

# The expected competitive ratio on a kind of stochastic-online flowtime scheduling with machine subject to an uncertain breakdown

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**Abstract:** We consider the problem of scheduling tasks on a single machine subject to an uncertain breakdown to minimize the flowtime. Assume the machine is unavailable during the breakdown; the starting time of the breakdown is a random variable  $s$  with distribution function  $D(s)$  and the terminating time of the breakdown has no any other information; jobs are non-resumable. Under the assumptions, starting from the perspective of statistical optimization, we first define the expected competitive ratio of an algorithm to find the optimized solution with the considered problem. Further, we propose and prove a certain result on the expected competitive ratio of the SPT rule. In particular, we show that under a quite loose condition the expected competitive ratio of the SPT rule is no more than  $\frac{5}{4}$ . Meanwhile, we also make some discussions about our studies. What we have done will rich and improve the studying results on the area of scheduling to minimize flowtime and advance the development of online optimization and statistical optimization.

**Keywords:** single machine; single breakdown; schedule; flowtime; stochastic optimization; online optimization

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## 1. Introduction

As is known to all, machines may occur some breakdowns or say unavailability periods. And many breakdowns have a lot of uncertainty. To meet the actual needs, many scholars have paid much attention to the study of the scheduling problem with the machine subject to breakdowns. In the first place, in 1984, Glazebrook [1] studied a kind of scheduling problem for a machine with a stochastic breakdown. In 1989, Adiri et al [2] considered the problem of scheduling tasks on a single machine to minimize the flowtime (total completion time); they mainly discussed the problem that the SPT policy minimizes the follow time in the case of a single breakdown and the more general case of multiple breakdowns, respectively. In 1990, Birge et al [3] studied a kind of single-machine scheduling problem with the machine subject to a sequence of stochastic breakdowns. After this, the scheduling work with the machine subject to a sequence of stochastic breakdowns has constantly appeared. For instance, in 1993, Mittenthal and Raghavachair [4] investigated the problem of quadratic early-tardy penalties with the machine subject to a sequence of stochastic breakdowns. In 1996, Cai and Tu [5] made a further investigation into the problem that was considered by the paper [4] under the condition of jobs with random processing times. In 2008, Tang et al [6] studied a stochastic JIT scheduling problem with a machine subject to a sequence of stochastic breakdowns. In 2008, Cheng et al [7] studied the problem that scheduling jobs on a machine subject to stochastic breakdowns to minimize absolute early-tardy penalties. In 2022, Li and Chen[8] explored the problem of minimizing total weighted late work on a single machine with non-availability intervals. In 2022, Choi and Park [9] considered a single-machine

scheduling problem with resource-dependent processing times and multiple unavailability periods.

Since this century, the study of online scheduling with machine subject to breakdowns has been gradually thriving. In 2002, Tan and He [10] studied the optimal online algorithm for scheduling on two identical machines with unavailable periods. In 2014, Huo et al [11] investigated the scheduling problem on a single machine with a single breakdown, or say with a single unavailable period; they considered the online optimization problem of minimizing total weighted completion time under the condition assumed that every job has a weight proportional to its processing time. In 2016, Kacem and Kellerer [12] investigated three single-machine scheduling problems with the so-called semi-online scenario, which are actually the scheduling problem on a single machine with an unexpected breakdown. In 2020, Tian [13] improved a part of the work of literature [11].

Up to now, most research for the online scheduling problem with the machine subject to a single breakdown focuses on the following research setting. The breakdown results in an unavailable period or hole  $[s, t]$  during which the machine cannot perform the processing of any job. At the beginning of the scheduling horizon, except the breakdown can occur, no other information is known, in other words, its state is unknown, or say, the starting time  $s$  and the terminating time  $t$ , including the distributions of theirs, are unknown and independent; a sequence of the jobs must be determined in advance and the sequence cannot be changed in the whole process of production. Jobs are non-resumable, that is, the job which has been interrupted by the breakdown has to be restarted to process after the machine is recovered later. The objective is to find an optimization sequence or schedule.

However, in many cases, we have known some states of the related processing situation in advance, it is a very meaningful work to discuss the optimization scheme of problems under a certain statistical information of the breakdown. In the present work, we address the problem to minimize flowtime (minimize total completion time) with the above setting integrated the condition that the starting time is a random variable with known distribution. We will introduce and investigate the expected competitive ratio of online minimizing flowtime with the case described. In addition, we will also make some discussions on our study.

The remainder contents are organized as follows. Sect.2 gives preliminary notations and knowledge. Sect.3 introduces the related works. Sect.4 pays attention to our main work. Sect.5 makes some discussions on our work. Finally, we draw our conclusions in Sect.6.

## 2. Formulation

Process  $n$  jobs on a single machine  $M$  subject to a single breakdown. Assume the starting time and the terminating time of the breakdown are  $s$  and  $t$ , respectively; machine  $M$  cannot work in the period of time  $[s, t]$  for the breakdown. Jobs are non-resumable, i.e., when a job is interrupted by the unavailable interval, it has to be restarted after the machine is recovered later; and all jobs are ready to be scheduled at the time  $0$ . Unless there is a breakdown during the processing of a task, tasks must be processed without interruption. Each job  $J_j$  ( $j = 1, 2, \dots, n$ ) has a determined processing time  $p_j > 0$ .  $w_j$  is a positive real number, called the weight of job  $J_j$ . The process starts from the time  $0$  and one job can be processed at a time. A processing schedule, or say sequence,  $S$  must be made before the time  $0$  and the schedule  $S$  cannot be changed in the whole process. Let  $C_j$  be the completion time of the job  $J_j$ . Consider the problem to find a schedule  $S: J_1, J_2, \dots, J_n$  before time  $0$  such that the (weighted) followtime, or say, the

total (weighted) completion time, is minimized, namely  $\sum_{j=1}^n C_j$  ( $\sum_{j=1}^n w_j C_j$ ) is minimized.

When  $s = t$ , we believe no breakdown occurs, which is called the degenerate case. The problem is just the classical scheduling problem of minimizing the (weighted) followtime, which is denoted by

$$1|nr - a|\sum w_j C_j. \quad (2.1)$$

When  $s < t$ , the problem can be divided into the following three types. If  $[s, t]$  is given before time 0, we call the problem as minimizing weighted followtime for the machine with a non-availability interval or hole. If  $[s, t]$  is unknown before time 0, we call the problem as online minimizing weighted flowtime for the machine with a non-availability interval or hole. If a part of distribution on interval  $[s, t]$  is known before time 0, we call the problem as the scheduling problem of stochastic online minimizing weighted followtime for the machine with a non-availability interval or hole. In the present work, by the three-field notation, we respectively denote the three problems as

$$1, h_1|nr - a|\sum w_j C_j, \quad (2.2)$$

$$1, h_1, \text{online}|nr - a|\sum w_j C_j, \quad (2.3)$$

$$1, h_1, \text{stochastic} - \text{online}|nr - a|\sum w_j C_j. \quad (2.4)$$

When  $w_j = 1$ ,  $j = 1, 2, \dots, n$ , for convenience, we denote the problems (2.2), (2.3) and (2.4) respectively as

$$1, h_1|nr - a|\sum C_j, \quad (2.5)$$

$$1, h_1, \text{online}|nr - a|\sum C_j, \quad (2.6)$$

$$1, h_1, \text{stochastic} - \text{online}|nr - a|\sum C_j. \quad (2.7)$$

For problem (2.7), assume the starting time is a random variable  $\mathcal{S}$  and there is no any other information on the terminating time  $t$ ; assume also

$$Pr(\{\mathcal{S} < 0\}) = 0, \quad Pr(\{\mathcal{S} = 0\}) = p, \quad Pr(\{\mathcal{S} > P\}) = q;$$

$$D(s) = p + \int_0^s \varphi(x) dx, \quad 0 < s \leq P,$$

where  $D(s)$  is the distribution function of  $\mathcal{S}$ ;  $\varphi(x)$  expresses the density of  $\mathcal{S}$  on the interval  $(s, P]$ ; the integral is Riemann integral. In the case, we denote the problem as HSONRP (Hole Stochastic Online Non-Resumable Problem).

In briefness, we denote the objective function of the schedule  $S$  by  $F(S)$ , that is,  $F(S) = \sum w_j C_j$ .

For the problem (2.1) ((2.2)), let  $S$  be a schedule and  $S^*$  be the optimal schedule. Then we call  $\rho(S) = \frac{F(S)}{F(S^*)}$  as the competitive ratio of schedule  $S$ . Let  $\mathcal{A}$  be an algorithm to solve the problem,  $I$  be an instance and  $S_I$  be the schedule with  $I$  obtained by algorithm  $\mathcal{A}$ . Then

$$\rho(\mathcal{A}) = \sup\{\rho(S_I)|I\}$$

is called the competitive ratio, or the approximation ratio of algorithm  $\mathcal{A}$ , where  $I$  is any instance of (2.1) ((2.2)).

For the problem (2.3), let  $\mathcal{A}$  be an algorithm to solve the problem,  $I$  be an instance and for the instance  $I$ ,  $S_I$  be the schedule obtained by algorithm  $\mathcal{A}$ . Let also  $\rho(S_I, [s, t])$  be the competitive ratio of the schedule  $S_I$  for the problem (2.2) with hole  $[s, t]$ . Then we call

$$\rho(\mathcal{A}) = \sup\{\rho(S_I, [s, t])|I, [s, t]\}$$

as the competitive ratio of algorithm  $\mathcal{A}$  for the problem.

For the problem HSONRP, let  $\mathcal{A}$  be an algorithm to solve the problem,  $I$  be an instance and  $S_I$  be the schedule obtained by algorithm  $\mathcal{A}$  for the instance  $I$ . Given  $s \in (0, P)$ , for any  $t > s$ , let also  $\rho(S_I, [s, t])$  be the competitive ratio of the schedule  $S_I$

for the problem (2.2) with the hole  $[s, t]$ . Define  $\rho(s, \mathcal{A}) = \sup\{\rho(S_I, [s, t]) | I, t > s\}$ . When  $s = 0$  or  $s \geq P$ , we define  $\rho(s, \mathcal{A}) = \sup\{\rho(S_I) | I\}$ , where  $\rho(S_I)$  is the competitive ratio of the schedule  $S_I$  for the problem (2.1). Since, if the starting time  $s = 0$ , we can arrange the schedule after the breakdown; and if the starting time  $s \geq P$ , the breakdown does not disturb the arranged schedule, we can believe no breakdown occurs in the two cases. That is, when  $s = 0$  or  $s \geq P$ , it is reasonable that we define

$$\rho(s, \mathcal{A}) = \sup\{\rho(S_I) | I\}.$$

When the starting time is the random variable  $\mathcal{S}$ , we call the mean value of random variable  $E[\rho(s, \mathcal{A})]$  as the expected competitive ratio of algorithm  $\mathcal{A}$  for the problem.

In order to convenience, we also introduce the following notations and assumptions. Given  $[s, t]$ , let  $S: J_1, J_2, \dots, J_n$  be a schedule. We use  $J_k$  express the interrupted task. Let also

$$P(i, j) = \sum_{l=i}^j p_l, P_j = P(1, j), 1 \leq i \leq j \leq n; P_0 = 0; FP = \sum_{j=1}^n P_j.$$

Denote the schedule obtained by SPT rule as  $S': J'_1, J'_2, \dots, J'_n$ . Since the case that  $n = 1$  is trivial, we assume  $n > 1$  in the following. Moreover, we also assume that the event  $\{s = 0\}$  includes the two cases the time 0 is in the breakdown period and there is no breakdown.

### 3. Related work

In the first place, (2.1) is the classical scheduling problem of minimizing the total weighted completion time, which has the famous WSPT (weighted shortest processing time) rule, that is, arranging jobs as the order  $\frac{w_j}{p_j}$  non-decreasing, the objective is minimized.

(2.2) is a quite difficult problem. Substantial works have discussed this problem over the last couple of decades, see the related work of Huo et al. [11]. In particular, in 1996, Lee [14] showed that the error bound of the WSPT rule can be arbitrarily large. 2008, Kacem and Chu [15] showed that both the WSPT rule and MWSPT (modified weighted shortest processing time) rules have a tight error bound of 3 under some conditions. 2008, Kacem [16] proposed a 2-approximation algorithm with  $O(n^2)$  time complexity and showed bound 2 is tight. In 2014, Huo et al. [11] proposed the so-called the FF-LPT rule, which is a modified version of the LPT rule; and they showed both the LPT rule and FF-LPT rule can give a tight approximation ratio of 2 in the case of  $w_j = p_j$ .

(2.5) is the special case of (2.2), which is called the minimizing flowtime problem. For the problem, in 1989, Adiri et al. [2] showed the problem is NP-complete, and the SPT problem has the approximation ratio  $\frac{5}{4}$  (the relative error for the SPT schedule is less than or equal to  $\frac{1}{4}$ ). In 1992, Lee and Liman [17] proved that SPT rule has a tight approximation ratio of  $\frac{9}{7}$ , instead of  $\frac{5}{4}$ . Later, in 2005, Sadfi et al. [18] gave the so-called Modified SPT Police algorithm that has a tight approximation ratio of  $\frac{20}{7}$ . In 2006, He et al. [19] also proposed a polynomial time approximation scheme. In 2007, Breit [20] developed a parametric  $O(n \log n)$ -algorithm, in which better worst-case error bounds can be obtained.

For problem (2.3), in 2014, Huo et al. [10] first showed in the case  $w_i = p_i$ , there is no algorithm  $\mathcal{A}$  such that  $\rho(\mathcal{A}) > \frac{\sqrt{5}+1}{2}$ , and LPT rule has a tight competitive ratio 2. (2.6) is the special case of (2.3), which is called online minimizing the flowtime for the

machine with a non-availability interval or hole, and closely contact with (2.5). Since the SPT rule and the above stated results of competitive ratio of the rule for problem (2.5) have no relation with specific conditions of  $s$  and  $t$ , these results are also true for the problem (2.6).

For the problem (2.7), assume the starting time to breakdown is a random variable  $S$ . Adiri et al. [2] proposed a definition of a schedule to stochastically minimize the flowtime and showed that if the distribution function of the starting time  $S$  is concave, then the SPT rule stochastically minimizes the flow time in the sense of their definition. In addition, for problems (2.4) and (2.7), to the best of our knowledge, there are no other works until now.

In the present work, we address problem HSONRP, which is a kind of problem (2.7). We mainly study the expected competitive ratio of the SPT rule, and make some discussions about the work we study.

## 4. Main work

First of all, given  $[s, t]$ , let  $S: J_1, J_2, \dots, J_n$  be a schedule. Then we have

$$\begin{aligned}
 F(S) &= \sum_{j=1}^{k-1} C_j + \sum_{j=k}^n C_j = \sum_{j=1}^{k-1} C_j + \sum_{j=k}^n [t + P(k, j)] \\
 &= \sum_{j=1}^{k-1} C_j + \sum_{j=k}^n [(t - C_{k-1}) + C_{k-1} + P(k, j)] \\
 &= \sum_{j=1}^{k-1} P_j + (n - k + 1)(t - P_{k-1}) + \sum_{j=k}^n [P_{k-1} + P(k, j)] \\
 &= \sum_{j=1}^{k-1} P_j + (n - k + 1)(t - P_{k-1}) + \sum_{j=k}^n P_j \\
 &= \sum_{j=1}^n P_j + (n - k + 1)(t - P_{k-1}) \\
 &= \sum_{j=1}^n P_j + (n - k + 1)[(t - s) + (s - P_{k-1})].
 \end{aligned}$$

Here,  $(s - P_{k-1})$  is the idle time immediately before the breakdown in the schedule  $S$ .

**Lemma 1.** For the problem (2.5)((2.6) and (2.7) with given  $[s, t]$ ), let  $S': J'_1, J'_2, \dots, J'_n$  be the SPT schedule. Then we have the following two conclusions. (1) For any schedule  $S: J_1, J_2, \dots, J_n$ , we have:  $k \leq k'$ , where  $J_k$  and  $J'_{k'}$  are the interrupted tasks respectively in the schedule  $S$  and the schedule  $S'$ . (2) For the optimal schedule  $S^*: J^*_1, J^*_2, \dots, J^*_n$ , we have:  $P'_{k'-1} \leq P^*_{k^*-1} \leq s$ .

**Proof.** For  $S': J'_1, J'_2, \dots, J'_n$  is the SPT schedule, we have  $p'_1 \leq p'_2 \leq \dots \leq p'_n$ . Hence (1) holds. Now we prove (2) as follows. By the above expressions of  $F(S)$ , we have

$$\begin{aligned}
 F(S') &= \sum_{j=1}^n P'_j + (n - k' + 1)[(t - s) + (s - P'_{k'-1})]; \\
 F(S^*) &= \sum_{j=1}^n P^*_j + (n - k^* + 1)[(t - s) + (s - P^*_{k^*-1})].
 \end{aligned}$$

From Lemma 1, we have  $(n - k^* + 1) \geq (n - k' + 1)$ . Due to  $S^*$  is the optimal schedule, we also have  $F(S^*) \leq F(S')$ . Due to  $S'$  is the SPT schedule, we have  $\sum_{j=1}^n P'_j \leq \sum_{j=1}^n P^*_j$ . Combining these results, we have

$$\begin{aligned}
& \sum_{j=1}^n P_j^* + (n - k^* + 1) \left[ (t - s) + (s - P_{k^*-1}^*) \right] \leq \sum_{j=1}^n P_j' + (n - k' + 1) \left[ (t - s) + (s - P_{k'-1}') \right] \\
& \Rightarrow \left( \sum_{j=1}^n P_j^* - \sum_{j=1}^n P_j' \right) + (k' - k^*)(t - s) + (n - k^* + 1)(s - P_{k^*-1}^*) \leq (n - k' + 1)(s - P_{k'-1}') \\
& \Rightarrow (n - k^* + 1)(s - P_{k^*-1}^*) \leq (n - k' + 1)(s - P_{k'-1}') \\
& \Rightarrow (s - P_{k^*-1}^*) \leq (s - P_{k'-1}') \Rightarrow P_{k'-1}' \leq P_{k^*-1}^*. \\
& P_{k^*-1}^* \leq s \text{ is obvious. Thus (2) holds, and we have completed the proof. } \square
\end{aligned}$$

Lemma 2. For problem (2.5), we have

$$\begin{aligned}
\rho(\text{SPT}) & \leq 1 + \frac{(n - k' + 1)(s - P_{k'-1}')}{FP' + (n - k' + 1)(t - s)} \\
& \leq 1 + \frac{(n - k' + 1)(s - P_{k'-1}')}{FP'};
\end{aligned} \tag{4.1}$$

$$\begin{aligned}
\rho(\text{SPT}) & \leq 1 + \frac{(n - k' + 1)p_{k'}'}{FP' + (n - k' + 1)(t - s)} \\
& \leq 1 + \frac{(n - k' + 1)p_{k'}'}{FP'}.
\end{aligned} \tag{4.2}$$

For problem (2.3), we have

$$\rho(\text{SPT}) \leq 1 + \frac{(n - k' + 1)p_n'}{FP'}. \tag{4.3}$$

Here SPT denotes the algorithm of SPT rule, sequence  $S': J_1', J_2', \dots, J_n'$  is the SPT schedule.

**Proof.** Let  $S^*$  be the optimal schedule. Since  $S'$  be the SPT schedule, we have  $\sum_{j=1}^n P_j^* \geq \sum_{j=1}^n P_j'$ . By Lemma1, we also have  $k \leq k'$ . Therefore, we obtain

$$\begin{aligned}
& \sum_{j=1}^n P_j^* + (n - k^* + 1) \left[ (t - s) + (s - P_{k^*-1}^*) \right] \\
& \geq \sum_{j=1}^n P_j' + (n - k' + 1)(t - s) + (n - k' + 1)(s - P_{k^*-1}^*) \\
& \geq \sum_{j=1}^n P_j' + (n - k' + 1)(t - s).
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{F(S')}{F(S^*)} &= \frac{\sum_{j=1}^n P'_j + (n-k'+1)[(t-s) + (s - P'_{k'-1})]}{\sum_{j=1}^n P_j^* + (n-k^*+1)[(t-s) + (s - P_{k^*-1}^*)]} \\
&\leq \frac{\left[ \sum_{j=1}^n P'_j + (n-k'+1)(t-s) \right] + (n-k'+1)(s - P'_{k'-1})}{\sum_{j=1}^n P'_j + (n-k'+1)(t-s)} \\
&= 1 + \frac{(n-k'+1)(s - P'_{k'-1})}{\sum_{j=1}^n P'_j + (n-k'+1)(t-s)}.
\end{aligned}$$

(4.1) holds. For  $J'_k$  is the interrupted task, we have  $P'_{k'-1} \leq s < P'_k$ . This leads to  $0 \leq s - P'_{k'-1} < P'_k - P'_{k'-1} = p'_k$ . Bring the result in (4.1), we obtain (4.2). (4.3) is straight obtained from (4.2). The proof is finished.  $\square$

**Theorem 1.** For problem HSONRP, we have

$$E[\rho(s, \text{SPT})] \leq 1 + \frac{1}{FP'} \left[ \sum_{k=1}^n (n-k+1) \int_{P'_{k-1}}^{P'_k} (s - P'_{k-1}) \varphi(s) ds \right]. \quad (4.4)$$

When  $S$  is uniform distribution on the interval  $(0, P]$ , we further have

$$E[\rho(s, \text{SPT})] \leq 1 + \frac{(1-p-q)}{2P \cdot FP'} \left[ \sum_{k=1}^n (n-k+1) p_k'^2 \right]. \quad (4.5)$$

$$E[\rho(s, \text{SPT})] \leq 1 + \frac{(1-p-q)p_n'}{2P}. \quad (4.6)$$

Moreover, if  $p'_n \leq \frac{1}{2}P$  or  $p'_n \leq \sum_{k=1}^{n-1} (n-k+1)p'_k$ , then we have

$$E[\rho(s, \text{SPT})] \leq 1 + \frac{(1-p-q)}{4}.$$

**Proof.** We can easily know  $\rho(s, \text{SPT}) = 1$  when  $s = 0$  or  $s \geq P$  and in terms of (4.1), we have

$$\rho(s, \text{SPT}) \leq 1 + \frac{(n-k+1)(s - P'_{k-1})}{FP'}, P'_{k-1} < s \leq P'_k, k = 1, 2, \dots, n. \quad (4.7)$$

Further,

$$\begin{aligned}
E[\rho(s, \text{SPT})] &= \int_{-\infty}^{+\infty} \rho(s, \text{SPT}) dD(s) \\
&= \int_{-\infty}^0 \rho(s, \text{SPT}) dD(s) + \int_0^P \rho(s, \text{SPT}) dD(s) + \int_P^{+\infty} \rho(s, \text{SPT}) dD(s) \quad (4.8) \\
&= p + q + \int_0^P \rho(s, \text{SPT}) dD(s),
\end{aligned}$$

where the integrals are all the Lebesgue-Stieltjes integral of the function  $\rho(s, \text{SPT})$  on the function  $D(s)$ . For  $D(s) = p + \int_0^s \varphi(x) dx$ ,  $0 < s \leq P$ , by (4.7), we have

$$\begin{aligned}
\int_0^P \rho(s, \text{SPT}) dD(s) &= \int_0^P \rho(s, \text{SPT}) \varphi(s) ds \\
&= \sum_{k=1}^n \int_{P'_{k-1}}^{P'_k} \rho(s, \text{SPT}) \varphi(s) ds \\
&\leq \sum_{k=1}^n \left[ \int_{P'_{k-1}}^{P'_k} \varphi(s) ds + \int_{P'_{k-1}}^{P'_k} \frac{(n-k+1)(s-P'_{k-1})}{FP'} \varphi(s) ds \right] \quad (4.9) \\
&\leq \int_0^P \varphi(s) ds + \sum_{k=1}^n \int_{P'_{k-1}}^{P'_k} \frac{(n-k+1)(s-P'_{k-1})}{FP'} \varphi(s) ds \\
&= 1-p-q + \sum_{k=1}^n \frac{(n-k+1)}{FP'} \int_{P'_{k-1}}^{P'_k} (n-k+1)(s-P'_{k-1}) \varphi(s) ds.
\end{aligned}$$

Combining (4.8) and (4.9), we obtain (4.4). When  $\mathfrak{S}$  is uniform distribution on the interval  $(0, P]$ , we have

$$\varphi(s) = \frac{1-p-q}{P}, \quad s \in (0, P].$$

This leads to

$$\int_{P'_{k-1}}^{P'_k} (s-P'_{k-1}) \varphi(s) ds = \frac{(1-p-q)}{P} \int_{P'_{k-1}}^{P'_k} (s-P'_{k-1}) ds = \frac{(1-p-q)p_k'^2}{2P}.$$

Bring it in (4.4), we obtain

$$E[\rho(\mathbf{s}, \text{SPT})] \leq 1 + \frac{(1-p-q)}{2P \cdot FP'} \left[ \sum_{k=1}^n (n-k+1) p_k'^2 \right].$$

That is, (4.5) holds. Note that  $p'_k \leq p'_n, k=1, 2, \dots, n$ ; and

$$FP' = \sum_{k=1}^n P'_k = \sum_{k=1}^n (n-k+1) p'_k.$$

We know that (4.6) is true from (4.5). If  $p'_n \leq \frac{1}{2}P$ , from (4.6),

$$E[\rho(\mathbf{s}, \text{SPT})] \leq 1 + \frac{(1-p-q)}{4}$$

is obvious. If  $p'_n > \frac{1}{2}P$  and  $p'_n \leq \sum_{k=1}^{n-1} (n-k+1) p'_k$ , we have

$$\begin{aligned}
& E[\rho(s, \text{SPT})] \\
& \leq 1 + \frac{(1-p-q)}{2P \cdot FP'} \left[ \sum_{k=1}^n (n-k+1) p_k'^2 \right] \\
& = 1 + \frac{(1-p-q)}{2P \cdot FP'} \left[ \sum_{k=1}^{n-1} (n-k+1) p_k'^2 + p_n'^2 \right] \\
& = 1 + \frac{(1-p-q)}{2P \cdot FP'} \left[ \sum_{k=1}^{n-1} (n-k+1) p_k'^2 + p_n' \left( p_n' - \frac{1}{2}P \right) + p_n' \left( \frac{1}{2}P \right) \right] \\
& \leq 1 + \frac{(1-p-q)}{2P \cdot FP'} \left\{ \sum_{k=1}^{n-1} (n-k+1) p_k'^2 + \left[ \sum_{k=1}^{n-1} (n-k+1) p_k' \right] \left( p_n' - \frac{1}{2}P \right) + p_n' \left( \frac{1}{2}P \right) \right\} \\
& \leq 1 + \frac{(1-p-q)}{2P \cdot FP'} \left\{ \left[ \sum_{k=1}^{n-1} (n-k+1) p_k' \left( p_k' + p_n' - \frac{1}{2}P \right) \right] + p_n' \left( \frac{1}{2}P \right) \right\} \\
& \leq 1 + \frac{(1-p-q)}{2P \cdot FP'} \left\{ \left[ \sum_{k=1}^{n-1} (n-k+1) p_k' \left( P - \frac{1}{2}P \right) \right] + p_n' \left( \frac{1}{2}P \right) \right\} \\
& \leq 1 + \frac{(1-p-q)}{2P \cdot FP'} \left\{ \left[ \sum_{k=1}^n (n-k+1) p_k' \right] \left( \frac{1}{2}P \right) \right\} = 1 + \frac{(1-p-q)}{4}.
\end{aligned}$$

The proof is completed.  $\square$

## 5. Discussions

(1) When the system possesses some information on statistics, such as the distributions of certain variables, it is quite useful in practice to find the high probability laws. Hence, facing the era of big data, we should work hard to develop statistical optimization, whose meaning is mainly to catch the optimization law of large probability. On the other hand, when a system, which has a large amount of uncertainty, possesses not enough information on statistics; or the information on statistics is too complex, it is impossible or quite difficult to find an effective optimization method by statistical methods alone. Hence, facing the era of big data, we should also work hard to develop methods of combining statistical optimization with online optimization. The present work is conducted under the promotion of this idea. The definition of the expected competitive ratio is proposed in the light of this idea.

(2) Sometimes, considering the problem under appropriate conditions is easy to find a good method. For example, under some conditions, we can obtain the result

$$E[\rho(s, \text{SPT})] \leq 1 + \frac{(1-p-q)}{4}.$$

The present work fully demonstrates this viewpoint.

(3) For problem HSONRP, when  $\mathcal{S}$  is uniform distribution on the interval  $(0, P]$ , by the related result of Lee and Liman [17], that is, SPT has a tight approximation ratio of  $\frac{9}{7}$ , we have

$$\rho(s, \text{SPT}) \leq \frac{9}{7}, 0 < s \leq P.$$

In terms of (4.8) and (4.9), we can easily obtain the following conclusion.

$$E[\rho(s, \text{SPT})] \leq 1 + \frac{2}{7}(1-p-q).$$

(4) For problem (1.5), in 1989, Adiri et al. [2] showed the SPT rule has approximation

ratio  $\frac{5}{4}$ . In 1992, Lee and Liman [17] proved the SPT rule has a tight approximation ratio of  $\frac{9}{7}$ , instead of  $\frac{5}{4}$ . When  $p + q = 0$ , which is the setting of the previous work in this research line, from the result  $E[\rho(s, \text{SPT})] \leq 1 + \frac{(1-p-q)}{4}$ , we obtain  $E[\rho(s, \text{SPT})] \leq \frac{5}{4}$ . For  $\frac{5}{4} < \frac{9}{7}$  and the condition of this conclusion is quite loose, to some extent, the result demonstrates the advantage of the present work.

(5) Moreover, from (4.6), we can obtain

$$E[\rho(s, \text{SPT})] \leq 1 + \frac{p'_n}{2P} = 1 + \frac{\max\{p_i\}}{2P}.$$

In terms of the result, we can quickly estimate  $E[\rho(s, \text{SPT})]$  by  $\frac{\max\{p_i\}}{P}$ . This is also an advantage of the present work.

(6) In practice, we can approximately calculate the expected competitive ratio  $E[\rho(s, \mathcal{A})]$  by the method of numerical evaluations. This will fully develop the efficiency of big data and statistical optimization, widen the investigating horizon of the scheduling area and strengthen the effect of the scheduling research. The formulae (4.1)-(4.6) can strongly support the numerical evaluations. In the actual estimate, we can choose the formulae from (4.1)-(4.6) making calculation according to the specific needs and conditions.

## 6. Conclusions

In the present work, we study problem HSONRP. We have first proposed the definition of the expected competitive ratio  $E[\rho(s, \mathcal{A})]$  for algorithm  $\mathcal{A}$  to find the optimized solution to the problem. And then we studied  $E[\rho(s, \text{SPT})]$ . In particular, under a quite loose conditions, we proved  $E[\rho(s, \text{SPT})] \leq \frac{5}{4}$ . Through this work we have developed the approaches to study the scheduling problem which includes determinate information, stochastic information and other uncertain information. For the expected competitive ratio, which can be extended to other models, there are much work to do. In the future, this is an interesting direction to pursue.

## Author Contributions

Z.P. LI: Conceptualization, project administration, and writing—original draft. C.D. CHENG: Conceptualization, formal analysis, and writing—review and editing. All authors have read and agreed to the published version of the manuscript.

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Data availability is not applicable to this article as no new data were created or analyzed in this study.

## Conflicts of Interest

The authors have no financial or proprietary interests in any material discussed in this article.

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