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Associated Lie Algebras of One-Variable and Bivariate Hermite Polynomials and New Generating Functions

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Associated Lie Algebras of one-variable and Bivariate Hermite polynomials and New Generating Functions

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Abstract: This paper presents the symmetries of differential equations associated with one-variable and Bivariate Hermite polynomials by proposing a representation of Lie algebra $\mathfrak{sl}(2,R)$ for these differential operators. Applying the Baker-Campbell-Hausdorff formula to these algebras, results in new relations and generating functions in one-variable and Bivariate Hermite polynomials. A general form of $\mathfrak{sl}(2,R)$ representation for other orthogonal polynomials such as Laguerre polynomials is introduced.

Keywords: Bivariate Hermite polynomial; Lie Algebra; Baker-Campbell-Hausdorff formula; generating function; sl (2, R) algebra

1. Introduction

Hermite polynomials are among the most applicable special functions. These polynomials arise in diverse fields of probability, physics, numerical analysis, and signal processing. As an example, in quantum mechanics, eigenfunction solutions to quantum harmonic oscillator are described in terms of Hermite polynomials. Bivariate Hermite polynomials are useful in algebraic geometry and two-dimensional quantum harmonic oscillator [3–5]. Respect to the field of applications Hermite polynomials in one variable are divided into probabilist and physicist versions. In present paper we focus on one and two dimensional probabilist Hermite polynomials. We prove the symmetries of associated differential equations are compatible with $\mathfrak{sl}(2,R)$ algebra. By introducing isomorphic Lie algebras whose Cartan sub-algebras are the Hermite differential operators, applying Baker-Campbell-Hausdorff formula yields new relations for one variable and bivariate Hermite polynomials. Without exception, all known generating functions for Hermite polynomials contains a factorial term in denominator. We introduce a new generating function without factorial denominator.

2. Lie algebra of Hermite polynomials of one variable

Probabilistic Hermite polynomials, presented as:

$$H_{en} = e^{\frac{-D^2}{2}} x^n \qquad (1)$$

H_{en} are the solutions to Hermite differential equations:

$$\mathbb{D}_H H_{en} = (xD - D^2) H_{en} = n H_{en} \qquad (2)$$

where D denoted as $\frac{d}{dx}$ and $\mathbb{D}_H$ as Hermite differential operator. This is an eigenvalue problem with positive integer eigenvalues n.

The equation (1) is the transformation of basis $(1, x, x^2, x^3, \dots)$ under the action of operator $e^{\frac{-D^2}{2}}$ which is compatible with Rodrigues' formula and results in probabilistic Hermite polynomials H_{en}.

The monomials x^n expand a polynomial vector space $\mathbb{V}$. The operator $e^{\frac{-D^2}{2}}$ changes the basis x^n into the basis H_{en}. Let $\mathfrak{gl}\,(\mathbb{V})$ denote the linear transformation that maps vector space $\mathbb{V}$ onto itself. We present isomorphic Lie algebras to $\mathfrak{sl}(2,R)$ defined by $\mathfrak{sl}(2,R)$ module on vector space $\mathbb{V}$ which is a linear map ϕ defined by $\phi : \mathfrak{sl}(2,R) \rightarrow \mathfrak{gl}\,(\mathbb{V})$ that preserves the commutator relations of $\mathfrak{sl}(2,R)$ algebra [1,2].

$$\phi[a,b] = [\phi(a),\phi(b)] \qquad a,b \in \mathfrak{sl}(2,R) \tag{3}$$

This representation is $\mathfrak{sl}(2,R)$ module on vector space $\mathbb{V}$.

First, we review the structure of irreducible vector field representation of $\mathfrak{sl}(2,R)$.The generators of this algebra in matrix representation are as follows:

$$H = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad X = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{4}$$

The commutation relations for this representation of $\mathfrak{sl}(2,R)$ are:

$$[X,Y] = 2H \quad , \quad [H,X] = -X \quad , \quad [H,Y] = Y \tag{5}$$

Let define a representation of $\mathfrak{sl}(2,R)$ as its module on $\mathbb{V}$ that preserves commutation relations by differential operators as its generators [1]:

$$\boldsymbol{h} = xD - \frac{n}{2} \quad , \qquad \boldsymbol{e} = D = \partial_x \quad , \qquad \boldsymbol{f} = x^2D - nx \tag{6}$$

With the same commutation relations

$$[\boldsymbol{e},\boldsymbol{f}] = 2\boldsymbol{h} \quad , \quad [\boldsymbol{h},\boldsymbol{e}] = -\boldsymbol{e} \quad , \quad [\boldsymbol{h},\boldsymbol{f}] = \boldsymbol{f} \tag{7}$$

The Cartan sub-algebra $H = \boldsymbol{h}$ produces a decomposition of representation space:

$$\mathbb{V} = \bigoplus \mathbb{V}_j \tag{8}$$

$\mathbb{V}_j$ are the eigenspace (eigenfunction) of generator $\boldsymbol{h}$ as Cartan sub-algebra of $\mathfrak{sl}(2,R)$ and provide the solutions to the related differential equation.

$$\boldsymbol{h}\mathbb{V}_j = f(j)\mathbb{V}_j \tag{9}$$

As an example, monomials x^n are eigenfunctions or eigenspaces of generator $\boldsymbol{h}$, realized as eigenspace $\mathbb{V}_n$. The eigenvalues $f(j)$ in most cases equals an integer j or $j(j+1)$ as we observe in Hermite, Laguerre and Legendre differential equations.

We search for a Lie algebra $\mathfrak{L}_H$ isomorphic to $\mathfrak{sl}(2,R)$ algebra that its generators to be defined based on Hermite differential operators. Here we apply the transformation operator $e^{\frac{-D^2}{2}}$ as described in (1) for Hermite polynomials to derive similarity transformations (conjugation) of $\mathfrak{sl}(2,R)$ bases as follows:

$$X_1 = e^{\frac{-D^2}{2}}\boldsymbol{h}\, e^{\frac{D^2}{2}} \; , \quad X_2 = e^{\frac{-D^2}{2}}\boldsymbol{e}\, e^{\frac{D^2}{2}} \; , \quad X_3 = e^{\frac{-D^2}{2}}\boldsymbol{f}\, e^{\frac{D^2}{2}} \tag{10}$$

Respect to a theorem in Lie algebra theory, these generators constitute an isomorphic Lie algebra to $\mathfrak{sl}(2,R)$ with similar commutation relations. We call this algebra as “ Hermite operator Lie algebra". Due to the equation (1) that implies the change of basis x^n to H_{en}, the operator xD with eigenfunctions x^n corresponds the operator $\mathbb{D}_H$ with eigenfunctions H_{en} and common eigenvalues n through a similarity transformation described by

$$\mathbb{D}_H = e^{\frac{-D^2}{2}}(xD)\, e^{\frac{D^2}{2}} \qquad (11)$$

Therefor we have:

$$X_1 = e^{\frac{-D^2}{2}}\boldsymbol{h}\, e^{\frac{D^2}{2}} = e^{\frac{-D^2}{2}}\left(xD - \frac{n}{2}\right) e^{\frac{D^2}{2}} = \mathbb{D}_H - \frac{n}{2} \qquad (12)$$

Generator X_2 simply be calculated as $\partial_x = D$.

Proposition 1. For X_3 we have:

$$X_3 = (x - \partial_x)(\mathbb{D}_H - n)$$

Proof: by Equation (11) we have the identity:

$$\mathbb{D}_H = e^{\frac{-D^2}{2}}(xD)e^{\frac{D^2}{2}} = \left(e^{\frac{-D^2}{2}} x\, e^{\frac{D^2}{2}}\right)\left(e^{\frac{-D^2}{2}} D\, e^{\frac{D^2}{2}}\right) = e^{\frac{-D^2}{2}} x e^{\frac{D^2}{2}} D \qquad (13)$$

Thus:

$$\mathbb{D}_H D^{-1} = e^{\frac{-D^2}{2}} x e^{\frac{D^2}{2}} \qquad (14)$$

By equation (2) we have:

$$(xD - D^2)D^{-1} = (x - D) = e^{\frac{-D^2}{2}} x e^{\frac{D^2}{2}} \qquad (15)$$

Now for X_3 from (10) we have:

$$X_3 = e^{\frac{-D^2}{2}}\boldsymbol{f}\, e^{\frac{D^2}{2}} = e^{\frac{-D^2}{2}}(x^2D - nx)\, e^{\frac{D^2}{2}} = e^{\frac{-D^2}{2}}(x^2D)e^{\frac{D^2}{2}} - ne^{\frac{-D^2}{2}} x e^{\frac{D^2}{2}}$$

By equations (13), (14) and (15) we get

$$X_3 = \left[e^{\frac{-D^2}{2}} x e^{\frac{D^2}{2}}\right]\left[e^{\frac{-D^2}{2}}(xD)e^{\frac{D^2}{2}}\right] - n(x - D) =$$

$$= (x - D)\mathbb{D}_H - n(x - D) = (x - \partial_x)(\mathbb{D}_H - n)$$

and for generators of this Lie algebra, we have:

$$X_1 = x\partial_x - \partial_x^2 - \frac{n}{2} = \mathbb{D}_H - \frac{n}{2} \quad , \quad X_2 = \partial_x \quad , \quad X_3 = (x - \partial_x)(\mathbb{D}_H - n) \qquad (16)$$

where $\mathbb{D}_H$ denotes the Hermite differential operator i.e., $x\partial_x - \partial_x^2$. The commutation relations coincide the Lie algebra $\mathfrak{sl}(2, R)$ and are as follows:

$$[X_1, X_2] = -X_2 \quad , \quad [X_1, X_3] = X_3 \quad , \quad [X_2, X_3] = 2X_1 \qquad (17)$$

Proposition 2. Hermite polynomials, satisfies the equation:

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$$e^{\frac{-\partial_x^2}{2}}(e^{-1}+x)^n = e^{(\mathbb{D}_H - n - \frac{\partial_x}{1-e})}\, H_n^e(x) \tag{18}$$

proof: Due to a theorem for BCH formula, if $[X, Y] = sY$ for $s \in \mathbb{R}$, we have:

$$e^X e^Y = e^{X + \frac{s}{1-e^{-s}}Y} \tag{19}$$

The BCH formula for X_1 and X_2 generators gives:

$$e^{[x\partial_x - \partial_x^2]}\, e^{\partial_x} = e^{\left[x\partial_x - \partial_x^2 - \frac{\partial_x}{1-e}\right]} \tag{20}$$

The term $-\frac{n}{2}$ in X_1 omitted because it has no role in commutation relation $[X_1, X_2]$. Multiplying both side by $H_n^e(x)$

$$e^{[x\partial_x - \partial_x^2]}\, e^{\partial_x} H_n^e(x) = e^{\left[x\partial_x - \partial_x^2 - \frac{\partial_x}{1-e}\right]}\, H_n^e(x) \tag{21}$$

For $e^{\partial_x} H_n^e(x)$ we obtain

$$e^{\partial_x} H_n^e(x) = \left(1 + \partial_x + \frac{\partial_x^2}{2!} + \dots\right) H_n^e(x) = H_n^e(x) + nH_{n-1}^e(x) + \frac{n(n-1)}{2!} H_{n-k}^e(x) + \dots + \frac{H_0^e(x)}{n!}$$

Thus

$$e^{\partial_x} H_n^e(x) = \sum_{k=0}^{n} \binom{n}{k} H_{n-k}^e(x) \tag{22}$$

Substituting in equation (21) and replacing Hermite differential operator $x\partial_x - \partial_x^2$ with $\mathbb{D}_H$ gives

$$e^{\mathbb{D}_H} \sum_{k=0}^{n} \binom{n}{k} H_{n-k}^e(x) = e^{\mathbb{D}_H - \frac{\partial_x}{1-e}} H_n^e(x) \tag{23}$$

$$\sum_{k=0}^{n} \binom{n}{k} e^{\mathbb{D}_H} H_{n-k}^e(x) = e^{\mathbb{D}_H - \frac{\partial_x}{1-e}} H_n^e(x) \tag{24}$$

$$\sum_{k=0}^{n} \binom{n}{k} e^{n-k}\, H_{n-k}^e(x) = e^{\mathbb{D}_H - \frac{\partial_x}{1-e}} H_n^e(x)$$

$$\sum_{k=0}^{n} \binom{n}{k} e^{-k}\, H_{n-k}^e(x) = e^{\mathbb{D}_H - n - \frac{\partial_x}{1-e}} H_n^e(x)$$

$$\sum_{k=0}^{n} \binom{n}{k} e^{-k}\, e^{\frac{-\partial_x^2}{2}} x^{n-k} = e^{(\mathbb{D}_H - n - \frac{\partial_x}{1-e})} H_n^e(x)$$

$$e^{\frac{-\partial_x^2}{2}} \sum_{k=0}^{n} \binom{n}{k} (e^{-1})^k\, x^{n-k} = e^{(\mathbb{D}_H - n - \frac{\partial_x}{1-e})} H_n^e(x) \tag{25}$$

$$e^{\frac{-\partial_x^2}{2}}(e^{-1}+x)^n = e^{(\mathbb{D}_H - n - \frac{\partial_x}{1-e})}\, H_n^e(x)$$

or

$$e^{-(\mathbb{D}_H - n - \frac{\partial_x}{1-e})} e^{\frac{-\partial_x^2}{2}}(e^{-1}+x)^n = H_n^e(x) \tag{26}$$

It is notable to compare this equation with

$$e^{\frac{-\partial_x^2}{2}} x^n = H_n^e(x) \tag{27}$$

3. Bivariate Hermite Polynomials

An ordinary definition for bivariate Hermite polynomials is as follows [3,4].

$$H_{n,m}(x,y) = \sum_{k=0}^{\min(n,m)} (-1)^k\, k! \binom{m}{k}\binom{n}{k} a^{(n-k)/2} b^k c^{(m-k)/2} H_{n-k}^e(x) H_{m-k}^e(y) \tag{28}$$

With $a, c > 0$ and $ac - b^2 > 0$. These polynomials satisfy the partial differential equation:

$$\left[\left(x\frac{\partial}{\partial x} - \frac{\partial^2}{\partial x^2}\right) + \left(y\frac{\partial}{\partial y} - \frac{\partial^2}{\partial y^2}\right) - 2\frac{b}{\sqrt{ac}}\frac{\partial^2}{\partial x \partial y}\right] H_{m,n}(x,y) = (m+n) H_{m,n}(x,y) \tag{29}$$

Let denote $\mathfrak{D}$ as the differential operator in equation (2)

$$\mathfrak{D} = \left(x\frac{\partial}{\partial x} - \frac{\partial^2}{\partial x^2}\right) + \left(y\frac{\partial}{\partial y} - \frac{\partial^2}{\partial y^2}\right) - 2\frac{b}{\sqrt{ac}}\frac{\partial^2}{\partial x \partial y} \tag{30}$$

If we denote $\partial_x = \frac{\partial}{\partial x}$ and $\partial_y = \frac{\partial}{\partial y}$, with the identities:

$$e^{\frac{-\partial_x^2}{2}} x^{m-k} = H_{m-k}^e(x) \quad , \quad e^{\frac{-\partial_y^2}{2}} y^{n-k} = H_{n-k}^e(y) \tag{31}$$

The equation (28) converts to

$$H_{n,m}(x,y) = e^{\frac{-\partial_x^2}{2}} e^{\frac{-\partial_y^2}{2}} \sum_{k=0}^{\min(n,m)} (-1)^k\, k! \binom{n}{k}\binom{m}{k} a^{(n-k)/2} b^k c^{(m-k)/2} x^{n-k} y^{m-k} \tag{32}$$

We denote the new polynomials as $u_{m,n}(x,y)$

$$u_{n,m}(x,y) = \sum_{k=0}^{\min(n,m)} (-1)^k\, k! \binom{n}{k}\binom{m}{k} a^{(n-k)/2} b^k c^{(m-k)/2} x^{n-k} y^{m-k} \tag{33}$$

If these polynomials are assumed as linearly independent basis, the transformation from these basis to $H_{n,m}(x,y)$ is as follows:

$$H_{n,m}(x,y) = e^{\frac{-\partial_x^2}{2}} e^{\frac{-\partial_y^2}{2}} u_{n,m}(x,y) = e^{-\frac{\partial_x^2+\partial_y^2}{2}} u_{n,m}(x,y) \tag{34}$$

Therefor the corresponding differential operator with $u_{n,m}(x,y)$ as its eigenfunctions could be derived by similarity transformation:

$$\mathfrak{D}' = e^{\frac{\partial_x^2+\partial_y^2}{2}}\, \mathfrak{D}\, e^{-\frac{\partial_x^2+\partial_y^2}{2}} \tag{35}$$

$\mathfrak{D}$ denoted as the differential operator given in eigenvalue equation (29). Thus, we have

$$\mathfrak{D}' = e^{\frac{\partial_x^2+\partial_y^2}{2}} \left[(x\partial_x - \partial_x^2) + (y\partial_y - \partial_y^2) - 2\frac{b}{\sqrt{ac}}\partial_x\partial_y\right] e^{-\frac{\partial_x^2+\partial_y^2}{2}} \tag{36}$$

Then due to commutativity of we have ∂_x^2 and ∂_y^2 we get

$$\mathfrak{D}' = \left[e^{\frac{\partial_x^2}{2}}(x\partial_x - \partial_x^2)e^{-\frac{\partial_x^2}{2}} + e^{\frac{\partial_y^2}{2}}(y\partial_y - \partial_y^2)e^{-\frac{\partial_y^2}{2}} - 2\frac{b}{\sqrt{ac}}(e^{\frac{\partial_x^2}{2}}\partial_x e^{-\frac{\partial_x^2}{2}})(e^{\frac{\partial_y^2}{2}}\partial_y e^{-\frac{\partial_y^2}{2}})\right] \tag{37}$$

Respect to (11) and (12) this reduces to

$$\mathfrak{D}' = \left[x\partial_x + y\partial_y - 2\frac{b}{\sqrt{ac}}\partial_x\partial_y\right] \tag{38}$$

Therefor the differential operator $\mathfrak{D}'$ satisfy the differential equation:

$$\mathfrak{D}' u_{n,m}(x,y) = (m+n) u_{n,m}(x,y) \tag{39}$$

Its eigenvalues are the same as the differential equation (24), because $\mathfrak{D}$ and $\mathfrak{D}'$ related by the similarity relation (35).

4. Bivariate Hermite Polynomials as $\mathfrak{sl}(2,\boldsymbol{R})$ Modules

4.1. In this section we introduce an associated Lie algebra of bivariate Hermite differential operator. First, we search for the compatible $\mathfrak{sl}(2,R)$ algebra in terms of differential operators of two variables. Respect to equations (6) and (11) the Cartan sub-algebra of $\mathfrak{sl}(2,R)$ can be taken as:

$$\boldsymbol{h} = \frac{1}{2}\left(x\partial_x + y\partial_y + 1\right) + \alpha\partial_x\partial_y \tag{40}$$

The additional term $\alpha\partial_x\partial_y$ has been chosen to satisfy the required commutation relations. The other generators are proposed as

$$\boldsymbol{e} = \alpha\, \partial_x\partial_y \quad , \quad \boldsymbol{f} = \frac{1}{2\alpha} xy + \frac{1}{2}\left(x\partial_x + y\partial_y\right) + \frac{\alpha}{4}\, \partial_x\partial_y \tag{41}$$

These generators satisfy the commutation relations of $\mathfrak{sl}(2,R)$:

$$[\boldsymbol{h}\,, \boldsymbol{e}] = -\boldsymbol{e} \quad , \quad [\boldsymbol{h}\,, \boldsymbol{f}] = \boldsymbol{f} \quad , \quad [\boldsymbol{e}\,, \boldsymbol{f}] = 2\boldsymbol{h} \tag{42}$$

By substituting $\alpha = \frac{-b}{\sqrt{ac}}$, the differential operator $\boldsymbol{h}$ satisfies the differential equation:

$$\boldsymbol{h}\, u_{n,m}(x,y) = \frac{1}{2}\left[\left(x\partial_x + y\partial_y + 1\right) - \frac{2b}{\sqrt{ac}}\partial_x\partial_y\right] u_{n,m}(x,y) \tag{43}$$

Respect (39) and (40) we have:

$$\boldsymbol{h}\, u_{n,m}(x,y) = \frac{1}{2}\left[(m+n)u_{n,m}(x,y) - u_{n,m}(x,y)\right] = \frac{1}{2}(m+n-1)u_{n,m}(x,y) \tag{44}$$

Thus $u_{n,m}(x,y)$ are eigenfunctions or weight vectors of $\boldsymbol{h}$ as Cartan sub-algebra of $\mathfrak{sl}(2,R)$. According to the equation

$$e^{-\frac{\partial_x^2+\partial_y^2}{2}} u_{n,m}(x,y) = H_{n,m}(x,y) \tag{45}$$

Respect to Equation (11) and (12), similarity transformation of generators $\boldsymbol{h}$, $\boldsymbol{e}$ and $\boldsymbol{f}$ by operator $e^{-\frac{\partial_x^2+\partial_y^2}{2}}$ yields:

$$\boldsymbol{h}' = \frac{1}{2}\left(\mathbb{D}_H(x) + \mathbb{D}_H(y) + 1\right) - \frac{b}{\sqrt{ac}}\partial_x\partial_y \tag{46}$$

$$\boldsymbol{e}' = -\frac{b}{\sqrt{ac}}\partial_x\partial_y$$

$$\boldsymbol{f}' = -\frac{\sqrt{ac}}{2b}(x-D)(y-D) + \frac{1}{2}\left[\mathbb{D}_H(x) + \mathbb{D}_H(y)\right] - \frac{b}{4\sqrt{ac}}\partial_x\partial_y$$

The bivariate Hermite polynomials $H_{n,m}(x,y)$ are eigenfunctions of $\boldsymbol{h}'$ with eigenvalues $\frac{1}{2}(m+n-1)$.

Lowering operator in this algebra is given by:

$$A^- = \boldsymbol{e}' = -\frac{b}{\sqrt{ac}}\, \partial_x\partial_y \tag{47}$$

$\boldsymbol{h}'$ represents the Cartan subalgebra of related Lie algebra. On of the commutator relations is

$$[\boldsymbol{h}', A^-] = -A^- = \frac{b}{\sqrt{ac}}\partial_x\partial_y \tag{48}$$

4.2. Due to a theorem for BCH formula, if $[X, Y] = sY$ *then we have:*

$$e^X e^Y = e^{X+\frac{s}{1-e^{-s}}Y} \tag{49}$$

we can modify $\boldsymbol{h}$ and $\boldsymbol{e}$ in such a way that BCH formula simplified to equations that gives rise to new relations of Hermite polynomials. If we assume $\boldsymbol{h}$ and $\boldsymbol{e}$ in a modified from

$$\tilde{\boldsymbol{h}} = x\partial_x + y\partial_y - \frac{2b}{\sqrt{ac}}\partial_x\partial_y \quad , \quad \tilde{\boldsymbol{e}} = \frac{-2(1-e)b}{\sqrt{ac}}\partial_x\partial_y \tag{50}$$

Respect to the commutation relation

$$[\tilde{\boldsymbol{h}}, \tilde{\boldsymbol{e}}] = -\tilde{\boldsymbol{e}} = \frac{2(1-e)b}{\sqrt{ac}}\partial_x\partial_y \tag{51}$$

Then, we have

$$\exp\tilde{\boldsymbol{h}}\exp\tilde{\boldsymbol{e}} = \exp\left(\tilde{\boldsymbol{h}} - \frac{\tilde{\boldsymbol{e}}}{1-e}\right) = \exp\left(\tilde{\boldsymbol{h}} + \frac{2b}{\sqrt{ac}}\partial_x\partial_y\right) \tag{52}$$

$$\exp(x\partial_x + y\partial_y - \frac{2b}{\sqrt{ac}}\partial_x\partial_y)\exp(\frac{-2(1-e)b}{\sqrt{ac}}\partial_x\partial_y) = \exp(x\partial_x + y\partial_y - \frac{2b}{\sqrt{ac}}\partial_x\partial_y + \frac{2b}{\sqrt{ac}}\partial_x\partial_y)$$

$$\exp(x\partial_x + y\partial_y - \frac{2b}{\sqrt{ac}}\partial_x\partial_y)\exp(\frac{-2(1-e)b}{\sqrt{ac}}\partial_x\partial_y) = \exp(x\partial_x + y\partial_y)$$

$$\exp(\tilde{\boldsymbol{h}})\exp(\frac{-2b(1-e)}{\sqrt{ac}}\partial_x\partial_y) = \exp(x\partial_x + y\partial_y) \tag{53}$$

Similarity transformation of both side with $e^{-\frac{D_x^2+D_y^2}{2}}$ yields

$$\left(e^{-\frac{\partial_x^2+\partial_y^2}{2}} e^{\tilde{\boldsymbol{h}}} e^{\frac{\partial_x^2+\partial_y^2}{2}}\right)\left(e^{-\frac{\partial_x^2+\partial_y^2}{2}} e^{(\frac{-2b(1-e^2)}{\sqrt{ac}}\partial_x\partial_y)} e^{\frac{\partial_x^2+\partial_y^2}{2}}\right) = e^{-\frac{\partial_x^2+\partial_y^2}{2}} e^{(xD_x+yD_y)} e^{\frac{\partial_x^2+\partial_y^2}{2}}$$

$$e^{\mathfrak{D}} e^{(\frac{-b(1-e^2)}{\sqrt{ac}}\partial_x\partial_y)} = e^{(\mathbb{D}_{H(x)}+\mathbb{D}_{H(y)})} \tag{54}$$

Where we used $\mathbb{D}_{H(x)} = e^{-\frac{\partial_x^2}{2}}(x\partial_x)e^{\frac{\partial_x^2}{2}}$ and $\mathbb{D}_{H(y)} = e^{-\frac{\partial_y^2}{2}}(y\partial_y)e^{\frac{\partial_y^2}{2}}$ and

$$\mathfrak{D} = e^{-\frac{\partial_x^2+\partial_y^2}{2}}\tilde{\boldsymbol{h}}\, e^{-\frac{\partial_x^2+\partial_y^2}{2}} \tag{55}$$

a. Multiplying both sides of (54) by $H_n^e(x)H_m^e(y)$ yields

$$e^{\mathfrak{D}} e^{\left(\frac{-b(1-e^2)}{\sqrt{ac}}\partial_x\partial_y\right)} H_n^e(x)H_m^e(y) = e^{(\mathbb{D}_{H(x)}+\mathbb{D}_{H(y)})} H_n^e(x)H_m^e(y) \tag{56}$$

$$e^{\mathfrak{D}} e^{\left(\frac{-b(1-e^2)}{\sqrt{ac}}\partial_x\partial_y\right)} H_n^e(x)H_m^e(y) = e^{m+n} H_n^e(x)H_m^e(y) \tag{57}$$

$$e^{\mathfrak{D}} \sum_k^n \sum_{k'}^m \binom{m}{k'}\binom{n}{k}\left(\frac{-b(1-e^2)}{\sqrt{ac}}\right)^{k+k'} H_{n-k}^e(x)H_{n-k'}^e(y) = e^{m+n} H_n^e(x)H_m^e(y) \tag{58}$$

where $x\partial_x-\partial_x^2$ and $y\partial_y-\partial_y^2$ denoted as $\mathbb{D}_{H(x)}$ and $\mathbb{D}_{H(y)}$ respectively. These are Hermite differential operators with single variable. Substitution of $H_{n-k}^e(x)$ and $H_{n-k'}^e(y)$ through equation (31) gives:

$$e^{\mathfrak{D}}\sum_k^n\sum_{k'}^m\binom{m}{k'}\binom{n}{k}\left(\frac{-b(1-e^2)}{\sqrt{ac}}\right)^{k+k'}(e^{\frac{-\partial_x^2}{2}}x^{m-k})(e^{\frac{-\partial_y^2}{2}}y^{n-k'})=e^{m+n}H_n^e(x)H_m^e(y) \tag{59}$$

$$e^{\mathfrak{D}}e^{\frac{-\partial_x^2-\partial_y^2}{2}}\sum_k^n\sum_{k'}^m\binom{m}{k'}\binom{n}{k}\left(\frac{-b(1-e^2)}{\sqrt{ac}}\right)^{k+k'}x^{m-k}y^{n-k'}=e^{m+n}H_n^e(x)H_m^e(y) \tag{60}$$

$$e^{\mathfrak{D}}e^{\frac{-\partial_x^2-\partial_y^2}{2}}\sum_k^n\left[\binom{n}{k}\left(\frac{b(e^2-1)}{\sqrt{ac}}\right)^k x^{m-k}\right]\sum_{k'}^m\left[\binom{m}{k'}\left(\frac{b(e^2-1)}{\sqrt{ac}}\right)^{k'}y^{n-k'}\right]=e^{m+n}H_n^e(x)H_m^e(y) \tag{61}$$

$$e^{\mathfrak{D}}e^{\frac{-\partial_x^2-\partial_y^2}{2}}\left(x+\frac{b(e^2-1)}{\sqrt{ac}}\right)^n\left(y+\frac{b(e^2-1)}{\sqrt{ac}}\right)^m=e^{m+n}H_n^e(x)H_m^e(y) \tag{62}$$

By changing the variables

$$X=x+\frac{b(e^2-1)}{\sqrt{ac}}\quad,\quad Y=y+\frac{b(e^2-1)}{\sqrt{ac}} \tag{63}$$

we have

$$e^{\mathfrak{D}}e^{\frac{-\partial_x^2-\partial_y^2}{2}}X^nY^m=e^{m+n}H_n^e\left(X-\frac{b(e^2-1)}{\sqrt{ac}}\right)H_m^e\left(Y-\frac{b(e^2-1)}{\sqrt{ac}}\right) \tag{64}$$

Taking into account the identities

$$\partial_x=\partial_X\quad,\quad\partial_y=\partial_Y \tag{65}$$

we get

$$e^{\mathfrak{D}}e^{\frac{-\partial_X^2-\partial_Y^2}{2}}X^nY^m=e^{m+n}H_n^e\left(X-\frac{b(e^2-1)}{\sqrt{ac}}\right)H_m^e\left(Y-\frac{b(e^2-1)}{\sqrt{ac}}\right) \tag{66}$$

$$e^{\mathfrak{D}}e^{\frac{-\partial_Y^2}{2}}H_n^e(X)Y^m=e^{m+n}H_n^e\left(X-\frac{b(e^2-1)}{\sqrt{ac}}\right)H_m^e\left(Y-\frac{b(e^2-1)}{\sqrt{ac}}\right) \tag{67}$$

Let denote $t=Y$

$$e^{\mathfrak{D}}e^{\frac{-\partial_Y^2}{2}}H_n^e(X)t^m=e^{m+n}H_n^e\left(X-\frac{b(e^2-1)}{\sqrt{ac}}\right)H_m^e\left(t-\frac{b(e^2-1)}{\sqrt{ac}}\right) \tag{68}$$

or

$$e^{-(m+n)}e^{\mathfrak{D}}H_n^e(X)H_m^e(Y)=H_n^e\left(X-\frac{b(e^2-1)}{\sqrt{ac}}\right)H_m^e\left(Y-\frac{b(e^2-1)}{\sqrt{ac}}\right) \tag{69}$$

Thus, the operator $e^{-(m+n)}e^{\mathfrak{D}}$ acts as a shift operator for $H_n^e(X)H_n^e(Y)$.

b. Applying the Equation (54)

$$e^{\mathfrak{D}}\,e^{\left(\frac{-b(1-e^2)}{\sqrt{ac}}\partial_x\partial_y\right)}=e^{(\mathbb{D}_{H(x)}+\mathbb{D}_{H(y)})} \tag{70}$$

Multiplying both side by $u_{n,m}(x,y)$ gives

$$e^{\mathfrak{D}}\,e^{\left(\frac{-b(1-e^2)}{\sqrt{ac}}\partial_x\partial_y\right)}u_{n,m}(x,y)=e^{(\mathbb{D}_{H(x)}+\mathbb{D}_{H(y)})}u_{n,m}(x,y) \tag{71}$$

$$e^{\left(\frac{-b(1-e^2)}{\sqrt{ac}}\partial_x\partial_y\right)}u_{n,m}(x,y)=e^{-\mathfrak{D}}H_{n,m}(x,y)$$

$$e^{(\frac{-b(1-e^2)}{\sqrt{ac}}\partial_x\partial_y)} u_{n,m}(x,y) = e^{-(n+m)} H_{n,m}(x,y)$$

$$e^{(m+n-\frac{b(1-e^2)}{\sqrt{ac}}\partial_x\partial_y)} u_{n,m}(x,y) = H_{n,m}(x,y) \quad (72)$$

Comparing this equation with

$$e^{-\frac{\partial_x^2+\partial_y^2}{2}} u_{n,m}(x,y) = H_{n,m}(x,y) \quad (73)$$

c. Multiplying both side by $e^{-\mathfrak{D}}$ from left and $H_{n,m}(x,y)$ from right yields.

$$e^{(\frac{-b(1-e^2)}{\sqrt{ac}}\partial_x\partial_y)} H_{n,m}(x,y) = e^{-\mathfrak{D}} e^{(\mathbb{D}_{H(x)}+\mathbb{D}_{H(y)})} H_{n,m}(x,y) \quad (74)$$

$$e^{(\frac{-b(1-e^2)}{\sqrt{ac}}\partial_x\partial_y)} H_{n,m}(x,y) =$$

$$e^{-\mathfrak{D}} \sum_{k=0}^{\min(n,m)} (-1)^k\, k! \binom{m}{k}\binom{n}{k} a^{(n-k)/2} b^k c^{(m-k)/2}\, e^{m+n-2k}\, H_{n-k}^e(x) H_{m-k}^e(y)$$

This equation could be read as:.

$$e^{(\frac{-b(1-e^2)}{\sqrt{ac}}\partial_x\partial_y)} H_{n,m}(x,y) = e^{-\mathfrak{D}} \sum_{k=0}^{\min(n,m)} (-1)^k\, k! \binom{m}{k}\binom{n}{k} (ae^2)^{(n-k)/2} b^k (ce^2)^{(m-k)/2}\, H_{n-k}^e(x) H_{m-k}^e(y) \quad (75)$$

The sum on the right side is a version of $H_{n,m}(x,y)$ with:

$$a' = ae^2 \quad , \quad b' = b \quad , \quad c' = ce^2 \quad (76)$$

5. General form of differential operator representation of $\mathfrak{sl}(2,R)$ and BCH formula

Denote $B(x)$ and its integers exponents form a set $[1, B(x), B^2(x), \dots, B^n(x)]$ of independent basis in polynomial space. Introducing the differential operator generators that construct an isomorphic algebra to $\mathfrak{sl}(2,R)$, defined as

$$\boldsymbol{h} = \frac{B}{B'} D - \frac{n}{2} \quad , \quad \boldsymbol{e} = \frac{D}{B'} \quad , \quad \boldsymbol{f} = \frac{B^2}{B'} D - nB \quad (77)$$

It is straight forward to prove these bases satisfy the commutation relations of $\mathfrak{sl}(2,R)$ in (7). We apply the specific case of BCH formula [6]:

$$e^{X_1} e^{X_2} = e^{X_1 + \frac{sX_2}{1-e^{-s}}} \quad (78)$$

When the generators X_1 and X_2 satisfy the commutation relation

$$[X_1, X_2] = sX_2 \quad (79)$$

with $s \in \mathbb{R}$.

Due to the commutation relation

$$[\boldsymbol{h}, \boldsymbol{e}] = -\boldsymbol{e} \quad (80)$$

with $s = -1$, and $\boldsymbol{e}' = (1-e)\boldsymbol{e}$ The commutation relation becomes:

$$[\boldsymbol{h}, \boldsymbol{e}'] = (1-e)[\boldsymbol{h}, \boldsymbol{e}] = -\boldsymbol{e}' = -(1-e)\boldsymbol{e} \quad (81)$$

BCH formula reads as:

$$\exp \boldsymbol{h} \exp \boldsymbol{e}' = \exp\left(\boldsymbol{h} - \frac{\boldsymbol{e}'}{1-e}\right) = \exp(\boldsymbol{h} - \boldsymbol{e}) \quad (82)$$

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If $\boldsymbol{h}$ is written as:

$$\boldsymbol{h} = \frac{B-1}{B\prime}D - \frac{n}{2} + \frac{D}{B'} = \frac{B-1}{B'}D - \frac{n}{2} + \boldsymbol{e} \tag{83}$$

Then the BCH formula reads as:

$$\exp\left(\frac{B}{B'}D - \frac{n}{2}\right)\exp\left((1-e)\frac{D}{B'}\right) = \exp\left(\frac{B-1}{B'}D - \frac{n}{2}\right)$$

$$\exp\left(\frac{B}{B\prime}D\right)\exp\left((1-e)\frac{D}{B\prime}\right) = \exp\left(\frac{B-1}{B'}D\right) \tag{84}$$

inverse of both side yields

$$\exp\left((e-1)\frac{D}{B\prime}\right)\exp\left(\frac{-B}{B\prime}D\right) = \exp\left(\frac{1-B}{B'}D\right) \tag{85}$$

Example :

For algebra of Laguerre differential operator, the equivalent generators to $\boldsymbol{h}$ and $\boldsymbol{e}'$ are:

$$Y_1 = \mathfrak{D}'_L - \frac{n}{2} \quad , \qquad Y_2 = (1-e)(\mathfrak{D}'_L - xD) \tag{86}$$

where the Laguerre differential operator $\mathfrak{D}'_L$ whose eigenfunctions are Laguerre polynomials is defined as:

$$\mathfrak{D}' = -(xD^2 - xD + D) \tag{87}$$

commutation relation reads as:

$$[Y_1\,, Y_2] = -Y_2 \tag{88}$$

Thus, for BCH formula we have:

$$\exp\left(\mathfrak{D}'_L - \frac{n}{2}\right)\exp(\mathfrak{D}'_L - xD) = \exp\left(\mathfrak{D}'_L - \frac{n}{2} - \mathfrak{D}'_L + xD\right)$$

$$\exp\left(\mathfrak{D}'_L - \frac{n}{2}\right)\exp(\mathfrak{D}'_L - xD) = \exp\left(-\frac{n}{2} + xD\right)$$

6. A new generating function for Hermite polynomials

Using operator $O = e^{\frac{-D_x^2}{2}}$ on the series $\sum_n t^n x^n = \frac{1}{1-xt}$ due to ( ) yields (89)

$$e^{\frac{-D_x^2}{2}}\sum_n t^n x^n = e^{\frac{-D_x^2}{2}}\left(\frac{1}{1-xt}\right) \tag{90}$$

$$g(x,t) = \sum_n t^n H_n^e = \sum_j (-1)^j \frac{(1-xt)^{-2j-1}(2j)!}{2^j j!} \tag{91}$$

Respect to the identity

$$\frac{(2j)!}{2^j j!} = \frac{2^j}{\sqrt{\pi}}\Gamma\left(j + \frac{1}{2}\right) \tag{92}$$

For $t = 1$ we have:

$$\sum_{n=1}^{\infty} H_n^e = \frac{1}{\sqrt{\pi}}\sum_j \left(\frac{\sqrt{2}}{1-x}\right)^{2j+1}\Gamma\left(j + \frac{1}{2}\right) \tag{93}$$

This series is convergent for $x < 0$.

Multiplying two sides by $e^{\frac{D_x^2}{2}}$ gives

$$e^{\frac{\partial_x^2}{2}} \sum_{n=1}^{\infty} H_n^e = e^{\frac{\partial_x^2}{2}} \frac{1}{\sqrt{\pi}(1-x)} \sum_j \left(\frac{\sqrt{2}}{1-x}\right)^{2j} \Gamma(j+\tfrac{1}{2}) \tag{94}$$

$$\sum_{n=1}^{\infty} x^n = e^{\frac{\partial_x^2}{2}} \frac{1}{\sqrt{\pi}} \sum_j \left(\frac{\sqrt{2}}{1-x}\right)^{2j+1} \Gamma\left(j+\frac{1}{2}\right) = \frac{1}{1-x} \qquad \text{for} \quad -1 \le x < 1$$

$$\frac{1}{\sqrt{\pi}} \sum_j \left(\frac{\sqrt{2}}{1-x}\right)^{2j+1} \Gamma\left(j+\frac{1}{2}\right) = e^{\frac{-\partial_x^2}{2}} \left(\frac{1}{1-x}\right) \tag{95}$$

Let denote $y = 1 - xt$, then the equation ( ) reads as

$$g(x,t) = (1-xt)^{-1} \sum_j (-1)^j \frac{(2j)!}{2^j j!} y^{-2j} \tag{96}$$

By the identity (92)

$$(2j-1)!! = \frac{(2j)!}{2^j j!} = \frac{2^j}{\sqrt{\pi}} \Gamma(j+\tfrac{1}{2}) \tag{97}$$

The equation (96) converts to

$$g(x,t) = \frac{1}{\sqrt{\pi} y} \sum_j (-1)^j 2^j \, \Gamma\left(j+\frac{1}{2}\right) y^{-2j} \tag{98}$$

$$g(x,t) = \frac{1}{\sqrt{\pi} y} \sum_j (-1)^j \, \Gamma\left(j+\frac{1}{2}\right) \left(\frac{y^2}{2}\right)^{-j} \tag{99}$$

By the identity for n-th derivative of $(1+z)^{-s}$ i.e.

$$\frac{d^n}{dz^n}(1+z)^{-s} = D_z^n (1+z)^{-s} = (-1)^n \frac{\Gamma(n+s)}{\Gamma(s)} (1+z)^{-n-s} \tag{100}$$

For $s = \frac{3}{2}$ we have:

$$D_z^n (1+z)^{-\frac{3}{2}} = (-1)^n \frac{\Gamma(n+\frac{3}{2})}{\Gamma(\frac{3}{2})} (1+z)^{-n-\frac{3}{2}} \tag{101}$$

$$D_z^n (1+z)^{-\frac{3}{2}} = \sum_n (-1)^n \frac{(1+z)^{\frac{-3}{2}}}{\Gamma\left(\frac{3}{2}\right)} \Gamma(n+\tfrac{3}{2})(1+z)^{-n} \tag{102}$$

$$\sum_n D_z^n (1+z)^{-\frac{3}{2}} = \frac{(1+z)^{\frac{-3}{2}}}{\Gamma\left(\frac{3}{2}\right)} \sum_n (-1)^n \Gamma(n+\tfrac{3}{2})(1+z)^{-n} \tag{103}$$

$$g(x,t) = \frac{1}{\sqrt{\pi} y} \sum_j (-1)^j \, \Gamma\left(j-1+\frac{3}{2}\right) \left(\frac{y^2}{2}\right)^{-j} \tag{104}$$

Denote $k = j - 1$

$$g(x,t) = \frac{1}{\sqrt{\pi} y} \sum_{k=-1} (-1)^{k+1} \, \Gamma\left(k+\frac{3}{2}\right) \left(\frac{y^2}{2}\right)^{-(k+1)} \tag{105}$$

$$g(x,t) = \frac{1}{\sqrt{\pi} y} \sum_{k=-1} (-1)^{k+1} \, \Gamma\left(k+\frac{3}{2}\right) \left(\frac{y^2}{2}\right)^{-(k+1)} \tag{106}$$

$$= \Gamma\left(\frac{1}{2}\right) + \frac{1}{\sqrt{\pi} y} \sum_{k=0} (-1)^{k+1} \, \Gamma\left(k+\frac{3}{2}\right) \left(\frac{y^2}{2}\right)^{-(k+1)}$$

$$= \Gamma\left(\frac{1}{2}\right) + \frac{-1}{\sqrt{\pi}y}\left(\frac{y^2}{2}\right)^{-1} \sum_{k=0} (-1)^k\, \Gamma\left(k+\frac{3}{2}\right)\left(\frac{y^2}{2}\right)^{-k} \qquad (107)$$

By $\frac{y^2}{2} = 1+z$ and identities $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, $\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$

$$g(x,t) = \sqrt{\pi} + \frac{-1}{\sqrt{2\pi}}(1+z)^{-\frac{3}{2}} \sum_{k=0} (-1)^k\, \Gamma\left(k+\frac{3}{2}\right)(1+z)^{-k} \qquad (108)$$

On the other hand, we have:

$$\sum_n D_z^n (1+z)^{-\frac{3}{2}} = \frac{1}{1-D_z}(1+z)^{-\frac{3}{2}} = \frac{(1+z)^{\frac{-3}{2}}}{\Gamma\left(\frac{3}{2}\right)} \sum_n (-1)^n \Gamma\left(n+\frac{3}{2}\right)(1+z)^{-n} \qquad (109)$$

Thus, we obtain:

$$g(x,t) = \sqrt{\pi} + \frac{-\frac{\sqrt{\pi}}{2}}{\sqrt{2\pi}} \frac{1}{1-D_z}(1+z)^{-\frac{3}{2}} = \sqrt{\pi} - \frac{1}{\sqrt{2}}\, e^z \int e^{-z}\,(1+z)^{-\frac{3}{2}} dz \qquad (110)$$

By the identity

$$\frac{1}{1-D_z}(1+z)^{-\frac{3}{2}} = e^z \int e^{-z}\,(1+z)^{-\frac{3}{2}} dz \qquad (111)$$

Calculation of the integral results in:

$$\int e^{-z}\,(1+z)^{-\frac{3}{2}} dz = 2e\Gamma\left(\frac{1}{2}, z+1\right) - \frac{2e^{-z}}{\sqrt{z+1}} \qquad (112)$$

Finally, we have:

$$g(x,t) = \sqrt{\pi} - \frac{1}{\sqrt{2}}\, e^z \Gamma\left(\frac{1}{2}, z+1\right) - \frac{2e^{-z}}{\sqrt{z+1}} \qquad (113)$$

Substitution of $\frac{(1-xt)^2}{2} = 1+z$, gives the explicit closed form of $g(x,t)$ in terms of x and t.

$$g(x,t) = \sqrt{\pi} - \frac{1}{\sqrt{2}}\, e^{\left[\frac{(1-xt)^2}{2}\right]} \Gamma\left(\frac{1}{2}, \frac{(1-xt)^2}{2}\right) - \frac{2\sqrt{2}e^{\left[1-\frac{(1-xt)^2}{2}\right]}}{(1-xt)} \qquad (114)$$

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