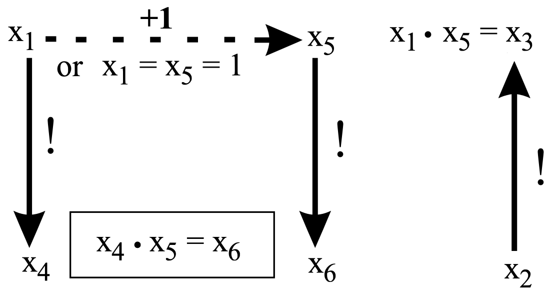
has exactly three solutions in positive integers, namely $(1,\ldots,1)$, $(1,1,2,2,2,2)$, and $(2,2,3,6,720,720!)$.

Hypothesis 3. *If a system of equations $\mathcal{S} \subseteq C$ has at most finitely many solutions in positive integers $x_1,\ldots,x_6$, then each such solution $(x_1,\ldots,x_6)$ satisfies $x_1,\ldots,x_6 \leqslant 720!$.*

Theorem 3. *Hypothesis 3 implies that if the equation $x_1(x_1+1) = x_2!$ has at most finitely many solutions in positive integers x_1 and x_2, then each such solution (x_1,x_2) belongs to the set $\{(1,2),(2,3)\}$.*

Proof. The following system of equations C_1

$$\begin{cases} x_1! = x_4 \\ x_5! = x_6 \\ x_4 \cdot x_5 = x_6 \\ x_2! = x_3 \\ x_1 \cdot x_5 = x_3 \end{cases}$$



is a subsystem of C. By Lemma 2, in positive integers, the system C_1 expresses that $x_1 = \ldots = x_6 = 1$ or

$$\begin{cases} x_1 \cdot (x_1+1) &= x_2! \\ x_3 &= x_1 \cdot (x_1+1) \\ x_4 &= x_1! \\ x_5 &= x_1+1 \\ x_6 &= (x_1+1)! \end{cases}$$

If the equation $x_1(x_1+1) = x_2!$ has at most finitely many solutions in positive integers x_1 and x_2, then C_1 has at most finitely many solutions in positive integers $x_1,\ldots,x_6$ and Hypothesis 3 implies that every tuple $(x_1,\ldots,x_6)$ of positive integers that solves C_1 satisfies $x_2! = x_3 \leqslant 720!$. Hence, $x_2 \in \{1,\ldots,720\}$. If $x_2 \in \{1,\ldots,720\}$, then $x_2!$ is a product of two consecutive positive integers only for $x_2 \in \{2,3\}$ because the following *MuPAD* program

```
for x2 from 1 to 720 do
x1:=round(sqrt(x2!+(1/4))-(1/2)):
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```
if x1*(x1+1)=x2! then print(x2) end_if:
end_for:
```

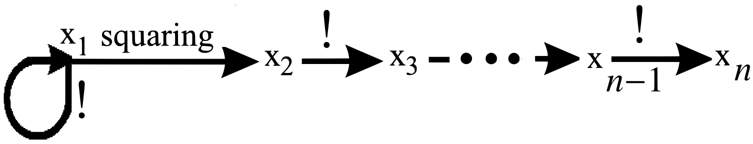
returns 2 and 3. □

The question of solving the equation $x(x+1) = y!$ was posed by P. Erdös, see [1]. F. Luca proved that the *abc* conjecture implies that the equation $x(x+1) = y!$ has only finitely many solutions in positive integers, see [4].

4. Hypotheses 2 and 3 cannot be generalized to an arbitrary number of variables

Let $f(1) = 2$, $f(2) = 4$, and let $f(n+1) = f(n)!$ for every integer $n \geqslant 2$. Let $\mathcal{W}_1$ denote the system of equations $\{x_1! = x_1$. For an integer $n \geqslant 2$, let $\mathcal{W}_n$ denote the following system of equations:

$$\begin{cases} x_1! = x_1 \\ x_1 \cdot x_1 = x_2 \\ \forall i \in \{2, \ldots, n-1\}\ x_i! = x_{i+1} \end{cases}$$



For every positive integer n, the system $\mathcal{W}_n$ has exactly two solutions in positive integers $x_1, \ldots, x_n$, namely $(1, \ldots, 1)$ and $(f(1), \ldots, f(n))$. For a positive integer n, let Ψ_n denote the following statement: *if a system of equations*

$$\mathcal{S} \subseteq \{x_i \cdot x_j = x_k : i, j, k \in \{1, \ldots, n\}\} \cup \{x_j! = x_k : j, k \in \{1, \ldots, n\}\}$$

has at most finitely many solutions in positive integers $x_1, \ldots, x_n$*, then each such solution* $(x_1, \ldots, x_n)$ *satisfies* $x_1, \ldots, x_n \leqslant f(n)$.

Theorem 4. *Every factorial Diophantine equation can be algorithmically transformed into an equivalent system of equations of the forms* $x_i \cdot x_j = x_k$ *and* $x_j! = x_k$*. It means that this system of equations satisfies a modified version of Lemma 4 in [7].*

Proof. It follows from Lemmas 2–4 in [7] and Lemma 2. □

The statement $\forall n \in \mathbb{N} \setminus \{0\}\ \Psi_n$ is dubious. By Theorem 4, this statement implies that there is an algorithm which takes as input a factorial Diophantine equation and returns an integer which is greater than the solutions in positive integers, if these solutions form a finite set. This conclusion is strange because properties of factorial Diophantine equations are similar to properties of exponential Diophantine equations and a computable upper bound on non-negative integer solutions does not exist for exponential Diophantine equations with a finite number of solutions, see [5].

5. Equivalent forms of Hypotheses 1–3

If $k \in [10^{19}, 10^{20} - 1] \cap \mathbb{N}$, then there are uniquely determined non-negative integers $a(0), \ldots, a(19) \in \{0, \ldots, 9\}$ such that

$$\Big(a(19) \geqslant 1\Big) \wedge \Big(k = a(19) \cdot 10^{19} + a(18) \cdot 10^{18} + \ldots + a(1) \cdot 10^1 + a(0) \cdot 10^0\Big)$$

Definition 1. *For an integer* $k \in [10^{19}, 10^{20} - 1]$, $\mathcal{S}_k$ *stands for the smallest system of equations* $\mathcal{S}$ *satisfying conditions* (1) *and* (2).

(1) *If* $i \in \{0, 4, 8, 16\}$ *and* $a(i) \in \{0, 1, 2, 3, 4\}$*, then the equation* $x_{a(i+1)} \cdot x_{a(i+2)} = x_{a(i+3)}$ *belongs to* $\mathcal{S}$ *when it belongs to* $\mathcal{A}$.

(2) *If* $i \in \{0, 4, 8, 16\}$ *and* $a(i) \in \{5, 6, 7, 8, 9\}$*, then the equation* $2^{2^{x_{a(i+1)}}} = x_{a(i+2)}$ *belongs to* $\mathcal{S}$ *when it belongs to* $\mathcal{A}$.

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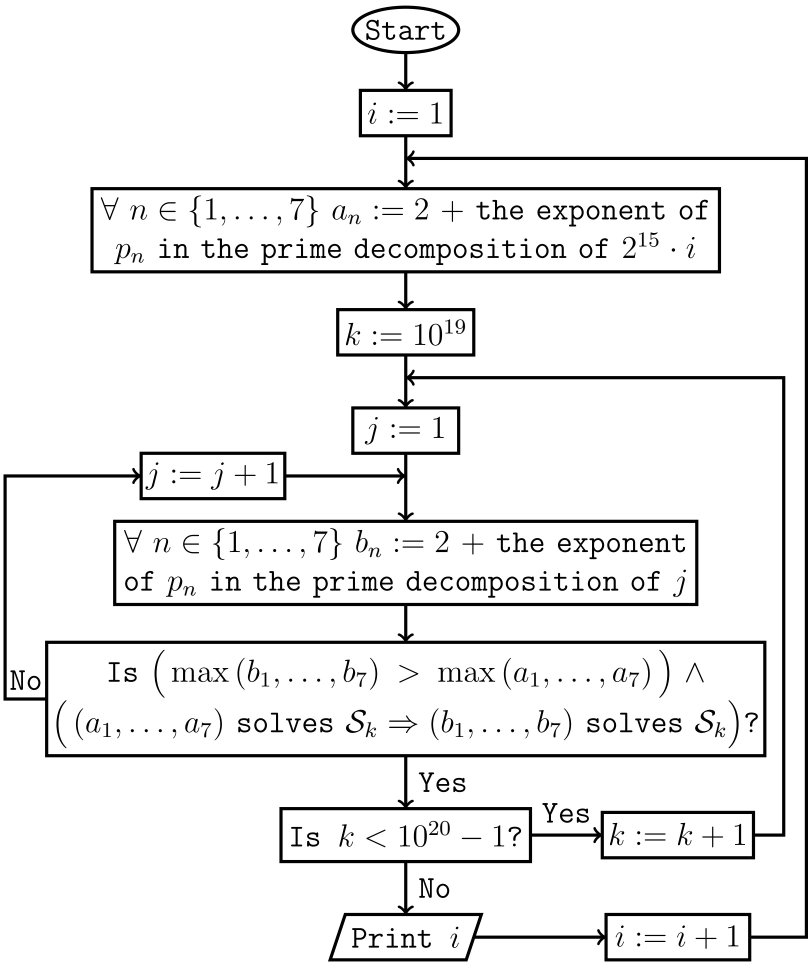
Lemma 3. $\{\mathcal{S}_k : k \in [10^{19}, 10^{20} - 1] \cap \mathbb{N}\} = \{\mathcal{S} : (\mathcal{S} \subseteq \mathcal{A}) \wedge (\mathrm{card}(\mathcal{S}) \leqslant 5)\}$.

Proof. It follows from the equality $5 \cdot 4 = 20$. □

For a positive integer n, let p_n denote the n-th prime number.

Theorem 5. *The following semi-algorithm* $\texttt{sem}_1$ *never terminates.*



If Hypothesis 1 is true, then $\texttt{sem}_1$ *endlessly prints consecutive positive integers starting from* 1. *If Hypothesis 1 is false, then* $\texttt{sem}_1$ *prints a finite number (including zero) of consecutive positive integers starting from* 1.

Proof. It follows from Lemma 3. □

If $k \in [10^{1007}, 10^{1008} - 1] \cap \mathbb{N}$, then there are uniquely determined non-negative integers $c(0), \dots, c(1007) \in \{0, \dots, 9\}$ such that

$$\Big(c(1007 \geqslant 1)\Big) \wedge \Big(k = c(1007) \cdot 10^{1007} + c(1006) \cdot 10^{1006} + \ldots + c(1) \cdot 10^1 + c(0) \cdot 10^0\Big)$$

Definition 2. *For an integer* $k \in [10^{1007}, 10^{1008} - 1]$, $\mathcal{T}_k$ *stands for the smallest system of equations* $\mathcal{T}$ *satisfying conditions* (3) *and* (4).

(3) *If* $i \in \{0, 4, 8, \dots, 1004\}$ *and* $c(i) \in \{0, 1, 2, 3, 4\}$, *then the equation* $x_{c(i+1)} \cdot x_{c(i+2)} = x_{c(i+3)}$ *belongs to* $\mathcal{T}$ *when it belongs to* $\mathcal{B}$.

(4) *If* $i \in \{0, 4, 8, \dots, 1004\}$ *and* $c(i) \in \{5, 6, 7, 8, 9\}$, *then the equation* $x_{c(i+1)}! = x_{c(i+2)}$ *belongs to* $\mathcal{T}$ *when it belongs to* $\mathcal{B}$.

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Lemma 4. $\{\mathcal{T}_k : k \in [10^{1007}, 10^{1008} - 1] \cap \mathbb{N}\} = \{\mathcal{T} : \mathcal{T} \subseteq \mathcal{B}\}$.

Proof. It follows from the equality $(6^3 + 6^2) \cdot 4 = 1008$. □

Definition 3. *For an integer $k \in [10^{1007}, 10^{1008} - 1]$, $\mathcal{U}_k$ stands for the smallest system of equations $\mathcal{U}$ satisfying conditions* (5) *and* (6)*.*

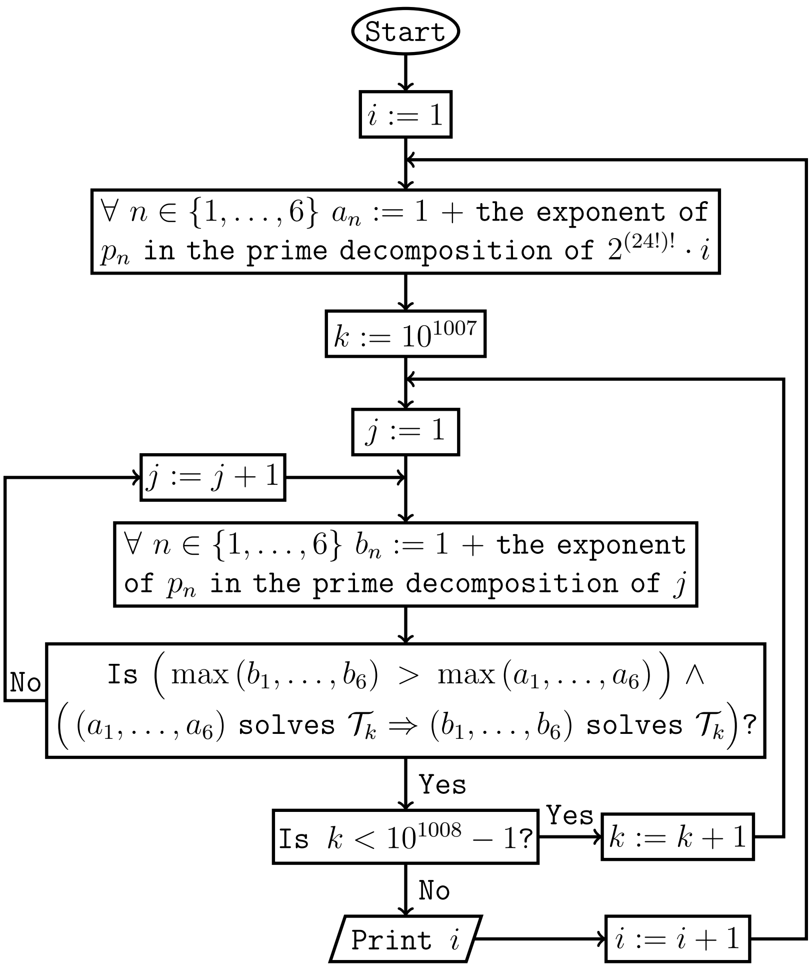
(5) *If $i \in \{0, 4, 8, \ldots, 1004\}$ and $c(i) \in \{0, 1, 2, 3, 4\}$, then the equation $x_{c(i+1)} \cdot x_{c(i+2)} = x_{c(i+3)}$ belongs to $\mathcal{U}$ when it belongs to $\mathcal{C}$.*

(6) *If $i \in \{0, 4, 8, \ldots, 1004\}$ and $c(i) \in \{5, 6, 7, 8, 9\}$, then the equation $x_{c(i+1)}! = x_{c(i+2)}$ belongs to $\mathcal{U}$ when it belongs to $\mathcal{C}$.*

Lemma 5. $\{\mathcal{U}_k : k \in [10^{1007}, 10^{1008} - 1] \cap \mathbb{N}\} = \{\mathcal{U} : \mathcal{U} \subseteq \mathcal{C}\}$.

Proof. It follows from the equality $(6^3 + 6^2) \cdot 4 = 1008$. □

Theorem 6. *The following semi-algorithm* $\texttt{sem}_2$ *never terminates.*



If Hypothesis 2 is true, then $\texttt{sem}_2$ *endlessly prints consecutive positive integers starting from* 1*. If Hypothesis 2 is false, then* $\texttt{sem}_2$ *prints a finite number (including zero) of consecutive positive integers starting from* 1.

Proof. It follows from Lemma 4. □

Theorem 7. *The following semi-algorithm* $\texttt{sem}_3$ *never terminates.*

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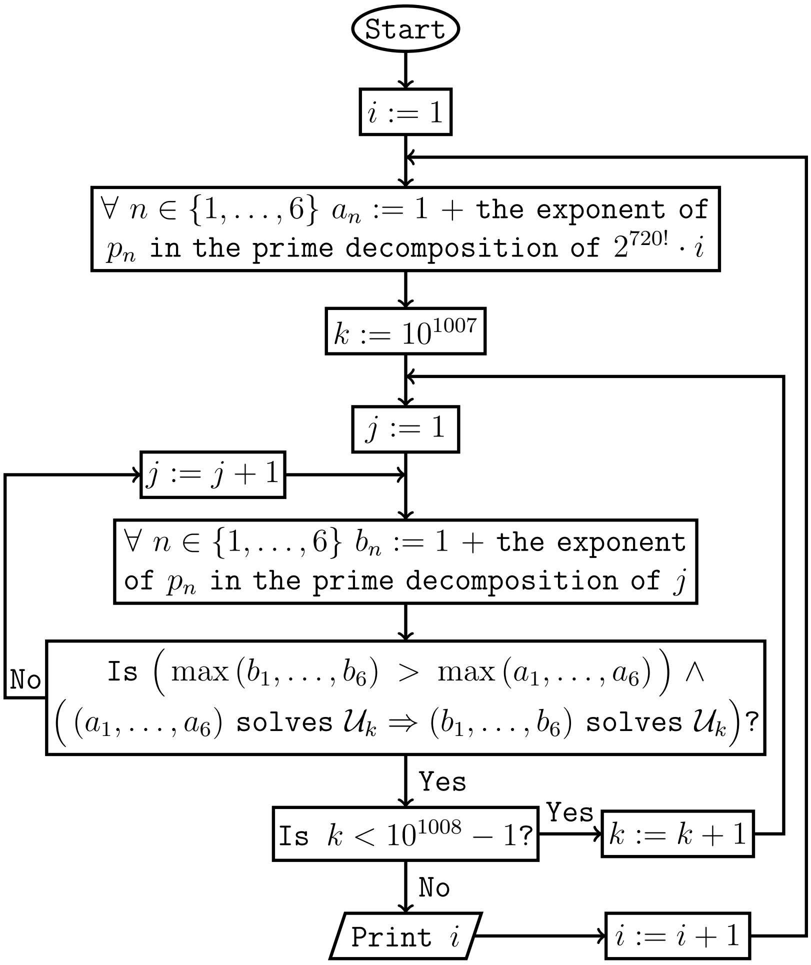

If Hypothesis 3 is true, then sem$_3$ *endlessly prints consecutive positive integers starting from* 1. *If Hypothesis 3 is false, then* sem$_3$ *prints a finite number (including zero) of consecutive positive integers starting from* 1.

Proof. It follows from Lemma 5. □

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