Article

On an Indefinite Metric on a 4-Dimensional Riemannian Manifold

Dimitar Razpopov [1], Georgi Dzhelepov [2] and Iva Dokuzova [3]*

[1 2] Department of Mathematics and Informatics, Faculty of Economics, Agricultural University of Plovdiv, 12 Mendeleev Blvd., 4000 Plovdiv, Bulgaria; razpopov@au-plovdiv.bg, dzhelepov@abv.bg
[3] Department of Algebra and Geometry, Faculty of Mathematics and Informatics, University of Plovdiv Paisii Hilendarski, 24 Tzar Asen, 4000 Plovdiv, Bulgaria; dokuzova@uni-plovdiv.bg
* Correspondence: dokuzova@uni-plovdiv.bg
† Current address: Affiliation 3.
‡ These authors contributed equally to this work.

Abstract: Our research is in the tangent space of a point on a 4-dimensional Riemannian manifold. Besides the positive definite metric, the manifold is endowed with a tensor structure of type $(1,1)$, whose fourth power is minus the identity. Both structures are compatible and they define an indefinite metric on the manifold. With the help of the indefinite metric we determine a circle in different 2-planes in the tangent space on the manifold, we also calculate the length and area of the circle. On a smooth closed curve such as a circle, we define a vector force field. Further, we obtain the circulation done by the vector force field along the curve, as well as the flux of the curl of this vector force field across the curve. Finally, we find a relation between these two values, which is an analogue of the well known Green's formula in the Euclidean space.

Keywords: Riemannian manifold; indefinite metric tensor; length; area; Green's formula

MSC: 53B30; 53A04; 26B15; 26B20

1. Introduction

If k is a simple closed curve in a plane, then it surrounds some region in the plane. Green's theorem transforms the line integral around k into a double integral over the region inside k. In physics, it gives a relationship between the circulation $C = \oint_k F.ds$ of the vector force field F around the path k and the flux, done by *curl* F, across the region inside k.

Green's theorem is a special case of Stokes' theorem. Both theorems are widely used during the study of electric and magnetic fields. The modern approach to these theorems on manifolds using differential forms is exhibited, for example, in [2,3,9,12,13]. A theorem similar to the theorem of Green, in a special 2-plane of the tangent space on a 3-dimensional Riemannian manifold with circulant structures, is obtained in [5].

We consider a 4-dimensional Riemannian manifold M with an additional tensor field S of type $(1,1)$, whose fourth power is minus the identity. The structure S is compatible with the metric g such that an isometry is induced in every tangent space T_pM on M. Both structures g and S define an indefinite metric $\tilde{g}$ [4]. The metric $\tilde{g}$ determines space-like, isotropic and time-like vectors in T_pM. In special 2-planes β_i of T_pM, constructed on space-like and time-like vectors, we consider circles k_i with respect to $\tilde{g}$. We calculate their length and area (with respect to $\tilde{g}$), which in some cases are imaginary or negative numbers. It turns out that these measures are the same as in the Euclidean space. We note that some problems related to circles, their length or area, considered in terms of indefinite metrics, are given in [1,7,8,10,11]. Finally, we obtain analogues of Green's theorem that give a relation between circulation of the vector force field F around a closed curve (in particular a circle) k_i in β_i and the flux, done by the curl of F, across the region inside k_i.

The paper is organized as follows. In Section 2, we give some facts, definitions and statements, which are necessary for the present considerations. Some of them are obtained in [4], [6] and [14]. In Section 3, we introduce a special 2-plane β_1 of T_pM and determine an equation of a circle k_1 with respect to $\tilde{g}$. In Subsections 3.1 and 3.2 we calculate the length and the area of k_1. In Subsection 3.3, we find the circulation of a vector force field F around smooth closed curve k_1 and the flux, done by the curl of F, across the region inside k_1. In Section 4, we introduce a 2-plane β_2 of T_pM and determine an equation of a circle k_2 with respect to $\tilde{g}$. Further we calculate the length and the area of k_2. We get the circulation of a vector force field F around a smooth closed curve k_2 and the flux, done by the curl of F, across the region inside k_2. We find a relationship between the circulation and the flux in both cases. All values obtained in Sections 3 and Sections 4 are calculated with respect to $\tilde{g}$.

2. Preliminaries

We consider a 4-dimensional Riemannian manifold M with an additional tensor structure S of type $(1,1)$. In a local coordinate system (x^1, x^2, x^3, x^4) the coordinates of S form the following circulant matrix:

$$S = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

Thus we have

$$S^4 = -\mathrm{id}. \tag{1}$$

Let g be a positive definite metric on M, which satisfies the equality

$$g(Su, Sv) = g(u, v), \quad u,\ v \in \mathfrak{X}M. \tag{2}$$

Such a manifold (M, g, S) is introduced in [4].

Further u, v, w, e_1, e_2 will stand for arbitrary smooth vector fields on M or arbitrary vectors in the tangent space T_pM, $p \in M$.

Let the vector u induce a basis of type $\{S^3u, S^2u, Su, u\}$. In [4] it is called an S-basis and the following statements about the angles between the basis vectors are obtained.

(i) The angle φ, determined by $\varphi = \angle(u, Su)$, satisfies inequalities

$$\frac{\pi}{4} < \varphi < \frac{3\pi}{4}.$$

(ii) For the angles between the basis vectors we have

$$\begin{gathered} \angle(u, Su) = \angle(Su, S^2u) = \angle(S^2u, S^3u) = \varphi, \\ \angle(S^3u, u) = \pi - \varphi,\ \angle(u, S^2u) = \angle(Su, S^3u) = \tfrac{\pi}{2}. \end{gathered} \tag{3}$$

The associated metric $\tilde{g}$ on (M, g, S), determined by

$$\tilde{g}(u, v) = g(u, Sv) + g(Su, v), \tag{4}$$

is necessary indefinite ([4]). Consequently, for an arbitrary vector v it is valid:

$$\tilde{g}(v, v) = 2g(v, Sv) = R^2,\ R^2 \in \mathbb{R}. \tag{5}$$

The norm of every vector u and the cosine of φ are given by the following equalities:

$$\|u\| = \sqrt{g(u, u)}, \quad \cos\varphi = \frac{g(u, Su)}{g(u, u)}. \tag{6}$$

 doi:10.20944/preprints202303.0223.v1

In rest of this paper, we assume that $\|u\| = 1$ and using (6) we have

$$\cos\varphi = g(u, Su). \tag{7}$$

Due to (2), (4), (6) and (7) we state that the normal basis $\{S^3u, S^2u, Su, u\}$ satisfies the following equalities:

$$\begin{aligned}
&\tilde{g}(u,u) = \tilde{g}(Su,Su) = \tilde{g}(S^2u,S^2u) = \tilde{g}(S^3u,S^3u) = 2\cos\varphi,\\
&\tilde{g}(u,Su) = \tilde{g}(Su,S^2u) = \tilde{g}(S^2u,S^3u) = -\tilde{g}(S^3u,u) = 1.\\
&\tilde{g}(u,S^2u) = \tilde{g}(Su,S^3u) = 0.
\end{aligned} \tag{8}$$

A circle k in a 2-plane of T_pM of a radius R centered at the origin $p \in T_pM$, with respect to the associated metric $\tilde{g}$ on (M, g, S), is determined by (5), where v is the radius vector of an arbitrary point on k.

Farther, we consider circles k_1 and k_2, and the regions D_1 and D_2 inside them, in two different subspaces β_1 and β_2, spanned by 2-planes $\{u, S^2u\}$ and $\{u, Su\}$, respectively.

3. Circles in the 2-plane β_1

Because of (3), it is true that the vectors u and S^2u form an orthonormal basis of β_1. The coordinate system p_{xy} on β_1, such that u is on the axis p_x and S^2u is on the axis p_y, is an orthonormal coordinate system of β_1.

A circle k_1 in β_1 centered at the origin p, with respect to $\tilde{g}$ on (M, g, S), is defined by (5). The equation of k_1 with respect to p_{xy} is obtained by the following

Theorem 1. *[6] Let $\tilde{g}$ be the associated metric on (M, g, S) and let β_1 be a 2-plane in T_pM with a basis $\{u, S^2u\}$. If p_{xy} is a coordinate system such that $u \in p_x$, $S^2u \in p_y$, then the equation of the circle (5) in β_1 is given by*

$$2\cos\varphi x^2 + 2\cos\varphi y^2 = R^2. \tag{9}$$

The curve k_1, determined by (9), is a circle in terms of g if:

Case (A) $\varphi \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ and $R^2 > 0$;

Case (B) $\varphi \in \left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$ and $R^2 < 0$.

3.1. *Length of a circle with respect to $\tilde{g}$*

Firstly, we consider Case (A). The circle (5) has a radius $R > 0$ and φ satisfies

$$\frac{\pi}{4} < \varphi < \frac{\pi}{2}. \tag{10}$$

Theorem 2. *The circle k_1 with (10) and a radius $R > 0$ has a length*

$$L = 2\pi R. \tag{11}$$

Proof. Let $v = xu + yS^2u$ be a radius vector of an arbitrary point on the circle k_1. Then $\mathrm{d}v = \mathrm{d}xu + \mathrm{d}yS^2u$ is a tangent vector on k_1. The length L of k_1 with respect to $\tilde{g}$ is determined as usual by

$$\mathrm{d}L = \sqrt{\tilde{g}(\mathrm{d}v, \mathrm{d}v)}.$$

Then, using (8) and (9), we obtain

$$L = \oint_{k_1} \sqrt{2\cos\varphi \mathrm{d}x^2 + 2\cos\varphi \mathrm{d}y^2}. \tag{12}$$

We substitute

$$x = \frac{R}{\sqrt{2\cos\varphi}}\cos t,\ y = \frac{R}{\sqrt{2\cos\varphi}}\sin t,\ t \in [0, 2\pi],$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 13 March 2023 doi:10.20944/preprints202303.0223.v1

into (12) and find (11). □

Now, we consider Case (B). The circle k_1 has a radius $R = ri,\ r > 0,\ i^2 = -1$ and φ satisfies

$$\frac{\pi}{2} < \varphi < \frac{3\pi}{4}. \tag{13}$$

Therefore, the equation (9) transforms into

$$2\cos\varphi x^2 + 2\cos\varphi y^2 = -r^2. \tag{14}$$

By calculations similar to Case (A), but considering that R is an imaginary number, we obtain that the circle k_1 with (13) has an imaginary length (11).

3.2. Area of a circle with respect $\tilde{g}$

For Case (A) we state the following

Theorem 3. *The area* $\mathtt{A}$ *of the circle* k_1 *with* (10) *and a radius* $R > 0$ *is*

$$\mathtt{A} = \pi R^2. \tag{15}$$

Proof. We denote by $\widetilde{\cos}\angle(u, S^2u)$ and $\widetilde{\sin}\angle(u, S^2u)$ the cosine and the sine of the angle $\angle(u, S^2u)$ with respect to $\tilde{g}$. Considering $\tilde{g}(u, S^2u) = 0$ (presented in (8)), we have

$$\widetilde{\cos}\angle(u, S^2u) = 0,$$

and hence

$$\widetilde{\sin}\angle(u, S^2u) = 1. \tag{16}$$

In the coordinate plane p_{xy}, we construct a parallelogram with locus vectors $\mathrm{d}xu$ and $\mathrm{d}ySu$. For its area $\mathtt{A}$ with respect to $\tilde{g}$ we get

$$\mathrm{d}\mathtt{A} = \sqrt{\tilde{g}(\mathrm{d}xu, \mathrm{d}xu)}\sqrt{\tilde{g}(\mathrm{d}ySu, \mathrm{d}ySu)}\widetilde{\sin}\angle(u, S^2u).$$

We apply (8) and (16) in the latter equality and find

$$\mathrm{d}\mathtt{A} = 2\cos\varphi \mathrm{d}x\mathrm{d}y. \tag{17}$$

We integrate (17) over the region D_1 inside k_1 and calculate

$$\mathtt{A} = 2\cos\varphi \int\int_{D_1} \mathrm{d}x\mathrm{d}y, \tag{18}$$

with

$$D_1 :\ 2\cos\varphi x^2 + 2\cos\varphi y^2 \leq R^2.$$

We substitute

$$x = \frac{R}{\sqrt{2\cos\varphi}}\rho\cos t,\ y = \frac{R}{\sqrt{2\cos\varphi}}\rho\sin t,\ t \in [0, 2\pi],\ \rho \in [0, 1],$$

and Jacobian $\triangle = \frac{R^2}{2\cos\varphi}\rho$ into the integral (18) and obtain (15). □

Now, we consider Case (B). The circle k_1 has an equation (14) with conditions (13) and a radius $R = ri,\ r > 0,\ i^2 = -1$. By calculations similar to Case (A), we find that the area of k_1 is given by (15). In this case, $\mathtt{A}$ has a negative value.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 13 March 2023 doi:10.20944/preprints202303.0223.v1

3.3. Circulation and flux with respect to the metric $\tilde{g}$

We consider a closed curve k_1 in β_1, given by

$$x = x(t), \quad y = y(t), \quad t \in [\alpha, \beta], \tag{19}$$

where $x(\alpha) = x(\beta), y(\alpha) = y(\beta)$.

Let

$$F(x, y) = P(x, y)u + Q(x, y)S^2u \tag{20}$$

be a vector force field on the curve k_1.

For the circulation C of a vector field F along a curve k we assume the following definition

$$C = \oint_k \tilde{g}(F, \mathrm{d}v), \tag{21}$$

where v is the radius vector of a point on k.

We denote by D_1 the region inside k_1. For both cases (A) and (B) of the circle (9) the following statements are valid.

Theorem 4. *The circulation C, done by the force* (20) *along the curve* (19)*, is expressed by*

$$C = 2\cos\varphi \oint_{k_1} (P(x, y)\mathrm{d}x + Q(x, y)\mathrm{d}y), \tag{22}$$

where $\varphi \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$.

Proof. Let $v = xu + yS^2u$ be the radius vector of a point on k_1. By virtue of (8) and (20), and bearing in mind $\mathrm{d}v = \mathrm{d}xu + \mathrm{d}yS^2u$, we obtain

$$\tilde{g}(F, \mathrm{d}v) = 2\cos\varphi(P(x, y)\mathrm{d}x + Q(x, y)\mathrm{d}y). \tag{23}$$

Evidently (22) follows from (19), (21) and (23). □

We determine a vector w in T_pM by the equality

$$w = \frac{1}{\sqrt{1 - 2\cos^2\varphi}}\Big(\cos\varphi u - Su + \cos\varphi S^2u\Big), \tag{24}$$

where $\varphi \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$. By using (1), (2) and (7) it is easy to verify that

$$g(w, u) = g(w, S^2u) = 0, \quad g(w, w) = 1.$$

We construct an orthonormal coordinate system $Oxyz$, such that $u \in Ox$, $S^2u \in Oy$, $w \in Oz$.

We suppose that the curl of F, determined by (20), with respect to $Oxyz$ is

$$\mathrm{curl}F = (Q_x - P_y)w.$$

The flux T of the vector field *curl F* across the region D_1 inside the curve k_1 is given by

$$T = \int\int_{D_1} \tilde{g}(curlF, w)\mathtt{d}\mathtt{A}. \tag{25}$$

With the help of (8) and (24) we get $\tilde{g}(w, w) = -2\cos\varphi$. Then from (17) and (25) we state the following

Theorem 5. *The flux T of the vector field curlF across the region D_1 inside* (19) *is expressed by*

$$T = -4\cos^2\varphi \int\int_{D_1} (Q_x - P_y)\mathrm{d}x\mathrm{d}y, \tag{26}$$

where $\varphi \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$.

On the other hand, due to Green's formula, we have

$$\int\int_D (Q_x - P_y)\mathrm{d}x\mathrm{d}y = \oint_k (P\mathrm{d}x + Q\mathrm{d}y).$$

Bearing in mind the above formula we obtain the following statement.

Theorem 6. *The relation between the circulation* (22) *and the flux* (26) *is determined by*

$$T = -2\cos\varphi C.$$

Corollary 1. *The relation between the circulation C and the flux T is*

a) $T = -C$, *in case* $\varphi = \frac{\pi}{3}$;

b) $T = C$, *in case* $\varphi = \frac{2\pi}{3}$.

4. Circles in the 2-plane β_2

Lemma 1. *[6] Let* β_2 *be the* 2*-plane spanned by unit vectors* u *and* Su*. The system of vectors* $\{e_1, e_2\}$*, determined by the equalities*

$$e_1 = \frac{1}{\sqrt{2(1+\cos\varphi)}}(u + Su), \quad e_2 = \frac{1}{\sqrt{2(1-\cos\varphi)}}(-u + Su), \tag{27}$$

is an orthonormal basis of β_2 *with respect to* g.

The coordinate system p_{xy} on β_2, such that e_1 is on the axis p_x and e_2 is on the axis p_y, is orthonormal. Due to (8) we obtain that the system $\{e_1, e_2\}$ satisfies the following equalities:

$$\tilde{g}(e_1, e_1) = \frac{2\cos\varphi + 1}{1+\cos\varphi}, \quad \tilde{g}(e_2, e_2) = \frac{2\cos\varphi - 1}{1-\cos\varphi}, \quad \tilde{g}(e_1, e_2) = 0. \tag{28}$$

A circle k_2 in β_2 centered at the origin p, with respect to $\tilde{g}$ on (M, g, S), is defined by (5). The equation of k_2 with respect to p_{xy} is obtained in the following

Theorem 7. *[6] Let* $\tilde{g}$ *be the associated metric on* (M, g, S) *and let* $\beta_2 = \{u, Su\}$ *be a* 2*-plane in* T_pM *with an orthonormal basis* (27)*. If* p_{xy} *is a coordinate system such that* $e_1 \in p_x$, $e_2 \in p_y$*, then the equation of the circle* (5) *in* β_2 *is given by*

$$\frac{2\cos\varphi + 1}{1+\cos\varphi}x^2 + \frac{2\cos\varphi - 1}{1-\cos\varphi}y^2 = R^2. \tag{29}$$

The curve k_2, determined by (29), is an ellipse in terms of g if:

Case (A) $\varphi \in \left(\frac{\pi}{4}, \frac{\pi}{3}\right)$ and $R^2 > 0$;

Case (B) $\varphi \in \left(\frac{2\pi}{3}, \frac{3\pi}{4}\right)$ and $R^2 < 0$.

Firstly, we consider Case (A). The circle (5) has a radius $R > 0$ and φ satisfies

$$\frac{\pi}{4} < \varphi < \frac{\pi}{3}. \tag{30}$$

Theorem 8. *The circle* k_2 *with* (30) *and a radius* $R > 0$ *has a length*

$$L = 2\pi R. \tag{31}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 13 March 2023 doi:10.20944/preprints202303.0223.v1

Proof. The radius vector v of an arbitrary point on the curve k_2 is $v = xe_1 + ye_2$. Then $\mathrm{d}v = \mathrm{d}xe_1 + \mathrm{d}ye_2$ is a tangent vector on k_2. The length L of k_2 with respect to $\tilde{g}$ is

$$\mathrm{d}L = \sqrt{\tilde{g}(\mathrm{d}v, \mathrm{d}v)}.$$

From (28) we find

$$\tilde{g}(\mathrm{d}v, \mathrm{d}v) = \sqrt{\frac{2\cos\varphi + 1}{1 + \cos\varphi}\mathrm{d}x^2 + \frac{2\cos\varphi - 1}{1 - \cos\varphi}\mathrm{d}y^2}.$$

Then we obtain 140

$$L = \oint_{k_2} \sqrt{\frac{2\cos\varphi + 1}{1 + \cos\varphi}\mathrm{d}x^2 + \frac{2\cos\varphi - 1}{1 - \cos\varphi}\mathrm{d}y^2}. \quad (32)$$

We substitute

$$x = R\sqrt{\frac{1 + \cos\varphi}{2\cos\varphi + 1}}\cos t, \; y = R\sqrt{\frac{1 - \cos\varphi}{2\cos\varphi - 1}}\sin t, \; t \in [0, 2\pi]$$

into (32) and get

$$L = \int_0^{2\pi} \sqrt{R^2 \sin^2 t + R^2 \cos^2 t} dt,$$

which implies (31). □ 141

Now, we consider Case (B). The circle k_2 has a radius $R = ri$, $r > 0$, $i^2 = -1$ and φ 142
satisfies 143

$$\frac{2\pi}{3} < \varphi < \frac{3\pi}{4}. \quad (33)$$

By calculations similar to Case (A), but taking into account that R is an imaginary number, 144
we find that the circle k_2 with (33) has an imaginary length (31). 145

4.1. Area of a circle with respect $\tilde{g}$ 146

For Case (A) we state the following 147

Theorem 9. *The area* $\mathtt{A}$ *of the circle* k_2 *with* (30) *and radius* $R > 0$ *is* 148

$$\mathtt{A} = \pi R^2. \quad (34)$$

Proof. Let us denote $\theta = \angle(e_1, e_2)$. The cosine of θ with respect to $\tilde{g}$ is

$$\widetilde{\cos}\theta = \frac{\tilde{g}(e_1, e_2)}{\sqrt{\tilde{g}(e_1, e_1)}\sqrt{\tilde{g}(e_2, e_2)}}.$$

Then, using (28), we get $\widetilde{\cos}\theta = 0$, which implies 149

$$\widetilde{\sin}\theta = 1. \quad (35)$$

In the coordinate plane p_{xy}, we construct a parallelogram with locus vectors $\mathrm{d}xe_1$ and $\mathrm{d}ye_2$. For its area $\mathtt{A}$ with respect to $\tilde{g}$ we get

$$\mathrm{d}\mathtt{A} = \sqrt{\tilde{g}(\mathrm{d}xe_1, \mathrm{d}xe_1)}\sqrt{\tilde{g}(\mathrm{d}ye_2, \mathrm{d}ye_2)}\widetilde{\sin}\theta.$$

We apply (28) and (35) in the above equality and get 150

$$\mathrm{d}\mathtt{A} = \frac{\sqrt{4\cos^2\varphi - 1}}{\sin\varphi}\mathrm{d}x\mathrm{d}y. \quad (36)$$

 doi:10.20944/preprints202303.0223.v1

We integrate (36) over the region D_2 inside k_2 and calculate

$$\mathtt{A} = \frac{\sqrt{4\cos^2\varphi - 1}}{\sin\varphi} \int\int_{D_2} \mathrm{d}x\mathrm{d}y, \tag{37}$$

with

$$D_2 : \frac{2\cos\varphi + 1}{1+\cos\varphi}x^2 + \frac{2\cos\varphi - 1}{1-\cos\varphi}y^2 \le R^2.$$

We substitute

$$x = \sqrt{\frac{1+\cos\varphi}{2\cos\varphi + 1}}R\rho\cos t,\ y = \sqrt{\frac{1-\cos\varphi}{2\cos\varphi - 1}}R\rho\sin t,\ t \in [0, 2\pi],\ \rho \in [0,1],$$

and Jacobian $\triangle = \dfrac{\sin\varphi}{\sqrt{4\cos^2\varphi - 1}}R^2\rho$ into (37). Finally we get (34). □

Now, we consider Case (B). The circle k_2 has an equation (29) with conditions (33) and a radius $R = ri,\ r > 0,\ i^2 = -1$. By calculations analogous to the previous case, we obtain that the area of k_2 is given in (34). This area $\mathtt{A}$ has a negative value.

4.2. Circulation and flux

We consider a closed curve k_2 in β_2, given by

$$x = x(t), \quad y = y(t), \quad t \in [\alpha, \beta], \tag{38}$$

where $x(\alpha) = x(\beta), y(\alpha) = y(\beta)$.

Let

$$F(x,y) = P(x,y)e_1 + Q(x,y)e_2 \tag{39}$$

be a vector force field on the curve k_2.

We denote by D_2 the region inside k_2. For both cases (A) and (B) of the ellipse (29) the following statements are valid.

Theorem 10. *The circulation C done by the force (39) along the curve (38) is expressed by*

$$\mathsf{C} = \oint_{k_2}\left(\frac{2\cos\varphi+1}{1+\cos\varphi}P(x,y)\mathrm{d}x + \frac{2\cos\varphi-1}{1-\cos\varphi}Q(x,y)\mathrm{d}y\right), \tag{40}$$

where $\varphi \in \left(\frac{\pi}{3}, \frac{\pi}{4}\right) \cup \left(\frac{2\pi}{3}, \frac{3\pi}{4}\right)$.

Proof. For the circulation C of a vector force field F acting along the curve (38) we use (21), where $v = xe_1 + ye_2$ is the radius vector of a point on k_2. Therefore we have

$$\mathsf{C} = \oint_{k_2} \tilde{g}(F, \mathrm{d}v), \tag{41}$$

with a tangent vector $\mathrm{d}v = \mathrm{d}xe_1 + \mathrm{d}ye_2$ on k_2. Then, by virtue of (28) and (39), we obtain

$$\tilde{g}(F, \mathrm{d}v) = \left(\frac{2\cos\varphi+1}{1+\cos\varphi}P(x,y)\mathrm{d}x + \frac{2\cos\varphi-1}{1-\cos\varphi}Q(x,y)\mathrm{d}y\right). \tag{42}$$

Hence (38), (41) and (42) imply (40). □

Theorem 11. *The flux T of the vector field curl$F$ across the region D_2 inside the curve (38) is expressed by*

$$T = -2\cot^3\varphi\sqrt{4\cos^2\varphi - 1}\int\int_{D_2}(Q_x - P_y)\mathrm{d}x\mathrm{d}y, \tag{43}$$

where $\varphi \in \left(\frac{\pi}{3}, \frac{\pi}{4}\right) \cup \left(\frac{2\pi}{3}, \frac{3\pi}{4}\right)$.

Proof. We determine a vector w in T_pM by the equality

$$w = \frac{1}{\sin\varphi\sqrt{1 - 2\cos^2\varphi}}\Big(\cos^2\varphi u - (\cos\varphi)Su + \sin^2\varphi S^2 u\Big), \tag{44}$$

where $\varphi \in \left(\frac{\pi}{3}, \frac{\pi}{4}\right) \cup \left(\frac{2\pi}{3}, \frac{3\pi}{4}\right)$. Using (1), (2), (7) and (27) we verify that

$$g(w, e_1) = g(w, e_2) = 0, \; g(w, w) = 1.$$

The coordinate system $Oxyz$, such that $e_1 \in Ox$, $e_2 \in Oy$, $w \in Oz$ is orthonormal.

We obtain the curl of F, determined by (39), using the equality curl$F = (Q_x - P_y)w$. For the flux T of the vector field *curl F* across the region D_2 inside the curve (38) we have

$$T = \int\int_{D_2} \tilde{g}(curlF, w)\mathrm{dA}. \tag{45}$$

With the help of (28) and (44) we calculate

$$\tilde{g}(w, w) = -\frac{2\cos^3\varphi}{\sin^2\varphi}.$$

Then, from (36) and (45), it follows (43). □

Now, we introduce the following notations:

$$c_1 = \frac{2\cos\varphi + 1}{1 + \cos\varphi}\oint_{k_2} Pdx, \quad c_2 = \frac{2\cos\varphi - 1}{1 - \cos\varphi}\oint_{k_2} Qdy. \tag{46}$$

On the other hand, due to Green's formula, we have

$$\int\int_D P_y \mathrm{d}x\mathrm{d}y = -\oint_k P\mathrm{d}x, \quad \int\int_D Q_x \mathrm{d}x\mathrm{d}y = \oint_k Q\mathrm{d}y.$$

Bearing in mind the latter equalities we obtain the following statement.

Theorem 12. *The relation between the circulation* (40) *and the flux* (43) *is determined by*

$$T = -2\cot^3\varphi\Big((1 + \cos\varphi)\sqrt{\frac{2\cos\varphi - 1}{2\cos\varphi + 1}}c_1 + (1 - \cos\varphi)\sqrt{\frac{2\cos\varphi + 1}{2\cos\varphi - 1}}c_2\Big),$$

where c_1 *and* c_2 *are given in* (46).

Author Contributions: "Conceptualization, Razpopov, D.; Dzhelepov, G.; Dokuzova, I.; methodology, Razpopov, D.; Dzhelepov, G.; Dokuzova, I.; investigation, Razpopov, D.; Dzhelepov, G.; Dokuzova, I.; writing—original draft preparation, Razpopov, D.; Dzhelepov, G.; Dokuzova, I.; writing—review and editing, Razpopov, D.; Dzhelepov, G.; Dokuzova, I.; funding acquisition, Razpopov, D.; Dzhelepov, G. All authors have read and agreed to the published version of the manuscript.".

Funding: This research was partially funded by project 17-12 "Support for publishing activities", Agricultural University of Plovdiv, Bulgaria

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: MDPI Research Data Policies at https://www.mdpi.com/ethics.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Abe N.; Nakanishi, Y.; Yamaguchi, S. Circles and spheres in pseudo-Riemannian geometry, *Aequationes Math.* **1990**, *39*, 134–145.
2. Boothby, W. M. *An Introduction to Differentiable Manifolds and Riemannian Geometry*, Second edition, Pure and Applied Mathematics, 120. Academic Press, Inc., Orlando FL, 1986.
3. do Carmo, M. P. *Differential Forms and Applications. Integration on Manifolds*, Universitext, Springer: Berlin/Heidelberg, 2012.
4. Dokuzova, I.; Razpopov, D. Four-dimensional almost Einstein manifolds with skew-circulant structures, *J. Geom.* **2020**, *111*, Paper No. 9, 18 pp.
5. Dzhelepov G., On an indefinite metric on a 3-dimensional Riemannian manifold, *Int J. Geom.* **2022**, *11*, 12–19.
6. Dzhelepov, G.; Dokuzova I.; Razpopov D. Spheres and circles with respect to an indefinite metric of a 4-dimensional Riemannian manifold with skew-circulant structures, https://doi.org/10.48550/arXiv.2301.03675.
7. Holmes, R. D.; Thompson, A. N-dimensional area and content in Minkowski spaces, *Pacific J. Math.* **1979**, *85*, 77–110.
8. Ikawa, T. On curves and submanifolds in an indefinite-Riemannian manifold, *Tsukuba J. Math.* **1985**, *9*, 353–371.
9. Gupta, V. G.; Sharma, P. Differential forms and its application, *Int. J. Math. Anal. (Ruse)* **2008**, *2*, 1051–1060.
10. Lopez, R. Differential geometry of curves and surfaces in Lorentz-Minkowski space, Int. Electron. J. Geom. **2014**, *7*, 44–107.
11. Mustafaev, Z. The ratio of the length of the unit circle to the area of the unit disk in Minkowski planes, *Proc. American Math. Soc.* **2005**, *133*, 1231–1237.
12. Parkinson, Ch. The elegance of differential forms in vector calculus and electromagnetics, MSc Thesis, University of Chester, United Kingdom, 2014.
13. Petrello, R. Stokes' theorem, MSc Thesis, California State University, USA, 1998.
14. Razpopov, D.; Dokuzova, I. A Riemannian manifold with skew-circulant structures and an associated locally conformal Kähler manifold, *Novi Sad J. Math.* 2023 *(accepted)*.