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A Common Approach to Three Open Problems in Number Theory

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Abstract: The following system of equations $\{x_1 \cdot x_1 = x_2,\ x_2 \cdot x_2 = x_3,\ 2^{2^{x_1}} = x_3,\ x_4 \cdot x_5 = x_2,\ x_6 \cdot x_7 = x_2\}$ has exactly one solution in $(\mathbb{N} \setminus \{0,1\})^7$, namely $(2,4,16,2,2,2,2)$. Hypothesis 1 states that if a system of equations $\mathcal{S} \subseteq \{x_i \cdot x_j = x_k : i,j,k \in \{1,\dots,7\}\} \cup \{2^{2^{x_j}} = x_k : j,k \in \{1,\dots,7\}\}$ has at most five equations and at most finitely many solutions in $(\mathbb{N} \setminus \{0,1\})^7$, then each such solution $(x_1,\dots,x_7)$ satisfies $x_1,\dots,x_7 \leqslant 16$. Hypothesis 1 implies that there are infinitely many composite numbers of the form $2^{2^n}+1$. Hypotheses 2 and 3 are of similar kind. Hypothesis 2 implies that if the equation $x!+1=y^2$ has at most finitely many solutions in positive integers x and y, then each such solution (x,y) belongs to the set $\{(4,5),(5,11),(7,71)\}$. Hypothesis 3 implies that if the equation $x(x+1)=y!$ has at most finitely many solutions in positive integers x and y, then each such solution (x,y) belongs to the set $\{(1,2),(2,3)\}$.

Keywords: Brocard's problem; Brocard-Ramanujan equation $x!+1=y^2$; composite Fermat numbers; composite numbers of the form $2^{2^n}+1$; Erdös' equation $x(x+1)=y!$

MSC: 11D61; 11D85

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1. Composite numbers of the form $2^{2^n}+1$

Let $\mathcal{A}$ denote the following system of equations:

$$\left\{x_i \cdot x_j = x_k :\ i,j,k \in \{1,\dots,7\}\right\} \cup \left\{2^{2^{x_j}} = x_k :\ j,k \in \{1,\dots,7\}\right\}$$

The following subsystem of $\mathcal{A}$

$$\begin{cases} x_1 \cdot x_1 = x_2 \\ x_2 \cdot x_2 = x_3 \\ 2^{2^{x_1}} = x_3 \\ x_4 \cdot x_5 = x_2 \\ x_6 \cdot x_7 = x_2 \end{cases}$$

$$\begin{tikzcd} x_1 \arrow[rr, "2^{2^{(\bullet)}}"] \arrow[dr, "\text{squaring}"'] & & x_3 \\ & x_4 \cdot x_5 = x_2 = x_6 \cdot x_7 \arrow[ur, "\text{squaring}"'] & \end{tikzcd}$$

has exactly one solution in $(\mathbb{N} \setminus \{0,1\})^7$, namely $(2,4,16,2,2,2,2)$.

Hypothesis 1. *If a system of equations $\mathcal{S} \subseteq \mathcal{A}$ has at most five equations and at most finitely many solutions in $(\mathbb{N} \setminus \{0,1\})^7$, then each such solution $(x_1,\dots,x_7)$ satisfies $x_1,\dots,x_7 \leqslant 16$.*

Lemma 1. *([7] (p. 109)). For every non-negative integers x and y, $x+1=y$ if and only if $2^{2^x} \cdot 2^{2^x} = 2^{2^y}$.*

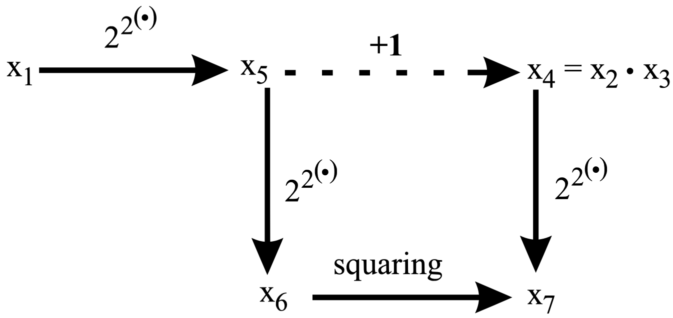
Theorem 1. *Hypothesis 1 implies that $2^{2^{x_1}}+1$ is composite for infinitely many integers x_1 greater than 1.*

Proof. Assume, on the contrary, that Hypothesis 1 holds and $2^{2^{x_1}}+1$ is composite for at most finitely many integers x_1 greater than 1. Then, the equation

$$x_2 \cdot x_3 = 2^{2^{x_1}} + 1$$

has at most finitely many solutions in $(\mathbb{N} \setminus \{0,1\})^3$. By Lemma 1, in positive integers greater than 1, the following subsystem of $\mathcal{A}$

$$\left\{\begin{array}{rcl} 2^{2^{x_1}} &=& x_5 \\ 2^{2^{x_5}} &=& x_6 \\ 2^{2^{x_4}} &=& x_7 \\ x_6 \cdot x_6 &=& x_7 \\ x_2 \cdot x_3 &=& x_4 \end{array}\right.$$



has at most finitely many solutions in $(\mathbb{N} \setminus \{0,1\})^7$ and expresses that $x_2 \cdot x_3 = 2^{2^{x_1}} + 1$. Since $641 \cdot 6700417 = 2^{2^5} + 1 > 16$, we get a contradiction. $\square$

Most mathematicians believe that $2^{2^n}+1$ is composite for every integer $n \geqslant 5$, see [2] (p. 23).

Open Problem 1. *([3] (p. 159)). Are there infinitely many composite numbers of the form $2^{2^n}+1$?*

Primes of the form $2^{2^n}+1$ are called Fermat primes, as Fermat conjectured that every integer of the form $2^{2^n}+1$ is prime, see [3] (p. 1). Fermat remarked that $2^{2^0}+1=3$, $2^{2^1}+1=5$, $2^{2^2}+1=17$, $2^{2^3}+1=257$, and $2^{2^4}+1=65537$ are all prime, see [3, p. 1].

Open Problem 2. *([3] (p. 158)). Are there infinitely many prime numbers of the form $2^{2^n}+1$?*

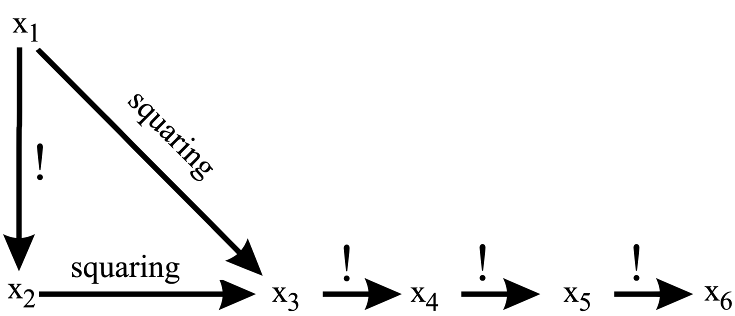
2. The Brocard-Ramanujan equation $x!+1=y^2$

Let $\mathcal{B}$ denote the following system of equations:

$$\{x_i \cdot x_j = x_k : i,j,k \in \{1,\dots,6\}\} \cup \{x_j! = x_k : (j,k \in \{1,\dots,6\}) \wedge (j \neq k)\}$$

The following subsystem of $\mathcal{B}$

$$\left\{\begin{array}{rcl} x_1! &=& x_2 \\ x_1 \cdot x_1 &=& x_3 \\ x_2 \cdot x_2 &=& x_3 \\ x_3! &=& x_4 \\ x_4! &=& x_5 \\ x_5! &=& x_6 \end{array}\right.$$



has exactly two solutions in positive integers, namely $(1,\dots,1)$ and $(2,2,4,24,24!,(24!)!)$.

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Hypothesis 2. *If a system of equations $\mathcal{S} \subseteq \mathcal{B}$ has at most finitely many solutions in positive integers $x_1, \ldots, x_6$, then each such solution $(x_1, \ldots, x_6)$ satisfies $x_1, \ldots, x_6 \leqslant (24!)!$.*

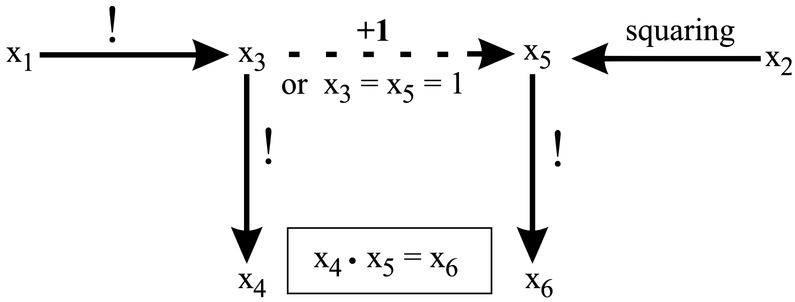
Lemma 2. *For every positive integers x and y, $x! \cdot y = y!$ if and only if*

$$(x + 1 = y) \vee (x = y = 1)$$

Theorem 2. *Hypothesis 2 implies that if the equation $x_1! + 1 = x_2^2$ has at most finitely many solutions in positive integers x_1 and x_2, then each such solution (x_1, x_2) belongs to the set $\{(4,5), (5,11), (7,71)\}$.*

Proof. The following system of equations $\mathcal{B}_1$

$$\begin{cases} x_1! = x_3 \\ x_3! = x_4 \\ x_5! = x_6 \\ x_4 \cdot x_5 = x_6 \\ x_2 \cdot x_2 = x_5 \end{cases}$$



is a subsystem of $\mathcal{B}$. By Lemma 2, the system $\mathcal{B}_1$ expresses that

$$(x_1! + 1 = x_2^2) \vee (x_1 = \ldots = x_6 = 1)$$

If the equation $x_1! + 1 = x_2^2$ has at most finitely many solutions in positive integers x_1 and x_2, then $\mathcal{B}_1$ has at most finitely many solutions in positive integers $x_1, \ldots, x_6$ and Hypothesis 2 implies that every tuple $(x_1, \ldots, x_6)$ of positive integers that solves $\mathcal{B}_1$ satisfies $x_6 \leqslant (24!)!$. Hence, $x_1 \in \{1, \ldots, 23\}$. If $x_1 \in \{1, \ldots, 23\}$, then $x_1! + 1$ is a square only for $x_1 \in \{4, 5, 7\}$. $\square$

It is conjectured that $x! + 1$ is a square only for $x \in \{4, 5, 7\}$, see [10] (p. 297). A weak form of Szpiro's conjecture implies that the equation $x! + 1 = y^2$ has only finitely many solutions in positive integers, see [6].

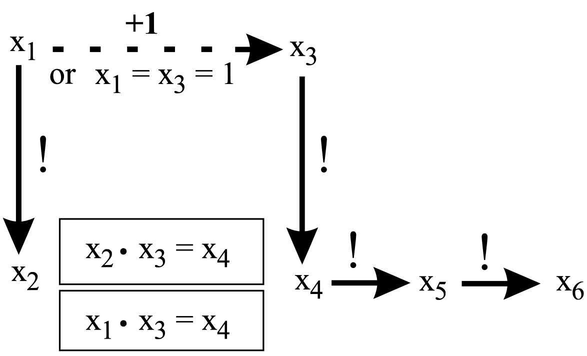
3. Erdös' equation $x(x + 1) = y!$

Let $\mathcal{C}$ denote the following system of equations:

$$\{x_i \cdot x_j = x_k : (i, j, k \in \{1, \ldots, 6\}) \wedge (i \neq j)\} \cup \{x_j! = x_k : (j, k \in \{1, \ldots, 6\}) \wedge (j \neq k)\}$$

The following subsystem of $\mathcal{C}$

$$\begin{cases} x_1! = x_2 \\ x_3! = x_4 \\ x_2 \cdot x_3 = x_4 \\ x_1 \cdot x_3 = x_4 \\ x_4! = x_5 \\ x_5! = x_6 \end{cases}$$



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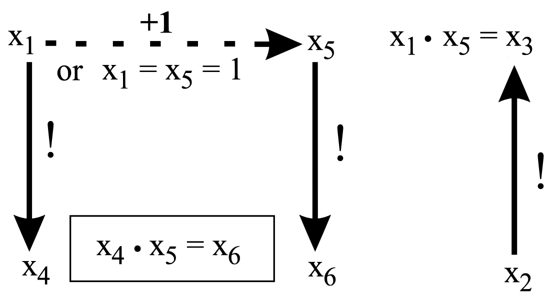
has exactly three solutions in positive integers, namely $(1,\dots,1)$, $(1,1,2,2,2,2)$, and $(2,2,3,6,720,720!)$.

Hypothesis 3. *If a system of equations $\mathcal{S} \subseteq \mathcal{C}$ has at most finitely many solutions in positive integers $x_1,\dots,x_6$, then each such solution $(x_1,\dots,x_6)$ satisfies $x_1,\dots,x_6 \leqslant 720!$.*

Theorem 3. *Hypothesis 3 implies that if the equation $x_1(x_1+1) = x_2!$ has at most finitely many solutions in positive integers x_1 and x_2, then each such solution (x_1,x_2) belongs to the set $\{(1,2),(2,3)\}$.*

Proof. The following system of equations $\mathcal{C}_1$

$$\begin{cases} x_1! = x_4 \\ x_5! = x_6 \\ x_4 \cdot x_5 = x_6 \\ x_2! = x_3 \\ x_1 \cdot x_5 = x_3 \end{cases}$$



is a subsystem of $\mathcal{C}$. By Lemma 2, the system $\mathcal{C}_1$ expresses that

$$(x_1(x_1+1) = x_2!) \vee (x_1 = \ldots = x_6 = 1)$$

If the equation $x_1(x_1+1) = x_2!$ has at most finitely many solutions in positive integers x_1 and x_2, then $\mathcal{C}_1$ has at most finitely many solutions in positive integers $x_1,\dots,x_6$ and Hypothesis 3 implies that every tuple $(x_1,\dots,x_6)$ of positive integers that solves $\mathcal{C}_1$ satisfies $x_3 \leqslant 720!$. Hence, $x_2 \in \{1,\dots,720\}$. If $x_2 \in \{1,\dots,720\}$, then $x_2!$ is a product of two consecutive positive integers only for $x_2 \in \{2,3\}$ because the following *MuPAD* program

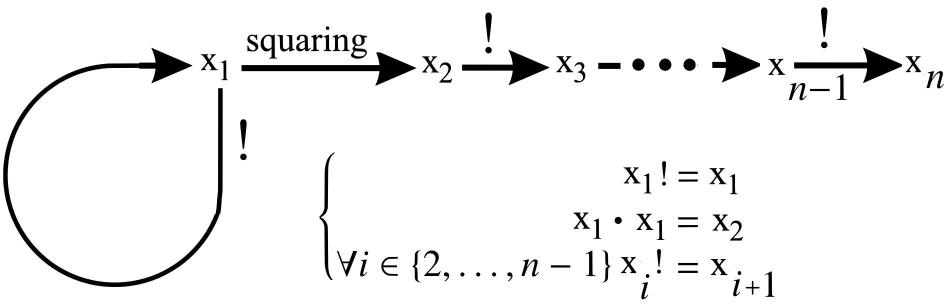
```
for x2 from 1 to 720 do
x1:=round(sqrt(x2!+(1/4))-(1/2)):
if x1*(x1+1)=x2! then print(x2) end_if:
end_for:
```

returns 2 and 3. □

The question of solving the equation $x(x+1) = y!$ was posed by P. Erdös, see [1]. F. Luca proved that the *abc* conjecture implies that the equation $x(x+1) = y!$ has only finitely many solutions in positive integers, see [4].

4. There is no hope for similar hypotheses with an arbitrary number of variables

Let $f(1) = 2$, $f(2) = 4$, and let $f(n+1) = f(n)!$ for every integer $n \geqslant 2$. Let $\mathcal{U}_1$ denote the system of equations $\{x_1! = x_1$. For an integer $n \geqslant 2$, let $\mathcal{U}_n$ denote the following system of equations:



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For every positive integer n, the system $\mathcal{U}_n$ has exactly two solutions in positive integers $x_1, \ldots, x_n$, namely $(1, \ldots, 1)$ and $(f(1), \ldots, f(n))$. For a positive integer n, let Ψ_n denote the following statement: *if a system of equations*

$$\mathcal{S} \subseteq \{x_i \cdot x_j = x_k : i, j, k \in \{1, \ldots, n\}\} \cup \{x_j! = x_k : j, k \in \{1, \ldots, n\}\}$$

has at most finitely many solutions in positive integers $x_1, \ldots, x_n$, then each such solution $(x_1, \ldots, x_n)$ satisfies $x_1, \ldots, x_n \leqslant f(n)$. The statements Ψ_n are discussed in [8,9].

Theorem 4. *Every factorial Diophantine equation can be algorithmically transformed into an equivalent system of equations of the forms $x_i \cdot x_j = x_k$ and $x_j! = x_k$.*

Proof. It follows from Lemmas 2–4 in [7] and Lemma 2. □

The statement $\forall n \in \mathbb{N} \setminus \{0\}\ \Psi_n$ is dubious. By Theorem 4, this statement implies that there is an algorithm which takes as input a factorial Diophantine equation and returns an integer which is greater than the solutions in positive integers, if these solutions form a finite set. This conclusion is strange because properties of factorial Diophantine equations are similar to properties of exponential Diophantine equations and a computable upper bound on non-negative integer solutions does not exist for exponential Diophantine equations with a finite number of solutions, see [5].

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