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Constructions of Mixed Reverse-Order Laws for the Products of Matrices Involving Moore–Penrose Inverses and Group Inverses

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Constructions of Mixed Reverse-Order Laws for the Products of Matrices Involving Moore–Penrose Inverses and Group Inverses

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Abstract: It is a classic topic in algebras to construct and verify equalities that are composed of various algebraic operations of elements and their inverses or generalized inverses. In this note, the author constructs a matrix equality $(AB)^{\dagger} = B^*(A^*ABB^*)^{\#}A^*$ (called a mixed reverse-order law), where A and B are two matrices of appropriate sizes, $(\cdot)^*$, $(\cdot)^{\dagger}$, and $(\cdot)^{\#}$ denote the conjugate transpose, the Moore–Penrose inverse, and the group inverse of a matrix, respectively, and shows that the mixed reverse-order law always holds. A list of variation forms of this matrix equality are also given, and necessary and sufficient conditions are obtained for them to hold. Especially, we show a remarkable fact that the two reverse-order laws $(AB)^{\dagger} = B^{\dagger}A^{\dagger}$ and $(A^*ABB^*)^{\#} = (BB^*)^{\#}(A^*A)^{\#}$ are equivalent.

Keywords: block matrix; group inverse; Moore–Penrose inverse; range; rank; reverse-order law
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1. Introduction

Throughout this article, let $\mathbb{C}^{m\times n}$ denote the collection of all $m \times n$ matrices over the field of complex numbers; A^* denote the conjugate transpose of $A \in \mathbb{C}^{m\times n}$; $r(A)$ and $\mathscr{R}(A)$ stand for the rank and the range (column space) of $A \in \mathbb{C}^{m\times n}$, respectively. A matrix $A \in \mathbb{C}^{m\times m}$ is said to be EP (range-Hermitian) if and only if $\mathscr{R}(A) = \mathscr{R}(A^*)$ holds, namely, the two linear subspaces spanned by the column vectors of A and A^* coincide. The Moore–Penrose inverse of $A \in \mathbb{C}^{m\times n}$, denoted by $A^{\dagger}$, is defined to be the unique matrix $X \in \mathbb{C}^{n\times m}$ that satisfies the following four Penrose equations

$$(1)\ AXA = A,\ \ (2)\ XAX = X,\ \ (3)\ (AX)^* = AX,\ \ (4)\ (XA)^* = XA. \tag{1.1}$$

A square matrix $A \in \mathbb{C}^{m\times m}$ is said to be group invertible if and only if there exists an $X \in \mathbb{C}^{m\times m}$ that satisfies the following three matrix equations

$$(1)\ AXA = A,\ \ (2)\ XAX = X,\ \ (5)\ AX = XA. \tag{1.2}$$

In such a case, the matrix X, called the group inverse of A, is unique and is denoted by $X = A^{\#}$. It has been recognized that Moore–Penrose inverses and group inverses of matrices are two typical kinds of generalized inverses, which were defined and approached in matrix algebra and applications in 1950s, and thereby belong to the core and influential part in the discipline of generalized inverses, see e.g., [1–3].

As we know, constructions and characterizations of matrix equalities have been classical and attractive topics in matrix analysis and applications. Given the above definitions of the Moore–Penrose inverse and group inverse, one of the primary work is to construct and describe various matrix expressions and matrix equalities that are composed of algebraic operations of matrices and generalized inverses from theoretical and applied points of view. As usual, matrix equalities involving mixed operations of two matrices A and B of appropriate sizes and their Moore–Penrose inverses and group inverses can be generally represented as

$$f(A, A^*, A^{\dagger}, A^{\#}, B, B^*, B^{\dagger}, B^{\#}) = 0.$$

According to specified demands and interests, we can construct a tremendous number of matrix equalities as such due to the noncommutativity of matrix algebras and the singularity of a matrix. Yet, algebraists' particular interest are of the equalities that composed of mixed products of matrices and

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their generalized inverses. Here, we mention just two fundamental and representative examples as follows

$$(AB)^{\dagger} = B^{\dagger}A^{\dagger}, \ \ (AB)^{\#} = B^{\#}A^{\#} \tag{1.3}$$

for the purpose of presenting some background details of the work. As we know that these two matrix equalities are usually called the reverse-order laws for the Moore–Penrose inverse and the group inverse of the product of two matrices, respectively, in the theory of generalized inverses. Obviously, the two reverse-order laws in (1.3) are certain extensions of the best-known basic equality $(AB)^{-1} = B^{-1}A^{-1}$ for the standard inverse of the product of two invertible matrices in linear algebra. The two equalities, however, do not necessarily hold because of singularity of A and B and the noncommutativity of matrix algebra. Hence, there have been strong interests in the discipline of generalized inverses to search various reasonable conditions under which the two reverse-order laws hold since 1960s; see ([1] p. 161) and [20,21], while necessary and sufficient conditions for the second the reverse-order law in (1.3) can be found, e.g., in [4,8].

In addition to (1.3), various reverse-order equalities of mixed or nested type for Moore–Penrose inverses and group inverses of matrix products can be constructed through various feasible algebraic operations of matrices and their generalized inverses, such as,

$$(AB)^{\dagger} = (A^{\dagger}AB)^{\dagger}(ABB^{\dagger})^{\dagger}, \ \ (AB)^{\dagger} = B^{\dagger}(A^{\dagger}ABB^{\dagger})^{\dagger}A^{\dagger}, \ \ (AB)^{\dagger} = B^{*}(A^{*}ABB^{*})^{\dagger}A^{*}, \tag{1.4}$$

$$(AB)^{\#} = B^{\dagger}(A^{\dagger}ABB^{\dagger})^{\dagger}A^{\dagger}, \ \ (AB)^{\#} = B^{\dagger}(A^{\dagger}ABB^{\dagger})^{\#}A^{\dagger}, \ \ (AB)^{\#} = B^{\#}(A^{\dagger}ABB^{\dagger})^{\#}A^{\#}, \tag{1.5}$$

etc.; see [9,11–13,16–18]. Generally speaking, reverse-order law problems belong to theoretic aspects of generalized inverses, while the studies of these matrix equalities have brought essential progress of fundamental theory of generalized inverses in the past several decades.

In this note, we show how to construct various reasonable matrix equalities involving mixed products of the Moore–Penrose inverse and group inverse of a matrix by definitions of generalized inverses and matrix rank equalities. The main work is to study the following mixed reverse-order law

$$(AB)^{\dagger} = B^{*}(A^{*}ABB^{*})^{\#}A^{*} \tag{1.6}$$

and its variations, where $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$. Although this matrix equality involves operations of two matrices and their conjugate transposes, Moore–Penrose inverse, and group inverse, it can be directly reduced to the form $(AB)^{-1} = B^{-1}A^{-1}$ when A and B are two invertible matrices of the same size. The author will show that (1.6) always holds using the matrix rank method. The author then constructs a variety of variation forms of (1.6) involving the Moore–Penrose inverse and group inverses, and derives necessary and sufficient conditions for them to hold. Especially, we shall show a new fact that the first reverse-order law in (1.3) and the following reverse-order law

$$(A^{*}ABB^{*})^{\#} = (BB^{*})^{\#}(A^{*}A)^{\#} \tag{1.7}$$

are equivalent.

The rest of this note is organized as follows. In Section 2, we introduce some preliminary results and facts on ranks, ranges, and generalized inverses of matrices. In Section 3, we construct a series of mixed reverse-order laws involving Moore–Penrose inverses and group inverses of matrices, and derive necessary and sufficient conditions for them to hold using the matrix rank formula in Lemma 5. Some conclusions and remarks are given Section 4.

2. Some Preliminaries

In this section, we present some lemmas which provide a variety of commonly-used formulas and facts, which can be found in the literature, see e.g., [1,3], or easy to prove by the definitions of ranks, ranges, and generalized inverses of matrices.

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Lemma 1. *Let $A \in \mathbb{C}^{m\times n}$. Then,*

$$(A^{\dagger})^* = (A^*)^{\dagger}, \quad (A^{\dagger})^{\dagger} = A, \tag{2.1}$$

$$A^{\dagger} = A^*(AA^*)^{\dagger} = (A^*A)^{\dagger}A^* = A^*(A^*AA^*)^{\dagger}A^*, \tag{2.2}$$

$$(A^*)^{\dagger}A^* = (AA^{\dagger})^* = AA^{\dagger}, \ A^*(A^*)^{\dagger} = (A^{\dagger}A)^* = A^{\dagger}A, \tag{2.3}$$

$$(AA^*)^{\dagger} = (A^{\dagger})^*A^{\dagger}, \ (A^*A)^{\dagger} = A^{\dagger}(A^{\dagger})^*, \ (AA^*A)^{\dagger} = A^{\dagger}(A^{\dagger})^*A^{\dagger}. \tag{2.4}$$

Lemma 2. *Let $A \in \mathbb{C}^{m\times m}$. Then, A is group invertible if and only if $r(A^2) = r(A)$. In this case,*

$$A^{\#} = A(A^3)^{\dagger}A. \tag{2.5}$$

In particular,

$$A^{\dagger} = A^{\#} \Leftrightarrow \mathscr{R}(A) = \mathscr{R}(A^*). \tag{2.6}$$

Recall that that a matrix is null if and only if its rank is zero. As a direct consequence of this elementary fact we see that two matrices A and B of the same size are equal if and only if $r(A-B) = 0$. In view of this fact, we may figure out that if certain nontrivial and analytical formulas for calculating the rank of $A - B$ are obtained, they can reasonably be utilized to interpret essential links between the two matrices and to characterize the matrix quality $A = B$ in a convenient manner. The usefulness of this proposed method is in that we are able to precisely calculate the rank of matrix by elementary operations of matrices. Matrix rank formulas now are highly recognized as useful and reliable techniques to construct and characterize various simple or complicated algebraic equalities for matrices and their operations. In particular, algebraists have realized that reverse-order law problems for generalized inverses of products of matrices can be reasonably described by the matrix method; see some recent papers [18–22] by the present author regarding the matrix rank method in the investigations of reverse-order laws for generalized inverses of matrix products. Below, we give some existing basic facts and formulas on ranks of matrices.

Lemma 3. *Let $A_1 \in \mathbb{C}^{m\times n_1}$, $A_2 \in \mathbb{C}^{m\times n_2}$, $B_1 \in \mathbb{C}^{m\times p_1}$, and $B_2 \in \mathbb{C}^{m\times p_2}$. If $\mathscr{R}(A_1) = \mathscr{R}(B_1)$ and $\mathscr{R}(A_2) = \mathscr{R}(B_2)$, then two equalities $\mathscr{R}[A_1, A_2] = \mathscr{R}[B_1, B_2]$ and $r[A_1, A_2] = r[B_1, B_2]$ hold.*

Lemma 4. *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times p}$, $P \in \mathbb{C}^{p\times m}$, and $Q \in \mathbb{C}^{q\times n}$. Then,*

$$r(AA^*A) = r(AA^*) = r(A^*A) = r(A^{\dagger}) = r(A^*) = r(A), \tag{2.7}$$

$$\mathscr{R}(AA^*A) = \mathscr{R}(AA^*) = \mathscr{R}(AA^{\dagger}) = \mathscr{R}((A^{\dagger})^*) = \mathscr{R}(A), \tag{2.8}$$

$$\mathscr{R}(A^*AA^*) = \mathscr{R}(A^*A) = \mathscr{R}(A^{\dagger}A) = \mathscr{R}(A^{\dagger}) = \mathscr{R}(A^*), \tag{2.9}$$

$$\mathscr{R}(AQ^{\dagger}Q) = \mathscr{R}(AQ^{\dagger}) = \mathscr{R}(AQ^*Q) = \mathscr{R}(AQ^*), \tag{2.10}$$

and

$$\mathscr{R}(A) \subseteq \mathscr{R}(B) \ \text{ and } \ r(A) = r(B) \Rightarrow \mathscr{R}(A) = \mathscr{R}(B). \tag{2.11}$$

It is trivial to see that

$$f(A,\, A^*,\, A^{\dagger},\, A^{\#}, B,\, B^*, B^{\dagger}, B^{\#}) = 0 \Leftrightarrow r(f(A,\, A^*,\, A^{\dagger},\, A^{\#}, B,\, B^*, B^{\dagger}, B^{\#})) = 0.$$

Thus, if a certain analytical formula are established for calculating the rank of $f(\cdot)$, we can derive certain necessary and sufficient conditions for $f(\cdot)$ to hold from the rank formula. This fact has been proved to be a feasible and effective methodology since 1970s to construct and describe various matrix equalities that involve generalized inverses of matrices. One of such representative formulas we shall use is given below.

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Lemma 5 ([15]). *Let $A, B \in \mathbb{C}^{m\times n}$, and assume that $AXA = A$ and $BXB = B$ hold for an $X \in \mathbb{C}^{n\times m}$, namely, A and B are outer inverses of X. Then, the following rank equality*

$$r(A - B) = r\begin{bmatrix} A \\ B \end{bmatrix} + r[A,\, B] - r(A) - r(B) \tag{2.12}$$

holds. Therefore,

$$A = B \Leftrightarrow r\begin{bmatrix} A \\ B \end{bmatrix} + r[A,\, B] = r(A) + r(B) \Leftrightarrow \mathscr{R}(A) = \mathscr{R}(B) \text{ and } \mathscr{R}(A^*) = \mathscr{R}(B^*). \tag{2.13}$$

Lemma 6 ([1]). *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{m\times p}$. Then*

$$(AB)^{\dagger} = B^{\dagger}A^{\dagger} \Leftrightarrow \mathscr{R}(A^*ABB^*) = \mathscr{R}(BB^*A^*A). \tag{2.14}$$

3. Main Results

In this section, we construct a series of mixed reverse-order laws involving Moore–Penrose inverses and group inverses of matrices, and derive necessary and sufficient conditions for them to hold using the matrix rank formula in Lemma 5.

Theorem 7. *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$. Then, the matrix product A^*ABB^* is group invertible and the mixed reverse-order law in (1.6) always holds.*

Proof. Since the matrix product A^*ABB^* is a square matrix, $(A^*ABB^*)^2$ is defined, and its rank satisfies

$$\begin{aligned} r((A^*ABB^*)^2) &\le r(A^*ABB^*), \\ r((A^*ABB^*)^2) &\ge r((A^*ABB^*)^2A^*A) \\ &= r(A^*ABB^*A^*(A^*ABB^*A^*)^*) \\ &= r(A^*ABB^*A^*) \\ &= r(A^*ABB^*) \end{aligned}$$

by (2.7), and thus, $r((A^*ABB^*)^2) = r(A^*ABB^*)$ holds. This rank equality implies that $(A^*ABB^*)^{\#}$ exists by Lemma 2. Also by (2.8)–(2.11),

$$\mathscr{R}((AB)^{\dagger}) = \mathscr{R}((AB)^*), \;\; \mathscr{R}(((AB)^{\dagger})^*) = \mathscr{R}(AB), \tag{3.1}$$

$$\mathscr{R}(B^*(A^*ABB^*)^{\#}A^*) = \mathscr{R}(B^*A^*), \;\; \mathscr{R}((B^*(A^*ABB^*)^{\#}A^*)^*) = \mathscr{R}(AB). \tag{3.2}$$

Further by the definitions of the Moore–Penrose inverse and the group inverse, $(AB)^{\dagger}$ and $B^*(A^*ABB^*)^{\#}A^*$ are outer inverses of AB. In this situation, applying (2.12) to the difference of $(AB)^{\dagger}$ and $B^*(A^*ABB^*)^{\#}A^*$ and simplifying by Lemma 3, (3.1), and (3.2), we obtain

$$\begin{aligned} r((AB)^{\dagger} - B^*(A^*ABB^*)^{\#}A^*) &= r\begin{bmatrix} (AB)^{\dagger} \\ B^*(A^*ABB^*)^{\#}A^* \end{bmatrix} + r[(AB)^{\dagger},\, B^*(A^*ABB^*)^{\#}A^*] \\ &\quad - r((AB)^{\dagger}) - r(B^*(A^*ABB^*)^{\#}A^*) \\ &= r\begin{bmatrix} (AB)^* \\ B^*A^* \end{bmatrix} + r[(AB)^*,\, B^*A^*] - 2r(AB) \\ &= r\begin{bmatrix} (AB)^* \\ 0 \end{bmatrix} + r[(AB)^*,\, 0] - 2r(AB) = 0, \end{aligned}$$

thus establishing the equality in (1.6). □

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Corollary 8. *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$. Then, the following mixed reverse-order laws always hold:*

$$(A^{\dagger}AB)^{\dagger} = B^{*}(A^{\dagger}ABB^{*})^{\#}A^{\dagger}A, \tag{3.3}$$
$$(ABB^{\dagger})^{\dagger} = BB^{\dagger}(A^{*}ABB^{\dagger})^{\#}A^{*}, \tag{3.4}$$
$$(A^{\dagger}ABB^{\dagger})^{\dagger} = BB^{\dagger}(A^{\dagger}ABB^{\dagger})^{\#}A^{\dagger}A, \tag{3.5}$$
$$(AB)^{\dagger} = B^{*}(A^{\dagger}ABB^{*})^{\#}A^{\dagger}ABB^{\dagger}(A^{*}ABB^{\dagger})^{\#}A^{*}, \tag{3.6}$$
$$(A^{*}ABB^{*})^{\dagger} = BB^{*}((A^{*}A)^{2}(BB^{*})^{2})^{\#}A^{*}A. \tag{3.7}$$

Proof. Replacing A and B in (1.6) with $A^{\dagger}A$ and $BB^{\dagger}$, respectively or simultaneously, and simplifying lead to (3.3), (3.4), and (3.5). Substitution of (3.3) and (3.4) into the well-known equality $(AB)^{\dagger} = (A^{\dagger}AB)^{\dagger}(ABB^{\dagger})^{\dagger}$ leads to (3.6). Replacing A and B in (1.6) with $A^{*}A$ and BB^{*} leads to (3.7). □

Theorem 9. *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$. Then, the following three rank equalities*

$$\begin{aligned} r((AB)^{\dagger} - B^{*}(A^{*}ABB^{*})^{\dagger}A^{*}) &= r((AB)^{\dagger} - B^{\dagger}(A^{\dagger}ABB^{\dagger})^{\dagger}A^{\dagger}) \\ &= r((AB)^{\dagger} - B^{\dagger}(A^{\dagger}ABB^{\dagger})^{\#}A^{\dagger}) \\ &= r\begin{bmatrix} ABB^{*}B \\ AB \end{bmatrix} + r[AA^{*}AB,\ AB] - 2r(AB) \end{aligned} \tag{3.8}$$

hold. Hence,

$$\begin{aligned} (AB)^{\dagger} = B^{*}(A^{*}ABB^{*})^{\dagger}A^{*} &\Leftrightarrow (AB)^{\dagger} = B^{\dagger}(A^{\dagger}ABB^{\dagger})^{\dagger}A^{\dagger} \\ &\Leftrightarrow (AB)^{\dagger} = B^{\dagger}(A^{\dagger}ABB^{\dagger})^{\#}A^{\dagger} \\ &\Leftrightarrow r\begin{bmatrix} ABB^{*}B \\ AB \end{bmatrix} = r[AA^{*}AB,\ AB] = r(AB). \end{aligned} \tag{3.9}$$

Proof. The first two equalities in (3.8) were given in [16]. Substitution of (3.5) into the second equality in (3.8) leads to the third equality in (3.8). □

Theorem 10. *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$. Then,*

$$r((AB)^{\dagger} - B^{*}(A^{\dagger}ABB^{*})^{\#}A^{\dagger}) = r[AA^{*}AB,\ AB] - r(AB), \tag{3.10}$$
$$r((AB)^{\dagger} - B^{\dagger}(A^{*}ABB^{\dagger})^{\#}A^{*}) = r\begin{bmatrix} ABB^{*}B \\ AB \end{bmatrix} - r(AB), \tag{3.11}$$
$$r((AB)^{\dagger} - B^{\dagger}(A^{\dagger}ABB^{\dagger})^{\#}A^{\dagger}) = r\begin{bmatrix} ABB^{*}B \\ AB \end{bmatrix} + r[AA^{*}AB,\ AB] - 2r(AB), \tag{3.12}$$
$$r((AB)^{\dagger} - B^{*}BB^{*}((A^{*}A)^{2}(BB^{*})^{2})^{\#}A^{*}AA^{*}) = r\begin{bmatrix} ABB^{*}B \\ AB \end{bmatrix} + r[AA^{*}AB,\ AB] - 2r(AB) \tag{3.13}$$

hold. Hence,

$$(AB)^{\dagger} = B^{*}(A^{\dagger}ABB^{*})^{\#}A^{\dagger} \Leftrightarrow r[AA^{*}AB,\ AB] = r(AB), \tag{3.14}$$
$$(AB)^{\dagger} = B^{\dagger}(A^{*}ABB^{\dagger})^{\#}A^{*} \Leftrightarrow r\begin{bmatrix} ABB^{*}B \\ AB \end{bmatrix} = r(AB), \tag{3.15}$$
$$\begin{aligned} (AB)^{\dagger} = B^{\dagger}(A^{\dagger}ABB^{\dagger})^{\#}A^{\dagger} &\Leftrightarrow (AB)^{\dagger} = B^{*}(A^{\dagger}ABB^{*})^{\#}A^{\dagger} = B^{\dagger}(A^{*}ABB^{\dagger})^{\#}A^{*} \\ &\Leftrightarrow (AB)^{\dagger} = B^{*}BB^{*}((A^{*}A)^{2}(BB^{*})^{2})^{\#}A^{*}AA^{*} \\ &\Leftrightarrow r\begin{bmatrix} ABB^{*}B \\ AB \end{bmatrix} = r[AA^{*}AB,\ AB] = r(AB). \end{aligned} \tag{3.16}$$

Proof. Eqs. (3.10)–(3.13) follow from (2.12) and simplifications of the corresponding matrix rank equalities. □

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Corollary 11. *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$. Then,*

$$(A^{\dagger}AB)^{\dagger} = B^{*}(A^{\dagger}ABB^{*})^{\#}A^{\dagger}A \Leftrightarrow r[AA^{*}AB,\ AB] = r(AB), \tag{3.17}$$

$$(A^{\dagger}AB)^{\dagger} = B^{\dagger}(A^{\dagger}ABB^{\dagger})^{\#}A^{\dagger}A \Leftrightarrow r\begin{bmatrix} ABB^{*}B \\ AB \end{bmatrix} = r[AA^{*}AB,\ AB] = r(AB), \tag{3.18}$$

$$(ABB^{\dagger})^{\dagger} = BB^{\dagger}(A^{*}ABB^{\dagger})^{\#}A^{*} \Leftrightarrow r[AA^{*}AB,\ AB] = r(AB), \tag{3.19}$$

$$(ABB^{\dagger})^{\dagger} = BB^{\dagger}(A^{\dagger}ABB^{\dagger})^{\#}A^{\dagger} \Leftrightarrow r\begin{bmatrix} ABB^{*}B \\ AB \end{bmatrix} = r[AA^{*}AB,\ AB] = r(AB). \tag{3.20}$$

Proof. Replacing A and B in (3.14), (3.15), and (3.16) with $A^{\dagger}A$ and $BB^{\dagger}$, respectively, and simplifying lead to (3.17)–(3.20). □

Below, we show the equivalence of the first equality in (1.3) and the one in (1.7).

Theorem 12. *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$. Then,*

$$(AB)^{\dagger} = B^{\dagger}A^{\dagger} \Leftrightarrow (A^{*}ABB^{*})^{\#} = (BB^{*})^{\#}(A^{*}A)^{\#}. \tag{3.21}$$

Proof. Since $A^{*}A$ and BB^{*} are Hermitian, it is easy to verify by the definitions of the Moore–Penrose inverse and the group inverse that $(A^{*}A)^{\#} = (A^{*}A)^{\dagger}$ and $(BB^{*})^{\#} = (BB^{*})^{\dagger}$ hold. Substitution of the two equalities into the second equality in (3.21) and then the second equality into (1.6) lead to

$$(AB)^{\dagger} = B^{*}(A^{*}ABB^{*})^{\#}A^{*} = B^{*}(BB^{*})^{\dagger}(A^{*}A)^{\dagger}A^{*} = B^{\dagger}A^{\dagger}$$

by (2.2). Conversely, we are first able to obtain from (2.14) that

$$(A^{*}ABB^{*})^{\dagger} = (BB^{*})^{\dagger}(A^{*}A)^{\dagger} \tag{3.22}$$

holds. Further, we obtain from (2.14), (2.6), and (3.22) that

$$(A^{*}ABB^{*})^{\#} = (A^{*}ABB^{*})^{\dagger} = (BB^{*})^{\dagger}(A^{*}A)^{\dagger} = (BB^{*})^{\#}(A^{*}A)^{\#},$$

as required for the second equality in (3.21). □

Corollary 13. *Let $A \in \mathbb{C}^{m\times m}$. Then, $(A^{*}A^{2}A^{*})^{\#}$ exists, and we have the following results.*

(a) *The following equalities always hold:*

$$\begin{aligned}
(A^{2})^{\dagger} &= A^{*}(A^{*}A^{2}A^{*})^{\#}A^{*}, \\
(A^{2})^{\dagger} &= A^{*}(A^{\dagger}A^{2}A^{*})^{\#}A^{\dagger}A^{2}A^{\dagger}(A^{*}A^{2}A^{\dagger})^{\#}A^{*}, \\
(A^{\dagger}A^{2})^{\dagger} &= A^{*}(A^{\dagger}A^{2}A^{*})^{\#}A^{\dagger}A, \\
(A^{2}A^{\dagger})^{\dagger} &= AA^{\dagger}(A^{*}A^{2}A^{\dagger})^{\#}A^{*}, \\
(A^{\dagger}A^{2}A^{\dagger})^{\dagger} &= AA^{\dagger}(A^{\dagger}A^{2}A^{\dagger})^{\#}A^{\dagger}A, \\
(A^{*}A^{2}A^{*})^{\dagger} &= AA^{*}((A^{*}A)^{2}(AA^{*})^{2})^{\#}A^{*}A.
\end{aligned}$$

(b) *If the condition $r(A^{2}) = r(A)$, then*

$$\begin{aligned}
(A^{2})^{\dagger} &= A^{*}(A^{*}A^{2}A^{*})^{\dagger}A^{*} = A^{\dagger}(A^{\dagger}A^{2}A^{\dagger})^{\dagger}A^{\dagger} \\
&= A^{\dagger}(A^{\dagger}A^{2}A^{\dagger})^{\#}A^{\dagger} = A^{*}AA^{*}((A^{*}A)^{2}(AA^{*})^{2})^{\#}A^{*}AA^{*}, \\
(A^{3})^{\dagger} &= A^{*}(A^{*}A^{2}A^{*})^{\dagger}A^{*}AA^{*}(A^{*}A^{2}A^{*})^{\dagger}A^{*} = A^{\dagger}(A^{\dagger}A^{2}A^{\dagger})^{\dagger}A^{\dagger}(A^{\dagger}A^{2}A^{\dagger})^{\dagger}A^{\dagger}.
\end{aligned}$$

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(c) *The following facts hold*:

$$\begin{aligned}(A^2)^{\dagger} = A^*(A^*A^2A^*)^{\dagger}A^* &\Leftrightarrow (A^2)^{\dagger} = A^{\dagger}(A^{\dagger}A^2A^{\dagger})^{\dagger}A^{\dagger}\\ &\Leftrightarrow (A^2)^{\dagger} = A^{\dagger}(A^{\dagger}A^2A^{\dagger})^{\#}A^{\dagger}\\ &\Leftrightarrow (A^2)^{\dagger} = A^*AA^*((A^*A)^2(AA^*)^2)^{\#}A^*AA^*\\ &\Leftrightarrow r\begin{bmatrix} A^2A^*A \\ A^2 \end{bmatrix} = r[AA^*A^2,\ A^2] = r(A^2).\end{aligned}$$

(d) $(A^2)^{\dagger} = (A^{\dagger})^2 \Leftrightarrow (A^*A^2A^*)^{\#} = (AA^*)^{\#}(A^*A)^{\#}$.

Proof. It follows from setting $A = B$ in the preceding theorems and corollaries. □

Finally, we present several results on mixed products of $A^{\dagger}$ and $A^{\#}$ and their generalized inverses.

Theorem 14. *Let $A \in \mathbb{C}^{m\times m}$ and assume that $r(A^2) = r(A)$. Then, we have the following results.*

(a) *The following four reverse-order laws always hold*:

$$(A^{\#}A^{\dagger})^{\dagger} = A(A^{\#})^{\dagger},\ \ (A^{\dagger}A^{\#})^{\dagger} = (A^{\#})^{\dagger}A,$$
$$(A^{\#}A^{\dagger}A^{\#})^{\dagger} = (A^{\#})^{\dagger}A(A^{\#})^{\dagger},\ \ (A^{\dagger}A^{\#}A^{\dagger})^{\dagger} = A(A^{\#})^{\dagger}A.$$

(b) *The following four matrix equalities always hold*:

$$(A^{\#}A^{\dagger})^{\dagger}A^{\#} = A^{\#}(A^{\dagger}A^{\#})^{\dagger} = A^{\#}(A^{\#}A^{\dagger}A^{\#})^{\dagger}A^{\#} = ((A^{\dagger}(A^{\dagger}A^{\#}A^{\dagger})^{\dagger}A^{\dagger})^{\dagger})^{\#} = A.$$

(c) *The following four reverse-order laws always hold*:

$$((A^{\#}A^{\dagger})^k)^{\dagger} = (A(A^{\#})^{\dagger})^k,\ \ ((A^{\dagger}A^{\#})^k)^{\dagger} = ((A^{\#})^{\dagger}A)^k,\ \ k \geq 2,$$
$$((A^{\#}A^{\dagger})^kA^{\#})^{\dagger} = ((A^{\#})^{\dagger}A)^k(A^{\#})^{\dagger},\ \ ((A^{\dagger}A^{\#})^kA^{\dagger})^{\dagger} = (A(A^{\#})^{\dagger})^kA,\ \ k \geq 2.$$

Proof. Follows from (2.14). □

4. Conclusions

We constructed a variety of matrix equalities for mixed products of two matrices and their generalized inverses, and developed a sequence of novel and pleasing results and equivalent facts. Since this work is conducted by means of ordinary algebraic operations of matrices and their generalized inverses, including the cogent use of the block matrix representation method and the matrix rank method, the findings in the preceding sections are easy to understand within the scope of generalized inverses of matrices, and thus bring convenience of extending this work to general algebraic settings, such as, rings and operator algebras, in which generalized inverses of objects, as well as reverse-order laws for generalized inverses can be defined accordingly.

References

1. A. Ben–Israel, T.N.E. Greville. Generalized Inverses: Theory and Applications. 2nd ed., Springer, New York, 2003.
2. D.S. Bernstein. Scalar, Vector, and Matrix Mathematics: Theory, Facts, and Formulas–Revised and Expanded Edition. Princeton University Press, Princeton, NJ, 2018.
3. S.L. Campbell, C.D. Meyer, Jr. Generalized Inverses of Linear Transformations. Pitman Publishing Limited, 1979; Dover Publications, 1991; SIAM, Philadelphia, 2009.
4. C. Cao, X. Zhang, X. Tang. Reverse order law of group inverses of products of two matrices. Appl. Math. Comput. 158(2004), 489–495.
5. J. Cen. On existence of weighted group inverse of rectangular matrix (in Chinese). Math. Numer. Sin. 29(2007), 39–48.
6. Y. Chen. Existence conditions and expressions for weighted group inverses of rectangular matrices. J. Nanjing Norm. Univ. Nat. Sci. Edn. 31-3(2008), 1–5.

7. R.E. Cline, T. Greville. A Drazin inverse for rectangular matrices. Linear Algebra Appl. 29(1980), 53–62.
8. C.-Y. Deng. Reverse order law for the group inverses. J. Anal. Math. Appl. 382(2011), 663–671.
9. S. Izumino. The product of operators with closed range and an extension of the reverse order law. Tôhoku Math. J. 34(1982), 43–52.
10. X. Liu, Z. Xu, Q. Zhao, H. Wei. On the perturbation of weighted group inverse of rectangular matrices. J Appl. Math. Comput. 42(2013), 441–454.
11. D. Mosić, D.S. Djordjević. Reverse order law for the Moore–Penrose inverse in C^*-algebras. Electron. J. Linear Algebra 22(2011), 92–111.
12. D. Mosić, D.S. Djordjević. Reverse order law $(ab)^{\dagger} = b^{\dagger}(a^{\dagger}abb^{\dagger})^{\dagger}a^{\dagger}$ in rings with involution. Filomat 28(2014), 1791–1815.
13. D. Mosić, D.S. Djordjević. The reverse order law $(ab)^{\#} = b^{\dagger}(a^{\dagger}abb^{\dagger})^{\dagger}a^{\dagger}$ in rings with involution. RACSAM 109(2015), 257–265.
14. X. Sheng, G. Chen. The computation and perturbation analysis for weighted group inverse of rectangular matrices. J. Appl. Math. Comput. 31(2008), 33–43.
15. Y. Tian. Rank equalities related to outer inverses of matrices and applications. Linear Multilinear Algebra 49(2002), 269–288.
16. Y. Tian. The reverse-order law $(AB)^{\dagger} = B^{\dagger}(A^{\dagger}ABB^{\dagger})^{\dagger}A^{\dagger}$ and its equivalent equalities. J. Math. Kyoto Univ. 45(2005), 841–850.
17. Y. Tian. The equivalence between $(AB)^{\dagger} = B^{\dagger}A^{\dagger}$ and other mixed-type reverse-order laws. Internat. J. Math. Edu. Sci. Tech. 37(2007), 331–339.
18. Y. Tian. A family of 512 reverse order laws for generalized inverses of a matrix product: a review. Heliyon 6(2020), e04924.
19. Y. Tian. Miscellaneous reverse order laws and their equivalent facts for generalized inverses of a triple matrix product. AIMS Math. 6(2021), 13845–13886.
20. Y. Tian. A study of range equalities for matrix expressions that involve matrices and their generalized inverses. Comput. Appl. Math. 41(2022), 384.
21. Y. Tian. Equivalence analysis of different reverse order laws for generalized inverses of a matrix product. Indian J. Pure Appl. Math. 53(2022), 939–947.
22. Y. Tian. Characterizations of the group invertibility of a matrix revisited. Demonstratio Math. 55(2022), 866–890.