Article

The Conservation Laws in Quantum Mechanics

Richard Joseph Oldani *

Posted Date: 20 March 2023

doi: 10.20944/preprints202303.0334.v1

Keywords: conservation laws; Hamilton's principle; energy; momentum; path integral formulation; matrix mechanics

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 20 March 2023 doi:10.20944/preprints202303.0334.v1

Article

The Conservation Laws in Quantum Mechanics

Richard Oldani

Illinois Institute of Technology; oldanirl@gmail.com

Abstract: When the theoretical foundations of Einstein's quantum theory of radiation are compared with Heisenberg's and Feynman's theories of nonrelativistic quantum mechanics they are found to violate the conservation laws. The deficiencies are corrected by introducing Hamilton's principle $S = \int Ldt$ and applying the methods of quantum field theory to derive relativistic equations of motion. Based on the calculus of variations proof is given that the wave function equals twice the allowable action minimum.

Keywords: conservation laws; quantum mechanics; Hamilton's principle

1. Introduction

It is often claimed, whether overtly or implied, that the most accurate theory is the best theory. A clock that was accurate to within one second over the age of the universe was later improved upon by a clock accurate to within 100 milliseconds. The most recent clock experiment can differentiate between gravitational potentials of one millimeter [1]. Clock accuracy is continuously being improved upon because it is believed that more accurate time measurement "offers new opportunities for tests of fundamental physics". In another area of research universities are competing to improve upon measurements of the g-2 factor of an electron [2]. Once again the goal is to find a discrepancy with the standard model that will lead to "new physics". Improved instrumentation, more accurate measurements, and better predictions are the procedures that many say will lead to an improved understanding of Nature. We will show in these pages that improved accuracy does not equate with improved understanding, and that it is precisely what cannot be observed that is the key to an improved understanding of natural phenomena.

The importance of the observer, and by inference the observation, in scientific inquiry is a sticking point that has a long history in theoretical physics. The majority of physicists side with Bohr, who said, "It is wrong to think that the task of physics is to find out how nature is. Physics concerns what we can say about nature." On the other side of the dispute is Einstein, who believed that, "On principle, it is quite wrong to try founding a theory on observable magnitudes alone." Our purpose is to determine which of the two approaches is the more accurate, the empirical or the intuitive.

2. Momentum conservation

2.1. Einstein's theory

Einstein's first attempt at a quantum theory was a derivation of Planck's law by statistical methods "in an amazingly simple and general manner" [3]. There he describes the dynamic equilibrium that exists between the thermal energy absorbed by molecules and its subsequent quantum mechanical emission as black body radiation. Heat energy that is absorbed according to classically defined Maxwell Boltzmann statistics is transformed at the molecular level and emitted according to the Planck radiation law. The *absorption* of energy by a molecule leads to momentum +E/c in the direction of propagation, while the *emission* of a photon causes a recoil momentum -E/c that is directed in the opposite direction of propagation. The sudden reversals and random nature of the impulses cause molecular trajectories to be discontinuous, as observed in Brownian motion.

Einstein approached the problem of radiation in a methodical way by using the coordinate system K to describe absorption by thermal radiation and K' to describe quantum mechanical emission. “Let a molecule of given kind be in uniform motion with speed **v** along the X-axis of the coordinate system K. We inquire about the momentum transferred on the average from the radiation to the molecule per unit time. To calculate this we must consider the radiation from a coordinate system K′ that is at rest with respect to the given molecule. For we have formulated our hypotheses about emission and absorption only for molecules at rest.”

The emission and absorption of energy is to be described quantum mechanically in the coordinate system K' which is “at rest with respect to the given molecule”, while the kinetic energy of molecules will be treated with “ordinary mechanics” in the coordinates of the system K. To determine the total energy of a molecule we sum a classical component due to kinetic energy, as determined by temperature; and a quantum mechanical component due to the energy of excited states. The inclusion of both classical and quantum mechanical energy forms in the same model of radiation is a feature that distinguishes Einstein’s methods from all others. This contrasts with the Bohr-Heisenberg method, which derives two independent expressions, one classical and one quantum mechanical, and then links them by using the correspondence principle.

2.2. Heisenberg’s theory

Matrix mechanics evolved from long-standing attempts to describe dispersion phenomena, the continuous change in the angle of refraction of different frequencies of light by a prism or other medium. Although light disperses continuously across the entire spectrum, at certain specific frequencies characteristic of the medium, it is completely absorbed forming lines. When complex sets of mathematical rules were discovered that describe the relationship between the observed frequencies and intensities of spectral lines it allowed Heisenberg to formulate a theory of quantum mechanics that reconciles the continuity of radiation fields with the discrete energy states of an atom by expressing electron transitions in the form of a matrix [4].

$$\sum_k \left(p_{nk} q_{km} - q_{nk} p_{km}\right) = \begin{matrix} i\hbar \, \text{for } n = m \\ 0 \, \text{for } n \neq m \end{matrix} \qquad (1)$$

To compare Einstein’s statistically defined measurements of energy fluctuation with the methods of matrix mechanics we refer to the energy matrix [5]. The diagonal elements of the matrix $m = n$ refer to all possible energy states of a quantum system. Because the theory only concerns quantum mechanical phenomena it represents a complete break from classical theory. Its weakness lies in the fact that it does not describe how an atom absorbs energy to enter into an excited state.

Off-diagonal elements of the energy matrix $m \neq n$ refer to the resonances of radiation with an atom’s valence electrons, where elements of the array E_{mn} that are above the diagonal have a plus sign because they represent energy absorption and elements of the array that are below the diagonal E_{nm} have a minus sign representing energy emission. Because the energy of an absorption offsets the energy of an emission except for a difference in phase a value of zero is assigned to these matrix elements when averaged over time. However changes in state do not account for all contributions of energy when examined microscopically. Spectral lines also broaden due to exchanges of momentum caused by thermal energy that cannot be avoided. Although the collisions cannot be observed individually they are evident statistically in measurements of gas temperature. Heisenberg did not take these microscopic interactions into account because he believed that quantum mechanics should be “founded exclusively upon relationships between quantities which in principle are observable” [6]. This caused him to focus exclusively upon the intensity and frequency of the spectral lines, which are the emission properties of a radiating atom or molecule.

Due to the conservation of energy an atom must first absorb energy from surrounding atoms before it can emit energy. Einstein realized the importance of unobserved momentum exchanges stating [3], “In general one is satisfied with a consideration of the *energy* exchange without consideration of the exchange of *momentum*. One feels easily justified to do so, because the smallness of the momenta transferred by radiation are almost always negligible when compared to other motion-generating causes. But in *theoretical* investigations these small effects are definitely as important as the more prominently appearing *energy* transfers by radiation, because energy and momenta are always intimately linked together." What Einstein did not say was that *energy and momentum are linked together by the conservation laws.* Thermal energy absorbed by oscillators during black body radiation due to momentum exchange is immediately emitted quantum mechanically as radiation. If the classically defined absorption energy in K is ignored because it is unobservable it invalidates energy emission in K' due to the violation of energy conservation.

2.3. Feynman's theory

In the path integral approach to quantum mechanics we can see violations of the conservation laws even more clearly. In the following passage Feynman considers absorption and emission by using the field approach of quantum field theory, but then rejects it in favor of a particle model of *only* emission [5]. "In empty space an atom emits light and yet there is no potential to perturb the systems and so for it to make a transition. The explanation of modern quantum mechanical electrodynamics is that the atom is perturbed by the zero-point fluctuations of the quantized radiation field. It is here that the theory of action at a distance gives us a different viewpoint. It says that an atom alone in empty space would, in fact, *not* radiate. Radiation is a consequence of the interaction with other atoms (namely, those in the matter which absorbs the radiation). We are then led to the possibility that the spontaneous radiation of an atom in quantum mechanics also, may not be spontaneous at all, but induced by the interaction with other atoms, and that all of the apparent quantum properties of light and the existence of photons may be nothing more than the result of matter interacting with matter directly and according to quantum mechanical laws."

Due to considerations of energy conservation Feynman's statement on radiation processes is accurate; that is, an atom in empty space would not radiate. However, that realization did not stimulate thoughts about how to incorporate absorption into a *complete* theory of radiation, rather it caused him to formulate emission theory as a sum over paths [7]. His theory describes only what is observable in a theory of radiation, in complete agreement with Heisenberg's theory. An entirely expected result of the path integral formulation is that particle paths are unbounded both in geometry, by curved and looping trajectories, and in spatial extent, by paths that exceed the energy of excited states. Both cases are clear violations of the conservation laws.

3. Relativistic quantum mechanics

3.1. Energy absorption

A theory of quantum mechanics compatible with special relativity is sought after by describing emission and absorption with the time integral of a Lagrangian and applying the calculus of variations. Consider a radiating atom with ground state |1> and excited state |2>. To describe the electron's equation of motion during energy absorption we make use of Hamilton's principle function S = ∫ Ldt, where L=T-V. It provides for a more economical expression of the laws of motion by specifying fixed boundary conditions for particle paths in both space and time rather than trajectories in the Cartesian coordinates of Newton's laws. The limitations of absolute space and time inherent to Newton's laws are thereby avoided.

Let the electron initiate its motion at a fixed point on the equipotential surface R_1 of the ground state at time t_1, proceed along a path r, and upon arriving at the excited state R_2 adopt the circular path 2πr of an orbital thereby assuming orbital angular momentum. The action minimum S[r(t)] for a path between the stationary points R_1 and R_2 yields not zero as in classical dynamics, but Planck's constant h.

$$S[r(t)] = \int_{R_1}^{R_2} L \int_{t_1(r)}^{t_2(2\pi r)} dt = h \tag{2}$$

The action S is a functional, S[r(t)], which means it has as its argument an infinite number of functions, the possible electron trajectories r(t) between R_1 and R_2. If we arbitrarily set the initial time $t_1(r)$ at zero, then the action integral for a path of unit length is evaluated as follows:

$$E_{12} 2\pi\tau = h$$

and simplifying, we have $E_{12}\ \tau = \hbar$.

The electron's path is described in a configuration space consisting of three coordinates that define the origin, or nucleus, and three coordinates that define the manifolds R_1 and R_2 upon which the electron is constrained to move; that is, the electron shells. In other words, the quantum oscillator is described in configuration space as having six parameters, three for the nucleus and three for the electron shells, a total of six degrees of freedom.

3.2. Energy emission

At the relativistic or high end of the energy spectrum in quantum field theory, particles are treated as excited states of the more fundamental underlying quantum fields. As Nobel laureate Frank Wilczek noted [8], "In quantum field theory, the primary elements of reality are not individual particles, but underlying fields." On the other hand, when we examine lower energy interactions in nonrelativistic theory we find that interpretations are almost exclusively about particles. It seems that theoreticians use fields when it is convenient to use fields, and particles when it is convenient to use particles. However, the physical principles that underlie mathematical models demand a level of consistency that goes beyond simple expediency.

In quantum field theory particles are described by an action functional $S[\psi(x_i)]$ where the fields $\psi(x_i)$ of particles are defined throughout space. Each of the elementary particles has a field and the action depends on all of the fields. The equations of motion are determined by minimizing the Lagrangian in a region of space-time. We will follow these same practices in order to extend the field interpretation to nonrelativistic theory. The region of space-time that is of interest for lower energy interactions lies between the two states of an electron transition. Therefore we specify field boundaries coincident with the atom's electron shells that prevent violations of energy conservation by containing particles and fields inside atomic space. Within the space-time region we define a Lagrangian density of the fields and their first derivatives $£(\phi_i,\ \phi_{i,\mu})$ which allows for a complete accounting of the energy interactions, where ϕ_i is the current density and $\phi_{i,\mu}$ is the electromagnetic field strength. The action integral for a quantum oscillator with an outer electron that occupies either of two allowable energy states may now be formulated, where emission initiates from the excited state $R_2 = (x_2,y_2,z_2)$ at time t_2 and it finalizes at the ground state $R_1 = (x_1,y_1,z_1)$ at time t_1. Applying Hamilton's principle we require the integral of the Lagrangian density over the region of space-time between the excited and ground states to be a minimum for all small variations of the coordinates inside the region, where the action minimum for a quantum system is defined to be h.

$$S[\phi_i(t)] = \int_{R_2}^{R_1}\int_{t_2}^{t_1} £(\phi_i\,\phi_{i,\mu})\,d^3x\,dt = h \tag{3}$$

The energy states |2> and |1> determine invariant field boundaries coinciding with the electron shells where photon fields vanish. The change in action yields a relativistic formulation of emission that is invariant, the same for all observers. The action $S[\phi_i(t)]$ is a functional, a function of the values of coordinates on the *discrete* boundaries of the space-time surfaces R_2 and R_1 which are in turn functions of the *continuous* space-time variables of the fields within the surface. The field boundaries are uniquely fixed in four-dimensions by the volume d^3x and the time interval t_2-t_1 causing photon emission to be described as a four-dimensional localization of fields.

4. Comparison of the relativistic and non-relativistic models

The overriding problem in quantum mechanics has been how to describe two physical processes, absorption and emission, with a single equation. According to the Schrödinger wave equation they occur as a single process that evolves symmetrically in time. The wave function ψ used to describe an electron oscillating between two energy states performs two complete rotations, or a total of 720 degrees, before returning to its original state. If the electron is represented mathematically by a vector in Hilbert space $|\psi>$ then one rotation of 2π results in a negative value $-|\psi>$ and a second rotation of 2π brings the electron back to its original state $|\psi>$. The rotation occurs in abstract space so no physical interpretation is possible.

The Lagrangian model of a quantum system describes quantization in real space and real time as a four-dimensional absorption of energy by the electron in 2), followed by a four-dimensional localization of fields in 3) and release of a photon. We interpret the first rotation of the wave function not as a rotation in *abstract space*, but as the change in phase of electromagnetic fields from 0 to 2π in *real space* during absorption. Thus one "rotation" of the wave function is interpreted as one full cycle of an electromagnetic wave and an increase in the electron's energy from the ground state to an excited state. The second rotation occurs as the electron returns to the ground state and is interpreted as a localization of electromagnetic field energy and emission of a photon. The dual wave-particle nature of the photon is thereby realized in a physical transformation.

The external appearances of a radiating atomic system, the frequency and intensity of its spectral lines, are observables described by the matrix mechanical formulation in 1), where each matrix includes both absorption and emission processes for all possible radiation processes. This is because, due to the conservation of energy, no observable can be realized unless absorption and emission are both present. Hermitian matrices are specifically chosen to represent quantum mechanical observables because it is a complex square matrix that is equal to its own conjugate transpose. As a result when upper elements, absorptions, are multiplied with lower elements, emissions, real values, eigenvalues, are obtained. Thus a single expression, or matrix, describes two physical events and noncommutation is the result of a difference in angular momentum between excited and ground states [9].

5. Conclusions

The purpose of Hamilton's principle function $S = \int L dt$ is to find the action minimum of an electron transition by recognizing the symmetries of the system, R_2 and R_1, and identifying initial and final stationary points, t_1 and t_2 in configuration space. The correct path is the one that minimizes the action between points in configuration space, not necessarily the path that is the most precisely determinable. The Schrödinger equation is our most accurate equation of motion, but it does not minimize the action. As noted in the previous section the wave function includes twice the minimum allowable action for a quantum oscillator. Similarly, in the case of matrix mechanics the action of each element of an infinite array is equal to h so the action is not a minimum, it is infinite.

References

1. T. Bothwell *et al*. (2022) "Resolving the gravitational red shift within a millimeter atomic sample"*Nature* **602**, Issue 7897, 420. arXiv:2109.12238
2. Guellati-Khelifa, S. *Physics* **16**, 22 (2023).
3. A. Einstein, (1917) *Phys Z* **18**, 121.**f** Einstein (1917)_Quantum theory of radiation W.
4. Heisenberg W. (1925), *Z Phys* **33**, 879 in B.L. van der Waerden (ed.), *Sources of Quantum Mechanics*, (Dover, 1968), p. 261.
5. Oldani, R. (2022) *Physics Essays* **3** (1), 51-54. doi:https://doi.org/10.4006/0836-1398-35.1.51
6. Brown, L.M., (2005) *Feynman's thesis: A new approach to quantum theory* (World Scientific), p. 4.
7. Feynman, R. (1948) "A space-time approach to non-relativistic quantum mechanics" *Rev Mod Phys* **20**, 367-387.

doi:10.20944/preprints202303.0334.v1

8. Wilczek, F. (1999) "Mass Without Mass I: Most of Matter," *Physics Today* **52** (11), 11-13.
9. Oldani, R. (2021) “Application of Einstein’s methods in a quantum theory of radiation” (Intechopen, London).