Towards an Analytical Theory for the Solution of the Nagel-Schreckenberg Model with $\mathrm{v_{max}} > 1$

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Abstract: **An exact solution for the Nagel-Schreckenberg model for $\mathrm{v_{max}} = 1$ is known since the 90s, yet for $\mathrm{v_{max}} > 1$ it remains an open question. It is conjectured here that this general solution can be generated from the basic solution (for $\mathrm{v_{max}} = 1$) by a linear coordinate transformation, the shear mapping, which requires no additional parameters in the deterministic case, and two additional parameters otherwise. The proposed solutions show remarkable agreement with existing numerical solutions, except in the congested regime where the Monte Carlo data shows more concavity than the more linear congested branches predicted by the theory. More research is needed to investigate this bias.**

Keywords: urban congestion; traffic flow theory; transportation; phase transitions; surface growth; Nonequilibrium physics; criticality

1. Introduction

The influential Nagel-Schreckenberg (NaSch) model [1] has become a pivotal model for traffic science to this day. In its deterministic form, it encapsulates the simplest *microscopic* driving rules: "Travel at the maximum speed $\mathrm{v_{max}}$ unless impeded by the downstream car". The resulting steady-state equilibrium relationship between the flow, f, and the density, ρ, of vehicles on a link–known as the fundamental diagram of traffic flow–is triangular in shape, and has been used extensively in the literature. These canonical driving rules are shared by a large family of well-known deterministic traffic flow models, both continuum and discrete, and therefore are fundamentally equivalent. These models include: the kinematic wave theory of [2, 3] with triangular fundamental diagram, or KWT for short, the celebrated Newell's car-following model [4], Daganzo's cellular automaton [5], and elementary cellular automata rule 184 [6] when $\mathrm{v_{max}} = 1$.

The NaSch model adds two more rules at each discrete time step: (i) bounded acceleration: a vehicle driving at a speed $v < \mathrm{v_{max}}$ may only increase its speed by one unit, and (ii) a probability p of slowing down by one speed unit for moving vehicles, $v > 0$.

The focus here is an analytical expression for the fundamental diagram of the NaSch model, namely $f^{u,r}(\rho)$, as a function of model parameters $\mathrm{v_{max}} = u$ and $p = r$. An exact solution for the NaSch model for $\mathrm{v_{max}} = 1$ has been known since the 70s [7, 8] thanks to the analogy to the TASEP with parallel dynamics [9, 10]:

$$f^{1,p}(\rho) = \frac{1}{2}\left(1 - \sqrt{1 - 4(1-p)(1-\rho)\rho}\right), \qquad (\mathrm{v_{max}} = 1). \tag{1}$$

Yet for $\mathrm{v_{max}} > 1$, this remains an open question. According to [11] the main reasons for the difficulties is that "For $\mathrm{v_{max}} > 1$ the particle-hole symmetry of the TASEP is lost and as a result the fundamental diagram is no longer symmetric around $\rho = 1/2$."

It is shown here that both the particle-hole symmetry and the symmetry of the fundamental diagram can be restored thanks to a linear coordinate transformation, the shear mapping, first described in the traffic flow context in [12]. This transformation constitutes a symmetry of conservation laws: a solution under a given set of transformation parameters can be mapped to the solution under another parameter set. The significance here is that depending on these parameters, one may slant the shape of the fundamental diagram until it becomes symmetric and Eq. (1) may be applied. Notice that the spirit of this solution method is in line with the intuition that "TASEP is the mother of all traffic flow models" per Schadschneider's book [13].

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2. The shear symmetry of conservation laws

The Galilean transformation $x \to x + v_0 t, t \to t$, expresses the invariance of the physical world to inertial frames of reference moving at a constant velocity v_0, by leaving most physical laws, including conservation laws, identical. A lesser-used linear symmetry that also leaves conservation laws invariant is the shear (or skewing) mapping:

$$x \to \epsilon x, \quad t \to t + x/v_0, \tag{2}$$

which can be interpreted as using asynchronous time [14], where clocks start running at each location according to the passage of an observer moving at a constant velocity v_0. The parameter ϵ represents a spatial scale parameter. [12] shows that applying the same transformation to flows and densities, i.e.:

$$f \to \epsilon f, \quad \rho \to \rho + f/v_0 \tag{3}$$

leaves delays and total distance traveled unchanged. The significance of the shear mapping Eq. (2)-Eq. (3) is that one can make the shape of fundamental diagrams mirror-symmetric with respect to the line $\rho = 1/2$ by using the appropriate value of v_0.

3. The conjecture

This paper entertains the conjecture that the general fundamental diagram $f^{u,r}(\rho)$ for model parameters $\mathrm{v}_{\max} = u$ and $p = r$ can be obtained from the basic solution $f^{1,p}(\rho)$ in Eq. (1) by the shear mapping Eq. (3), denoted by $\mathcal{S}(\cdot; v_0, \epsilon)$, with (u,r)-specific non-universal parameters $-\infty < v_0^{u,r} < -1, \epsilon^{u,r} > 1$ and $0 < p^{u,r} < 1$, which are allowed to be arbitrary functions of u, r in this paper, i.e.:

$$f^{u,r}(\rho) = \mathcal{S}(f^{1,p^{u,r}}(\rho); v_0^{u,r}, \epsilon^{u,r}) \tag{4}$$

Under this conjecture $f^{u,r}(\rho)$ corresponds to $\epsilon^{u,r} q$ where q solves $q = f^{1,p^{u,r}}(\rho - q/v_0^{u,r})$. It turns out that this solution can be expressed as:

$$f^{u,r}(\rho) = \epsilon^{u,r}(-b + \sqrt{b^2 - 4ac})/2a, \tag{5}$$

where $a = p^{u,r} + (v_0^{u,r})^2 - 1/v_0^{u,r}, b = 2(1 - p^{u,r})\rho + p^{u,r} - v_0^{u,r} - 1, c = (1-\rho)\rho v_0^{u,r}\,(1 - p^{u,r})$. ⋄

In the deterministic case this conjecture can be easily verified and the non-universal parameters easily determined; see the appendix. Unfortunately, this deterministic recipe does not work in the general case, and the challenge now is to formulate the non-universal parameters $v_0^{u,r}, \epsilon^{u,r}$ and $p^{u,r}$ as a function of both $\mathrm{v}_{\max} = u$ and $p = r$.

To understand the type of predictions made by the proposed framework, Fig. 1 shows Eq. (5) for a few values of $v_0^{u,r}$ and $p^{u,r}$, and using $\epsilon^{u,r} = 1 - 1/v_0^{u,r}$, a relationship that arises in the deterministic model (see appendix), and is used for illustration purposes only. It can be seen that: (i) the fundamental diagram gains curvature around the critical density as the slowdown probability increases, but the congested branch is mostly linear in accordance with empirical evidence, (ii) the capacity and wave speed (slope of the congested branch) decreases with the slowdown probability.

4. Comparison with existing results

Fig. 2 presents a comparison of the analytical solution proposed here (black curves) and the numerical results using Monte Carlo simulation (color thick lines) reported in [11] for $p = 0.25$ and for $\mathrm{v}_{\max} = 2, 3, 5$. In each case, the parameters $v_0^{u,r}, \epsilon^{u,r}, p^{u,r}$ were chosen to minimize the total sum of squared errors, and the result is shown in the bottom panels of the figure.

It can be seen that the fit is excellent under free-flow, but deteriorates under congestion. This is most visible for the case $\mathrm{v}_{\max} = 5$ where the model predicts a congested branch that is much more linear than the Monte Carlo simulation.

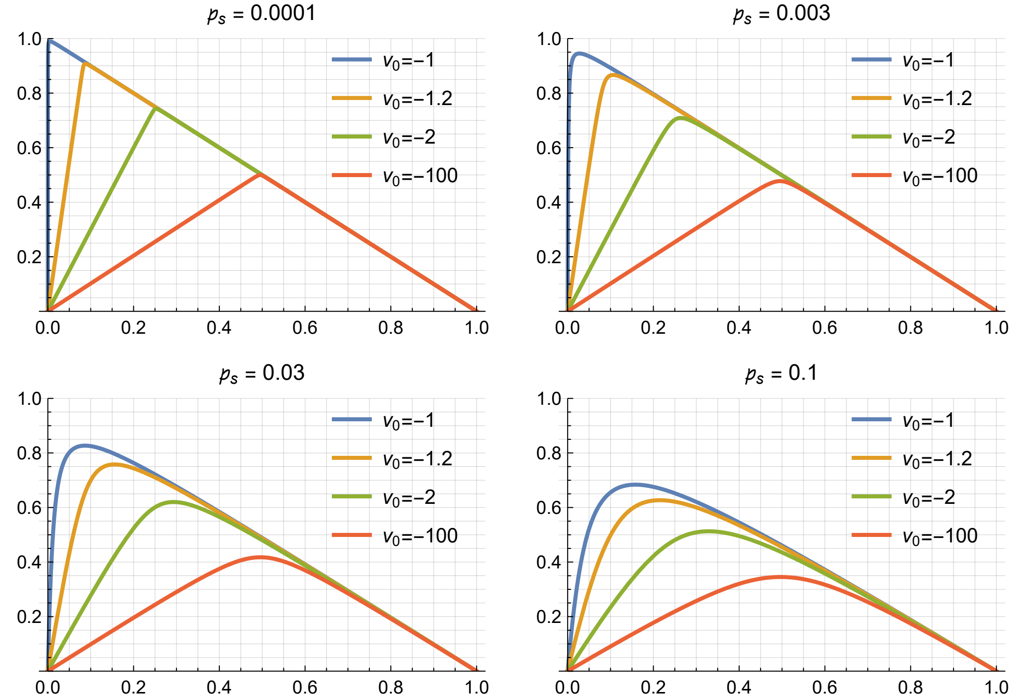

Fig. 1. $f^{u,r}(\rho)$ per of Eq. (5) for a few values of $v_0 = v_0^{u,r}$ and $p_s = p^{u,r}$ and $\epsilon^{u,0} = 1 - 1/v_0$ obtained by eliminating the maximum speed from Eq. (9)-Eq. (10).

5. Discussion in Outlook

This paper conjectures that the general solution for the NaSch model can be obtained from the basic solution for $\mathrm{v_{max}} = 1$ by a shear mapping. While this was not proven conclusively here, we can conclude that the shear mapping Eq. (3) will be a key ingredient of the solution, as it accounts for the universal properties of the problem across different values of $\mathrm{v_{max}}$. More precisely, the shear mapping captures the correct scaling behavior of the system with respect to the size parameter $\mathrm{v_{max}}$.

The challenge is to formulate the non-universal parameters $v_0^{u,r}, \epsilon^{u,r}$ and $p^{u,r}$ as a function of both $\mathrm{v_{max}} = u$ and $p = r$. From the optimization procedure in this paper, we can conjecture the following general forms of these functions for $r = 0.25$ only:

$$v_0^{u,r} = -(0.83 + 2.17/u) \quad [6a]$$
$$\epsilon^{u,r} = 1 + (1 - \exp(-1.14(u-1))/12.5 \quad [6b]$$
$$p^{u,r} = \exp(-1.15u - 0.23) \quad [6c]$$

as suggested by the lower panel in Fig. 2.

The Monte Carlo data reported in [11] shows more concavity than the more linear congested branches predicted by the theory. This indicates that there might be a missing ingredient in the theory. An important candidate at this point is the bounded acceleration that takes place in the model only for $u > 1$, currently not being captured in the theory. Another source of discrepancy might be the discrete nature of cellular automata compared to the continuum nature of the proposed theory. It may be possible that the discrepancies can be seen as a type of numerical error when approximating continuum problems; perhaps the system size of 10^4 cells used in [11] is not large enough?

Interestingly, Eq. (5) implies that $f^{u,r}$ is a solution to the elementary quadratic polynomial equation $aq^2 + bq + c = 0$, whose determinant is always positive. Naturally, the basic solution Eq. (1) also possesses this property, in which case $a = 1, b = -1, c = (1-p)(1-\rho)\rho$. This property of the solution might hint to a connection with the variational solution of traffic flow [15, 16], where flows are obtained by minimizing a cost function. It is possible that this quadratic equation represents the first-order optimality conditions.

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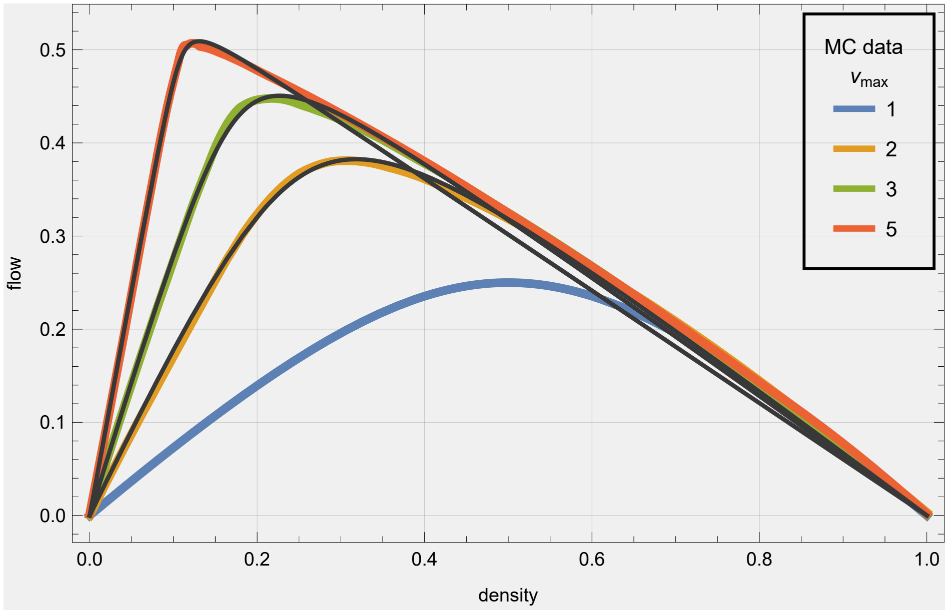

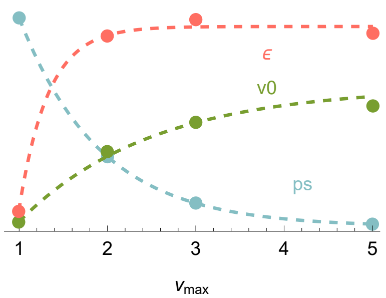

	p_s	v_0	ϵ
$v_{max}=2$	0.084	-1.956	1.077
$v_{max}=3$	0.028	-1.523	1.084
$v_{max}=5$	0.003	-1.28	1.078

$$v_0 = -3 + 2\left(1 - e^{-0.666446\,(-1+u)}\right)$$

$$p_s = e^{-0.294463-1.0919\,u}$$

$$\epsilon = 1 + 0.0806746\left(1 - e^{-3.0275\,(-1+u)}\right)$$

Fig. 2. Comparison of the analytical solution proposed here (black curves) and the numerical results using Monte Carlo simulation (color thick lines) reported in [11] for $p = 0.25$ and for $v_{max} = 2, 3, 5$. In each case, the parameters $v_0 = v_0^{u,r}$, $\epsilon = \epsilon^{u,r}$, $p_s = p^{u,r}$ were chosen to minimize the total sum of squared errors, and the result is shown in the bottom panels of the figure. (Superscripts u, r have been omitted for clarity.)

This connection should be explored, as it might pave the way for using the NaSch model as an aggregated model for cities.

A. Appendix

A. The deterministic case, $p \to 0$. In the deterministic case the fundamental diagram in the NaSch model with $v_{max} = u$ becomes triangular; Fig. 1:

$$f^{u,0}(\rho) = \min\{u\rho, 1-\rho\}, \qquad [7]$$

whose critical density and capacity are given by:

$$\rho_c^{u,0} = \frac{1}{1+u} \quad \text{and} \quad q_c^{u,0} = \frac{u}{1+u} \quad \text{respectevly.} \qquad [8]$$

To perform the shear transformation one might proceed in two steps. First, we slant the isosceles triangle to the left by increasing the value of the parameter v_0, which in this deterministic case also represents the speed scaling. This procedure yields $v_0^{u,0} \equiv (1/2)/(\rho_c^{u,0} - 1/2)$, or:

$$v_0^{u,0} = \frac{1+u}{1-u} \qquad [9]$$

The second step is to scale the flows appropriately by setting the parameter ϵ to:

$$\epsilon^{u,0} \equiv q_c^{u,0}/q_c^{1,0} = 2q_c^{u,0} \qquad [10]$$

With this, it is straightforward to check that the conjecture in this case is exact:

$$f^{u,0}(\rho) = \mathcal{S}(f^{1,0}(\rho); v_0^{u,0}, \epsilon^{u,0}) \qquad [11]$$

i.e., that $f^{u,0}(\rho)$ is given by the shear-transformed version of $f^{1,0}(\rho)$. To see this, the first step just described gives a function q that satisfies $q = f^{1,0}(\rho - q/v_0^{u,0})$, while the second steps simply says that $f^{u,0}(\rho)$ corresponds to $\epsilon^{u,0}q$.

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