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# Absolutely Summing Morphisms between Hilbert C*-Modules and Modular Pietsch Factorization Problem

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# Absolutely Summing Morphisms between Hilbert C*-Modules and Modular Pietsch Factorization Problem

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**Abstract:** Motivated from the theory of Hilbert-Schmidt morphisms between Hilbert C*-modules over commutative C*-algebras by Stern and van Suijlekom *[J. Funct. Anal., 2021]*, we introduce the notion of p-absolutely summing morphisms between Hilbert C*-modules over commutative C*-algebras. We show that an adjointable morphism between Hilbert C*-modules over monotone closed commutative C*-algebra is 2-absolutely summing if and only if it is Hilbert-Schmidt. We formulate version of Pietsch factorization problem for p-absolutely summing morphisms and solve partially.

**Keywords:** absolutely summing operator; commutative C*-algebra; Hilbert C*-module; Hilbert-Schmidt operator

**MSC:** 47B10, 47L20, 46L08, 46L05, 42C15

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## 1. Introduction

In his *Resume*, A. Grothendieck studied 1-absolutely and 2-absolutely summing operators between Banach spaces [1] (also see [2]). In 1967, for each 1≤ p< ∞, A. Pietsch introduced the notion of p-absolutely summing operators which became an area around the end of 20 century [3–20]. In 1979, Tomczak-Jaegermann studied p-summing operators by fixing by fixing number of points [21]. In 1970, Kwapien defined the notion of 0-summing operators [22]. In 2003, Farmer and Johnson introduced the notion of Lipschitz p-summing operators between metric spaces [23] (also see [24–27]).

**Definition 1.** *[7,28] Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces, $\mathcal{X}^*$ be the dual of $\mathcal{X}$ and $1 \leq p < \infty$. A bounded linear operator $T : \mathcal{X} \to \mathcal{Y}$ is said to be* ***p-absolutely summing*** *if there is a real constant $C > 0$ satisfying following: for every $n \in \mathbb{N}$ and for all $x_1, \dots, x_n \in \mathcal{X}$,*

$$\left(\sum_{j=1}^{n} \|Tx_j\|^p\right)^{\frac{1}{p}} \leq C \sup_{f \in \mathcal{X}^*, \|f\| \leq 1} \left(\sum_{j=1}^{n} |f(x_j)|^p\right)^{\frac{1}{p}}. \tag{1}$$

*In this case, the* ***p-absolutely summing norm*** *of $T$ is defined as*

$$\pi_p(T) := \inf\{C : C \text{satisfies Inequality} (1)\}.$$

*The set of all p-absolutely summing operators from $\mathcal{X}$ to $\mathcal{Y}$ is denoted by $\Pi_p(\mathcal{X}, \mathcal{Y})$.*

Following are most important results in the theory of p-absolutely summing operators.

**Theorem 1.** *[7,11] Let $1 \leq p < \infty$ and $\mathcal{X}, \mathcal{Y}$ be Banach spaces. Then $(\Pi_p(\mathcal{X}, \mathcal{Y}), \pi_p(\cdot))$ is an operator ideal.*

**Theorem 2.** *[7,28] Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. Then a bounded linear operator $T : \mathcal{H} \to \mathcal{K}$ is 2-absolutely summing if and only if it is Hilbert-Schmidt. Moreover, $\|T\|_{HS} = \pi_2(T)$.*

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**Theorem 3.** *[7,28] (**Pietsch Factorization Theorem**) Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces. A bounded linear operator $T : \mathcal{X} \to \mathcal{Y}$ is p-absolutely summing if and only if there is a real constant $C > 0$ and a regular Borel probability measure on $B_{\mathcal{X}^*} := \{f : f \in \mathcal{X}^*, \|f\| \leq 1\}$ in weak*-topology such that*

$$\|Tx\| \leq C \left( \int_{B_{\mathcal{X}^*}} |f(x)|^p \, d\mu_{B_{\mathcal{X}^*}}(f) \right)^{\frac{1}{p}}, \quad \forall x \in \mathcal{X}. \tag{2}$$

*Moreover, $\pi_p(T) = \inf\{C : C \text{ satisfies Inequality } (2)\}$.*

In this paper, we define the notion of p-absolutely summing morphisms between Hilbert C*-modules over commutative C*-algebras (Definition 2). We derive in Theorem 5 that an adjointable morphism between Hilbert C*-module over a monote closed C*-algebra is 2-summing if and only if modular Hilbert-Schmidt. We then formulate version of Pietsch factorization problem for p-absolutely summing morphisms and solve partially.

## 2. p-absolutely summing morphisms

We define modular version of Definition 1 as follows. For the theory of Hilbert C*-modules we refer [29–31].

**Definition 2.** *Let $1 \leq p < \infty$. Let $\mathcal{M}$ and $\mathcal{N}$ be Hilbert C*-modules over a commutative C*-algebra $\mathcal{A}$. An adjointable morphism $T : \mathcal{M} \to \mathcal{N}$ is said to be **modular p-absolutely summing** if there is a real constant $C > 0$ satisfying following: for every $n \in \mathbb{N}$ and for all $x_1, \dots, x_n \in \mathcal{M}$,*

$$\left\| \sum_{j=1}^{n} \langle Tx_j, Tx_j \rangle^{\frac{p}{2}} \right\|^{\frac{1}{p}} \leq C \sup_{x \in \mathcal{M}, \|x\| \leq 1} \left\| \sum_{j=1}^{n} (\langle x, x_j \rangle \langle x_j, x \rangle)^{\frac{p}{2}} \right\|^{\frac{1}{p}}. \tag{3}$$

*In this case, the p-absolutely summing norm of $T$ is defined as*

$$\pi_p(T) := \inf\{C : C \text{ satisfies Inequality } (3)\}.$$

*The set of all p-absolutely summing morphisms from $\mathcal{M}$ to $\mathcal{N}$ is denoted by $\Pi_p(\mathcal{M}, \mathcal{N})$.*

In 2021, Stern and van Suijlekom introduced the notion of modular Schatten class morphisms [32].

**Definition 3.** *[32] Let $1 \leq p < \infty$. Let $\mathcal{A}$ be a C*-algebra and $\widehat{\mathcal{A}}$ be its Gelfand spectrum. Let $\mathcal{M}$ and $\mathcal{N}$ be Hilbert C*-modules over $\mathcal{A}$. Let $T : \mathcal{M} \to \mathcal{N}$ be an adjointable morphism. We say that $T$ is in the **modular p-Schatten class** if the function*

$$\operatorname{Tr} |T|^p : \widehat{\mathcal{A}} \ni \chi \mapsto \operatorname{Tr} |\chi_* T|^p \in \mathbb{R} \cup \{\infty\}$$

*lies in $\mathcal{A}$. The modular p-Schatten norm of $T$ is defined as*

$$\|T\|_p := \| \operatorname{Tr} |T|^p \|_{\mathcal{A}}^{\frac{1}{p}}.$$

*Modular 2-Schatten (resp. 1-Schatten) class morphism is called as **modular Hilbert-Schmidt** (resp. **modular trace class**). We denote $\|T\|_2$ by $\|T\|_{HS}$.*

Using the theory of modular frames for Hilbert C*-modules (see [33]) Stern and van Suijlekom were able to derive following result.

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**Theorem 4.** *[32] Let $\mathcal{M}$ and $\mathcal{N}$ be Hilbert C*-modules over $\mathcal{A}$. Let $T : \mathcal{M} \to \mathcal{N}$ be an adjointable morphism. Then $T$ is modular Hilbert-Schmidt if and only if for every modular Parseval frame $\{\tau_n\}_n$ for $\mathcal{M}$, the series $\sum_{n=1}^{\infty}\langle |T|^p\tau_n, \tau_n\rangle$ converges in norm in $\mathcal{A}$ and*

$$\operatorname{Tr}|T|^p = \sum_{n=1}^{\infty}\langle |T|^p\tau_n, \tau_n\rangle.$$

We now derive modular version of Theorem 2 with the following notion.

**Definition 4.** *[34] A C*-algebra $\mathcal{A}$ is said to be **monotone closed** if every bounded increasing net in $\mathcal{A}$ has the least upper bound in $\mathcal{A}$.*

**Theorem 5.** *Let $\mathcal{M}$ and $\mathcal{N}$ be Hilbert C*-modules over a commutative C*-algebra $\mathcal{A}$. Assume $\mathcal{A}$ is monotone closed. Let $T : \mathcal{M} \to \mathcal{N}$ be an adjointable morphism. Then $T \in \Pi_2(\mathcal{M}, \mathcal{N})$ if and only if $T$ is modular Hilbert-Schmidt. Moreover, $\|T\|_{HS} = \pi_2(T)$.*

**Proof.** ($\Rightarrow$) Let $T \in \Pi_2(\mathcal{M}, \mathcal{N})$. Let $\{\tau_n\}_{n=1}^{\infty}$ be a modular Parseval frame for $\mathcal{M}$. Then

$$\langle x, x\rangle = \sum_{n=1}^{\infty}\langle x, \tau_n\rangle\langle \tau_n, x\rangle, \quad \forall x \in \mathcal{M}, \tag{4}$$

where the series converges in the norm of $\mathcal{A}$. To show $T$ is modular Hilbert-Schmidt, using Theorem 4, it suffices to show that the series $\sum_{n=1}^{\infty}\langle T\tau_n, T\tau_n\rangle$ converges in norm in $\mathcal{A}$. Note that the series $\sum_{n=1}^{\infty}\langle T\tau_n, T\tau_n\rangle$ is monotonically increasing. Since the C*-algebra is monotone closed, we are done if we show the sequence $\{\sum_{j=1}^{n}\langle T\tau_j, T\tau_j\rangle\}_{n=1}^{\infty}$ is bounded. Let $n \in \mathbb{N}$. Since $T$ is 2-summing, using Equation (4) we have

$$\begin{aligned}\left\|\sum_{j=1}^{n}\langle T\tau_j, T\tau_j\rangle\right\| &\leq \pi_2(T)^2 \sup_{x\in\mathcal{M}, \|x\|\leq 1}\left\|\sum_{j=1}^{n}\langle x, \tau_j\rangle\langle \tau_j, x\rangle\right\| \leq \pi_2(T)^2 \sup_{x\in\mathcal{M}, \|x\|\leq 1}\left\|\sum_{j=1}^{\infty}\langle x, \tau_j\rangle\langle \tau_j, x\rangle\right\|\\ &= \pi_2(T)^2 \sup_{x\in\mathcal{M}, \|x\|\leq 1}\|x\|^2 = \pi_2(T)^2.\end{aligned}$$

($\Leftarrow$) Let $n \in \mathbb{N}$ and $x_1, \dots, x_n \in \mathcal{M}$. Let $\{\omega_n\}_{n=1}^{\infty}$ be an orthonormal basis for $\mathcal{M}$. Define

$$S : \mathcal{M} \ni x \mapsto \sum_{j=1}^{n}\langle x, \omega_j\rangle x_j \in \mathcal{M}.$$

Then

$$\begin{aligned}\|S\|^2 = \|S^*\|^2 &= \sup_{x\in\mathcal{M}, \|x\|\leq 1}\|S^*x\|^2 = \sup_{x\in\mathcal{M}, \|x\|\leq 1}\left\|\sum_{n=1}^{\infty}\langle S^*x, \omega_n\rangle\omega_n\right\|^2 = \sup_{x\in\mathcal{M}, \|x\|\leq 1}\left\|\sum_{n=1}^{\infty}\langle x, S\omega_n\rangle\omega_n\right\|^2\\ &= \sup_{x\in\mathcal{M}, \|x\|\leq 1}\left\|\sum_{j=1}^{n}\langle x, x_j\rangle\omega_j\right\|^2 = \sup_{x\in\mathcal{M}, \|x\|\leq 1}\left\|\sum_{j=1}^{n}\langle x, x_j\rangle\langle x_j, x\rangle\right\|.\end{aligned}$$

Hence

$$\left\|\sum_{j=1}^{n}\langle Tx_j, Tx_j\rangle\right\| = \left\|\sum_{j=1}^{n}\langle TS\omega_j, TS\omega_j\rangle\right\| = \|TS\|_{\mathrm{HS}}^2 \leq \|T\|_{\mathrm{HS}}^2\|S\| = \|T\|_{\mathrm{HS}}^2 \sup_{x\in\mathcal{M}, \|x\|\leq 1}\left\|\sum_{j=1}^{n}\langle x, x_j\rangle\langle x_j, x\rangle\right\|.$$

$\square$

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Note that we have not used monotonic closedness of C*-algebra in "if" part. In view of Theorem 3, we formulate following problem.

**Question 6.** *Whether there exists a modular Pietsch factorization theorem?*

We solve Question (6) partially in the following theorem. Integrals in the following theorem is in the Kasparov sense [35].

**Theorem 7.** *Let $\mathcal{M}$ and $\mathcal{N}$ be Hilbert C*-modules over a commutative C*-algebra $\mathcal{A}$. Let $T : \mathcal{M} \to \mathcal{N}$ be an adjointable morphism. Assume that there exists a Lie group $G \subseteq B_{\mathcal{M}} := \{x : x \in \mathcal{M}, \|x\| \leq 1\}$ satisfying following.*

*(i)* $\mu_G(G) = 1$.
*(ii)* *For each $x \in \mathcal{M}$, the map $G \ni y \mapsto \langle x.y\rangle\langle y, x\rangle$ is continuous.*
*(iii)* *There exists a real $C > 0$ such that*

$$\langle Tx, Tx\rangle^{\frac{p}{2}} \leq C^p \int_G (\langle x.y\rangle\langle y, x\rangle)^{\frac{p}{2}}\, d\mu_G(y), \quad \forall x \in \mathcal{M}.$$

*Then $T$ modular $p$-absolutely summing and $\pi_p(T) = C$.*

**Proof.** Let $n \in \mathbb{N}$ and $x_1, \dots, x_n \in \mathcal{M}$. Then

$$\left\|\sum_{j=1}^n \langle Tx_j, Tx_j\rangle^{\frac{p}{2}}\right\| \leq C^p \left\|\sum_{j=1}^n \int_G (\langle x_j, y\rangle\langle y, x_j\rangle)^{\frac{p}{2}}\, d\mu_G(y)\right\| = C^p \left\|\int_G \sum_{j=1}^n (\langle x_j, y\rangle\langle y, x_j\rangle)^{\frac{p}{2}}\, d\mu_G(y)\right\|$$
$$\leq C^p \int_G \left\|\sum_{j=1}^n (\langle x_j, y\rangle\langle y, x_j\rangle)^{\frac{p}{2}}\right\| d\mu_G(y) \leq C^p \int_G \sup_{y\in\mathcal{M}, \|y\|\leq 1} \left\|\sum_{j=1}^n (\langle x_j, y\rangle\langle y, x_j\rangle)^{\frac{p}{2}}\right\| d\mu_G(y)$$
$$= C^p \sup_{y\in\mathcal{M}, \|y\|\leq 1} \left\|\sum_{j=1}^n (\langle x_j, y\rangle\langle y, x_j\rangle)^{\frac{p}{2}}\right\| \mu_G(G) = C^p \sup_{y\in\mathcal{M}, \|y\|\leq 1} \left\|\sum_{j=1}^n (\langle x_j, y\rangle\langle y, x_j\rangle)^{\frac{p}{2}}\right\|.$$

□

## 3. Appendix

In this appendix we formulate some problems for Banach modules over C*-algebras based on the results in Banach spaces which influenced a lot in the modern development of Functional Analysis. Our first kind of problems come from the Dvoretzky theorem [36–50]. Let $\mathcal{X}$ and $\mathcal{Y}$ be finite dimensional Banach spaces such that $\dim(\mathcal{X}) = \dim(\mathcal{Y})$. Remember that the Banach-Mazur distance between $\mathcal{X}$ and $\mathcal{Y}$ is defined as

$$d_{BM}(\mathcal{X}, \mathcal{Y}) := \inf\{\|T\|\|T^{-1}\| : T : \mathcal{X} \to \mathcal{Y} \text{ is invertible linear operator}\}.$$

For $n \in \mathbb{N}$, let $(\mathbb{R}^n, \langle\cdot,\cdot\rangle)$ be the standard Euclidean Hilbert space.

**Theorem 8.** *[28,51] **(John Theorem) If $\mathcal{X}$ is any $n$-dimensional real Banach space, then***

$$d_{BM}(\mathcal{Y}, (\mathbb{R}^n, \langle\cdot,\cdot\rangle)) \leq \sqrt{n}.$$

**Theorem 9.** *[28,52] **(Dvoretzky Theorem) There is a universal constant $C > 0$ satisfying the following property: If $\mathcal{X}$ is any $n$-dimensional real Banach space and $0 < \varepsilon < \frac{1}{3}$, then for every natural number***

$$k \leq C \log n \frac{\varepsilon^2}{|\log \varepsilon|},$$

*there exists a $k$-dimensional Banach subspace $\mathcal{Y}$ of $\mathcal{X}$ such that*

$$d_{BM}(\mathcal{Y}, (\mathbb{R}^k, \langle \cdot, \cdot \rangle)) < 1 + \varepsilon.$$

Let $\mathcal{A}$ be a unital C*-algebra with invariant basis number property (see [53] for a study on such C*-algebras) and $\mathcal{E}, \mathcal{F}$ be finite rank Banach modules over $\mathcal{A}$ such that $\text{rank}(\mathcal{E}) = \text{rank}(\mathcal{F})$. Modular Banach-Mazur distance between $\mathcal{E}$ and $\mathcal{F}$ is defined as

$$d_{MBM}(\mathcal{E}, \mathcal{F}) := \inf\{\|T\|\|T^{-1}\| : T : \mathcal{E} \to \mathcal{F} \text{ is invertible module homomorphism}\}.$$

Given a unital C*-algebra $\mathcal{A}$ and $n \in \mathbb{N}$, by $\mathcal{A}^n$ we mean the standard (left) module over $\mathcal{A}$. We equip $\mathcal{A}^n$ with the C*-valued inner product $\langle \cdot, \cdot \rangle : \mathcal{A}^n \times \mathcal{A}^n \to \mathcal{A}$ defined by

$$\langle (a_j)_{j=1}^n, (b_j)_{j=1}^n \rangle := \sum_{j=1}^n a_j b_j^*, \quad \forall (a_j)_{j=1}^n, (b_j)_{j=1}^n \in \mathcal{A}^n.$$

Hence norm on $\mathcal{A}^n$ is given by

$$\|(a_j)_{j=1}^n\| := \left\| \sum_{j=1}^n a_j a_j^* \right\|^{\frac{1}{2}}, \quad \forall (a_j)_{j=1}^n \in \mathcal{A}^n.$$

Then it is well-known that $\mathcal{A}^n$ is a Hilbert C*-module. We denote this Hilbert C*-module by $(\mathcal{A}^n, \langle \cdot, \cdot \rangle)$.

**Problem 1.** *(Modular Dvoretzky Problem) Let $\mathbb{A}$ be the set of all unital C*-algebras with invariant basis number property. What is the best function $\Psi : \mathbb{A} \times \left(0, \frac{1}{3}\right) \times \mathbb{N} \to (0, \infty)$ satisfying the following property: If $\mathcal{E}$ is any $n$-rank Banach module over a unital C*-algebra $\mathcal{A}$ with IBN property and $0 < \varepsilon < \frac{1}{3}$, then for every natural number*

$$k \leq \Psi(\mathcal{A}, \varepsilon, n),$$

*there exists a $k$-rank Banach submodule $\mathcal{F}$ of $\mathcal{E}$ such that*

$$d_{MBM}(\mathcal{F}, (\mathcal{A}^k, \langle \cdot, \cdot \rangle)) < 1 + \varepsilon.$$

A particular case of Problem 1 is the following conjecture.

**Conjecture 10.** *(Modular Dvoretzky Conjecture) Let $\mathcal{A}$ be a unital C*-algebra with IBN property. There is a universal constant $C > 0$ (which may depend upon $\mathcal{A}$) satisfying the following property: If $\mathcal{E}$ is any $n$-rank Banach module and $0 < \varepsilon < \frac{1}{3}$, then for every natural number*

$$k \leq C \log n \frac{\varepsilon^2}{|\log \varepsilon|},$$

*there exists a $k$-rank Banach submodule $\mathcal{F}$ of $\mathcal{E}$ such that*

$$d_{MBM}(\mathcal{F}, (\mathcal{A}^k, \langle \cdot, \cdot \rangle)) < 1 + \varepsilon.$$

Our second kind of problems come from the type-cotype theory of Banach spaces [8,12,13,18,19,28,54,55]. Let $\mathcal{H}$ be a Hilbert space, $n \in \mathbb{N}$. Recall that for any $n$ points $x_1, \ldots, x_n \in \mathcal{H}$, we have

$$\frac{1}{2^n} \sum_{\varepsilon_1, \ldots, \varepsilon_n \in \{-1,1\}} \left\| \sum_{j=1}^n \varepsilon_j x_j \right\|^2 = \sum_{j=1}^n \|x_j\|^2. \tag{5}$$

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It is Equality (5) which motivated the definition of type and cotype for Banach spaces.

**Definition 5.** *[28] Let $1 \leq p \leq 2$. A Banach space $\mathcal{X}$ is said to be of* **(Rademacher) type** *$p$ if there exists $T_p(\mathcal{X}) > 0$ such that*

$$\left(\frac{1}{2^n}\sum_{\varepsilon_1,\dots,\varepsilon_n\in\{-1,1\}}\left\|\sum_{j=1}^{n}\varepsilon_j x_j\right\|^p\right)^{\frac{1}{p}} \leq T_p(\mathcal{X})\left(\sum_{j=1}^{n}\|x_j\|^p\right)^{\frac{1}{p}}, \quad \forall x_1,\dots,x_n \in \mathcal{X},\ \forall n \in \mathbb{N}.$$

**Definition 6.** *[28] Let $2 \leq q < \infty$. A Banach space $\mathcal{X}$ is said to be of* **(Rademacher) cotype** *$q$ if there exists $C_q(\mathcal{X}) > 0$ such that*

$$\left(\sum_{j=1}^{n}\|x_j\|^q\right)^{\frac{1}{q}} \leq C_q(\mathcal{X})\left(\frac{1}{2^n}\sum_{\varepsilon_1,\dots,\varepsilon_n\in\{-1,1\}}\left\|\sum_{j=1}^{n}\varepsilon_j x_j\right\|^q\right)^{\frac{1}{q}}, \quad \forall x_1,\dots,x_n \in \mathcal{X},\ \forall n \in \mathbb{N}.$$

Let $\mathcal{E}$ be a (left) Hilbert C*-module over a unital C*-algebra $\mathcal{A}$, $n \in \mathbb{N}$. We see that for any $n$ points $x_1,\dots,x_n \in \mathcal{E}$, we have

$$\frac{1}{2^n}\sum_{\varepsilon_1,\dots,\varepsilon_n\in\{-1,1\}}\left\langle\sum_{j=1}^{n}\varepsilon_j x_j, \sum_{k=1}^{n}\varepsilon_k x_k\right\rangle = \sum_{j=1}^{n}\langle x_j, x_j\rangle. \tag{6}$$

**Problem 2.** ***(Modular Type-Cotype Problems) Whether there is a way to define type (we call modular-type) and cotype (we call modular-cotype) for Banach modules over C*-algebras which reduces to Equality (6) for Hilbert C*-modules?***

**Problem 3.** ***Whether there is a notion of type and cotype for Banach modules over C*-algebras such that Kwapien theorems holds?, In other words, whether following statements hold?***

*(i) A Banach module $\mathcal{M}$ over a unital C*-algebra $\mathcal{A}$ has modular-type 2 and modular-cotype 2 if and only if $\mathcal{M}$ is isomorphic to a Hilbert C*-module over $\mathcal{A}$.*

*(ii) If $\mathcal{M}$ and $\mathcal{N}$ are Banach modules over a unital C*-algebra $\mathcal{A}$ of modular-type 2 and modular-cotype 2, respectively, then a bounded module morphism $T : \mathcal{M} \to \mathcal{N}$ factors through a Hilbert C*-module over $\mathcal{A}$.*

**Problem 4.** ***(Modular Khinchin-Kahane Inequalities Problems) Whether there is a Khinchin-Kahane inequalities for Banach modules over C*-algebras which reduce to Equality (6) for Hilbert C*-modules?***

Our third kind of problems come from Grothendieck inequality [1,2,28,56–63].

**Theorem 11.** *[1,2,28,56–58]* ***(Grothendieck Inequality) There is a universal constant $K_G$ satisfying the following: For any Hilbert space $\mathcal{H}$ and any $m, n \in \mathbb{N}$, if a scalar matrix $[a_{j,k}]_{1\leq j\leq m, 1\leq k\leq n}$ satisfy***

$$\left|\sum_{j=1}^{m}\sum_{k=1}^{n}a_{j,k}s_j t_k\right| \leq 1, \quad \forall s_j, t_k \in \mathbb{K}, |s_j| \leq 1, |t_k| \leq 1,$$

***then***

$$\left|\sum_{j=1}^{m}\sum_{k=1}^{n}a_{j,k}\langle u_j, v_k\rangle\right| \leq K_G, \quad \forall u_j, v_k \in \mathcal{H}, \|u_j\| \leq 1, \|v_k\| \leq 1.$$

**Problem 5.** ***(Modular Grothendieck Inequality Problem - 1) Let $\mathbb{A}$ be the set of all unital C*-algebras. Let $\mathcal{E}$ be a Hilbert C*-module over a unital C*-algebra $\mathcal{A}$. Let $\mathcal{A}^+$ be the set of all positive elemnts***

*in $\mathcal{A}$. What is the best function $\Psi : \mathcal{A} \times \mathbb{N} \times \mathbb{N} \to \mathcal{A}^+$ satisfying the following property: If $[a_{j,k}]_{1\leq j\leq m,1\leq k\leq n} \in \mathbb{M}_{m\times n}(\mathcal{A})$ satisfy*

$$\left\langle \sum_{j=1}^{m}\sum_{k=1}^{n} a_{j,k}s_jt_k, \sum_{p=1}^{m}\sum_{q=1}^{n} a_{p,q}s_pt_q \right\rangle \leq 1, \quad \forall s_j, t_k \in \mathcal{A},$$
$$s_js_j^* = s_j^*s_j = 1, \forall 1 \leq j \leq m, t_kt_k^* = t_k^*t_k = 1, \forall 1 \leq k \leq n,$$

*then*

$$\left\langle \sum_{j=1}^{m}\sum_{k=1}^{n} a_{j,k}\langle u_j, v_k\rangle, \sum_{p=1}^{m}\sum_{q=1}^{n} a_{p,q}\langle u_p, v_q\rangle \right\rangle \leq \Psi(\mathcal{A}, m, n), \quad \forall u_j, v_k \in \mathcal{E},$$
$$\langle u_j, u_j\rangle = 1, \forall 1 \leq j \leq m, \langle v_k, v_k\rangle = 1, \forall 1 \leq k \leq n.$$

*In particular, whether $\Psi$ depends on $m$ and $n$?*

**Problem 6.** *(Modular Grothendieck Inequality Problem - 2) Let $\mathcal{A}$ be the set of all unital C*-algebras. Let $\mathcal{E}$ be a Hilbert C*-module over a unital C*-algebra $\mathcal{A}$. Let $\mathcal{A}^+$ be the set of all positive elemnts in $\mathcal{A}$. What is the best function $\Psi : \mathcal{A} \times \mathbb{N} \times \mathbb{N} \to \mathcal{A}^+$ satisfying the following property: If $[a_{j,k}]_{1\leq j\leq m,1\leq k\leq n} \in \mathbb{M}_{m\times n}(\mathcal{A})$ satisfy*

$$\left\langle \sum_{j=1}^{m}\sum_{k=1}^{n} a_{j,k}s_jt_k, \sum_{p=1}^{m}\sum_{q=1}^{n} a_{p,q}s_pt_q \right\rangle \leq 1, \quad \forall s_j, t_k \in \mathcal{A}, \|s_j\| \leq 1, \forall 1 \leq j \leq m, \|t_k\| \leq 1, \forall 1 \leq k \leq n,$$

*then*

$$\left\langle \sum_{j=1}^{m}\sum_{k=1}^{n} a_{j,k}\langle u_j, v_k\rangle, \sum_{p=1}^{m}\sum_{q=1}^{n} a_{p,q}\langle u_p, v_q\rangle \right\rangle \leq \Psi(\mathcal{A}, m, n), \quad \forall u_j, v_k \in \mathcal{E},$$
$$\|u_j\| \leq 1, \forall 1 \leq j \leq m, \|v_k\| \leq 1, \forall 1 \leq k \leq n.$$

*In particular, whether $\Psi$ depends on $m$ and $n$?*

We believe strongly that $\Psi$ depends on $\mathcal{A}$.

**Remark 1.** ***Modular Bourgain-Tzafriri restricted invertibility conjecture*** *and* ***Modular Johnson-Lindenstrauss flattening conjecture*** *have been stated in [64,65].*

## References


1. Grothendieck, A. Produits tensoriels topologiques et espaces nucléaires. *Mem. Amer. Math. Soc.* **1955**, *16*.
2. Diestel, J.; Fourie, J.H.; Swart, J. *The metric theory of tensor products: Grothendieck's resume revisited*; American Mathematical Society, Providence, RI, 2008; pp. x+278. doi:10.1090/mbk/052.
3. Tomczak-Jaegermann, N. *Banach-Mazur distances and finite-dimensional operator ideals*; Vol. 38, *Pitman Monographs and Surveys in Pure and Applied Mathematics*, Longman Scientific and Technical, Harlow, 1989; pp. xii+395.
4. Defant, A.; Floret, K. *Tensor norms and operator ideals*; Vol. 176, *North-Holland Mathematics Studies*, North-Holland Publishing Co., Amsterdam, 1993; pp. xii+566.
5. Wojtaszczyk, P. *Banach spaces for analysts*; Vol. 25, *Cambridge Studies in Advanced Mathematics*, Cambridge University Press, Cambridge, 1991; pp. xiv+382. doi:10.1017/CBO9780511608735.
6. Lindenstrauss, J.; Pelczynski, A. Absolutely summing operators in $L_p$-spaces and their applications. *Studia Math.* **1968**, *29*, 275–326. doi:10.4064/sm-29-3-275-326.

7. Pietsch, A. Absolut $p$-summierende Abbildungen in normierten Raumen. *Studia Math.* **1966/67**, *28*, 333–353. doi:10.4064/sm-28-3-333-353.
8. Diestel, J.; Jarchow, H.; Tonge, A. *Absolutely summing operators*; Vol. 43, *Cambridge Studies in Advanced Mathematics*, Cambridge University Press, Cambridge, 1995; pp. xvi+474. doi:10.1017/CBO9780511526138.
9. Pelczynski, A. A characterization of Hilbert-Schmidt operators. *Studia Math.* **1966/67**, *28*, 355–360. doi:10.4064/sm-28-3-355-360.
10. Johnson, W.B.; Schechtman, G. Computing $p$-summing norms with few vectors. *Israel J. Math.* **1994**, *87*, 19–31. doi:10.1007/BF02772980.
11. Pietsch, A. *Operator ideals*; Vol. 20, *North-Holland Mathematical Library*, North-Holland Publishing Co., Amsterdam-New York, 1980; p. 451.
12. Johnson, W.B.; Lindenstrauss, J., Eds. *Handbook of the geometry of Banach spaces. Vol. I*; North-Holland Publishing Co., Amsterdam, 2001; pp. x+1005.
13. Johnson, W.B.; Lindenstrauss, J., Eds. *Handbook of the geometry of Banach spaces. Vol. 2*; North-Holland, Amsterdam, 2003; pp. xii+1866.
14. Pietsch, A. *History of Banach spaces and linear operators*; Birkhäuser Boston, Inc., Boston, MA, 2007; pp. xxiv+855.
15. Hytonen, T.; van Neerven, J.; Veraar, M.; Weis, L. *Analysis in Banach spaces. Vol. II: Probabilistic methods and operator theory*; Vol. 67, *A Series of Modern Surveys in Mathematics*, Springer, 2017; pp. xxi+616. doi:10.1007/978-3-319-69808-3.
16. Jameson, G.J.O. *Summing and nuclear norms in Banach space theory*; Vol. 8, *London Mathematical Society Student Texts*, Cambridge University Press, Cambridge, 1987; pp. xii+174. doi:10.1017/CBO9780511569166.
17. Garling, D.J.H. *Inequalities: a journey into linear analysis*; Cambridge University Press, Cambridge, 2007; pp. x+335. doi:10.1017/CBO9780511755217.
18. Li, D.; Queffelec, H. *Introduction to Banach spaces: analysis and probability. Vol. 2*; Vol. 167, *Cambridge Studies in Advanced Mathematics*, Cambridge University Press, Cambridge, 2018; pp. xxx+374.
19. Li, D.; Queffelec, H. *Introduction to Banach spaces: analysis and probability. Vol. 1*; Vol. 166, *Cambridge Studies in Advanced Mathematics*, Cambridge University Press, Cambridge, 2018; pp. xxx+431.
20. Lindenstrauss, J.; Tzafriri, L. *Classical Banach spaces. I: Sequence spaces*; Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 92, Springer-Verlag, Berlin-New York, 1977; pp. xiii+188.
21. Tomczak-Jaegermann, N. Computing 2-summing norm with few vectors. *Ark. Mat.* **1979**, *17*, 273–277. doi:10.1007/BF02385473.
22. Kwapien, S. On a theorem of L. Schwartz and its applications to absolutely summing operators. *Studia Math.* **1970**, *38*, 193–201. doi:10.4064/sm-38-1-193-201.
23. Farmer, J.D.; Johnson, W.B. Lipschitz $p$-summing operators. *Proc. Amer. Math. Soc.* **2009**, *137*, 2989–2995. doi:10.1090/S0002-9939-09-09865-7.
24. Botelho, G.; Pellegrino, D.; Rueda, P. A unified Pietsch domination theorem. *J. Math. Anal. Appl.* **2010**, *365*, 269–276. doi:10.1016/j.jmaa.2009.10.025.
25. Pellegrino, D.; Santos, J. A general Pietsch domination theorem. *J. Math. Anal. Appl.* **2011**, *375*, 371–374. doi:10.1016/j.jmaa.2010.08.019.
26. Chen, D.; Zheng, B. Remarks on Lipschitz $p$-summing operators. *Proc. Amer. Math. Soc.* **2011**, *139*, 2891–2898. doi:10.1090/S0002-9939-2011-10720-2.
27. Botelho, G.; Maia, M.; Pellegrino, D.; Santos, J. A unified factorization theorem for Lipschitz summing operators. *Q. J. Math.* **2019**, *70*, 1521–1533. doi:10.1093/qmathj/haz030.
28. Albiac, F.; Kalton, N.J. *Topics in Banach space theory*; Vol. 233, *Graduate Texts in Mathematics*, Springer, [Cham], 2016; pp. xx+508. doi:10.1007/978-3-319-31557-7.
29. Paschke, W.L. Inner product modules over $B^*$-algebras. *Trans. Amer. Math. Soc.* **1973**, *182*, 443–468. doi:10.2307/1996542.
30. Rieffel, M.A. Induced representations of $C^*$-algebras. *Advances in Math.* **1974**, *13*, 176–257. doi:10.1016/0001-8708(74)90068-1.
31. Kaplansky, I. Modules over operator algebras. *Amer. J. Math.* **1953**, *75*, 839–858. doi:10.2307/2372552.
32. Stern, A.B.; van Suijlekom, W.D. Schatten classes for Hilbert modules over commutative $C^*$-algebras. *J. Funct. Anal.* **2021**, *281*, Paper No. 109042, 35. doi:10.1016/j.jfa.2021.109042.
33. Frank, M.; Larson, D.R. Frames in Hilbert $C^*$-modules and $C^*$-algebras. *J. Operator Theory* **2002**, *48*, 273–314.

34. Takesaki, M. *Theory of operator algebras. I*; Vol. 124, *Operator Algebras and Non-commutative Geometry: Encyclopaedia of Mathematical Sciences*, Springer-Verlag, Berlin, 2002; pp. xx+415.
35. Kasparov, G.G. Topological invariants of elliptic operators. I. *K*-homology. *Izv. Akad. Nauk SSSR Ser. Mat.* **1975**, *39*, 796–838.
36. Milman, V.D.; Schechtman, G. *Asymptotic theory of finite-dimensional normed spaces*; Vol. 1200, *Lecture Notes in Mathematics*, Springer-Verlag, Berlin, 1986; pp. viii+156.
37. Milman, V.D. A new proof of A. Dvoretzky's theorem on cross-sections of convex bodies. *Funkcional. Anal. i Priložen.* **1971**, *5*, 28–37.
38. Milman, V. Dvoretzky's theorem—thirty years later. *Geom. Funct. Anal.* **1992**, *2*, 455–479. doi:10.1007/BF01896663.
39. Figiel, T. A short proof of Dvoretzky's theorem on almost spherical sections of convex bodies. *Compositio Math.* **1976**, *33*, 297–301.
40. Szankowski, A. On Dvoretzky's theorem on almost spherical sections of convex bodies. *Israel J. Math.* **1974**, *17*, 325–338. doi:10.1007/BF02756881.
41. Matousek, J. *Lectures on discrete geometry*; Vol. 212, *Graduate Texts in Mathematics*, Springer-Verlag, New York, 2002; pp. xvi+481. doi:10.1007/978-1-4613-0039-7.
42. Matsak, I.; Plichko, A. Dvoretzky's theorem by Gaussian method. In *Functional analysis and its applications*; Elsevier Sci. B. V., Amsterdam, 2004; Vol. 197, *North-Holland Math. Stud.*, pp. 171–184. doi:10.1016/S0304-0208(04)80166-X.
43. Pisier, G. *The volume of convex bodies and Banach space geometry*; Vol. 94, *Cambridge Tracts in Mathematics*, Cambridge University Press, Cambridge, 1989; pp. xvi+250. doi:10.1017/CBO9780511662454.
44. Gordon, Y.; Guedon, O.; Meyer, M. An isomorphic Dvoretzky's theorem for convex bodies. *Studia Math.* **1998**, *127*, 191–200. doi:10.4064/sm-127-2-191-200.
45. Figiel, T. A short proof of Dvoretzky's theorem. In *Séminaire Maurey-Schwartz 1974–1975: Espaces $L^p$, applications radonifiantes et géométrie des espaces de Banach, Exp. No. XXIII*; 1975; pp. 6 pp. (erratum, p. 3).
46. Figiel, T. Some remarks on Dvoretzky's theorem on almost spherical sections of convex bodies. *Colloq. Math.* **1971/72**, *24*, 241–252. doi:10.4064/cm-24-2-241-252.
47. Figiel, T.; Lindenstrauss, J.; Milman, V.D. The dimension of almost spherical sections of convex bodies. *Acta Math.* **1977**, *139*, 53–94. doi:10.1007/BF02392234.
48. Schechtman, G. A remark concerning the dependence on $\epsilon$ in Dvoretzky's theorem. In *Geometric aspects of functional analysis (1987–88)*; Springer, Berlin, 1989; Vol. 1376, *Lecture Notes in Math.*, pp. 274–277. doi:10.1007/BFb0090061.
49. Gordon, Y. Gaussian processes and almost spherical sections of convex bodies. *Ann. Probab.* **1988**, *16*, 180–188.
50. Pisier, G. Probabilistic methods in the geometry of Banach spaces. In *Probability and analysis (Varenna, 1985)*; Springer, Berlin, 1986; Vol. 1206, *Lecture Notes in Math.*, pp. 167–241. doi:10.1007/BFb0076302.
51. John, F. Extremum problems with inequalities as subsidiary conditions. In *Studies and Essays Presented to R. Courant on his 60th Birthday, January 8, 1948*; Interscience Publishers, Inc., New York, N. Y., 1948; pp. 187–204.
52. Dvoretzky, A. Some results on convex bodies and Banach spaces. Proc. Internat. Sympos. Linear Spaces (Jerusalem, 1960). Jerusalem Academic Press, Jerusalem; Pergamon, Oxford, 1961, pp. 123–160.
53. Gipson, P.M. Invariant basis number for $C^*$-algebras. *Illinois J. Math.* **2015**, *59*, 85–98.
54. Latala, R.; Oleszkiewicz, K. On the best constant in the Khinchin-Kahane inequality. *Studia Math.* **1994**, *109*, 101–104.
55. Szarek, S.J. On the best constants in the Khinchin inequality. *Studia Math.* **1976**, *58*, 197–208. doi:10.4064/sm-58-2-197-208.
56. Blei, R.C. An elementary proof of the Grothendieck inequality. *Proc. Amer. Math. Soc.* **1987**, *100*, 58–60. doi:10.2307/2046119.
57. Friedland, S.; Lim, L.H.; Zhang, J. An elementary and unified proof of Grothendieck's inequality. *Enseign. Math.* **2018**, *64*, 327–351. doi:10.4171/LEM/64-3/4-6.
58. Rietz, R.E. A proof of the Grothendieck inequality. *Israel J. Math.* **1974**, *19*, 271–276. doi:10.1007/BF02757725.
59. Haagerup, U. A new upper bound for the complex Grothendieck constant. *Israel J. Math.* **1987**, *60*, 199–224. doi:10.1007/BF02790792.

 doi:10.20944/preprints202302.0146.v1



60. Jameson, G.J.O. The interpolation proof of Grothendieck's inequality. *Proc. Edinburgh Math. Soc. (2)* **1985**, *28*, 217–223. doi:10.1017/S0013091500022653.
61. Kaijser, S. A simple-minded proof of the Pisier-Grothendieck inequality. In *Banach spaces, harmonic analysis, and probability theory (Storrs, Conn., 1980/1981)*; Springer, Berlin, 1983; Vol. 995, *Lecture Notes in Math.*, pp. 33–44. doi:10.1007/BFb0061887.
62. Blei, R. The Grothendieck inequality revisited. *Mem. Amer. Math. Soc.* **2014**, *232*, vi+90. doi:10.1090/memo/1093.
63. Pisier, G. Grothendieck's theorem, past and present. *Bull. Amer. Math. Soc. (N.S.)* **2012**, *49*, 237–323. doi:10.1090/S0273-0979-2011-01348-9.
64. Krishna, K.M. Modular Bourgain-Tzafriri Restricted Invertibility Conjectures and Johnson-Lindenstrauss Flattening Conjecture. *arXiv:2208.05223v1 [math.FA], 10 August* **2022**.
65. Krishna, K.M. Modular Paulsen Problem and Modular Projection Problem. *arXiv:2207.12799.v1 [math.FA], 26 July* **2022**.