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Posted Date: 31 January 2023

doi: 10.20944/preprints202301.0582.v1

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 doi:10.20944/preprints202301.0582.v1

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Gevrey Asymptotics for Logarithmic Type Solutions to Singularly Perturbed Problems with Nonlocal Nonlinearities

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Abstract: We investigate a family of nonlinear partial differential equations which are singularly perturbed in a complex parameter ϵ and singular in a complex time variable t at the origin. These equations combine differential operators of Fuchsian type in time t and space derivatives on horizontal strips in the complex plane with a nonlocal operator acting on the parameter ϵ known as the formal monodromy around 0. Their coefficients and forcing terms comprise polynomial and logarithmic type functions in time and are bounded holomorphic in space. A set of logarithmic type solutions are shaped by means of Laplace transforms relatively to t and ϵ and Fourier integrals in space. Furthermore, a formal logarithmic type solution is modeled which represents the common asymptotic expansion of Gevrey type of the genuine solutions with respect to ϵ on bounded sectors at the origin.

Keywords: asymptotic expansion; Borel-Laplace transform; Fourier transform; initial value problem; formal power series; formal monodromy; singular perturbation

MSC: 35C10; 35C20

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1. Introduction

In this paper, we examine a family of singularly perturbed nonlinear partial differential equations modeled as

$$Q(\partial_z)u(t,z,\epsilon) = (\epsilon t)^{d_D}(t\partial_t)^{\delta_D}R_D(\partial_z)u(t,z,\epsilon) + P(t,z,\epsilon,t\partial_t,\partial_z)u(t,z,\epsilon) + f(t,z,\epsilon) \\ + H(\log(\epsilon t), z, \epsilon, \{P_j(\partial_z)u(t,z,\epsilon)\}_{j\in J_1}, \{Q_j(\partial_z)\gamma_\epsilon^* u(t,z,\epsilon)\}_{j\in J_2}) \quad (1)$$

for vanishing initial data $u(0,z,\epsilon) \equiv 0$. The constants $d_D, \delta_D \geq 1$ are natural numbers and $Q(X)$,$R_D(X)$,$P_j(X)$ for $j \in J_1$, $Q_j(X)$ for $j \in J_2$, where J_1, J_2 are two finite subsets of the positive integers $\mathbb{N}^*$, stand for polynomials with complex coefficients. The linear differential operator $P(t,z,\epsilon,t\partial_t,\partial_z)$ depends analytically in a perturbation parameter ϵ on a disc D_{ϵ_0} with radius $\epsilon_0 > 0$ centered at 0 and relies polynomially in the complex time t and holomorphically with respect to the space variable z on a horizontal strip framed as $H_\beta = \{z \in \mathbb{C}/|\mathrm{Im}(z)| < \beta\}$ in $\mathbb{C}$, for some given width $2\beta > 0$. The forcing term $f(t,z,\epsilon)$ is a map of logarithmic type represented as a sum

$$f(t,z,\epsilon) = f_1(t,z,\epsilon) + f_2(t,z,\epsilon)\log(\epsilon t)$$

were $f_j(t,z,\epsilon)$, $j = 1,2$, are polynomials in t, with holomorphic coefficients in z on H_β and in ϵ on D_{ϵ_0}. The map $H(v_0, z, \epsilon, \{v_j\}_{j\in J_1}, \{w_j\}_{j\in J_2})$ is a *specific* polynomial of degree at most 2 in its arguments v_0, $\{v_j\}_{j\in J_1}$ and $\{w_j\}_{j\in J_2}$, which relies holomorphically in z on H_β and in ϵ on D_{ϵ_0}. The precise shape of H is framed in (14).

The nonlinear term H of (1) involves not only powers of $P_j(\partial_z)u(t,z,\epsilon)$, $j \in J_1$, but also powers of derivatives of $\gamma_\epsilon^* u(t,z,\epsilon)$ where γ_ϵ^* is a nonlocal operator acting on $u(t,z,\epsilon)$ which represents the so-called *monodromy operator around 0* relatively to ϵ. In the literature, the concept of formal monodromy around a point a in $\mathbb{C}$ appears in the construction of formal fundamental solutions to linear systems of differential equations with so-called irregular singularity at the given point a, known as the Levelt-Turrittin theorem, see [1]. It asserts that a differential system of the form

$$x^r Y'(x) = A(x)Y(x) \tag{2}$$

for analytic coefficients matrix $A(x) \in M_n(\mathbb{C})\{x\}$ near 0 with $n \geq 1$, for an integer $r \geq 2$, with an irregular singularity at 0, possesses a formal fundamental solution with the shape

$$\hat{Y}(x) = \hat{P}(x^{1/e})x^C \exp\left(\varphi(x^{1/e})\right)$$

for some well chosen integer $e \geq 1$, where $\hat{P}(y) \in \mathrm{GL}_n(\mathbb{C}[[y]][1/y])$ is a formal meromorphic invertible matrix, $\varphi(x^{1/e})$ is a diagonal matrix whose coefficient are polynomials in $x^{-1/e}$ with complex coefficients and $C \in M_n(\mathbb{C})$ is related to the so-called *formal monodromy matrix* $M \in \mathrm{GL}_n(\mathbb{C})$ by the formula $M = \exp(2\pi i C)$. It is worth remarking that this formal monodromy matrix extends in the formal settings the so-called *monodromy matrix* that appear in the representation of fundamental matrix solutions to systems (2) with *regular singularity* of the form

$$Y(x) = H(x)x^E$$

where H is an invertible matrix with meromorphic coefficients near 0, for a matrix E giving rise to the monodromy matrix $N \in \mathrm{GL}_n(\mathbb{C})$ by means of $N = \exp(2\pi i E)$. The matrix N is obtained as analytic continuation of the fundamental matrix solution $Y(x)$ along a simple loop γ going counterclockwise around the origin 0 with base point x by means of the identity

$$\gamma^* Y(x) = Y(x)N$$

where $\gamma^* Y$ denotes the analytic continuation along γ, see [2]. In the same manner as the analytic continuation operator γ^* acting on analytic functions, a formal monodromy operator γ^* acting on various spaces and rings (such as the so-called Picard-Vessiot rings) through the formulas $\gamma^*(z^\lambda) = e^{2i\pi\lambda}z^\lambda$ for complex numbers $\lambda \in \mathbb{C}$ and $\gamma^*(l) = l + 2i\pi$ where l is the symbol for the Log function, has been introduced and studied from an abstract and algebraic point of view in the textbook [1].

In our context, the action of the formal monodromy γ_ϵ^* on $u(t,z,\epsilon)$ can be reformulated as a *shift mapping* on angles $\theta \mapsto \theta + 2\pi$ in polar coordinates by means of the change of functions

$$u(t,z,\epsilon) = v(t,z,r,\theta)$$

for $\epsilon = re^{\sqrt{-1}\theta}$, with radius $r > 0$ and angle $\theta \in \mathbb{R}$, through the formula

$$\gamma_\epsilon^* u(t,z,\epsilon) = v(t,z,r,\theta + 2\pi).$$

In this way, the main equation (1) can be recast as some nonlinear mixed type partial *difference-differential* equation for the map $v(t,z,r,\theta)$. In the framework of nonlinear difference equations in the complex domain with the shape

$$y(z+1) = F(z, y(z))$$

for $\mathbb{C}^n-$valued analytic maps F in a neighborhood of (∞, y_0) for some $y_0 \in \mathbb{C}^n$, we notice that important results concerning asymptotic features of their solutions have been obtained by several authors, see [3–5]. In comparison with these results, we do not reach asymptotic expansions as θ goes to infinity in the equation fulfilled by v but we rather plan to get exact asymptotics as the real singular perturbation parameter $r > 0$ approaches the origin.

We highlight our premise that the main equation (1) counts in powers of the basic differential operator $t\partial_t$ which is labelled of Fuchsian type. We refer to [6] for many sharp results about Fuchsian ordinary and partial differential equations. However, under the sufficient conditions required on

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(1) listed in Subsection 2.3 it pans out that (1) will be reduced throughout the work to a coupling of two partial differential equations, stated in (47) and (48), that comprise only powers of the basic differential operator $u_1^{k_1+1}\partial_{u_1}$, for a well chosen integer $k_1 \geq 1$, of irregular type in a complex variable u_1. The definition of irregular type differential operators is given in the classical textbook [7] in the ordinary differential equations settings displayed in (2) and in the work [8] in the framework of partial differential equations.

In the present contribution, we aim to cook up a set of holomorphic solutions to (1) and to describe their asymptotic expansions as ϵ tends to 0 (stated in Theorem 1 of Subsection 8.2). These solutions are shaped as logarithmic type maps that involve Fourier/Laplace transforms. Namely, under the list of requirements which mould (1) and detailed in Subsection 2.3, one can outline

- A set of properly selected bounded open sectors $\{\mathcal{E}_p\}_{p\in I_1}$ for some finite set $I_1 \subset \mathbb{N}$ and $\mathcal{T}$ centered at 0.
- A family of holomorphic functions $u_p(t,z,\epsilon)$, $p \in I_1$, which conform (1) on the domain $\mathcal{T} \times H_\beta \times \mathcal{E}_p$. Each solution u_p, $p \in I_1$, is expressed as a sum

$$u_p(t,z,\epsilon) = u_{1,p}(t,z,\epsilon) + u_{2,p}(t,z,\epsilon)\log(\epsilon t)$$

where each component $u_{j,p}(t,z,\epsilon)$, $j = 1,2$, is represented as a Fourier/Laplace transform

$$u_{j,p}(t,z,\epsilon) = \frac{k_1}{(2\pi)^{1/2}}\int_{L_{d_p}}\int_{-\infty}^{+\infty} \omega_{j,d_p}(\tau,m,\epsilon)\exp\big(-(\frac{\tau}{\epsilon t})^{k_1}\big)e^{\sqrt{-1}zm}\frac{d\tau}{\tau}dm$$

where the commonly named *Borel/Fourier map* $\omega_{j,d_p}(\tau,m,\epsilon)$ stands for a function

  - which is analytic near $\tau = 0$
  - with (at most) of exponential growth of some order $k_1 \geq 1$ on an infinite sector containing the halfline $L_{d_p} = [0,+\infty)e^{\sqrt{-1}d_p}$ with respect to τ for suitable direction $d_p \in \mathbb{R}$
  - continuous and subjected to exponential decay with respect to m on $\mathbb{R}$
  - with analytic dependence in ϵ on the punctured disc $D_{\epsilon_0} \setminus \{0\}$.

Furthermore, owing to their Laplace integral structure, the components $\{u_{j,p}\}_{p\in I_1}$ own asymptotic expansions of Gevrey type in the parameter ϵ. Indeed, for given $j = 1,2$, all the partial functions $\epsilon \mapsto u_{j,p}(t,z,\epsilon)$, $p \in I_1$, share a common asymptotic formal power series expansion

$$\hat{\mathbb{G}}_j(\epsilon) = \sum_{n\geq 0} \mathbb{G}_{n,j}(t,z)\frac{\epsilon^n}{n!}$$

on $\mathcal{E}_p$, with bounded holomorphic coefficients $\mathbb{G}_{n,j}$ on $\mathcal{T} \times H_\beta$. These asymptotic expansions turn out to be of Gevrey order $1/k_1$ on every sectors $\mathcal{E}_p$, meaning that constants $K_{p,j}, M_{p,j} > 0$ can be singled out for which the error bounds

$$|u_{j,p}(t,z,\epsilon) - \sum_{n=0}^{N} \mathbb{G}_{n,j}(t,z)\frac{\epsilon^n}{n!}| \leq K_{p,j}(M_{p,j})^{N+1}\Gamma(1+\frac{N+1}{k_1})|\epsilon|^{N+1}$$

hold for all integers $N \geq 0$, all $\epsilon \in \mathcal{E}_p$, uniformly in $t \in \mathcal{T}$ and $z \in H_\beta$. At last, we verify that the formal logarithmic type expression

$$\hat{\mathbb{G}}(\epsilon) = \hat{\mathbb{G}}_1(\epsilon) + \hat{\mathbb{G}}_2(\epsilon)\log(\epsilon t)$$

itself obeys the main equation (1).

Throughout the proof of our main result, we show that the components $u_{j,p}(t,z,\epsilon)$, $j = 1,2$ of the built up solutions u_p, $p \in I_1$, to (1) turn out to be *embedded* in a larger family of maps $u_{j,p}(t,z,\epsilon)$, $j = 1,2$, for all integers $0 \leq p \leq \varsigma - 1$ for some integer $\varsigma \geq 2$. These maps are bounded holomorphic

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on products $\mathcal{T} \times H_{\beta} \times \mathcal{E}_p$ where $\underline{\mathcal{E}} = \{\mathcal{E}_p\}_{0 \le p \le \varsigma - 1}$ stands for a set of bounded sectors, entailing $\mathcal{E}_p$ for $p \in I_1$, which represents a good covering in $\mathbb{C}^*$ (see Definition 7). Each map $u_{j,p}(t,z,\epsilon)$, $j = 1,2$, is modeled as a rescaled version of a bounded holomorphic map $(u_1, z) \mapsto U_{j,d_p}(u_1, z, \epsilon)$ through

$$u_{j,p}(t,z,\epsilon) = U_{j,d_p}(\epsilon t, z, \epsilon)$$

on domains $U_{1,d_p} \times H_{\beta}$ for any fixed $\epsilon \in D_{\epsilon_0} \setminus \{0\}$, where U_{1,d_p} are bounded sectors bisected by the direction d_p, depicted in Definition 8 of the work. The set of maps $\{U_{2,d_p}\}_{0 \le p \le \varsigma-1}$ is shown to solve a specific nonlinear partial differential equation with coefficients that are polynomial in u_1, holomorphic with respect to ϵ on D_{ϵ_0} and relatively to z on H_{β} displayed in (36). The set of maps $\{U_{1,d_p}\}_{0 \le p \le \varsigma-1}$ conforms a particular nonlinear partial differential equation stated in (37) whose coefficients and forcing term bring in not only polynomials in u_1 and holomorphic dependence relatively to ϵ on D_{ϵ_0} and to z on H_{β} but also polynomial reliance on the maps $\{U_{2,d_p}\}_{0 \le p \le \varsigma-1}$ and their derivatives with respect to u_1 and z. In this sense, the maps $\{U_{j,d_p}\}_{0 \le p \le \varsigma-1}$, $j = 1,2$, solve a *coupling* of nonlinear partial differential equations. The asymptotic property for the components $u_{j,p}(t,z,\epsilon)$, $j = 1,2$, of $u_p(t,z,\epsilon)$ stems from sharp exponential bound estimates for the differences of neighboring maps $u_{j,p+1} - u_{j,p}$ reached in Proposition 10, for which a classical statement for the existence of asymptotic expansions of Gevrey type can be applied, see Subsection 8.1.

In this work, as mentioned above, we restrict ourselves to quadratic nonlinearities. Besides, they are chosen in a way to respect the natural triangular structure of the systems of partial differential equations satisfied by the components $u_{j,p}(t,z,\epsilon)$, $j = 1,2$ stated in (193), (194), which stems from the linear part of (1). It means that its resolution is reduced to the study of a coupling of two equations which comprise one single equation satisfied by $u_{2,p}(t,z,\epsilon)$ and a second equation for $u_{1,p}(t,z,\epsilon)$ with coefficients and forcing term that involve $u_{2,p}(t,z,\epsilon)$. The treatement of a more general case with non triangular structure is postponed to a futur paper.

The approach developped in this work can be extended to the construction of both formal and genuine holomorphic solutions to comparable problems as (1) with higher order logarithmic terms

$$u(t,z,\epsilon) = \sum_{j=0}^{n} u_j(t,z,\epsilon)(\log(\epsilon t))^j$$

for $n \ge 2$, for suitable nonlinear terms and forcing terms chosen properly in a similar way as the ones in the present work. We focus on the complete description for the case $n = 1$ for the sake of simplicity in order to give the readers a clear idea of the main purpose of the study and avoiding cumbersome notations and computations.

Logarithmic type solutions have been extensively studied in the framework of nonlinear partial differential equations with so-called Fuchsian type and described in the Chapter 8 of the textbook by R. Gérard and H. Tahara [6]. Namely, these authors consider nonlinear partial differential equations with the shape

$$(t\partial_t)^m u(t,x) = F(t,x,\{(t\partial_t)^j \partial_x^{\alpha} u(t,x)\}_{(j,\alpha) \in I_m}) \tag{3}$$

where $I_m = \{(j,\alpha) \in \mathbb{N} \times \mathbb{N}^n / j + |\alpha| \le m, j < m\}$ for some integers $m, n \ge 1$, for analytic maps $F(t,x,Z)$ near the origin in $\mathbb{C} \times \mathbb{C}^n \times \mathbb{C}^{\mathrm{card}(I_m)}$. Under conditions of *non resonance* of the characteristic exponents at $x = 0$ combined with some Poincaré condition on the characteristic polynomial associated to (3), they have described the holomorphic solutions to (3) with at most polynomial growth in t on bounded sectors centered at 0, for x near the origin in $\mathbb{C}^n$ as the maps written in the form of a convergent logarithmic type expression

$$u(t,x) = u_0(t,x) + \sum_{(i,j,k) \in J_m} \varphi_{i,j,k}(x) t^{i + \sum_{l=1}^{\mu} j_l \rho_l(x)} (\log(t))^k$$

for $J_m = \{(i,j,k) \in \mathbb{N} \times \mathbb{N}^{\mu} \times \mathbb{N} / i + 2m|j| \ge k + 2m, |j| \ge 1\}$ where

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- u_0 stands for convergent power series near the origin
- $\rho_l(x)$, $1 \leq l \leq \mu$ are the characteristic exponents with positive real parts at $x = 0$
- $\varphi_{i,j,k}(x)$ are holomorphic coefficients near $x = 0$.

In the case of so-called equations of irregular type or non Fuchsian type, in which our present work falls, less results are known and represents a favourable breeding ground for upcoming research. Nonetheless, in that trend, we mention the remarkable recent general result [9] obtained by H. Tahara. This work extends a paper by H. Yamazawa which treats linear partial differential equations, see [10]. Therein, the author examines nonlinear partial differential equations

$$F(t,x,\{(t\partial_t)^j\partial_x^\alpha u(t,x)\}_{(j,\alpha)\in L_m}) = 0 \tag{4}$$

with $L_m = \{(j,\alpha) \in \mathbb{N} \times \mathbb{N}^K / j + |\alpha| \leq m\}$, for some integers $m, K \geq 1$, which possess a formal series (which is divergent in the generic situation)

$$\hat{u}(t,x) = \sum_{n\geq 1} u_n(t,x)$$

solution where each term u_n, $n \geq 1$, is analytic with respect to t on some appropriate bounded sector S centered at 0 in $\mathbb{C}$ and holomorphic near 0 relatively to x on some disc D_R in $\mathbb{C}^K$. In general, these expressions u_n might involve combinations of functions of the form $t^{\lambda(x)}$ for holomorphic maps λ, powers of t and $\log(t)$ and analytic functions with respect to x on D_R. The author introduced a so-called *Newton polygon* associated to the equation (4) *along* the formal solution $\hat{u}(t,x)$. In the case this Newton polygon possesses $p \geq 1$ slopes and under some additional technical requirements, the author builds up a new formal solution

$$\hat{w}(t,x) = \sum_{n\geq 1} w_n(t,x)$$

to (4) which is subjected to the next two features

- The formal series $\hat{u}$ and $\hat{w}$ are asymptotically equivalent in the sense that for any $A > 0$, there exists $N_0 \geq 1$, such that
$$\sup_{x\in D_R} |(t\partial_t)^j\partial_x^\alpha(\hat{u}_N - \hat{w}_N)| \leq C|t|^A$$
for all $t \in S$, $j + |\alpha| \leq m$, some constant $C > 0$, any $N \geq N_0$, where $\hat{u}_N$ and $\hat{w}_N$ denote the partial sums of the N first terms of $\hat{u}$ and $\hat{w}$.
- The formal series $\hat{w}$ is multisummable on S with respect to t, uniformly in x on D_R, in a sense that enhances the classical multisummability process described in [7] and gives rise to a genuine holomorphic solution $w(t,x)$ of (4) on $S \times D_R$ crafted as iterated analytic acceleration operators and Laplace integral of some Borel transform of $\hat{w}$.

Thereupon, it turns out that $w(t,x)$ admits $\hat{u}(t,x)$ as an asymptotic expansion as t tends to 0 on S in the sense that for any $A > 0$, there exists $N_0 \geq 1$ such that

$$\sup_{x\in D_R} |w(t,x) - \hat{u}_N(t,x)| \leq C|t|^A$$

for all $t \in S$, some constant $C > 0$, any $N \geq N_0$.

At last, in the linear setting, some general results reaching beyond the structure of logarithmic type solutions have been achieved. Namely, for Cauchy problems

$$a(x,D)u = v \ , \ D_{x_0}^h u_{|x_0=0} = 0 \ , \ 0 \leq h < m$$

involving linear differential operators $a(x,D)$ of order $m \geq 1$ with holomorphic coefficients in $x = (x_j)_{0\leq j\leq n}$ in $\mathbb{C}^{n+1}$, existence and uniqueness results for so-called *ramified solutions* around certain characteristic hypersurfaces K in $\mathbb{C}^{n+1}$, provided that v is ramified around K, have been obtained by several authors, see [11], [12], [13].

2. Layout of the main equation

2.1. Laplace transforms and Fourier inverse maps

In this brief subsection, we include some preliminary material about Laplace transforms and Fourier inverse maps that will be used in the ongoing sections.

Let $k \geq 1$ be an integer. We remind the reader the definition of the Laplace transform of order k as stated in [14].

Definition 1. *We set $S_{d,\delta} = \{\tau \in \mathbb{C}^* : |d - \arg(\tau)| < \delta\}$ as some unbounded sector with bisecting direction $d \in \mathbb{R}$ and aperture $2\delta > 0$ and D_ρ as a disc centered at 0 with radius $\rho > 0$. A holomorphic function $w : S_{d,\delta} \cup D_\rho \to \mathbb{C}$ is considered that vanishes at 0 and suffers the bounds : there exist $C > 0$ and $K > 0$ such that*

$$|w(\tau)| \leq C|\tau| \exp(K|\tau|^k) \tag{5}$$

for all $\tau \in S_{d,\delta}$. The Laplace transform of w of order k in the direction d is set up as the integral transform

$$\mathcal{L}_k^d(w)(T) = k \int_{L_\gamma} w(u) \exp(-(\frac{u}{T})^k) \frac{du}{u}$$

along a half-line $L_\gamma = [0, +\infty)e^{\sqrt{-1}\gamma} \subset S_{d,\delta} \cup \{0\}$, where γ hinges on T and is chosen in a way that $\cos(k(\gamma - \arg(T))) \geq \delta_1$, for some fixed real number $\delta_1 > 0$. The function $\mathcal{L}_k^d(w)(T)$ is well defined, holomorphic and bounded on any sector

$$S_{d,\theta,R^{1/k}} = \{T \in \mathbb{C}^* : |T| < R^{1/k} \ , \ |d - \arg(T)| < \theta/2\},$$

provided that $0 < \theta < \frac{\pi}{k} + 2\delta$ and $0 < R < \delta_1/K$.

From the above very definition the next practical feature is deduced : if $w(\tau) = \sum_{n\geq 1} w_n \tau^n$ represents an entire function w.r.t $\tau \in \mathbb{C}$ with the bounds (5), its Laplace transform $\mathcal{L}_k^d(w)(T)$ does not depend on the direction d in $\mathbb{R}$ and represents a bounded holomorphic function on $D_{R^{1/k}}$ whose Taylor expansion is represented by the convergent series $X(T) = \sum_{n\geq 1} w_n \Gamma(\frac{n}{k}) T^n$ on $D_{R^{1/k}}$, where $\Gamma(x)$ stands for the Gamma function.

The next Banach spaces have been introduced in [15] and used in several works by the author.

Definition 2. *Let $\beta, \mu \in \mathbb{R}$. We set $E_{(\beta,\mu)}$ as the vector space of continuous functions $h : \mathbb{R} \to \mathbb{C}$ such that*

$$||h(m)||_{(\beta,\mu)} = \sup_{m\in\mathbb{R}} (1 + |m|)^\mu \exp(\beta|m|)|h(m)|$$

is finite. The space $E_{(\beta,\mu)}$ endowed with the norm $||.||_{(\beta,\mu)}$ becomes a Banach space.

Finally, we restate the definition of the inverse Fourier transform acting on the latter Banach spaces and some of its handy formulas relative to derivation and convolution product as detailed in [14].

Definition 3. *Take $f \in E_{(\beta,\mu)}$ with $\beta > 0$, $\mu > 1$. The inverse Fourier transform of f is shaped as the integral map*

$$\mathcal{F}^{-1}(f)(x) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} f(m) \exp(\sqrt{-1}xm) dm$$

for all $x \in \mathbb{R}$. *The function* $\mathcal{F}^{-1}(f)$ *extends to an analytic bounded function on the strips*

$$H_{\beta'} = \{z \in \mathbb{C}/|\mathrm{Im}(z)| < \beta'\}. \tag{6}$$

for all given $0 < \beta' < \beta$.
a) The function $m \mapsto \phi(m) = \sqrt{-1} m f(m)$ *belongs to the space* $E_{(\beta,\mu-1)}$ *and the next identity*

$$\partial_z \mathcal{F}^{-1}(f)(z) = \mathcal{F}^{-1}(\phi)(z)$$

occurs on H_β.
b) Let $g \in E_{(\beta,\mu)}$ *and set*

$$\psi(m) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} f(m - m_1) g(m_1) dm_1$$

as the convolution product of f *and* g. *Then,* ψ *belongs to* $E_{(\beta,\mu)}$ *and moreover the product formula*

$$\mathcal{F}^{-1}(f)(z)\mathcal{F}^{-1}(g)(z) = \mathcal{F}^{-1}(\psi)(z)$$

holds for all $z \in H_\beta$.

2.2. Formal monodromy around the origin

In this subsection, we define the notion of formal monodromy operator around the origin acting on different classes of objects. Following the description of abstract formal monodromy operator as stated in Subsection 3.2 of [1], we first provide a definition of formal monodromy acting on logarithmic type expressions involving formal power series with coefficients in Banach spaces.

Definition 4. *Let* $\mathcal{T}$ *be a bounded open sector centered at 0 in* $\mathbb{C}^*$ *and let*

$$H_{\beta'} = \{z \in \mathbb{C}/|\mathrm{Im}(z)| < \beta'\} \tag{7}$$

be a strip with width $2\beta' > 0$. *We denote* $\mathcal{O}_b(\mathcal{T} \times H_{\beta'})$ *the Banach space of bounded holomorphic functions on* $\mathcal{T} \times H_{\beta'}$ *equipped with the sup norm and we set as* $\mathcal{O}_b(\mathcal{T} \times H_{\beta'})[[\epsilon]]$ *the vector space of all formal series*

$$\hat{a}(t,z,\epsilon) = \sum_{n \geq 0} a_n(t,z)\epsilon^n$$

with coefficients belonging to $\mathcal{O}_b(\mathcal{T} \times H_{\beta'})$. *Let* $\hat{u}_1(t,z,\epsilon)$, $\hat{u}_2(t,z,\epsilon)$ *be two elements of* $\mathcal{O}_b(\mathcal{T} \times H_{\beta'})[[\epsilon]]$, *we set the formal logarithmic type expression*

$$\hat{u}(t,z,\epsilon) = \hat{u}_1(t,z,\epsilon) + \hat{u}_2(t,z,\epsilon)\log(\epsilon t) \tag{8}$$

where $\log(x)$ *stands for the principal value of the logarithm of a complex number* $x \in \mathbb{C}^*$.
We define the formal monodromy operator around 0 relatively to ϵ, *denoted* γ_ϵ^* *as acting on* $\hat{u}$ *by means of*

$$\gamma_\epsilon^* \hat{u}(t,z,\epsilon) = \hat{u}_1(t,z,\epsilon) + 2\pi\sqrt{-1}\hat{u}_2(t,z,\epsilon) + \hat{u}_2(t,z,\epsilon)\log(\epsilon t) \tag{9}$$

The next definition of formal monodromy extends the concept of monodromy operator around 0 acting on analytic functions on a punctured neighborhood of 0 as analytic continuation along a simple loop aroung the origin as described in [2], Section 16.

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Definition 5. *Let $\mathcal{T},\mathcal{E}$ be bounded open sectors centered at 0 in $\mathbb{C}$ and $H_{\beta'}$ be a strip defined by (7). We set $\mathcal{O}_b(\mathcal{T} \times H_{\beta'} \times \mathcal{E})$ as the Banach space of bounded holomorphic functions on $\mathcal{T} \times H_{\beta'} \times \mathcal{E}$ endowed with the sup norm. Let $u_1(t,z,\epsilon), u_2(t,z,\epsilon)$ be two elements of $\mathcal{O}_b(\mathcal{T} \times H_{\beta'} \times \mathcal{E})$. We set*

$$u(t,z,\epsilon) = u_1(t,z,\epsilon) + u_2(t,z,\epsilon)\log(\epsilon t) \tag{10}$$

that represents a holomorphic function for all $(t,z,\epsilon) \in \mathcal{T} \times H_{\beta'} \times \mathcal{E}$ with $\epsilon t \notin (-\infty, 0]$. The formal monodromy operator around 0 relatively to ϵ denoted γ_ϵ^* acts on u through the formula

$$\gamma_\epsilon^* u(t,z,\epsilon) = u_1(t,z,\epsilon) + 2\pi\sqrt{-1}u_2(t,z,\epsilon) + u_2(t,z,\epsilon)\log(\epsilon t) \tag{11}$$

Notice that if u_1 and u_2 are holomorphic on a full punctured disc centered at 0 relatively to ϵ, the formal monodromy γ_ϵ^* given above coincides with the analytic continuation along a simple loop skirting counterclockwise the origin 0 with base point ϵ.

We observe that each components $\hat{u}_1$,$\hat{u}_2$ of (8) (resp. u_1,u_2 of (10)) can be expressed by means of $\hat{u}$ and $\gamma_\epsilon^*\hat{u}$ (resp. u and $\gamma_\epsilon^* u$) through the formulas

$$\left\{ \begin{array}{lcl} \hat{u}_2(t,z,\epsilon) & = & \frac{1}{2\sqrt{-1}\pi}(\gamma_\epsilon^* - \mathrm{id})\hat{u}(t,z,\epsilon) \\ \hat{u}_1(t,z,\epsilon) & = & \hat{u}(t,z,\epsilon) - [\frac{1}{2\sqrt{-1}\pi}(\gamma_\epsilon^* - \mathrm{id})\hat{u}(t,z,\epsilon)]\log(\epsilon t) \end{array} \right. \tag{12}$$

and

$$\left\{ \begin{array}{lcl} u_2(t,z,\epsilon) & = & \frac{1}{2\sqrt{-1}\pi}(\gamma_\epsilon^* - \mathrm{id})u(t,z,\epsilon) \\ u_1(t,z,\epsilon) & = & u(t,z,\epsilon) - [\frac{1}{2\sqrt{-1}\pi}(\gamma_\epsilon^* - \mathrm{id})u(t,z,\epsilon)]\log(\epsilon t) \end{array} \right. \tag{13}$$

where id represents the identity operator acting on $\mathcal{O}_b(\mathcal{T} \times H_{\beta'})[[\epsilon]]$ in (12) and on $\mathcal{O}_b(\mathcal{T} \times H_{\beta'} \times \mathcal{E})$ in (13).

2.3. Outline of the main problem

The principal problem under study in this work is shaped as follows

$$\begin{aligned}
Q(\partial_z)u(t,z,\epsilon) &= (\epsilon t)^{d_D}(t\partial_t)^{\delta_D}R_D(\partial_z)u(t,z,\epsilon) + \sum_{l=1}^{D-1}\epsilon^{\Delta_l}t^{d_l}a_l(z,\epsilon)(t\partial_t)^{\delta_l}R_l(\partial_z)u(t,z,\epsilon) \\
&+ f(t,z,\epsilon) + c_1(z,\epsilon)[\frac{1}{2\sqrt{-1}\pi}(\gamma_\epsilon^* - \mathrm{id})u(t,z,\epsilon)]\log(\epsilon t) \\
&+ b_1(z,\epsilon)\Big[u(t,z,\epsilon) - [\frac{1}{2\sqrt{-1}\pi}(\gamma_\epsilon^* - \mathrm{id})u(t,z,\epsilon)]\log(\epsilon t)\Big] + b_2(z,\epsilon)\frac{1}{2\sqrt{-1}\pi}(\gamma_\epsilon^* - \mathrm{id})u(t,z,\epsilon) \\
&+ c_{Q_1Q_2}Q_1(\partial_z)[\frac{1}{2\sqrt{-1}\pi}(\gamma_\epsilon^* - \mathrm{id})u(t,z,\epsilon)] \times Q_2(\partial_z)[\frac{1}{2\sqrt{-1}\pi}(\gamma_\epsilon^* - \mathrm{id})u(t,z,\epsilon)] \times \log(\epsilon t) \\
&+ c_{P_1P_2}P_1(\partial_z)\Big[u(t,z,\epsilon) - [\frac{1}{2\sqrt{-1}\pi}(\gamma_\epsilon^* - \mathrm{id})u(t,z,\epsilon)]\log(\epsilon t)\Big] \\
&\times P_2(\partial_z)[\frac{1}{2\sqrt{-1}\pi}(\gamma_\epsilon^* - \mathrm{id})u(t,z,\epsilon)] \\
&+ c_{P_3P_4}P_3(\partial_z)\Big[u(t,z,\epsilon) - [\frac{1}{2\sqrt{-1}\pi}(\gamma_\epsilon^* - \mathrm{id})u(t,z,\epsilon)]\log(\epsilon t)\Big] \\
&\times P_4(\partial_z)\Big[u(t,z,\epsilon) - [\frac{1}{2\sqrt{-1}\pi}(\gamma_\epsilon^* - \mathrm{id})u(t,z,\epsilon)]\log(\epsilon t)\Big] \\
&+ c_{P_5P_6}P_5(\partial_z)[\frac{1}{2\sqrt{-1}\pi}(\gamma_\epsilon^* - \mathrm{id})u(t,z,\epsilon)] \times P_6(\partial_z)[\frac{1}{2\sqrt{-1}\pi}(\gamma_\epsilon^* - \mathrm{id})u(t,z,\epsilon)]
\end{aligned} \tag{14}$$

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for vanishing initial data $u(0,z,\epsilon) \equiv 0$. On the way in reaching our main result Theorem 1, we need to impose a list of constraints on the building blocks of (14). Namely,

- The numbers $D \geq 2$, $d_D, \delta_D \geq 1$ and $\Delta_l, d_l, \delta_l \geq 1$, $1 \leq l \leq D-1$ are integers that are subjected to the next restrictions
  1. We assume the existence of an integer $k_1 \geq 1$ with
  $$d_D = \delta_D k_1. \tag{15}$$
  2. The inequalities
  $$d_l > \delta_l k_1 \tag{16}$$
  hold for all $1 \leq l \leq D-1$.
  3. The bounds
  $$k_1\delta_D - 1 \geq k_1\delta_l \tag{17}$$
  are asked for all $1 \leq l \leq D-1$.
  4. The lower estimates
  $$\Delta_l \geq 1 + \delta_l k_1 \tag{18}$$
  are mandatory for all $1 \leq l \leq D-1$.
- The constants $c_{Q_1Q_2}, c_{P_jP_{j+1}}, j = 1,3,5$ are non vanishing complex numbers that are chosen close enough to 0 (the precise constraints that these numbers are asked to obey are stated later on in the work, see Section 5 and Section 6).
- The maps $Q(X), R_l(X)$, $l = 1,\ldots,D$ and $Q_1(X), Q_2(X)$ along with $P_j(X)$, $1 \leq j \leq 6$ are polynomials with complex coefficients. We require that
  $$\deg(R_l) \leq \deg(R_D) \tag{19}$$
  for $1 \leq l \leq D-1$ and
  $$\deg(R_D) \geq \deg(Q_1) \;,\; \deg(R_D) \geq \deg(Q_2) \;,\; \deg(R_D) \geq \deg(P_j) \tag{20}$$
  for $1 \leq j \leq 6$. Furthermore, we require the existence of an unbounded sectorial annulus
  $$S_{Q,R_D} = \{z \in \mathbb{C}^*/r_{Q,R_D} < |z| \;,\; |\arg(z) - d_{Q,R_D}| \leq \eta_{Q,R_D}\} \tag{21}$$
  with bisecting direction $d_{Q,R_D} \in \mathbb{R}$, aperture $\eta_{Q,R_D} > 0$ and inner radius $r_{Q,R_D} > 0$ (prescribed later in the work), for which the next inclusion
  $$\{\frac{Q(\sqrt{-1}m)}{R_D(\sqrt{-1}m)}/m \in \mathbb{R}\} \subset S_{Q,R_D} \tag{22}$$
  occurs.

  The forcing term $f(t,z,\epsilon)$ is built up in the next manner. It is written as a sum
  $$f(t,z,\epsilon) = f_1(t,z,\epsilon) + f_2(t,z,\epsilon)\log(\epsilon t) \tag{23}$$

where the components f_1, f_2 are set up as follows. Let $J_1, J_2 \subset \mathbb{N}^*$ be finite subsets of the positive integers. For $l = 1,2$ and $j_l \in J_l$, we denote $m \mapsto \mathcal{F}_{l,j_l}(m,\epsilon)$ maps that

- appertain to the Banach space $E_{(\beta,\mu)}$ for some $\beta > 0$ and
  $$\mu > \deg(R_l) + 1 \;,\; \mu > \max(\deg(Q_1)+1, \deg(Q_2)+1),$$
  $$\mu > \max(\deg(P_j)+1, \deg(P_{j+1})+1) \tag{24}$$

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for all $1 \leq l \leq D-1$, $j = 1,3,5$

- rely analytically on ϵ on some disc D_{ϵ_0} with radius $\epsilon_0 > 0$ for which constants $\boldsymbol{F}_{l,j_l,\epsilon_0} > 0$ exist such that

$$\sup_{\epsilon \in D_{\epsilon_0}} ||\mathcal{F}_{l,j_l}(m,\epsilon)||_{(\beta,\mu)} \leq \boldsymbol{F}_{l,j_l,\epsilon_0}. \tag{25}$$

For $l = 1,2$, let us introduce the polynomials in the variable τ with coefficients in $E_{(\beta,\mu)}$,

$$\mathcal{F}_l(\tau,m,\epsilon) = \sum_{j_l \in J_l} \mathcal{F}_{l,j_l}(m,\epsilon)\tau^{j_l}$$

and set the integral representations

$$F_l(T,z,\epsilon) = \frac{k_1}{(2\pi)^{1/2}} \int_{L_{d_1}} \int_{-\infty}^{+\infty} \mathcal{F}_l(\tau,m,\epsilon) \exp\big(-(\frac{\tau}{T})^{k_1}\big) e^{\sqrt{-1}zm} \frac{d\tau}{\tau} dm$$

where $L_{d_1} = [0,+\infty)e^{\sqrt{-1}d_1}$ is a halfline in direction $d_1 \in \mathbb{R}$ that relies on T under the constraint $\cos(k_1(d_1 - \arg(T))) > 0$. According to Definition 1, we observe that F_1 and F_2 are polynomials in T and can be expanded in the form

$$F_l(T,z,\epsilon) = \sum_{j_l \in J_l} F_{l,j_l}(z,\epsilon)\Gamma(\frac{j_l}{k_1})T^{j_l}$$

for coefficients given by the inverse Fourier integral expressions

$$F_{l,j_l}(z,\epsilon) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \mathcal{F}_{l,j_l}(m,\epsilon)e^{\sqrt{-1}zm}dm$$

that are, according to Definition 3, bounded holomorphic on the product $H_{\beta'} \times D_{\epsilon_0}$, for any given $0 < \beta' < \beta$, where $H_{\beta'}$ is the horizontal strip given by (7), for $l = 1,2$. Eventually, we set the components

$$f_l(t,z,\epsilon) = F_l(\epsilon t,z,\epsilon) \tag{26}$$

of (23) as a time rescaled version of F_l, for $l = 1,2$, that represent bounded holomorphic functions on $\mathbb{C} \times H_{\beta'} \times D_{\epsilon_0}$.

The coefficients $a_l(z,\epsilon)$, $1 \leq l \leq D-1$, $c_1(z,\epsilon)$ and $b_j(z,\epsilon)$, $j = 1,2$ are manufactured as follows. Let $m \mapsto A_l(m,\epsilon)$, $1 \leq l \leq D-1$, $m \mapsto C_1(m,\epsilon)$ and $m \mapsto B_j(m,\epsilon)$, $j = 1,2$, be maps that

- belong to the Banach space $E_{(\beta,\mu)}$, for the real numbers $\beta > 0$ and $\mu > 1$ given above
- that depend analytically in ϵ on D_{ϵ_0} and for which positive constants $\boldsymbol{A}_{l,\epsilon_0}$, $1 \leq l \leq D-1$, $\boldsymbol{C}_{1,\epsilon_0}$, $\boldsymbol{B}_{j,\epsilon_0}$, $j = 1,2$ can be singled out with

$$\sup_{\epsilon \in D_{\epsilon_0}} ||A_l(m,\epsilon)||_{(\beta,\mu)} \leq \boldsymbol{A}_{l,\epsilon_0} \ , \ \sup_{\epsilon \in D_{\epsilon_0}} ||C_1(m,\epsilon)||_{(\beta,\mu)} \leq \boldsymbol{C}_{1,\epsilon_0},$$
$$\sup_{\epsilon \in D_{\epsilon_0}} ||B_j(m,\epsilon)||_{(\beta,\mu)} \leq \boldsymbol{B}_{j,\epsilon_0}. \tag{27}$$

We set

$$a_l(z,\epsilon) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} A_l(m,\epsilon)e^{\sqrt{-1}zm}dm \ , \ c_1(z,\epsilon) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} C_1(m,\epsilon)e^{\sqrt{-1}zm}dm,$$
$$b_j(z,\epsilon) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} B_j(m,\epsilon)e^{\sqrt{-1}zm}dm$$

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for $1 \leq l \leq D-1$, $j=1,2$. Owing to Definition 3, the maps a_l, $1 \leq l \leq D-1$, c_1 and b_j, $j=1,2$ represent bounded holomorphic maps on the product $H_{\beta'} \times D_{\epsilon_0}$, for any prescribed $0 < \beta' < \beta$.

3. Couplings of related initial value problems

3.1. *A coupling of associated partial differential equations*

We seek for solutions $u(t,z,\epsilon)$ to our main equation (14) in the form

$$u(t,z,\epsilon) = U(\epsilon t, \log(\epsilon t), z, \epsilon) \quad (28)$$

for some expression $U(u_1,u_2,z,\epsilon)$ in the four independent variables u_1,u_2,z,ϵ. We furthermore assume that U is an affine map relatively to u_2 meaning that U is polynomial of degree at most one in u_2.

We first disclose an equation fulfilled by $U(u_1,u_2,z,\epsilon)$ provided that $u(t,z,\epsilon)$ solves (14) given by (33). According to the usual chain rule *applied at a formal level* at this stage of the work, we first observe that

$$t\partial_t u(t,z,\epsilon) = t\left[\partial_t(\epsilon t)\right](\partial_{u_1}U)(\epsilon t,\log(\epsilon t),z,\epsilon) + \left[t\partial_t(\log(\epsilon t))\right](\partial_{u_2}U)(\epsilon t,\log(\epsilon t),z,\epsilon)$$
$$= \left[(u_1\partial_{u_1} + \partial_{u_2})U\right](\epsilon t,\log(\epsilon t),z,\epsilon). \quad (29)$$

Besides, owing to the assumption that U is affine in u_2, we can decompose U in the form

$$U(u_1,u_2,z,\epsilon) = U_1(u_1,z,\epsilon) + U_2(u_1,z,\epsilon)u_2 \quad (30)$$

for some expressions $U_j(u_1,z,\epsilon)$, $j=1,2$. If one sets

$$u_j(t,z,\epsilon) = U_j(\epsilon t,z,\epsilon) \quad (31)$$

for $j=1,2$, through (28), one arrives at the next expansion of u,

$$u(t,z,\epsilon) = u_1(t,z,\epsilon) + u_2(t,z,\epsilon)\log(\epsilon t). \quad (32)$$

As a result, in view of the formulas (12), (13) together with the identity (29) and the definitions (26), (31), we check that $u(t,z,\epsilon)$ *formally* solves the equation (14) **if** the expression $U(u_1,u_2,z,\epsilon)$ is subjected to the next equation

$$Q(\partial_z)U(u_1,u_2,z,\epsilon) = u_1^{d_D}(u_1\partial_{u_1} + \partial_{u_2})^{\delta_D}R_D(\partial_z)U(u_1,u_2,z,\epsilon)$$
$$+ \sum_{l=1}^{D-1} \epsilon^{\Delta_l - d_l} u_1^{d_l} a_l(z,\epsilon)(u_1\partial_{u_1} + \partial_{u_2})^{\delta_l}R_l(\partial_z)U(u_1,u_2,z,\epsilon) + F_1(u_1,z,\epsilon) + F_2(u_1,z,\epsilon)u_2$$
$$+ c_1(z,\epsilon)U_2(u_1,z,\epsilon)u_2 + b_1(z,\epsilon)U_1(u_1,z,\epsilon) + b_2(z,\epsilon)U_2(u_1,z,\epsilon)$$
$$+ c_{Q_1Q_2}\left[Q_1(\partial_z)U_2(u_1,z,\epsilon)\right] \times \left[Q_2(\partial_z)U_2(u_1,z,\epsilon)\right]u_2$$
$$+ c_{P_1P_2}\left[P_1(\partial_z)U_1(u_1,z,\epsilon)\right] \times \left[P_2(\partial_z)U_2(u_1,z,\epsilon)\right]$$
$$+ c_{P_3P_4}\left[P_3(\partial_z)U_1(u_1,z,\epsilon)\right] \times \left[P_4(\partial_z)U_1(u_1,z,\epsilon)\right]$$
$$+ c_{P_5P_6}\left[P_5(\partial_z)U_2(u_1,z,\epsilon)\right] \times \left[P_6(\partial_z)U_2(u_1,z,\epsilon)\right]. \quad (33)$$

In the next step, we derive some coupling of partial differential equations that the components U_1 and U_2 are asked to fulfill and displayed in (36), (37).

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Owing to the fact that the operators $u_1\partial_{u_1}$ and ∂_{u_2} commute to each other, the binomial formula helps us to rewrite (33) in the form

$$
\begin{aligned}
Q(\partial_z)U(u_1,u_2,z,\epsilon) &= u_1^{d_D}\Big[\sum_{p_1+p_2=\delta_D}\frac{\delta_D!}{p_1!p_2!}(u_1\partial_{u_1})^{p_1}\partial_{u_2}^{p_2}R_D(\partial_z)U(u_1,u_2,z,\epsilon)\Big] \\
&+\sum_{l=1}^{D-1}\epsilon^{\Delta_l-d_l}u_1^{d_l}a_l(z,\epsilon)\times\Big[\sum_{p_1+p_2=\delta_l}\frac{\delta_l!}{p_1!p_2!}(u_1\partial_{u_1})^{p_1}\partial_{u_2}^{p_2}R_l(\partial_z)U(u_1,u_2,z,\epsilon)\Big] \\
&+F_1(u_1,z,\epsilon)+F_2(u_1,z,\epsilon)u_2+c_1(z,\epsilon)U_2(u_1,z,\epsilon)u_2+b_1(z,\epsilon)U_1(u_1,z,\epsilon)+b_2(z,\epsilon)U_2(u_1,z,\epsilon) \\
&+c_{Q_1Q_2}\big[Q_1(\partial_z)U_2(u_1,z,\epsilon)\big]\times\big[Q_2(\partial_z)U_2(u_1,z,\epsilon)\big]u_2 \\
&+c_{P_1P_2}\big[P_1(\partial_z)U_1(u_1,z,\epsilon)\big]\times\big[P_2(\partial_z)U_2(u_1,z,\epsilon)\big] \\
&+c_{P_3P_4}\big[P_3(\partial_z)U_1(u_1,z,\epsilon)\big]\times\big[P_4(\partial_z)U_1(u_1,z,\epsilon)\big] \\
&+c_{P_5P_6}\big[P_5(\partial_z)U_2(u_1,z,\epsilon)\big]\times\big[P_6(\partial_z)U_2(u_1,z,\epsilon)\big]. \quad (34)
\end{aligned}
$$

Besides, from the decomposition (30), we observe that

$$\partial_{u_2}U(u_1,u_2,z,\epsilon)=U_2(u_1,z,\epsilon)\ ,\ \partial_{u_2}^{p_2}U(u_1,u_2,z,\epsilon)\equiv 0$$

whenever $p_2\geq 2$. We reach the next equation

$$
\begin{aligned}
Q(\partial_z)\big[U_1(u_1,z,\epsilon)+U_2(u_1,z,\epsilon)u_2\big] &= u_1^{d_D}\Big[(u_1\partial_{u_1})^{\delta_D}R_D(\partial_z)\big(U_1(u_1,z,\epsilon)+U_2(u_1,z,\epsilon)u_2\big) \\
&+\delta_D(u_1\partial_{u_1})^{\delta_D-1}R_D(\partial_z)U_2(u_1,z,\epsilon)\Big]+\sum_{l=1}^{D-1}\epsilon^{\Delta_l-d_l}u_1^{d_l}a_l(z,\epsilon)\Big[(u_1\partial_{u_1})^{\delta_l}R_l(\partial_z)\big(U_1(u_1,z,\epsilon) \\
&+U_2(u_1,z,\epsilon)u_2\big)+\delta_l(u_1\partial_{u_1})^{\delta_l-1}R_l(\partial_z)U_2(u_1,z,\epsilon)\Big]+F_1(u_1,z,\epsilon)+F_2(u_1,z,\epsilon)u_2 \\
&+c_1(z,\epsilon)U_2(u_1,z,\epsilon)u_2+b_1(z,\epsilon)U_1(u_1,z,\epsilon)+b_2(z,\epsilon)U_2(u_1,z,\epsilon) \\
&+c_{Q_1Q_2}\big[Q_1(\partial_z)U_2(u_1,z,\epsilon)\big]\times\big[Q_2(\partial_z)U_2(u_1,z,\epsilon)\big]u_2 \\
&+c_{P_1P_2}\big[P_1(\partial_z)U_1(u_1,z,\epsilon)\big]\times\big[P_2(\partial_z)U_2(u_1,z,\epsilon)\big] \\
&+c_{P_3P_4}\big[P_3(\partial_z)U_1(u_1,z,\epsilon)\big]\times\big[P_4(\partial_z)U_1(u_1,z,\epsilon)\big] \\
&+c_{P_5P_6}\big[P_5(\partial_z)U_2(u_1,z,\epsilon)\big]\times\big[P_6(\partial_z)U_2(u_1,z,\epsilon)\big]. \quad (35)
\end{aligned}
$$

Finally, by dint of identification of the powers of u_2 in the above equality, it turns out that this last equation (35) holds **if** the expressions U_1 and U_2 are asked to satisfy the next *coupling of two partial differential equations*

$$
\begin{aligned}
Q(\partial_z)U_2(u_1,z,\epsilon) &= u_1^{d_D}\Big[(u_1\partial_{u_1})^{\delta_D}R_D(\partial_z)U_2(u_1,z,\epsilon)\Big] \\
&+\sum_{l=1}^{D-1}\epsilon^{\Delta_l-d_l}u_1^{d_l}a_l(z,\epsilon)(u_1\partial_{u_1})^{\delta_l}R_l(\partial_z)U_2(u_1,z,\epsilon)+F_2(u_1,z,\epsilon)+c_1(z,\epsilon)U_2(u_1,z,\epsilon) \\
&+c_{Q_1Q_2}\big[Q_1(\partial_z)U_2(u_1,z,\epsilon)\big]\times\big[Q_2(\partial_z)U_2(u_1,z,\epsilon)\big] \quad (36)
\end{aligned}
$$

and

$$
\begin{aligned}
Q(\partial_z)U_1(u_1,z,\epsilon) &= u_1^{d_D}\Big[(u_1\partial_{u_1})^{\delta_D}R_D(\partial_z)U_1(u_1,z,\epsilon)\\
&+\delta_D(u_1\partial_{u_1})^{\delta_D-1}R_D(\partial_z)U_2(u_1,z,\epsilon)\Big] + \sum_{l=1}^{D-1}\epsilon^{\Delta_l-d_l}u_1^{d_l}a_l(z,\epsilon)\Big[(u_1\partial_{u_1})^{\delta_l}R_l(\partial_z)U_1(u_1,z,\epsilon)\\
&+\delta_l(u_1\partial_{u_1})^{\delta_l-1}R_l(\partial_z)U_2(u_1,z,\epsilon)\Big] + F_1(u_1,z,\epsilon) + b_1(z,\epsilon)U_1(u_1,z,\epsilon) + b_2(z,\epsilon)U_2(u_1,z,\epsilon)\\
&+c_{P_1P_2}[P_1(\partial_z)U_1(u_1,z,\epsilon)]\times[P_2(\partial_z)U_2(u_1,z,\epsilon)] + c_{P_3P_4}[P_3(\partial_z)U_1(u_1,z,\epsilon)]\times[P_4(\partial_z)U_1(u_1,z,\epsilon)]\\
&+c_{P_5P_6}[P_5(\partial_z)U_2(u_1,z,\epsilon)]\times[P_6(\partial_z)U_2(u_1,z,\epsilon)] \quad (37)
\end{aligned}
$$

3.2. A coupling of auxiliary convolution equations

We search for solutions to the coupling of partial differential equations (36), (37) in the form of a Laplace transform of some order $k_1 \geq 1$ and inverse Fourier integral

$$
U_{j,d_1}(u_1,z,\epsilon) = \frac{k_1}{(2\pi)^{1/2}}\int_{L_{d_1}}\int_{-\infty}^{+\infty}\omega_{j,d_1}(\tau,m,\epsilon)\exp\big(-(\frac{\tau}{u_1})^{k_1}\big)e^{\sqrt{-1}zm}\frac{d\tau}{\tau}dm \quad (38)
$$

for $j = 1,2$, where $L_{d_1} = [0,+\infty)e^{\sqrt{-1}d_1}$ stands for a halfline in suitable directions $d_1 \in \mathbb{R}$ which depend on τ in a way that $\cos(k_1(d_1 - \arg(u_1)))$ remains strictly positive.

Here, we assume that for all $\epsilon \in D_{\epsilon_0} \setminus \{0\}$, the so-called *Borel-Fourier maps* $(\tau,m) \mapsto \omega_{j,d_1}(\tau,m,\epsilon)$, $j = 1,2$, belong to the Banach space $F^{d_1}_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$ for well chosen constants $\nu,\rho > 0$ and for the prescribed constants β,μ in Subsection 2.3 that is described in the upcoming definition

Definition 6. *Let $\epsilon_0,\nu,\beta,\mu,\rho > 0$ be positive real numbers and $k_1 \geq 1$ be an integer. Let $\epsilon \in D_{\epsilon_0} \setminus \{0\}$. We set as S_{d_1} an unbounded sector centered at 0 with bisecting direction $d_1 \in \mathbb{R}$. We denote $F^{d_1}_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$ the vector space of all continuous maps $(\tau,m) \mapsto h(\tau,m)$ on $(S_{d_1} \cup D_\rho) \times \mathbb{R}$, holomorphic w.r.t τ on $S_{d_1} \cup D_\rho$, such that the norm*

$$
||h(\tau,m)||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} = \sup_{\tau\in S_{d_1}\cup D_\rho, m\in\mathbb{R}} (1+|m|)^{\mu}e^{\beta|m|}\frac{|\epsilon|}{|\tau|}(1+|\frac{\tau}{\epsilon}|^{2k_1})\exp\big(-\nu|\frac{\tau}{\epsilon}|^{k_1}\big)|h(\tau,m)|
$$

is finite. The vector space $F^{d_1}_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$ equipped with the norm $||.||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$ turns out to be a Banach space.

The main purpose of this subsection is to determine coupling convolution equations for the Borel-Fourier maps ω_{j,d_1} outlined in (49) and (50), (51). We depart from some features of the Laplace transforms under the action of multiplication by a monomial and differential operators that were already stated and proved in our foregoing work [16], Lemma 2.

Lemma 1. *The next identities hold.*

1. The action of the differential operator $u_1^{k_1+1}\partial_{u_1}$ on the integral representations U_{j,d_1} is given by

$$
\begin{aligned}
&u_1^{k_1+1}\partial_{u_1}U_{j,d_1}(u_1,z,\epsilon)\\
&\quad = \frac{k_1}{(2\pi)^{1/2}}\int_{L_{d_1}}\int_{-\infty}^{+\infty}[k_1\tau^{k_1}\omega_{j,d_1}(\tau,m,\epsilon)]\exp\big(-(\frac{\tau}{u_1})^{k_1}\big)e^{\sqrt{-1}zm}\frac{d\tau}{\tau}dm. \quad (39)
\end{aligned}
$$

2. Let $m' \geq 1$ be an integer. The multiplication by $u_1^{m'}$ acting on U_{j,d_1} is expressed through

$$
\begin{aligned}
u_1^{m'}U_{j,d_1}(u_1,z,\epsilon) &= \frac{k_1}{(2\pi)^{1/2}}\int_{L_{d_1}}\int_{-\infty}^{+\infty}\Big[\frac{\tau^{k_1}}{\Gamma(\frac{m'}{k_1})}\int_0^{\tau^{k_1}}(\tau^{k_1}-s)^{\frac{m'}{k_1}-1}\omega_{j,d_1}(s^{1/k_1},m,\epsilon)\frac{ds}{s}\Big]\\
&\times\exp\big(-(\frac{\tau}{u_1})^{k_1}\big)e^{\sqrt{-1}zm}\frac{d\tau}{\tau}dm. \quad (40)
\end{aligned}
$$

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3. Let $m \mapsto A(m)$ be a map that belongs to $E_{(\beta,\mu)}$. We set

$$a(z) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} A(m) e^{\sqrt{-1}zm} dm.$$

The action of multiplication by $a(z)$ on U_{j,d_1} is expressed by means of

$$a(z)U_{j,d_1}(u_1,z,\epsilon) = \frac{k_1}{(2\pi)^{1/2}} \int_{L_{d_1}} \int_{-\infty}^{+\infty} \Big[\frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} A(m-m_1)\omega_{j,d_1}(\tau,m_1,\epsilon)dm_1 \Big] \times \exp\big(-(\frac{\tau}{u_1})^{k_1}\big) e^{\sqrt{-1}zm} \frac{d\tau}{\tau} dm. \quad (41)$$

4. Let $H_k(X) \in \mathbb{C}[X]$, $k = 1,2$, be polynomials. The action of the differential operators $H_k(\partial_z)$ combined with the product of the resulting functions $H_k(\partial_z)U_{j,d_1}$ for $k = 1,2$, $j = 1,2$ maps U_{j,d_1} into a Fourier-Laplace transform,

$$\begin{aligned} &\big[H_1(\partial_z)U_{l,d_1}(u_1,z,\epsilon)\big] \times \big[H_2(\partial_z)U_{p,d_1}(u_1,z,\epsilon)\big] \\ &= \frac{k_1}{(2\pi)^{1/2}} \int_{L_{d_1}} \int_{-\infty}^{+\infty} \Big[\frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \tau^{k_1} \int_0^{\tau^{k_1}} H_1(\sqrt{-1}(m-m_1))\omega_{l,d_1}((\tau^{k_1}-s)^{1/k_1}, m-m_1,\epsilon) \\ &\quad \times H_2(\sqrt{-1}m_1)\omega_{p,d_1}(s^{1/k_1},m_1,\epsilon) \frac{1}{(\tau^{k_1}-s)s} ds dm_1 \Big] \\ &\quad \times \exp\big(-(\frac{\tau}{u_1})^{k_1}\big) e^{\sqrt{-1}zm} \frac{d\tau}{\tau} dm. \quad (42) \end{aligned}$$

for given $1 \leq l,p \leq 2$.

The next useful lemma already stated in the previous work by A. Lastra and the author [17] will show up in the process.

Lemma 2. *For all integers $p_1 \geq 1$, positive integers $a_{q,p_1} \geq 1$, for $1 \leq q \leq p_1$ can be singled out such that*

$$(u_1\partial_{u_1})^{p_1} = \sum_{q=1}^{p_1} a_{q,p_1} u_1^q \partial_{u_1}^q$$

with $a_{1,p_1} = a_{p_1,p_1} = 1$.

With the help of this lemma, the equations (36) and (37) can be remodeled in the form

$$\begin{aligned} Q(\partial_z)U_2(u_1,z,\epsilon) &= u_1^{d_D} \Big[\big(\sum_{q=1}^{\delta_D} a_{q,\delta_D} u_1^q \partial_{u_1}^q \big) R_D(\partial_z) U_2(u_1,z,\epsilon) \Big] \\ &+ \sum_{l=1}^{D-1} \epsilon^{\Delta_l - d_l} u_1^{d_l} a_l(z,\epsilon) \big(\sum_{q=1}^{\delta_l} a_{q,\delta_l} u_1^q \partial_{u_1}^q \big) R_l(\partial_z) U_2(u_1,z,\epsilon) + F_2(u_1,z,\epsilon) + c_1(z,\epsilon) U_2(u_1,z,\epsilon) \\ &+ c_{Q_1Q_2} \big[Q_1(\partial_z) U_2(u_1,z,\epsilon) \big] \times \big[Q_2(\partial_z) U_2(u_1,z,\epsilon) \big] \quad (43) \end{aligned}$$

and

$$
\begin{aligned}
Q(\partial_z)U_1(u_1,z,\epsilon) = u_1^{d_D}\Big[(\sum_{q=1}^{\delta_D} a_{q,\delta_D} u_1^q \partial_{u_1}^q) R_D(\partial_z) U_1(u_1,z,\epsilon) \\
+ \delta_D (\sum_{q=1}^{\delta_D - 1} a_{q,\delta_D - 1} u_1^q \partial_{u_1}^q) R_D(\partial_z) U_2(u_1,z,\epsilon) \Big] \\
+ \sum_{l=1}^{D-1} \epsilon^{\Delta_l - d_l} u_1^{d_l} a_l(z,\epsilon) \Big[(\sum_{q=1}^{\delta_l} a_{q,\delta_l} u_1^q \partial_{u_1}^q) R_l(\partial_z) U_1(u_1,z,\epsilon) \\
+ \delta_l (\sum_{q=1}^{\delta_l - 1} a_{q,\delta_l - 1} u_1^q \partial_{u_1}^q) R_l(\partial_z) U_2(u_1,z,\epsilon) \Big] + F_1(u_1,z,\epsilon) + b_1(z,\epsilon) U_1(u_1,z,\epsilon) + b_2(z,\epsilon) U_2(u_1,z,\epsilon) \\
+ c_{P_1P_2} \big[P_1(\partial_z) U_1(u_1,z,\epsilon) \big] \times \big[P_2(\partial_z) U_2(u_1,z,\epsilon) \big] + c_{P_3P_4} \big[P_3(\partial_z) U_1(u_1,z,\epsilon) \big] \times \big[P_4(\partial_z) U_1(u_1,z,\epsilon) \big] \\
+ c_{P_5P_6} \big[P_5(\partial_z) U_2(u_1,z,\epsilon) \big] \times \big[P_6(\partial_z) U_2(u_1,z,\epsilon) \big] \quad (44)
\end{aligned}
$$

The upcoming identity will also be called into play for the derivation of the coupling convolution equations. This technical formula was introduced in the work [18].

Lemma 3. *Let $k_1, \delta \geq 1$ be integers. Real numbers $A_{\delta,p}$, for $1 \leq p \leq \delta - 1$ can be found such that*

$$
u_1^{\delta(k_1+1)} \partial_{u_1}^{\delta} = (u_1^{k_1+1} \partial_{u_1})^{\delta} + \sum_{1 \leq p \leq \delta - 1} A_{\delta,p} u_1^{k_1(\delta - p)} (u_1^{k_1+1} \partial_{u_1})^p
$$

holds, where we assume by convention that the sum $\sum_{1 \leq p \leq \delta - 1} [..]$ vanishes for $\delta = 1$.

Owing to the assumption (15), the splitting

$$
d_D + q = q(k_1 + 1) + d_{D,q} \tag{45}
$$

holds for suitable integers $d_{D,q} \geq 1$, provided that $1 \leq q \leq \delta_D - 1$. Furthermore, under the constraint (16), the decomposition

$$
d_l + q = q(k_1 + 1) + d_{l,q} \tag{46}
$$

occurs for well chosen integers $d_{l,q} \geq 1$, as long as $1 \leq l \leq D - 1$ and $1 \leq q \leq \delta_l$.

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Ultimately, by means of the above two relations (45) and (46), the lemma 3 can be applied in order to rewrite both equations (43), (44) only with the help of the basic irregular differential operator $u_1^{k_1+1}\partial_{u_1}$. Namely,

$$
\begin{aligned}
&Q(\partial_z)U_2(u_1,z,\epsilon)\\
&= \Big(\sum_{q=1}^{\delta_D-1} a_{q,\delta_D}u_1^{d_{D,q}}\Big[(u_1^{k_1+1}\partial_{u_1})^q + \sum_{1\le p\le q-1} A_{q,p}u_1^{k_1(q-p)}(u_1^{k_1+1}\partial_{u_1})^p\Big]R_D(\partial_z)U_2(u_1,z,\epsilon)\Big)\\
&\quad + \Big[(u_1^{k_1+1}\partial_{u_1})^{\delta_D} + \sum_{1\le p\le \delta_D-1} A_{\delta_D,p}u_1^{k_1(\delta_D-p)}(u_1^{k_1+1}\partial_{u_1})^p\Big]R_D(\partial_z)U_2(u_1,z,\epsilon)\\
&\quad + \Big(\sum_{l=1}^{D-1}\epsilon^{\Delta_l-d_l}a_l(z,\epsilon)\\
&\quad \times \Big[\sum_{q=1}^{\delta_l} a_{q,\delta_l}u_1^{d_{l,q}}[(u_1^{k_1+1}\partial_{u_1})^q + \sum_{1\le p\le q-1} A_{q,p}u_1^{k_1(q-p)}(u_1^{k_1+1}\partial_{u_1})^p]R_l(\partial_z)U_2(u_1,z,\epsilon)\Big]\Big)\\
&\quad + F_2(u_1,z,\epsilon) + c_1(z,\epsilon)U_2(u_1,z,\epsilon)\\
&\quad + c_{Q_1Q_2}[Q_1(\partial_z)U_2(u_1,z,\epsilon)]\times[Q_2(\partial_z)U_2(u_1,z,\epsilon)] \qquad (47)
\end{aligned}
$$

together with

$$
\begin{aligned}
&Q(\partial_z)U_1(u_1,z,\epsilon)\\
&= \Big(\sum_{q=1}^{\delta_D-1} a_{q,\delta_D}u_1^{d_{D,q}}\Big[(u_1^{k_1+1}\partial_{u_1})^q + \sum_{1\le p\le q-1} A_{q,p}u_1^{k_1(q-p)}(u_1^{k_1+1}\partial_{u_1})^p\Big]R_D(\partial_z)U_1(u_1,z,\epsilon)\Big)\\
&\quad + \Big[(u_1^{k_1+1}\partial_{u_1})^{\delta_D} + \sum_{1\le p\le \delta_D-1} A_{\delta_D,p}u_1^{k_1(\delta_D-p)}(u_1^{k_1+1}\partial_{u_1})^p\Big]R_D(\partial_z)U_1(u_1,z,\epsilon)\\
&\quad + \delta_D\sum_{q=1}^{\delta_D-1} a_{q,\delta_D-1}u_1^{d_{D,q}}\Big[(u_1^{k_1+1}\partial_{u_1})^q + \sum_{1\le p\le q-1} A_{q,p}u_1^{k_1(q-p)}(u_1^{k_1+1}\partial_{u_1})^p\Big]R_D(\partial_z)U_2(u_1,z,\epsilon)\\
&\quad + \Big(\sum_{l=1}^{D-1}\epsilon^{\Delta_l-d_l}a_l(z,\epsilon)\\
&\quad \times \Big[\sum_{q=1}^{\delta_l} a_{q,\delta_l}u_1^{d_{l,q}}[(u_1^{k_1+1}\partial_{u_1})^q + \sum_{1\le p\le q-1} A_{q,p}u_1^{k_1(q-p)}(u_1^{k_1+1}\partial_{u_1})^p]R_l(\partial_z)U_1(u_1,z,\epsilon)\\
&\quad + \delta_l\sum_{q=1}^{\delta_l-1} a_{q,\delta_l-1}u_1^{d_{l,q}}[(u_1^{k_1+1}\partial_{u_1})^q + \sum_{1\le p\le q-1} A_{q,p}u_1^{k_1(q-p)}(u_1^{k_1+1}\partial_{u_1})^p]R_l(\partial_z)U_2(u_1,z,\epsilon)\Big]\Big)\\
&\quad + F_1(u_1,z,\epsilon) + b_1(z,\epsilon)U_1(u_1,z,\epsilon) + b_2(z,\epsilon)U_2(u_1,z,\epsilon)\\
&\quad + c_{P_1P_2}[P_1(\partial_z)U_1(u_1,z,\epsilon)]\times[P_2(\partial_z)U_2(u_1,z,\epsilon)]\\
&\quad + c_{P_3P_4}[P_3(\partial_z)U_1(u_1,z,\epsilon)]\times[P_4(\partial_z)U_1(u_1,z,\epsilon)]\\
&\quad + c_{P_5P_6}[P_5(\partial_z)U_2(u_1,z,\epsilon)]\times[P_6(\partial_z)U_2(u_1,z,\epsilon)] \qquad (48)
\end{aligned}
$$

On the ground of the identities disclosed in Lemma 1, this hindmost coupling of equations (47) and (48) allows us to reach the next statement.

The maps $U_{j,d_1}(u_1,z,\epsilon)$, $j=1,2$, displayed in (38) solve the closing coupling (47) and (48) **if** the Borel maps $\omega_{j,d_1}(\tau,m,\epsilon)$, $j=1,2$, fulfill the next coupling of convolution equations

$$
\begin{aligned}
&Q(\sqrt{-1}m)\omega_{2,d_1}(\tau,m,\epsilon)\\
&\quad = \Big(\sum_{q=1}^{\delta_D-1} a_{q,\delta_D}\Big[\frac{\tau^{k_1}}{\Gamma(\frac{d_{D,q}}{k_1})}\int_0^{\tau^{k_1}}(\tau^{k_1}-s)^{\frac{d_{D,q}}{k_1}-1}(k_1(s^{1/k_1})^{k_1})^q\omega_{2,d_1}(s^{1/k_1},m,\epsilon)\frac{ds}{s}\\
&+\sum_{1\le p\le q-1}A_{q,p}\frac{\tau^{k_1}}{\Gamma(\frac{d_{D,q}+k_1(q-p)}{k_1})}\int_0^{\tau^{k_1}}(\tau^{k_1}-s)^{\frac{d_{D,q}+k_1(q-p)}{k_1}-1}(k_1(s^{1/k_1})^{k_1})^p\omega_{2,d_1}(s^{1/k_1},m,\epsilon)\frac{ds}{s}\Big]\\
&\qquad \times R_D(\sqrt{-1}m)\Big) + \Big[(k_1\tau^{k_1})^{\delta_D}\omega_{2,d_1}(\tau,m,\epsilon)\\
&+\sum_{1\le p\le \delta_D-1}A_{\delta_D,p}\frac{\tau^{k_1}}{\Gamma(\frac{k_1(\delta_D-p)}{k_1})}\int_0^{\tau^{k_1}}(\tau^{k_1}-s)^{\frac{k_1(\delta_D-p)}{k_1}-1}(k_1(s^{1/k_1})^{k_1})^p\omega_{2,d_1}(s^{1/k_1},m,\epsilon)\frac{ds}{s}\Big]\\
&\qquad \times R_D(\sqrt{-1}m)\\
&+\sum_{l=1}^{D-1}\epsilon^{\Delta_l-d_l}\Big[\sum_{q=1}^{\delta_l}a_{q,\delta_l}\Big[\frac{\tau^{k_1}}{\Gamma(\frac{d_{l,q}}{k_1})}\int_0^{\tau^{k_1}}(\tau^{k_1}-s)^{\frac{d_{l,q}}{k_1}-1}\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty}A_l(m-m_1,\epsilon)(k_1(s^{1/k_1})^{k_1})^q\\
&\times R_l(\sqrt{-1}m_1)\omega_{2,d_1}(s^{1/k_1},m_1,\epsilon)\frac{ds}{s}dm_1+\sum_{1\le p\le q-1}A_{q,p}\frac{\tau^{k_1}}{\Gamma(\frac{d_{l,q}+k_1(q-p)}{k_1})}\int_0^{\tau^{k_1}}(\tau^{k_1}-s)^{\frac{d_{l,q}+k_1(q-p)}{k_1}-1}\\
&\times\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty}A_l(m-m_1,\epsilon)(k_1(s^{1/k_1})^{k_1})^pR_l(\sqrt{-1}m_1)\omega_{2,d_1}(s^{1/k_1},m_1,\epsilon)\frac{ds}{s}dm_1\Big]\Big]\\
&\qquad +\mathcal{F}_2(\tau,m,\epsilon)+\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty}C_1(m-m_1,\epsilon)\omega_{2,d_1}(\tau,m_1,\epsilon)dm_1\\
&+c_{Q_1Q_2}\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty}\tau^{k_1}\int_0^{\tau^{k_1}}Q_1(\sqrt{-1}(m-m_1))\omega_{2,d_1}((\tau^{k_1}-s)^{1/k_1},m-m_1,\epsilon)\\
&\qquad\qquad \times Q_2(\sqrt{-1}m_1)\omega_{2,d_1}(s^{1/k_1},m_1,\epsilon)\frac{1}{(\tau^{k_1}-s)s}dsdm_1 \qquad (49)
\end{aligned}
$$

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along with

$$
\begin{aligned}
&Q(\sqrt{-1}m)\omega_{1,d_1}(\tau,m,\epsilon)\\
&= \Big(\sum_{q=1}^{\delta_D-1} a_{q,\delta_D}\Big[\frac{\tau^{k_1}}{\Gamma(\frac{d_{D,q}}{k_1})}\int_0^{\tau^{k_1}}(\tau^{k_1}-s)^{\frac{d_{D,q}}{k_1}-1}(k_1(s^{1/k_1})^{k_1})^q\omega_{1,d_1}(s^{1/k_1},m,\epsilon)\frac{ds}{s}\\
&+\sum_{1\le p\le q-1}A_{q,p}\frac{\tau^{k_1}}{\Gamma(\frac{d_{D,q}+k_1(q-p)}{k_1})}\int_0^{\tau^{k_1}}(\tau^{k_1}-s)^{\frac{d_{D,q}+k_1(q-p)}{k_1}-1}(k_1(s^{1/k_1})^{k_1})^p\omega_{1,d_1}(s^{1/k_1},m,\epsilon)\frac{ds}{s}\Big]\\
&\times R_D(\sqrt{-1}m)\Big) + \Big[(k_1\tau^{k_1})^{\delta_D}R_D(\sqrt{-1}m)\omega_{1,d_1}(\tau,m,\epsilon)\\
&+\sum_{1\le p\le \delta_D-1}A_{\delta_D,p}\frac{\tau^{k_1}}{\Gamma(\frac{k_1(\delta_D-p)}{k_1})}\int_0^{\tau^{k_1}}(\tau^{k_1}-s)^{\frac{k_1(\delta_D-p)}{k_1}-1}(k_1(s^{1/k_1})^{k_1})^p\omega_{1,d_1}(s^{1/k_1},m,\epsilon)\frac{ds}{s}\\
&\times R_D(\sqrt{-1}m)\Big]\\
&+\Big(\delta_D\sum_{q=1}^{\delta_D-1}a_{q,\delta_D-1}\Big[\frac{\tau^{k_1}}{\Gamma(\frac{d_{D,q}}{k_1})}\int_0^{\tau^{k_1}}(\tau^{k_1}-s)^{\frac{d_{D,q}}{k_1}-1}(k_1(s^{1/k_1})^{k_1})^q\omega_{2,d_1}(s^{1/k_1},m,\epsilon)\frac{ds}{s}\\
&+\sum_{1\le p\le q-1}A_{q,p}\frac{\tau^{k_1}}{\Gamma(\frac{d_{D,q}+k_1(q-p)}{k_1})}\int_0^{\tau^{k_1}}(\tau^{k_1}-s)^{\frac{d_{D,q}+k_1(q-p)}{k_1}-1}(k_1(s^{1/k_1})^{k_1})^p\omega_{2,d_1}(s^{1/k_1},m,\epsilon)\frac{ds}{s}\Big]\\
&\times R_D(\sqrt{-1}m)\Big)\\
&+\sum_{l=1}^{D-1}\epsilon^{\Delta_l-d_l}\Big[\Big(\sum_{q=1}^{\delta_l}a_{q,\delta_l}\Big[\frac{\tau^{k_1}}{\Gamma(\frac{d_{l,q}}{k_1})}\int_0^{\tau^{k_1}}(\tau^{k_1}-s)^{\frac{d_{l,q}}{k_1}-1}\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty}A_l(m-m_1,\epsilon)(k_1(s^{1/k_1})^{k_1})^q\\
&\times R_l(\sqrt{-1}m_1)\omega_{1,d_1}(s^{1/k_1},m_1,\epsilon)\frac{ds}{s}dm_1+\sum_{1\le p\le q-1}A_{q,p}\frac{\tau^{k_1}}{\Gamma(\frac{d_{l,q}+k_1(q-p)}{k_1})}\int_0^{\tau^{k_1}}(\tau^{k_1}-s)^{\frac{d_{l,q}+k_1(q-p)}{k_1}-1}\\
&\times\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty}A_l(m-m_1,\epsilon)(k_1(s^{1/k_1})^{k_1})^pR_l(\sqrt{-1}m_1)\omega_{1,d_1}(s^{1/k_1},m_1,\epsilon)\frac{ds}{s}dm_1\Big]\Big)\\
&+\Big(\delta_l\sum_{q=1}^{\delta_l-1}a_{q,\delta_l-1}\Big[\frac{\tau^{k_1}}{\Gamma(\frac{d_{l,q}}{k_1})}\int_0^{\tau^{k_1}}(\tau^{k_1}-s)^{\frac{d_{l,q}}{k_1}-1}\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty}A_l(m-m_1,\epsilon)(k_1(s^{1/k_1})^{k_1})^qR_l(\sqrt{-1}m_1)\\
&\times\omega_{2,d_1}(s^{1/k_1},m_1,\epsilon)\frac{ds}{s}dm_1+\sum_{1\le p\le q-1}A_{q,p}\frac{\tau^{k_1}}{\Gamma(\frac{d_{l,q}+k_1(q-p)}{k_1})}\int_0^{\tau^{k_1}}(\tau^{k_1}-s)^{\frac{d_{l,q}+k_1(q-p)}{k_1}-1}\\
&\times\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty}A_l(m-m_1,\epsilon)(k_1(s^{1/k_1})^{k_1})^pR_l(\sqrt{-1}m_1)\omega_{2,d_1}(s^{1/k_1},m_1,\epsilon)\frac{ds}{s}dm_1\Big]\Big)\Big]\\
&+\mathcal{A}(\tau,m,\epsilon) \qquad (50)
\end{aligned}
$$

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where

$$
\begin{aligned}
\mathcal{A}(\tau,m,\epsilon) := \mathcal{F}_1(\tau,m,\epsilon) + \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty} B_1(m-m_1,\epsilon)\omega_{1,d_1}(\tau,m_1,\epsilon)dm_1 \\
+ \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty} B_2(m-m_1,\epsilon)\omega_{2,d_1}(\tau,m_1,\epsilon)dm_1 \\
+ c_{P_1P_2}\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty}\tau^{k_1}\int_0^{\tau^{k_1}} P_1(\sqrt{-1}(m-m_1))\omega_{1,d_1}((\tau^{k_1}-s)^{1/k_1},m-m_1,\epsilon) \\
\times P_2(\sqrt{-1}m_1)\omega_{2,d_1}(s^{1/k_1},m_1,\epsilon)\frac{1}{(\tau^{k_1}-s)s}dsdm_1 \\
+ c_{P_3P_4}\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty}\tau^{k_1}\int_0^{\tau^{k_1}} P_3(\sqrt{-1}(m-m_1))\omega_{1,d_1}((\tau^{k_1}-s)^{1/k_1},m-m_1,\epsilon) \\
\times P_4(\sqrt{-1}m_1)\omega_{1,d_1}(s^{1/k_1},m_1,\epsilon)\frac{1}{(\tau^{k_1}-s)s}dsdm_1 \\
+ c_{P_5P_6}\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty}\tau^{k_1}\int_0^{\tau^{k_1}} P_5(\sqrt{-1}(m-m_1))\omega_{2,d_1}((\tau^{k_1}-s)^{1/k_1},m-m_1,\epsilon) \\
\times P_6(\sqrt{-1}m_1)\omega_{2,d_1}(s^{1/k_1},m_1,\epsilon)\frac{1}{(\tau^{k_1}-s)s}dsdm_1
\end{aligned} \quad (51)
$$

4. Linear and bilinear convolution operators acting on Banach spaces

In this section, we examine continuity properties of several linear and bilinear convolutions operators that are applied on the Banach spaces given in Definition 6 and that unfold in the above coupled equations (49) and (50), (51).

Proposition 1. *Let $\gamma_1 \geq 0$, $\gamma_3 \geq -1$ be integers and set $\gamma_2 \in \mathbb{R}$. Let S_{d_1} be an unbounded sector centered at 0 with bisecting direction $d_1 \in \mathbb{R}$ and fix $\rho > 0$ as some positive real number. Let $a_{\gamma_1}(\tau,m)$ be a continuous map on the closure $(\bar{S_{d_1}} \cup \bar{D}_\rho) \times \mathbb{R}$ subjected to the upper bounds*

$$
|a_{\gamma_1}(\tau,m)| \leq \frac{M_{\gamma_1}}{(1+|\tau|)^{\gamma_1}} \quad (52)
$$

provided that $\tau \in S_{d_1} \cup D_\rho$, all $m \in \mathbb{R}$, for some constant $M_{\gamma_1} > 0$. We take for granted that

$$
\gamma_1 \geq k_1(\gamma_3+1) \ , \ \gamma_2 > -1 \ , \ \gamma_2+\gamma_3+\frac{1}{k_1}+1 \geq 0. \quad (53)
$$

Then, we can single out a constant $C_1 > 0$ (relying on γ_j, $j=1,2,3$, k_1 and ν) for which

$$
\begin{aligned}
||a_{\gamma_1}(\tau,m)\tau^{k_1}\int_0^{\tau^{k_1}}(\tau^{k_1}-s)^{\gamma_2}s^{\gamma_3}f(s^{1/k_1},m)ds||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \\
\leq C_1 M_{\gamma_1}|\epsilon|^{k_1(\gamma_2+1)}||f(\tau,m)||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}
\end{aligned} \quad (54)
$$

holds as long as f belongs to the Banach space $F^{d_1}_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$.

Proof. Let $f \in F^{d_1}_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$. By definition, the bounds

$$
|f(\tau,m)| \leq ||f||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}|\frac{\tau}{\epsilon}|\frac{1}{1+|\tau/\epsilon|^{2k_1}}\exp(\nu|\frac{\tau}{\epsilon}|^{k_1})(1+|m|)^{-\mu}e^{-\beta|m|} \quad (55)
$$

ensue provided that $\tau \in S_{d_1} \cup D_\rho$ and $m \in \mathbb{R}$. According to the assumption (52), the latter bounds warrant the next estimates

$$\mathcal{B}(\tau,m) := |a_{\gamma_1}(\tau,m)\tau^{k_1}\int_0^{\tau^{k_1}}(\tau^{k_1}-s)^{\gamma_2}s^{\gamma_3}f(s^{1/k_1},m)ds| \\ \leq \frac{M_{\gamma_1}||f||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}}{(1+|\tau|)^{\gamma_1}}|\tau|^{k_1}\int_0^{|\tau|^{k_1}}(|\tau|^{k_1}-h)^{\gamma_2}h^{\gamma_3}\frac{h^{1/k_1}}{|\epsilon|}\frac{1}{1+\frac{h^2}{|\epsilon|^{2k_1}}}\exp\left(\nu\frac{h}{|\epsilon|^{k_1}}\right)dh \\ \times(1+|m|)^{-\mu}e^{-\beta|m|} \quad (56)$$

for all $\tau \in S_{d_1} \cup D_\rho$, all $m \in \mathbb{R}$.

We further perform the change of variable $g = h/|\epsilon|^{k_1}$ in the above integral and get

$$\mathcal{B}(\tau,m) \leq \frac{M_{\gamma_1}||f||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}}{(1+|\tau|)^{\gamma_1}}|\tau|^{k_1}\int_0^{\frac{|\tau|^{k_1}}{|\epsilon|^{k_1}}}\left(\frac{|\tau|^{k_1}}{|\epsilon|^{k_1}}-g\right)^{\gamma_2}g^{\gamma_3+\frac{1}{k_1}}\frac{1}{1+g^2}e^{\nu g}dg \\ \times|\epsilon|^{k_1(\gamma_2+\gamma_3+1)}(1+|m|)^{-\mu}e^{-\beta|m|} \quad (57)$$

as long as $\tau \in S_{d_1} \cup D_\rho$ and $m \in \mathbb{R}$.

We introduce the function

$$G(x) = \int_0^x (x-g)^{\gamma_2}g^{\gamma_3+\frac{1}{k_1}}\frac{1}{1+g^2}e^{\nu g}dg$$

for all $x \geq 0$. In the next lemma, we uncover upper bounds for G for large values of x.

Lemma 4. *The function $G(x)$ is well defined and continuous for all $x \geq 0$. Furthermore, there exists a constant $K_G > 0$ for which*

$$G(x) \leq K_G\frac{x^{\gamma_3+\frac{1}{k_1}}}{1+x^2}e^{\nu x} \quad (58)$$

for all $x \geq 1$.

Proof. We first explain why $G(x)$ is well defined and continuous for $x \geq 0$. Indeed, by means of the change of variable $g = xg_1$ for $0 \leq g_1 \leq 1$, we can recast $G(x)$ in the form

$$G(x) = x^{\gamma_2+\gamma_3+\frac{1}{k_1}+1}\int_0^1(1-g_1)^{\gamma_2}g_1^{\gamma_3+\frac{1}{k_1}}\frac{1}{1+(xg_1)^2}e^{\nu x g_1}dg_1$$

which is a finite quantity for all $x \geq 0$ and represents a continuous map w.r.t x, according to the last inequality of (53).

In order to reach bounds for large $x \geq 1$, we apply a strategy stemming from Proposition 1 in our joint work [19]. Namely, we split $G(x)$ into two pieces,

$$G(x) = G_1(x) + G_2(x) \quad (59)$$

where

$$G_1(x) = \int_0^{x/2}(x-g)^{\gamma_2}g^{\gamma_3+\frac{1}{k_1}}\frac{1}{1+g^2}e^{\nu g}dg$$

and

$$G_2(x) = \int_{x/2}^x (x-g)^{\gamma_2}g^{\gamma_3+\frac{1}{k_1}}\frac{1}{1+g^2}e^{\nu g}dg$$

We first focus on bounds for $G_1(x)$. Two cases arise.

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- Assume that $-1 < \gamma_2 \leq 0$. In that situation, we observe that $(x-g)^{\gamma_2} \leq (x/2)^{\gamma_2}$ provided that $0 \leq g \leq x/2$, for $x \geq 0$. Therefore, bearing in mind the constraints (53),

$$G_1(x) \leq (\frac{x}{2})^{\gamma_2} e^{\nu x/2} \int_0^{x/2} g^{\gamma_3 + \frac{1}{k_1}} dg = \frac{1}{\gamma_3 + \frac{1}{k_1} + 1} (x/2)^{\gamma_2 + \gamma_3 + \frac{1}{k_1} + 1} e^{\nu x/2} \quad (60)$$

for all $x \geq 0$.
- Suppose that $\gamma_2 > 0$. We check that $(x-g)^{\gamma_2} \leq x^{\gamma_2}$ for any $0 \leq g \leq x/2$. Hence, paying regard to (53),

$$G_1(x) \leq x^{\gamma_2} e^{\nu x/2} \int_0^{x/2} g^{\gamma_3 + \frac{1}{k_1}} dg = (1/2)^{\gamma_3 + \frac{1}{k_1} + 1} \frac{1}{\gamma_3 + \frac{1}{k_1} + 1} x^{\gamma_2 + \gamma_3 + \frac{1}{k_1} + 1} e^{\nu x/2} \quad (61)$$

whenever $x \geq 0$.

In a second step, we provide upper estimates for $G_2(x)$. We notice that $1 + g^2 \geq 1 + (x/2)^2$, for $x/2 \leq g \leq x$. Hence,

$$G_2(x) \leq \frac{1}{1 + (x/2)^2} \int_{x/2}^{x} (x-g)^{\gamma_2} g^{\gamma_3 + \frac{1}{k_1}} e^{\nu g} dg \leq \frac{\tilde{G}_2(x)}{1 + (x/2)^2} \quad (62)$$

where

$$\tilde{G}_2(x) = \int_0^x (x-g)^{\gamma_2} g^{\gamma_3 + \frac{1}{k_1}} e^{\nu g} dg$$

for all $x \geq 0$. From the sharp bounds established in Proposition 1 of [16], we can pinpoint a constant $K_1 > 0$ (depending on $\gamma_2, \gamma_3, k_1, \nu$) with

$$\check{G}_2(x) \leq K_1 x^{\gamma_3 + \frac{1}{k_1}} e^{\nu x}$$

for all $x \geq 1$, under the conditions (53). As a result, we get that

$$G_2(x) \leq K_1 \frac{x^{\gamma_3 + \frac{1}{k_1}}}{1 + (x/2)^2} e^{\nu x} \quad (63)$$

provided that $x \geq 1$.

At last, gathering the bounds (60), (61) and (63), we deduce the awaited bounds (58) from the splitting (59). □

We turn to the bounds for the map $\mathcal{B}(\tau, m)$. We identify two alternatives.

- Assume that $\tau \in S_{d_1} \cup D_\rho$ is chosen such that

$$\frac{|\tau|^{k_1}}{|\epsilon|^{k_1}} > 1. \quad (64)$$

Owing to Lemma 4 and the first constraint of (53), we get from the upper bounds (57) some constant $C_{1.1} > 0$ with

$$\mathcal{B}(\tau, m) \leq \frac{M_{\gamma_1} ||f||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}}{(1+|\tau|)^{\gamma_1}} |\tau|^{k_1} |\epsilon|^{k_1(\gamma_2+\gamma_3+1)} K_G \frac{(|\tau/\epsilon|^{k_1})^{\gamma_3 + \frac{1}{k_1}}}{1 + |\tau/\epsilon|^{2k_1}} \exp\left(\nu |\frac{\tau}{\epsilon}|^{k_1}\right)$$
$$\times (1+|m|)^{-\mu} e^{-\beta|m|} \leq C_{1.1} M_{\gamma_1} |\epsilon|^{k_1(\gamma_2+1)} ||f||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \frac{|\tau/\epsilon|}{1 + |\tau/\epsilon|^{2k_1}} \exp\left(\nu |\frac{\tau}{\epsilon}|^{k_1}\right)$$
$$\times (1+|m|)^{-\mu} e^{-\beta|m|} \quad (65)$$

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for all $\tau \in S_{d_1} \cup D_\rho$ chosen under (64).

- Suppose that $\tau \in S_{d_1} \cup D_\rho$ fufills

$$0 \leq \frac{|\tau|^{k_1}}{|\epsilon|^{k_1}} \leq 1. \tag{66}$$

Based on (57), we arrive at some constant $C_{1.2} > 0$ such that

$$\begin{aligned}
\mathcal{B}(\tau,m) \leq & \frac{M_{\gamma_1}||f||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}}{(1+|\tau|)^{\gamma_1}}|\tau|^{k_1}\int_0^{\frac{|\tau|^{k_1}}{|\epsilon|^{k_1}}}(\frac{|\tau|^{k_1}}{|\epsilon|^{k_1}}-g)^{\gamma_2}g^{\gamma_3+\frac{1}{k_1}}\frac{1}{1+g^2}dg \\
& \times \exp\left(\nu|\frac{\tau}{\epsilon}|^{k_1}\right)|\epsilon|^{k_1(\gamma_2+\gamma_3+1)}(1+|m|)^{-\mu}e^{-\beta|m|} \\
& \leq ||f||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}\frac{|\tau/\epsilon|}{1+|\tau/\epsilon|^{2k_1}}\exp\left(\nu|\frac{\tau}{\epsilon}|^{k_1}\right)(1+|m|)^{-\mu}e^{-\beta|m|} \\
& \times [C_{1.2}|\tau|^{k_1-1}|\epsilon|^{k_1(\gamma_2+1)}|\epsilon|^{1+k_1\gamma_3}M_{\gamma_1}(1+|\tau/\epsilon|^{2k_1})] \\
\leq & [C_{1.2}M_{\gamma_1}\epsilon_0^{k_1(\gamma_3+1)}2]|\epsilon|^{k_1(\gamma_2+1)}||f||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}\frac{|\tau/\epsilon|}{1+|\tau/\epsilon|^{2k_1}}\exp\left(\nu|\frac{\tau}{\epsilon}|^{k_1}\right)(1+|m|)^{-\mu}e^{-\beta|m|}
\end{aligned} \tag{67}$$

whenever $\tau \in S_{d_1} \cup D_\rho$ is restricted to (66).

Eventually, the combination of the above bounds (65) and (67) yields the expected result (54). □

Proposition 2. *Let $Q(X), R(X) \in \mathbb{C}[X]$ be polynomials and $\mu > 0$ be a real number subjected to the constraints*

$$\deg(R) \geq \deg(Q) \ , \ R(\sqrt{-1}m) \neq 0 \ , \ \mu > \deg(Q) + 1 \tag{68}$$

for all $m \in \mathbb{R}$. Then, a constant $C_2 > 0$ (depending on Q,R and μ) can be selected such that

$$\begin{aligned}
&||\frac{1}{R(\sqrt{-1}m)}\int_{-\infty}^{+\infty}f(m-m_1)Q(\sqrt{-1}m_1)g(\tau,m_1)dm_1||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \\
&\qquad \leq C_2||f(m)||_{(\beta,\mu)}||g(\tau,m)||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}
\end{aligned} \tag{69}$$

holds provided that $f \in E_{(\beta,\mu)}$ and $g \in F^{d_1}_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$.

Proof. The proof mirrors the one of Proposition 2 in our recent work [20]. Indeed, let us choose g in $F^{d_1}_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$. The very definition of the norms displayed in Definitions 2 and 6 allows the bounds

$$|g(\tau,m_1)| \leq ||g||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}|\frac{\tau}{\epsilon}|\frac{1}{1+|\tau/\epsilon|^{2k_1}}\exp\left(\nu|\frac{\tau}{\epsilon}|^{k_1}\right)(1+|m_1|)^{-\mu}e^{-\beta|m_1|} \tag{70}$$

provided that $\tau_1 \in S_{d_1} \cup D_\rho$ and $m_1 \in \mathbb{R}$ together with

$$|f(m)| \leq ||f(m)||_{(\beta,\mu)}(1+|m|)^{-\mu}e^{-\beta|m|} \tag{71}$$

for all $m \in \mathbb{R}$. These two bounds (70) and (71) yield the next estimates

$$\begin{aligned}
|\mathcal{C}(\tau,m)| &:= |\frac{1}{R(\sqrt{-1}m)}\int_{-\infty}^{+\infty}f(m-m_1)Q(\sqrt{-1}m_1)g(\tau,m_1)dm_1| \\
&\leq ||f(m)||_{(\beta,\mu)}||g(\tau,m)||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}|\frac{\tau}{\epsilon}|\frac{1}{1+|\tau/\epsilon|^{2k_1}}\exp\left(\nu|\frac{\tau}{\epsilon}|^{k_1}\right)(1+|m|)^{-\mu}e^{-\beta|m|}C_{2.1}
\end{aligned} \tag{72}$$

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where

$$C_{2.1} = (1+|m|)^{\mu}e^{\beta|m|}\frac{1}{|R(\sqrt{-1}m)|}\int_{-\infty}^{+\infty}\frac{e^{-\beta|m-m_1|}}{(1+|m-m_1|)^{\mu}}\frac{|Q(\sqrt{-1}m_1)|}{(1+|m_1|)^{\mu}}e^{-\beta|m_1|}dm_1.$$

According to the triangular inequality, we observe that

$$|m| \leq |m-m_1| + |m_1| \tag{73}$$

for all real numbers $m, m_1 \in \mathbb{R}$ and by construction of the polynomials R, Q asked to fulfill (68), two constants $\mathfrak{Q}, \mathfrak{R} > 0$ can be pinpointed such that

$$|Q(\sqrt{-1}m_1)| \leq \mathfrak{Q}(1+|m_1|)^{\deg(Q)} \ , \ |R(\sqrt{-1}m)| \geq \mathfrak{R}(1+|m|)^{\deg(R)}$$

whenever $m, m_1 \in \mathbb{R}$. Thereby, the next upper bounds

$$C_{2.1} \leq \frac{\mathfrak{Q}}{\mathfrak{R}}\sup_{m\in\mathbb{R}}(1+|m|)^{\mu-\deg(R)}\int_{-\infty}^{+\infty}\frac{1}{(1+|m-m_1|)^{\mu}(1+|m_1|)^{\mu-\deg(Q)}}dm_1 \tag{74}$$

are reached whose right handside is a finite quantity under the restrictions (68), owing to Lemma 2.2 from [15] or Lemma 4 of [21].

Eventually, gathering (72) and (74) yields the foretold bounds (69). □

Proposition 3. *Let $k_1 \geq 1$ be an integer. Let $Q_1(X), Q_2(X)$ and $R(X)$ be polynomials with complex coefficients such that*

$$\deg(R) \geq \deg(Q_1) \ , \ \deg(R) \geq \deg(Q_2) \ , \ R(\sqrt{-1}m) \neq 0 \tag{75}$$

for all $m \in \mathbb{R}$. We require the positive real number $\mu > 0$ to satisfy

$$\mu > \max(\deg(Q_1)+1, \deg(Q_2)+1). \tag{76}$$

Let $m \mapsto b(m)$ be a continuous function on $\mathbb{R}$ such that

$$|b(m)| \leq \frac{1}{|R(\sqrt{-1}m)|} \tag{77}$$

for all $m \in \mathbb{R}$. Then, one can find a constant $C_3 > 0$ (relying on Q_1,Q_2,R,μ,k_1 and ν) such that

$$\begin{aligned}||b(m)\tau^{k_1}\int_0^{\tau^{k_1}}\int_{-\infty}^{+\infty} &Q_1(\sqrt{-1}(m-m_1))f((\tau^{k_1}-s)^{1/k_1}, m-m_1)\\ &\times Q_2(\sqrt{-1}m_1)g(s^{1/k_1}, m_1)\frac{1}{(\tau^{k_1}-s)s}dsdm_1||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}\\ &\leq C_3||f(\tau,m)||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}||g(\tau,m)||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}\end{aligned} \tag{78}$$

for all $f, g \in F^{d_1}_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$.

Proof. Take f, g in the space $F^{d_1}_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$. According to the definition of the norm, the next two bounds

$$|f(\tau,m)| \leq ||f||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}|\frac{\tau}{\epsilon}|\frac{1}{1+|\tau/\epsilon|^{2k_1}}\exp(\nu|\frac{\tau}{\epsilon}|^{k_1})(1+|m|)^{-\mu}e^{-\beta|m|} \tag{79}$$

and

$$|g(\tau,m)| \leq ||g||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}|\frac{\tau}{\epsilon}|\frac{1}{1+|\tau/\epsilon|^{2k_1}}\exp(\nu|\frac{\tau}{\epsilon}|^{k_1})(1+|m|)^{-\mu}e^{-\beta|m|} \tag{80}$$

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hold provided that $\tau \in S_{d_1} \cup D_\rho$ and $m \in \mathbb{R}$. These bounds together with the assumption (77) prompt

$$\mathcal{D}(\tau,m) := \left| b(m)\tau^{k_1} \int_0^{\tau^{k_1}} \int_{-\infty}^{+\infty} Q_1(\sqrt{-1}(m-m_1)) f((\tau^{k_1}-s)^{1/k_1}, m-m_1) \times Q_2(\sqrt{-1}m_1) g(s^{1/k_1}, m_1) \frac{1}{(\tau^{k_1}-s)s} ds dm_1 \right|$$
$$\leq \frac{1}{|R(\sqrt{-1}m)|} \int_{-\infty}^{+\infty} |Q_1(\sqrt{-1}(m-m_1))||Q_2(\sqrt{-1}m_1)|(1+|m-m_1|)^{-\mu} e^{-\beta|m-m_1|} \times (1+|m_1|)^{-\mu} e^{-\beta|m_1|} dm_1 ||f||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} ||g||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$$
$$\times |\tau|^{k_1} \int_0^{|\tau|^{k_1}} \frac{(|\tau|^{k_1}-h)^{1/k_1}}{|\epsilon|} \frac{1}{1+\frac{(|\tau|^{k_1}-h)^2}{|\epsilon|^{2k_1}}} \frac{h^{1/k_1}}{|\epsilon|} \frac{1}{1+\frac{h^2}{|\epsilon|^{2k_1}}} \frac{1}{(|\tau|^{k_1}-h)h} dh \times \exp\left(\nu\left|\frac{\tau}{\epsilon}\right|^{k_1}\right) \quad (81)$$

for all $\tau \in S_{d_1} \cup D_\rho$ and $m \in \mathbb{R}$.

By construction, we check that some positive constants $\mathfrak{Q}_1$,$\mathfrak{Q}_2$ and $\mathfrak{R}$ can be picked out in a way that

$$|Q_1(\sqrt{-1}(m-m_1))| \leq \mathfrak{Q}_1(1+|m-m_1|)^{\deg(Q_1)} \ , \ |Q_2(\sqrt{-1}m_1)| \leq \mathfrak{Q}_2(1+|m_1|)^{\deg(Q_2)}, \quad |R(\sqrt{-1}m)| \geq \mathfrak{R}(1+|m|)^{\deg(R)} \quad (82)$$

for all $m, m_1 \in \mathbb{R}$. As a result and keeping in mind the inequality (73), we deduce the next bounds for the first piece of the right handside of (81), namely

$$\frac{1}{|R(\sqrt{-1}m)|} \int_{-\infty}^{+\infty} |Q_1(\sqrt{-1}(m-m_1))||Q_2(\sqrt{-1}m_1)|(1+|m-m_1|)^{-\mu} e^{-\beta|m-m_1|} \times (1+|m_1|)^{-\mu} e^{-\beta|m_1|} dm_1 \leq \frac{\mathfrak{Q}_1\mathfrak{Q}_2}{\mathfrak{R}} \mathcal{D}(1+|m|)^{-\mu} e^{-\beta|m|} \quad (83)$$

where

$$\mathcal{D} := \sup_{m \in \mathbb{R}} (1+|m|)^{\mu-\deg(R)} \int_{-\infty}^{+\infty} \frac{1}{(1+|m-m_1|)^{\mu-\deg(Q_1)}(1+|m_1|)^{\mu-\deg(Q_2)}} dm_1$$

is a finite quantity under the conditions (75), (76), as explained in Lemma 2.2 from [15] or Lemma 4 of [21]. Besides, according to Lemma 3 of our recent work [22], there exists a constant K_{k_1} (relying on k_1) such that

$$|\tau|^{k_1} \int_0^{|\tau|^{k_1}} \frac{(|\tau|^{k_1}-h)^{1/k_1}/|\epsilon|}{1+\frac{(|\tau|^{k_1}-h)^2}{|\epsilon|^{2k_1}}} \frac{h^{1/k_1}/|\epsilon|}{1+\frac{h^2}{|\epsilon|^{2k_1}}} \frac{1}{(|\tau|^{k_1}-h)h} dh \leq K_{k_1} \frac{|\tau/\epsilon|}{1+|\tau/\epsilon|^{2k_1}} \quad (84)$$

for all $\tau \in S_{d_1} \cup D_\rho$, all $\epsilon \in D_{\epsilon_0} \setminus \{0\}$.

Counting up the above two bounds (83), (84), it results from (81) that

$$\mathcal{D}(\tau,m) \leq \frac{\mathfrak{Q}_1\mathfrak{Q}_2}{\mathfrak{R}} \mathcal{D} K_{k_1} ||f||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} ||g||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \frac{|\tau/\epsilon|}{1+|\tau/\epsilon|^{2k_1}} (1+|m|)^{-\mu} e^{-\beta|m|} \times \exp\left(\nu\left|\frac{\tau}{\epsilon}\right|^{k_1}\right) \quad (85)$$

whenever $\tau \in S_{d_1} \cup D_\rho$ and $m \in \mathbb{R}$. The estimates (78) follow. □

5. Solving the first convolution equation (49)

In this section we uniquely solve the auxiliary convolution equation (49) stated in Subsection 3.2 within the Banach spaces displayed in Definition 6. Our approach consists in rearranging (49) into a fixed point equation (disclosed later on in (129)). In a first stage, we ask to perform a division by the next parameter depending polynomial

$$P_m(\tau) = Q(\sqrt{-1}m) - R_D(\sqrt{-1}m)k_1^{\delta_D}\tau^{k_1\delta_D} \tag{86}$$

provided that $\tau \in S_{d_1} \cup D_\rho$. Decisive lower bounds concerning P_m are displayed in the next lemma.

Lemma 5. *For a convenient choice of the inner radius $r_{Q,R_D} > 0$ and aperture $\eta_{Q,R_D} > 0$ of the sector S_{Q,R_D} (introduced in (21)), unbounded sectors S_{d_1} centered at 0 with bisecting direction $d_1 \in \mathbb{R}$ and a small radius $\rho > 0$ can be distinguished in a way that the next lower estimates*

$$|P_m(\tau)| \geq C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}|R_D(\sqrt{-1}m)|(1+|\tau|)^{k_1\delta_D-1} \tag{87}$$

hold for some well chosen constant $C_P > 0$, provided that $\tau \in S_{d_1} \cup D_\rho$, for all $m \in \mathbb{R}$.

Proof. Owing to the fact that the complex roots $q_l(m)$, $0 \leq l \leq k_1\delta_D - 1$ of $\tau \mapsto P_m(\tau)$ can be explicitely computed, we factorize the polynomial as follows

$$P_m(\tau) = -R_D(\sqrt{-1}m)k_1^{\delta_D}\Pi_{l=0}^{k_1\delta_D-1}(\tau - q_l(m)) \tag{88}$$

with

$$q_l(m) = (\frac{|Q(\sqrt{-1}m)|}{|R_D(\sqrt{-1}m)|k_1^{\delta_D}})^{\frac{1}{k_1\delta_D}} \exp\left(\sqrt{-1}(\arg(\frac{Q(\sqrt{-1}m)}{R_D(\sqrt{-1}m)k_1^{\delta_D}})\frac{1}{k_1\delta_D} + \frac{2\pi l}{k_1\delta_D})\right)$$

for all $0 \leq l \leq k_1\delta_D - 1$, for any $\tau \in \mathbb{C}$ and $m \in \mathbb{R}$.

We pinpoint an unbounded sector S_{d_1} centered at 0, a small disc D_ρ and we position the sector S_{Q,R_D} given in (21) in a way that the next two properties hold:
1) A constant $M_1 > 0$ can be found such that

$$|\tau - q_l(m)| \geq M_1(1+|\tau|) \tag{89}$$

for all $0 \leq l \leq k_1\delta_D - 1$, all $m \in \mathbb{R}$, whenever $\tau \in S_{d_1} \cup D_\rho$.
2) There exists a constant $M_2 > 0$ with

$$|\tau - q_{l_0}(m)| \geq M_2|q_{l_0}(m)| \tag{90}$$

for some $0 \leq l_0 \leq \delta_D k_1 - 1$, all $m \in \mathbb{R}$, all $\tau \in S_{d_1} \cup D_\rho$.

We now explain how the above two bounds can be established.

- We deem the first inequality (89) in observing that under the hypothesis (22), the roots $q_l(m)$ are bounded from below and obey $|q_l(m)| \geq 2\rho$ for all $m \in \mathbb{R}$, all $0 \leq l \leq \delta_D k_1 - 1$ for a suitable choice of the radii $r_{Q,R_D}, \rho > 0$. Furthermore, for all $m \in \mathbb{R}$, all $0 \leq l \leq \delta_D k_1 - 1$, these roots are penned inside an union $\mathcal{Q}$ of unbounded sectors centered at 0 that do not cover a full neighborhood of 0 in $\mathbb{C}^*$ whenever the aperture $\eta_{Q,R_D} > 0$ of S_{Q,R_D} is taken small enough. Hence, a sector S_{d_1} may be chosen such that
$$S_{d_1} \cap \mathcal{Q} = \emptyset.$$
Such a sector satisfies in particular that for all $0 \leq l \leq \delta_D k_1 - 1$, the quotients $q_l(m)/\tau$ lay outside some small disc centered at 1 in $\mathbb{C}$ for all $\tau \in S_{d_1}$, all $m \in \mathbb{R}$. Eventually, (89) follows.
- The sector S_{d_1} and disc D_ρ are selected as above. The second lower bound (90) ensues from the fact that for any fixed $0 \leq l_0 \leq \delta_D k_1 - 1$, the quotient $\tau/q_{l_0}(m)$ stays apart a small disc centered at 1 in $\mathbb{C}$ for all $\tau \in S_{d_1} \cup D_\rho$, all $m \in \mathbb{R}$.

Departing from the factorization (88) and paying regard to the two lower bounds (89), (90) reached overhead, we arrive at

$$|P_m(\tau)| \geq M_1^{k_1\delta_D-1}M_2|R_D(\sqrt{-1}m)|k_1^{\delta_D}(\frac{|Q(\sqrt{-1}m)|}{|R_D(\sqrt{-1}m)|k_1^{\delta_D}})^{\frac{1}{k_1\delta_D}}(1+|\tau|)^{k_1\delta_D-1}$$
$$\geq C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}|R_D(\sqrt{-1}m)|(1+|\tau|)^{k_1\delta_D-1} \quad (91)$$

as long as $\tau \in S_{d_1} \cup D_\rho$, for all $m \in \mathbb{R}$. □

We introduce the next nonlinear map

$$\mathcal{H}_\epsilon(\omega(\tau,m)) :=$$
$$\Big(\sum_{q=1}^{\delta_D-1} a_{q,\delta_D}\Big[\frac{\tau^{k_1}}{P_m(\tau)\Gamma(\frac{d_{D,q}}{k_1})}\int_0^{\tau^{k_1}}(\tau^{k_1}-s)^{\frac{d_{D,q}}{k_1}-1}k_1^q s^q \omega(s^{1/k_1},m)\frac{ds}{s}$$
$$+\sum_{1\leq p\leq q-1} A_{q,p}\frac{\tau^{k_1}}{P_m(\tau)\Gamma(\frac{d_{D,q}+k_1(q-p)}{k_1})}\int_0^{\tau^{k_1}}(\tau^{k_1}-s)^{\frac{d_{D,q}+k_1(q-p)}{k_1}-1}k_1^p s^p\omega(s^{1/k_1},m)\frac{ds}{s}\Big]$$
$$\times R_D(\sqrt{-1}m)\Big)$$
$$+\Big[\sum_{1\leq p\leq \delta_D-1} A_{\delta_D,p}\frac{\tau^{k_1}}{P_m(\tau)\Gamma(\delta_D-p)}\int_0^{\tau^{k_1}}(\tau^{k_1}-s)^{\delta_D-p-1}k_1^p s^p\omega(s^{1/k_1},m)\frac{ds}{s}\Big]$$
$$\times R_D(\sqrt{-1}m)$$
$$+\sum_{l=1}^{D-1}\epsilon^{\Delta_l-d_l}\Big[\sum_{q=1}^{\delta_l} a_{q,\delta_l}\Big[\frac{\tau^{k_1}}{P_m(\tau)\Gamma(\frac{d_{l,q}}{k_1})}\int_0^{\tau^{k_1}}(\tau^{k_1}-s)^{\frac{d_{l,q}}{k_1}-1}\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty}A_l(m-m_1,\epsilon)k_1^q s^q$$
$$\times R_l(\sqrt{-1}m_1)\omega(s^{1/k_1},m_1)\frac{ds}{s}dm_1$$
$$+\sum_{1\leq p\leq q-1} A_{q,p}\frac{\tau^{k_1}}{P_m(\tau)\Gamma(\frac{d_{l,q}+k_1(q-p)}{k_1})}\int_0^{\tau^{k_1}}(\tau^{k_1}-s)^{\frac{d_{l,q}+k_1(q-p)}{k_1}-1}$$
$$\times\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty}A_l(m-m_1,\epsilon)k_1^p s^p R_l(\sqrt{-1}m_1)\omega(s^{1/k_1},m_1)\frac{ds}{s}dm_1\Big]\Big]$$
$$+\frac{\mathcal{F}_2(\tau,m,\epsilon)}{P_m(\tau)}+\frac{1}{(2\pi)^{1/2}P_m(\tau)}\int_{-\infty}^{+\infty}C_1(m-m_1,\epsilon)\omega(\tau,m_1)dm_1$$
$$+c_{Q_1Q_2}\frac{1}{(2\pi)^{1/2}P_m(\tau)}\int_{-\infty}^{+\infty}\tau^{k_1}\int_0^{\tau^{k_1}}Q_1(\sqrt{-1}(m-m_1))\omega((\tau^{k_1}-s)^{1/k_1},m-m_1)$$
$$\times Q_2(\sqrt{-1}m_1)\omega(s^{1/k_1},m_1)\frac{1}{(\tau^{k_1}-s)s}dsdm_1 \quad (92)$$

In the next proposition, we establish that $\mathcal{H}_\epsilon$ represents a shrinking map on some suitable ball of the Banach space mentioned in Definition 6.

Proposition 4. *Let us select a well chosen inner radius $r_{Q,R_D} > 0$ and aperture $\eta_{Q,R_D} > 0$ of the sector S_{Q,R_D} jointly with an unbounded sector S_{d_1} and radius $\rho > 0$ that heed the requirements of Lemma 5 and obey the additional condition*

$$-1 \notin S_{d_1} \cup D_\rho. \quad (93)$$

Then, one can single out a radius $\epsilon_0 > 0$ small enough, constants $\mathbf{C}_{1,\epsilon_0} > 0$ and $c_{Q_1,Q_2} \in \mathbb{C}^*$ close enough to 0 and a fitting radius $\varpi_2 > 0$ in a way that for all $\epsilon \in D_{\epsilon_0} \setminus \{0\}$, the map $\mathcal{H}_\epsilon$ enjoys the next two features

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- *The inclusion*
$$\mathcal{H}_{\epsilon}(\bar{B}_{\varpi_2}) \subset \bar{B}_{\varpi_2} \tag{94}$$
holds, where we denote $\bar{B}_{\varpi_2}$ *the closed ball of radius* $\varpi_2 > 0$ *centered at 0 in the space* $F^{d_1}_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$.
- *The* $1/2-$*Lipschitz condition*
$$||\mathcal{H}_{\epsilon}(\omega_1) - \mathcal{H}_{\epsilon}(\omega_2)||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \leq \frac{1}{2}||\omega_1 - \omega_2||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \tag{95}$$
occurs for all $\omega_1, \omega_2 \in F^{d_1}_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$.

Proof. We take aim at the first item stating the inclusion (94). We prescribe some real number $\varpi_2 > 0$ and take $\omega(\tau, m)$ in $F^{d_1}_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$, for given $\epsilon \in D_{\epsilon_0} \setminus \{0\}$, such that
$$||\omega||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \leq \varpi_2.$$
We provide explicit bounds for each term of the map $\mathcal{H}_{\epsilon}$ applied to ω.

According to Proposition 1 and Lemma 5, we observe that
$$||\frac{\tau^{k_1}}{P_m(\tau)} \int_0^{\tau^{k_1}} (\tau^{k_1} - s)^{\frac{d_{D,q}}{k_1}-1} s^q R_D(\sqrt{-1}m)\omega(s^{1/k_1}, m)\frac{ds}{s}||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$$
$$\leq \frac{C_1}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}}|\epsilon|^{(\delta_D - q)k_1}||\omega(\tau,m)||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \tag{96}$$
for $1 \leq q \leq \delta_D - 1$ along with
$$||\frac{\tau^{k_1}}{P_m(\tau)} \int_0^{\tau^{k_1}} (\tau^{k_1} - s)^{\frac{d_{D,q}+k_1(q-p)}{k_1}-1} s^p R_D(\sqrt{-1}m)\omega(s^{1/k_1}, m)\frac{ds}{s}||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$$
$$\leq \frac{C_1}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}}|\epsilon|^{(\delta_D - p)k_1}||\omega(\tau,m)||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \tag{97}$$
for $1 \leq p \leq q-1$ with $1 \leq q \leq \delta_D - 1$ and
$$||\frac{\tau^{k_1}}{P_m(\tau)} \int_0^{\tau^{k_1}} (\tau^{k_1} - s)^{\delta_D - p - 1} s^p R_D(\sqrt{-1}m)\omega(s^{1/k_1}, m)\frac{ds}{s}||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$$
$$\leq \frac{C_1}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}}|\epsilon|^{(\delta_D - p)k_1}||\omega(\tau,m)||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \tag{98}$$
as long as $1 \leq p \leq \delta_D - 1$. In order to handle the next piece, under the constraint (93), we can recast
$$\mathcal{E}_1(\tau,m,\epsilon) := \frac{\tau^{k_1}}{P_m(\tau)} \int_0^{\tau^{k_1}} (\tau^{k_1} - s)^{\frac{d_{l,q}}{k_1}-1} \int_{-\infty}^{+\infty} A_l(m - m_1,\epsilon) s^q$$
$$\times R_l(\sqrt{-1}m_1)\omega(s^{1/k_1}, m_1)\frac{ds}{s}dm_1 = \frac{R_D(\sqrt{-1}m)(1+\tau)^{k_1\delta_D - 1}}{P_m(\tau)}$$
$$\times \frac{1}{R_D(\sqrt{-1}m)} \int_{-\infty}^{+\infty} A_l(m - m_1,\epsilon) R_l(\sqrt{-1}m_1)$$
$$\times \Big[\frac{\tau^{k_1}}{(1+\tau)^{k_1\delta_D - 1}} \int_0^{\tau^{k_1}} (\tau^{k_1} - s)^{\frac{d_{l,q}}{k_1}-1} s^q \omega(s^{1/k_1}, m_1)\frac{ds}{s}\Big] dm_1. \tag{99}$$
for all $\tau \in S_{d_1} \cup D_\rho$, $m \in \mathbb{R}$ with $1 \leq l \leq D-1$ and $1 \leq q \leq \delta_l$. Based on Lemma 5, we check that

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$$\left|\frac{R_D(\sqrt{-1}m)(1+\tau)^{k_1\delta_D-1}}{P_m(\tau)}\right| \leq \frac{1}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}} \tag{100}$$

provided that $\tau \in S_{d_1} \cup D_\rho$, $m \in \mathbb{R}$. Owing to the assumptions (19) and (**??**), the proposition 2 together with (100) yield

$$||\mathcal{E}_1(\tau,m,\epsilon)||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \leq \frac{1}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}} C_2||A_l(m,\epsilon)||_{(\beta,\mu)} \\ \times ||\frac{\tau^{k_1}}{(1+\tau)^{k_1\delta_D-1}}\int_0^{\tau^{k_1}}(\tau^{k_1}-s)^{\frac{d_{l,q}}{k_1}-1}s^q\omega(s^{1/k_1},m)\frac{ds}{s}||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}. \tag{101}$$

Besides, a constant $M_{k_1,\delta_D} > 0$ can be pick up such that

$$\left|\frac{1}{(1+\tau)^{k_1\delta_D-1}}\right| \leq \frac{M_{k_1,\delta_D}}{(1+|\tau|)^{k_1\delta_D-1}} \tag{102}$$

for all $\tau \in S_{d_1} \cup D_\rho$, assuming the condition (93). The condition (17) together with (102) enable us to apply Proposition 1 and prompt

$$||\frac{\tau^{k_1}}{(1+\tau)^{k_1\delta_D-1}}\int_0^{\tau^{k_1}}(\tau^{k_1}-s)^{\frac{d_{l,q}}{k_1}-1}s^q\omega(s^{1/k_1},m_1)\frac{ds}{s}||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \\ \leq C_1 M_{k_1,\delta_D}|\epsilon|^{d_{l,q}}||\omega(\tau,m)||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}. \tag{103}$$

Eventually, bearing in mind (27), we deduce from (101) complemented by (103) that

$$||\mathcal{E}_1(\tau,m,\epsilon)||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \leq \frac{1}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}} C_2 \mathbf{A}_{l,\epsilon_0} C_1 M_{k_1,\delta_D}|\epsilon|^{d_{l,q}}||\omega(\tau,m)||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}. \tag{104}$$

The ensuing block is remodeled as

$$\mathcal{E}_2(\tau,m,\epsilon) := \frac{\tau^{k_1}}{P_m(\tau)}\int_0^{\tau^{k_1}}(\tau^{k_1}-s)^{\frac{d_{l,q}+k_1(q-p)}{k_1}-1} \\ \times \int_{-\infty}^{+\infty} A_l(m-m_1,\epsilon)s^p R_l(\sqrt{-1}m_1)\omega(s^{1/k_1},m_1)\frac{ds}{s}dm_1 = \frac{R_D(\sqrt{-1}m)(1+\tau)^{k_1\delta_D-1}}{P_m(\tau)} \\ \times \frac{1}{R_D(\sqrt{-1}m)}\int_{-\infty}^{+\infty} A_l(m-m_1,\epsilon)R_l(\sqrt{-1}m_1) \\ \times \left[\frac{\tau^{k_1}}{(1+\tau)^{k_1\delta_D-1}}\int_0^{\tau^{k_1}}(\tau^{k_1}-s)^{\frac{d_{l,q}+k_1(q-p)}{k_1}-1}s^p\omega(s^{1/k_1},m_1)\frac{ds}{s}\right]dm_1. \tag{105}$$

for all $\tau \in S_{d_1} \cup D_\rho$, $m \in \mathbb{R}$ with $1 \leq l \leq D-1$, $1 \leq q \leq \delta_l$ and $1 \leq p \leq q-1$, under (93).

The assumptions (19), (**??**) and the upper bounds (100) warrant the application of Proposition 2 which triggers

$$||\mathcal{E}_2(\tau,m,\epsilon)||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \leq \frac{1}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}} C_2||A_l(m,\epsilon)||_{(\beta,\mu)} \\ \times ||\frac{\tau^{k_1}}{(1+\tau)^{k_1\delta_D-1}}\int_0^{\tau^{k_1}}(\tau^{k_1}-s)^{\frac{d_{l,q}+k_1(q-p)}{k_1}-1}s^p\omega(s^{1/k_1},m)\frac{ds}{s}||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}. \tag{106}$$

The condition (17) coupled with (102) grant the use of Proposition 1 and beget

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$$||\frac{\tau^{k_1}}{(1+\tau)^{k_1\delta_D-1}}\int_0^{\tau^{k_1}}(\tau^{k_1}-s)^{\frac{d_{l,q}+k_1(q-p)}{k_1}-1}s^p\omega(s^{1/k_1},m_1)\frac{ds}{s}||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$$
$$\leq C_1 M_{k_1,\delta_D}|\epsilon|^{d_{l,q}+k_1(q-p)}||\omega(\tau,m)||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}. \quad (107)$$

At last, not forgetting (27), we deduce from the joint bounds (106), (107) that

$$||\mathcal{E}_2(\tau,m,\epsilon)||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$$
$$\leq \frac{1}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}}C_2\mathbf{A}_{l,\epsilon_0}C_1M_{k_1,\delta_D}|\epsilon|^{d_{l,q}+k_1(q-p)}||\omega(\tau,m)||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}. \quad (108)$$

We control now the piece $\mathcal{F}_2(\tau,m,\epsilon)/P_m(\tau)$. In accordance with Lemma 5, we notice that

$$\left|\frac{1}{P_m(\tau)}\right| \leq \frac{1}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}}\max_{m\in\mathbb{R}}\left|\frac{1}{R_D(\sqrt{-1}m)}\right| \quad (109)$$

provided that $\tau \in S_{d_1}\cup D_\rho$ and $m\in\mathbb{R}$, whose right handside is a finite quantity since $R_D(\sqrt{-1}m)\neq 0$ holds from (22) for all $m\in\mathbb{R}$. Besides, owing to the definition of $\mathcal{F}_2$ given in Subsection 2.3 and the bounds (25), we deduce

$$|\mathcal{F}_2(\tau,m,\epsilon)| \leq \sum_{j_2\in J_2}\mathbf{F}_{2,j_2,\epsilon_0}(1+|m|)^{-\mu}e^{-\beta|m|}|\tau|^{j_2} \quad (110)$$

for all $\tau\in\mathbb{C}$, $m\in\mathbb{R}$. The combination of the bounds (109) and (110) grants

$$||\mathcal{F}_2(\tau,m,\epsilon)/P_m(\tau)||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \leq \frac{1}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}}\max_{m\in\mathbb{R}}\left|\frac{1}{R_D(\sqrt{-1}m)}\right|$$
$$\times \sup_{\tau\in S_{d_1}\cup D_\rho, m\in\mathbb{R}}(1+|m|)^{\mu}e^{\beta|m|}|\frac{\epsilon}{\tau}|(1+|\frac{\tau}{\epsilon}|^{2k_1})\exp\left(-\nu|\frac{\tau}{\epsilon}|^{k_1}\right)$$
$$\times\left(\sum_{j_2\in J_2}\mathbf{F}_{2,j_2,\epsilon_0}(1+|m|)^{-\mu}e^{-\beta|m|}|\tau|^{j_2}\right)$$
$$\leq \frac{1}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}}\max_{m\in\mathbb{R}}\left|\frac{1}{R_D(\sqrt{-1}m)}\right|\sup_{\tau\in S_{d_1}\cup D_\rho}\exp\left(-\nu|\frac{\tau}{\epsilon}|^{k_1}\right)(1+|\frac{\tau}{\epsilon}|^{2k_1})$$
$$\times\left(\sum_{j_2\in J_2}\mathbf{F}_{2,j_2,\epsilon_0}|\epsilon|^{j_2}|\frac{\tau}{\epsilon}|^{j_2-1}\right)$$
$$\leq \frac{1}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}}\max_{m\in\mathbb{R}}\left|\frac{1}{R_D(\sqrt{-1}m)}\right|\epsilon_0\times\sup_{x\geq 0}e^{-\nu x^{k_1}}(1+x^{2k_1})\sum_{j_2\in J_2}\mathbf{F}_{2,j_2,\epsilon_0}\epsilon_0^{j_2-1}x^{j_2-1} \quad (111)$$

which represents a finite quantity bearing in mind that $J_2\subset\mathbb{N}^*$ contains only positive integers.

We address the ensuing linear part of $\mathcal{H}_\epsilon$. Paying regard to (109) and the bounds (27), Proposition 2 prompts

$$||\frac{1}{P_m(\tau)}\int_{-\infty}^{+\infty}C_1(m-m_1,\epsilon)\omega(\tau,m_1)dm_1||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$$
$$\leq \frac{1}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}}\max_{m\in\mathbb{R}}\left|\frac{1}{R_D(\sqrt{-1}m)}\right|C_2||C_1(m,\epsilon)||_{(\beta,\mu)}||\omega(\tau,m)||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$$
$$\leq \frac{1}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}}\max_{m\in\mathbb{R}}\left|\frac{1}{R_D(\sqrt{-1}m)}\right|C_2\mathbf{C}_{1,\epsilon_0}||\omega(\tau,m)||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}. \quad (112)$$

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At last, we manage the nonlinear tail piece of $\mathcal{H}_\epsilon$. We first factorize

$$\frac{1}{P_m(\tau)} = \frac{1}{R_D(\sqrt{-1}m)}\mathcal{G}(\tau,m) \tag{113}$$

where

$$|\mathcal{G}(\tau,m)| \leq \frac{1}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}} \tag{114}$$

for all $\tau \in S_{d_1} \cup D_\rho$ and $m \in \mathbb{R}$, according to (87). This latter decomposition together with the assumption (20) enable the application of Proposition 3 which yields

$$\begin{aligned} ||\frac{1}{P_m(\tau)} \int_{-\infty}^{+\infty} \tau^{k_1} \int_0^{\tau^{k_1}} Q_1(\sqrt{-1}(m-m_1))\omega((\tau^{k_1}-s)^{1/k_1}, m-m_1) \\ \times Q_2(\sqrt{-1}m_1)\omega(s^{1/k_1},m_1)\frac{1}{(\tau^{k_1}-s)s}dsdm_1||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \\ \leq \frac{C_3}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}}||\omega(\tau,m)||^2_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \end{aligned} \tag{115}$$

We select $\epsilon_0 > 0$, $\boldsymbol{C}_{1,\epsilon_0} > 0$ and $c_{Q_1,Q_2} \in \mathbb{C}^*$ close enough to 0 and take suitably $\varpi_2 > 0$ in a proper way that the next inequality

$$\begin{aligned} \Big(\sum_{q=1}^{\delta_D-1} |a_{q,\delta_D}|\Big[\frac{1}{\Gamma(\frac{d_{D,q}}{k_1})}k_1^q\frac{C_1}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}}\epsilon_0^{(\delta_D-q)k_1}\varpi_2 \\ + \sum_{1\leq p\leq q-1}|A_{q,p}|\frac{1}{\Gamma(\frac{d_{D,q}+k_1(q-p)}{k_1})}k_1^p\frac{C_1}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}}\epsilon_0^{(\delta_D-p)k_1}\varpi_2\Big]\Big) \\ + \Big[\sum_{1\leq p\leq \delta_D-1}|A_{\delta_D,p}|\frac{k_1^p}{\Gamma(\delta_D-p)}\frac{C_1}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}}\epsilon_0^{(\delta_D-p)k_1}\varpi_2\Big] \\ + \sum_{l=1}^{D-1}\epsilon_0^{\Delta_l-d_l}\Big[\sum_{q=1}^{\delta_l}|a_{q,\delta_l}|\Big[\frac{k_1^q}{\Gamma(\frac{d_{l,q}}{k_1})}\frac{1}{(2\pi)^{1/2}}\frac{1}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}}C_2\boldsymbol{A}_{l,\epsilon_0}C_1M_{k_1,\delta_D}\epsilon_0^{d_{l,q}}\varpi_2 \\ + \sum_{1\leq p\leq q-1}|A_{q,p}|\frac{k_1^p}{\Gamma(\frac{d_{l,q}+k_1(q-p)}{k_1})}\frac{1}{(2\pi)^{1/2}}\frac{1}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}}C_2\boldsymbol{A}_{l,\epsilon_0}C_1M_{k_1,\delta_D}\epsilon_0^{d_{l,q}+k_1(q-p)}\varpi_2\Big]\Big] \\ + \frac{1}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}}\max_{m\in\mathbb{R}}\Big|\frac{1}{R_D(\sqrt{-1}m)}\Big|\epsilon_0 \times \sup_{x\geq 0} e^{-\nu x^{k_1}}(1+x^{2k_1})\sum_{j_2\in J_2}\boldsymbol{F}_{2,j_2,\epsilon_0}\epsilon_0^{j_2-1}x^{j_2-1} \\ + \frac{1}{(2\pi)^{1/2}}\frac{1}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}}\max_{m\in\mathbb{R}}\Big|\frac{1}{R_D(\sqrt{-1}m)}\Big|C_2\boldsymbol{C}_{1,\epsilon_0}\varpi_2 \\ + |c_{Q_1,Q_2}|\frac{1}{(2\pi)^{1/2}}\frac{C_3}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}}\varpi_2^2 \leq \varpi_2 \end{aligned} \tag{116}$$

holds. Observe that the first six blocks of the left handside of (116) can be made small since they contain positive powers of ϵ_0, owing in particular to the constraint (18) imposed on (14) and its last two terms can be dwindled provided that the positive constants $\boldsymbol{C}_{1,\epsilon_0}$ and c_{Q_1,Q_2} are chosen nearby the origin.

Eventually, the collection of all the bounds overhead (96), (97), (98), (104), (108), (111), (112), (115) restricted by (116) gives rise to the inclusion (94).

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We mind the second item addressing the $1/2-$Lipschitz feature. Take ω_1, ω_2 inside the ball $\bar{B}_{\varpi_2}$ of the space $F^{d_1}_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$ whose radius ϖ_2 has been prescribed in the first item discussed above. We display norm estimates for each block of the difference $\mathcal{H}_\epsilon(\omega_1) - \mathcal{H}_\epsilon(\omega_2)$. Based on the bounds reached formerly in the proof of the first item, we check the next list of six estimates. Namely,

$$||\frac{\tau^{k_1}}{P_m(\tau)}\int_0^{\tau^{k_1}}(\tau^{k_1}-s)^{\frac{d_{D,q}}{k_1}-1}s^q R_D(\sqrt{-1}m)(\omega_1(s^{1/k_1},m)-\omega_2(s^{1/k_1},m))\frac{ds}{s}||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$$
$$\leq \frac{C_1}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}}|\epsilon|^{(\delta_D-q)k_1}||\omega_1(\tau,m)-\omega_2(\tau,m)||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \quad (117)$$

for $1 \leq q \leq \delta_D - 1$ along with

$$||\frac{\tau^{k_1}}{P_m(\tau)}\int_0^{\tau^{k_1}}(\tau^{k_1}-s)^{\frac{d_{D,q}+k_1(q-p)}{k_1}-1}s^p R_D(\sqrt{-1}m)$$
$$\times(\omega_1(s^{1/k_1},m)-\omega_2(s^{1/k_1},m))\frac{ds}{s}||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$$
$$\leq \frac{C_1}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}}|\epsilon|^{(\delta_D-p)k_1}||\omega_1(\tau,m)-\omega_2(\tau,m)||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \quad (118)$$

for $1 \leq p \leq q-1$ with $1 \leq q \leq \delta_D - 1$ and

$$||\frac{\tau^{k_1}}{P_m(\tau)}\int_0^{\tau^{k_1}}(\tau^{k_1}-s)^{\delta_D-p-1}s^p R_D(\sqrt{-1}m)(\omega_1(s^{1/k_1},m)-\omega_2(s^{1/k_1},m))\frac{ds}{s}||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$$
$$\leq \frac{C_1}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}}|\epsilon|^{(\delta_D-p)k_1}||\omega_1(\tau,m)-\omega_2(\tau,m)||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \quad (119)$$

as long as $1 \leq p \leq \delta_D - 1$. Furthermore,

$$||\frac{\tau^{k_1}}{P_m(\tau)}\int_0^{\tau^{k_1}}(\tau^{k_1}-s)^{\frac{d_{l,q}}{k_1}-1}\int_{-\infty}^{+\infty}A_l(m-m_1,\epsilon)s^q R_l(\sqrt{-1}m_1)$$
$$\times(\omega_1(s^{1/k_1},m_1)-\omega_2(s^{1/k_1},m_1))\frac{ds}{s}dm_1||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$$
$$\leq \frac{1}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}}C_2\mathbf{A}_{l,\epsilon_0}C_1M_{k_1,\delta_D}|\epsilon|^{d_{l,q}}||\omega_1(\tau,m)-\omega_2(\tau,m)||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}. \quad (120)$$

holds for $1 \leq l \leq D-1$ and $1 \leq q \leq \delta_l$ together with

$$||\frac{\tau^{k_1}}{P_m(\tau)}\int_0^{\tau^{k_1}}(\tau^{k_1}-s)^{\frac{d_{l,q}+k_1(q-p)}{k_1}-1}\int_{-\infty}^{+\infty}A_l(m-m_1,\epsilon)s^p R_l(\sqrt{-1}m_1)$$
$$\times(\omega_1(s^{1/k_1},m_1)-\omega_2(s^{1/k_1},m_1))\frac{ds}{s}dm_1||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$$
$$\leq \frac{1}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}}C_2\mathbf{A}_{l,\epsilon_0}C_1M_{k_1,\delta_D}|\epsilon|^{d_{l,q}+k_1(q-p)}||\omega_1(\tau,m)-\omega_2(\tau,m)||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \quad (121)$$

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for $1 \leq l \leq D-1, 1 \leq q \leq \delta_l$ and $1 \leq p \leq q-1$ in a row with

$$||\frac{1}{P_m(\tau)}\int_{-\infty}^{+\infty} C_1(m-m_1,\epsilon)(\omega_1(\tau,m_1)-\omega_2(\tau,m_1))dm_1||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$$
$$\leq \frac{1}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}}\max_{m\in\mathbb{R}}\left|\frac{1}{R_D(\sqrt{-1}m)}\right|C_2\mathbf{C}_{1,\epsilon_0}||\omega_1(\tau,m)-\omega_2(\tau,m)||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}. \quad (122)$$

Upper estimates for the rear part of $\mathcal{H}_\epsilon(\omega_1)-\mathcal{H}_\epsilon(\omega_2)$ ask some groundwork. Indeed, according to the classical identity $ab-cd=(a-c)b+c(b-d)$, we reshape

$$\Delta(\tau,m) := \frac{1}{P_m(\tau)}\int_{-\infty}^{+\infty}\tau^{k_1}\int_0^{\tau^{k_1}} Q_1(\sqrt{-1}(m-m_1))\omega_1((\tau^{k_1}-s)^{1/k_1},m-m_1)$$
$$\times Q_2(\sqrt{-1}m_1)\omega_1(s^{1/k_1},m_1)\frac{1}{(\tau^{k_1}-s)s}dsdm_1$$
$$-\frac{1}{P_m(\tau)}\int_{-\infty}^{+\infty}\tau^{k_1}\int_0^{\tau^{k_1}} Q_1(\sqrt{-1}(m-m_1))\omega_2((\tau^{k_1}-s)^{1/k_1},m-m_1)$$
$$\times Q_2(\sqrt{-1}m_1)\omega_2(s^{1/k_1},m_1)\frac{1}{(\tau^{k_1}-s)s}dsdm_1$$
$$=\frac{1}{P_m(\tau)}\int_{-\infty}^{+\infty}\tau^{k_1}\int_0^{\tau^{k_1}}\Big[Q_1(\sqrt{-1}(m-m_1))[\omega_1((\tau^{k_1}-s)^{1/k_1},m-m_1)-\omega_2((\tau^{k_1}-s)^{1/k_1},m-m_1)]$$
$$\times Q_2(\sqrt{-1}m)\omega_1(s^{1/k_1},m_1)+Q_1(\sqrt{-1}(m-m_1))\omega_2((\tau^{k_1}-s)^{1/k_1},m-m_1)Q_2(\sqrt{-1}m_1)$$
$$\times[\omega_1(s^{1/k_1},m_1)-\omega_2(s^{1/k_1},m_1)]\Big]\frac{1}{(\tau^{k_1}-s)s}dsdm_1. \quad (123)$$

Keeping in mind the factorization (113) with (114), the proposition 3 sparks of a constant $C_3>0$ with

$$||\frac{1}{P_m(\tau)}\int_{-\infty}^{+\infty}\tau^{k_1}\int_0^{\tau^{k_1}} Q_1(\sqrt{-1}(m-m_1))$$
$$\times[\omega_1((\tau^{k_1}-s)^{1/k_1},m-m_1)-\omega_2((\tau^{k_1}-s)^{1/k_1},m-m_1)]$$
$$\times Q_2(\sqrt{-1}m)\omega_1(s^{1/k_1},m_1)\frac{1}{(\tau^{k_1}-s)s}dsdm_1||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$$
$$\leq \frac{1}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}}C_3||\omega_1(\tau,m)-\omega_2(\tau,m)||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}||\omega_1(\tau,m)||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \quad (124)$$

and

$$||\frac{1}{P_m(\tau)}\int_{-\infty}^{+\infty}\tau^{k_1}\int_0^{\tau^{k_1}} Q_1(\sqrt{-1}(m-m_1))\omega_2((\tau^{k_1}-s)^{1/k_1},m-m_1)Q_2(\sqrt{-1}m_1)$$
$$\times[\omega_1(s^{1/k_1},m_1)-\omega_2(s^{1/k_1},m_1)]\Big]\frac{1}{(\tau^{k_1}-s)s}dsdm_1||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$$
$$\leq \frac{1}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}}C_3||\omega_1(\tau,m)-\omega_2(\tau,m)||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}||\omega_2(\tau,m)||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \quad (125)$$

The remodeling (123) of $\Delta(\tau, m)$ together with (124), (125) lead to

$$
\begin{aligned}
&||\Delta(\tau,m)||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \\
&\quad \leq \frac{1}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}} C_3\big(||\omega_1(\tau,m)||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} + ||\omega_2(\tau,m)||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}\big) \\
&\qquad\qquad \times ||\omega_1(\tau,m) - \omega_2(\tau,m)||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \\
&\qquad\qquad \leq \frac{1}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}} C_3 2\varpi_2 ||\omega_1(\tau,m) - \omega_2(\tau,m)||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}. \qquad (126)
\end{aligned}
$$

We enclose the constants $\epsilon_0 > 0$, $\boldsymbol{C}_{1,\epsilon_0} > 0$ and $c_{Q_1,Q_2} \in \mathbb{C}^*$ in the vicinity of the origin allowing the next inequality

$$
\begin{aligned}
&\Big(\sum_{q=1}^{\delta_D-1} |a_{q,\delta_D}| \Big[\frac{1}{\Gamma(\frac{d_{D,q}}{k_1})} k_1^q \frac{C_1}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}} \epsilon_0^{(\delta_D-q)k_1} \\
&\qquad + \sum_{1\leq p\leq q-1} |A_{q,p}| \frac{1}{\Gamma(\frac{d_{D,q}+k_1(q-p)}{k_1})} k_1^p \frac{C_1}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}} \epsilon_0^{(\delta_D-p)k_1}\Big]\Big) \\
&\qquad + \Big[\sum_{1\leq p\leq \delta_D-1} |A_{\delta_D,p}| \frac{k_1^p}{\Gamma(\delta_D-p)} \frac{C_1}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}} \epsilon_0^{(\delta_D-p)k_1}\Big] \\
&\quad + \sum_{l=1}^{D-1} \epsilon_0^{\Delta_l-d_l} \Big[\sum_{q=1}^{\delta_l} |a_{q,\delta_l}| \Big[\frac{k_1^q}{\Gamma(\frac{d_{l,q}}{k_1})} \frac{1}{(2\pi)^{1/2}} \frac{1}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}} C_2 \boldsymbol{A}_{l,\epsilon_0} C_1 M_{k_1,\delta_D} \epsilon_0^{d_{l,q}} \\
&\quad + \sum_{1\leq p\leq q-1} |A_{q,p}| \frac{k_1^p}{\Gamma(\frac{d_{l,q}+k_1(q-p)}{k_1})} \frac{1}{(2\pi)^{1/2}} \frac{1}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}} C_2 \boldsymbol{A}_{l,\epsilon_0} C_1 M_{k_1,\delta_D} \epsilon_0^{d_{l,q}+k_1(q-p)}\Big]\Big] \\
&\qquad + \frac{1}{(2\pi)^{1/2}} \frac{1}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}} \max_{m\in\mathbb{R}} \Big|\frac{1}{R_D(\sqrt{-1}m)}\Big| C_2 \boldsymbol{C}_{1,\epsilon_0} \\
&\qquad\qquad + |c_{Q_1,Q_2}| \frac{1}{(2\pi)^{1/2}} \frac{C_3}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}} 2\varpi_2 \leq 1/2. \qquad (127)
\end{aligned}
$$

The merging of the above bounds (117), (118), (119), (120), (121), (122), (126) subjected to (127) triggers the $1/2-$Lipschitz attribute of $\mathcal{H}_\epsilon$. Notice that the foremost five blocks of the left handside of (127) can be taken small scaled since they contain positive powers of ϵ_0 due to the constraint (18) imposed on (14) and its two tail terms can be downsized provided that the positive constants $\boldsymbol{C}_{1,\epsilon_0}$ and c_{Q_1,Q_2} are chosen close to the origin.

In the closing part of the proof, we fix the radius $\varpi_2 > 0$ and select the quantities $\epsilon_0 > 0$, $\boldsymbol{C}_{1,\epsilon_0} > 0$ together with $c_{Q_1,Q_2} \in \mathbb{C}^*$ close enough to 0 that conform both (116) and (127). For these values, the map $\mathcal{H}_\epsilon$ is endowed with both inclusion and shrinking properties (94), (95) for all $\epsilon \in D_{\epsilon_0} \setminus \{0\}$. Proposition 4 follows. □

The forthcoming proposition displays a solution to the first convolution equation (49) shaped in the Banach spaces described in Definition 6.

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Proposition 5. *Let us choose an appropriate inner radius $r_{Q,R_D} > 0$ and aperture $\eta_{Q,R_D} > 0$ of the sector S_{Q,R_D} together with an unbounded sector S_{d_1} and radius $\rho > 0$ that conform the requirements of Lemma 5. Then, a radius $\epsilon_0 > 0$ and constants $\mathbf{C}_{1,\epsilon_0} > 0$, $c_{Q_1,Q_2} \in \mathbb{C}^*$ can be pinpointed sufficiently close to 0 together with a proper radius $\varpi_2 > 0$ in a manner that for all $\epsilon \in D_{\epsilon_0} \setminus \{0\}$, a unique solution $\omega_{2,d_1}(\tau, m, \epsilon)$ to (49) exists such that*

- *the map $(\tau, m) \mapsto \omega_{2,d_1}(\tau, m, \epsilon)$ appertains to $F^{d_1}_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$ under the constraint*

$$\sup_{\epsilon \in D_{\epsilon_0} \setminus \{0\}} ||\omega_{2,d_1}(\tau, m, \epsilon)||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \leq \varpi_2 \tag{128}$$

- *the partial map $\epsilon \mapsto \omega_{2,d_1}(\tau, m, \epsilon)$ stands for an analytic map from $D_{\epsilon_0} \setminus \{0\}$ into $\mathbb{C}$, for any prescribed $\tau \in S_{d_1} \cup D_\rho$ and $m \in \mathbb{R}$.*

Proof. We take the constants $\epsilon_0 > 0$, $\mathbf{C}_{1,\epsilon_0} > 0$, $c_{Q_1,Q_2} \in \mathbb{C}^*$ together with $\varpi_2 > 0$ reached in Proposition 4. We observe that the closed ball $\bar{B}_{\varpi_2} \subset F^{d_1}_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$ represents a complete metric space for the distance $d(x, y) = ||x - y||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$. The proposition 4 claims that $\mathcal{H}_\epsilon$ induces a contractive map from $(\bar{B}_{\varpi_2}, d)$ into itself. It follows from the classical Banach fixed point theorem that $\mathcal{H}_\epsilon$ possesses a unique fixed point $\omega_{2,d_1}(\tau, m, \epsilon)$ inside the ball $\bar{B}_{\varpi_2}$, for all $\epsilon \in D_{\epsilon_0} \setminus \{0\}$, meaning that

$$\mathcal{H}_\epsilon(\omega_{2,d_1}(\tau, m, \epsilon)) = \omega_{2,d_1}(\tau, m, \epsilon) \tag{129}$$

holds. Furthermore, the map $\omega_{2,d_1}(\tau, m, \epsilon)$ relies analytically on ϵ since $\mathcal{H}_\epsilon$ does on the domain $D_{\epsilon_0} \setminus \{0\}$. On the other hand, we check that the convolution equation (49) can be rearranged as the equation (129) by shifting the term

$$(k_1 \tau^{k_1})^{\delta_D} R_D(\sqrt{-1}m)\omega_{2,d_1}(\tau, m, \epsilon)$$

from the right to the left handside of (49) and dividing by the resulting equation by the map $P_m(\tau)$ given by (86). As a result, the unique fixed point $\omega_{2,d_1}(\tau, m, \epsilon)$ of $\mathcal{H}_\epsilon$ enclosed in $\bar{B}_{\varpi_2}$ precisely solves (49). The result follows. □

6. Building up a solution to the second convolution equation (50) with (51)

In this section, we cook up a unique solution to the auxiliary convolution equation reached in (50) with (51) inside the Banach spaces described in Definition 6.

The roadmap follows the one of the previous section and consists in recasting (50) with (51) into a fixed point equation for a certain nonlinear map $\mathcal{G}_\epsilon$, stated in Proposition 7.

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The map $\mathcal{G}_{\epsilon}$ is set up as follows. We mind the map $\omega_{2,d_1}(\tau,m,\epsilon)$ stemming from Proposition 5 and the polynomial $P_m(\tau)$ displayed in (86). Let

$$
\begin{aligned}
\mathcal{G}_{\epsilon}(\omega(\tau,m)) := \Big(\sum_{q=1}^{\delta_D-1} a_{q,\delta_D} \Big[\frac{\tau^{k_1}}{P_m(\tau)\Gamma(\frac{d_{D,q}}{k_1})} \int_0^{\tau^{k_1}} (\tau^{k_1}-s)^{\frac{d_{D,q}}{k_1}-1} k_1^q s^q \omega(s^{1/k_1},m) \frac{ds}{s} \\
+ \sum_{1 \leq p \leq q-1} A_{q,p} \frac{\tau^{k_1}}{P_m(\tau)\Gamma(\frac{d_{D,q}+k_1(q-p)}{k_1})} \int_0^{\tau^{k_1}} (\tau^{k_1}-s)^{\frac{d_{D,q}+k_1(q-p)}{k_1}-1} k_1^p s^p \omega(s^{1/k_1},m) \frac{ds}{s} \Big] \\
\times R_D(\sqrt{-1}m) \Big) \\
+ \Big[\sum_{1 \leq p \leq \delta_D-1} A_{\delta_D,p} \frac{\tau^{k_1}}{P_m(\tau)\Gamma(\delta_D-p)} \int_0^{\tau^{k_1}} (\tau^{k_1}-s)^{\delta_D-p-1} k_1^p s^p \omega(s^{1/k_1},m) \frac{ds}{s} \\
\times R_D(\sqrt{-1}m) \Big] \\
+ \Big(\delta_D \sum_{q=1}^{\delta_D-1} a_{q,\delta_D-1} \Big[\frac{\tau^{k_1}}{P_m(\tau)\Gamma(\frac{d_{D,q}}{k_1})} \int_0^{\tau^{k_1}} (\tau^{k_1}-s)^{\frac{d_{D,q}}{k_1}-1} k_1^q s^q \omega_{2,d_1}(s^{1/k_1},m,\epsilon) \frac{ds}{s} \\
+ \sum_{1 \leq p \leq q-1} A_{q,p} \frac{\tau^{k_1}}{P_m(\tau)\Gamma(\frac{d_{D,q}+k_1(q-p)}{k_1})} \int_0^{\tau^{k_1}} (\tau^{k_1}-s)^{\frac{d_{D,q}+k_1(q-p)}{k_1}-1} k_1^p s^p \omega_{2,d_1}(s^{1/k_1},m,\epsilon) \frac{ds}{s} \Big] \\
\times R_D(\sqrt{-1}m) \Big) \\
+ \sum_{l=1}^{D-1} \epsilon^{\Delta_l - d_l} \Big[\Big(\sum_{q=1}^{\delta_l} a_{q,\delta_l} \Big[\frac{\tau^{k_1}}{P_m(\tau)\Gamma(\frac{d_{l,q}}{k_1})} \int_0^{\tau^{k_1}} (\tau^{k_1}-s)^{\frac{d_{l,q}}{k_1}-1} \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} A_l(m-m_1,\epsilon) k_1^q s^q \\
\times R_l(\sqrt{-1}m_1)\omega(s^{1/k_1},m_1) \frac{ds}{s} dm_1 + \sum_{1 \leq p \leq q-1} A_{q,p} \frac{\tau^{k_1}}{P_m(\tau)\Gamma(\frac{d_{l,q}+k_1(q-p)}{k_1})} \int_0^{\tau^{k_1}} (\tau^{k_1}-s)^{\frac{d_{l,q}+k_1(q-p)}{k_1}-1} \\
\times \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} A_l(m-m_1,\epsilon) k_1^p s^p R_l(\sqrt{-1}m_1)\omega(s^{1/k_1},m_1) \frac{ds}{s} dm_1 \Big] \Big) \\
+ \Big(\delta_l \sum_{q=1}^{\delta_l-1} a_{q,\delta_l-1} \Big[\frac{\tau^{k_1}}{P_m(\tau)\Gamma(\frac{d_{l,q}}{k_1})} \int_0^{\tau^{k_1}} (\tau^{k_1}-s)^{\frac{d_{l,q}}{k_1}-1} \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} A_l(m-m_1,\epsilon) k_1^q s^q R_l(\sqrt{-1}m_1) \\
\times \omega_{2,d_1}(s^{1/k_1},m_1,\epsilon) \frac{ds}{s} dm_1 + \sum_{1 \leq p \leq q-1} A_{q,p} \frac{\tau^{k_1}}{P_m(\tau)\Gamma(\frac{d_{l,q}+k_1(q-p)}{k_1})} \int_0^{\tau^{k_1}} (\tau^{k_1}-s)^{\frac{d_{l,q}+k_1(q-p)}{k_1}-1} \\
\times \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} A_l(m-m_1,\epsilon) k_1^p s^p R_l(\sqrt{-1}m_1)\omega_{2,d_1}(s^{1/k_1},m_1,\epsilon) \frac{ds}{s} dm_1 \Big] \Big) \Big] \\
+ \mathcal{A}_{\mathcal{G}_{\epsilon}}(\tau,m,\epsilon) \qquad (130)
\end{aligned}
$$

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where

$$
\begin{aligned}
\mathcal{A}_{\mathcal{G}_{\epsilon}}(\tau,m,\epsilon) := & \frac{\mathcal{F}_1(\tau,m,\epsilon)}{P_m(\tau)} + \frac{1}{P_m(\tau)(2\pi)^{1/2}}\int_{-\infty}^{+\infty} B_1(m-m_1,\epsilon)\omega(\tau,m_1)dm_1 \\
& + \frac{1}{P_m(\tau)(2\pi)^{1/2}}\int_{-\infty}^{+\infty} B_2(m-m_1,\epsilon)\omega_{2,d_1}(\tau,m_1,\epsilon)dm_1 \\
& + c_{P_1P_2}\frac{1}{P_m(\tau)(2\pi)^{1/2}}\int_{-\infty}^{+\infty}\tau^{k_1}\int_0^{\tau^{k_1}} P_1(\sqrt{-1}(m-m_1))\omega((\tau^{k_1}-s)^{1/k_1},m-m_1) \\
& \times P_2(\sqrt{-1}m_1)\omega_{2,d_1}(s^{1/k_1},m_1,\epsilon)\frac{1}{(\tau^{k_1}-s)s}dsdm_1 \\
& + c_{P_3P_4}\frac{1}{P_m(\tau)(2\pi)^{1/2}}\int_{-\infty}^{+\infty}\tau^{k_1}\int_0^{\tau^{k_1}} P_3(\sqrt{-1}(m-m_1))\omega((\tau^{k_1}-s)^{1/k_1},m-m_1) \\
& \times P_4(\sqrt{-1}m_1)\omega(s^{1/k_1},m_1)\frac{1}{(\tau^{k_1}-s)s}dsdm_1 \\
& + c_{P_5P_6}\frac{1}{P_m(\tau)(2\pi)^{1/2}}\int_{-\infty}^{+\infty}\tau^{k_1}\int_0^{\tau^{k_1}} P_5(\sqrt{-1}(m-m_1))\omega_{2,d_1}((\tau^{k_1}-s)^{1/k_1},m-m_1,\epsilon) \\
& \times P_6(\sqrt{-1}m_1)\omega_{2,d_1}(s^{1/k_1},m_1,\epsilon)\frac{1}{(\tau^{k_1}-s)s}dsdm_1 \qquad (131)
\end{aligned}
$$

In the next proposition we discuss the $1/2-$Lipschitz feature of $\mathcal{G}_{\epsilon}$ on some well chosen ball in the Banach spaces depicted in Definition 6.

Proposition 6. *Let a timely inner radius $r_{Q,R_D} > 0$ and aperture $\eta_{Q,R_D} > 0$ of the sector S_{Q,R_D} in a row with an unbounded sector S_{d_1} and radius $\rho > 0$ chosen to fulfill the specifications of Lemma 5. We also take for granted the additional condition (93) required for the sector S_{d_1} and the disc D_{ρ}.*

Then, one can target a small radius $\epsilon_0 > 0$ along with constants $\mathbf{B}_{j,\epsilon_0} > 0$, $c_{P_k,P_{k+1}} \in \mathbb{C}^*$, for $j = 1,2$ and $k = 1,3,5$ proximate to 0, coupled to a fitted radius $\varpi_1 > 0$ in a way that for all $\epsilon \in D_{\epsilon_0} \setminus \{0\}$, the map $\mathcal{G}_{\epsilon}$ boasts the next two properties

- *$\mathcal{G}_{\epsilon}$ maps $\bar{B}_{\varpi_1}$ into itself, where $\bar{B}_{\varpi_1}$ stands for the closed ball of radius ϖ_1 centered at 0 in the space $F^{d_1}_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$.*
- *The norm downsizing condition*

$$||\mathcal{G}_{\epsilon}(\omega_1) - \mathcal{G}_{\epsilon}(\omega_2)||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \leq \frac{1}{2}||\omega_1 - \omega_2||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \qquad (132)$$

holds whenever $\omega_1, \omega_2 \in F^{d_1}_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$.

Proof. We heed the first item asserting the inclusion. We fix some real number $\varpi_1 > 0$ and pick up an element $\omega(\tau,m)$ in $F^{d_1}_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$, for $\epsilon \in D_{\epsilon_0} \setminus \{0\}$, with

$$||\omega||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \leq \varpi_1.$$

Concrete bounds are presented for each piece of the map $\mathcal{G}_{\epsilon}$ applied to ω.

The estimates for the first three blocks of $\mathcal{G}_{\epsilon}$ are merely the same as the ones obtained in (96), (97) and (98). Namely, owing to Proposition 1 and Lemma 5, we observe that

$$
\begin{aligned}
& ||\frac{\tau^{k_1}}{P_m(\tau)}\int_0^{\tau^{k_1}}(\tau^{k_1}-s)^{\frac{d_{D,q}}{k_1}-1}s^q R_D(\sqrt{-1}m)\omega(s^{1/k_1},m)\frac{ds}{s}||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \\
& \qquad \leq \frac{C_1}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}}|\epsilon|^{(\delta_D-q)k_1}||\omega(\tau,m)||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \qquad (133)
\end{aligned}
$$

for $1 \leq q \leq \delta_D - 1$ along with

$$||\frac{\tau^{k_1}}{P_m(\tau)}\int_0^{\tau^{k_1}}(\tau^{k_1}-s)^{\frac{d_{D,q}+k_1(q-p)}{k_1}-1}s^p R_D(\sqrt{-1}m)\omega(s^{1/k_1},m)\frac{ds}{s}||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \leq \frac{C_1}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}}|\epsilon|^{(\delta_D-p)k_1}||\omega(\tau,m)||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \quad (134)$$

for $1 \leq p \leq q-1$ with $1 \leq q \leq \delta_D - 1$ and

$$||\frac{\tau^{k_1}}{P_m(\tau)}\int_0^{\tau^{k_1}}(\tau^{k_1}-s)^{\delta_D-p-1}s^p R_D(\sqrt{-1}m)\omega(s^{1/k_1},m)\frac{ds}{s}||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \leq \frac{C_1}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}}|\epsilon|^{(\delta_D-p)k_1}||\omega(\tau,m)||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \quad (135)$$

as long as $1 \leq p \leq \delta_D - 1$.

The next two pieces of $\mathcal{G}_\epsilon$ follow from Proposition 1 and Lemma 5 together with the estimates (128) reached in Proposition 5. Indeed, we arrive at

$$||\frac{\tau^{k_1}}{P_m(\tau)}\int_0^{\tau^{k_1}}(\tau^{k_1}-s)^{\frac{d_{D,q}}{k_1}-1}s^q R_D(\sqrt{-1}m)\omega_{2,d_1}(s^{1/k_1},m,\epsilon)\frac{ds}{s}||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \leq \frac{C_1}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}}|\epsilon|^{(\delta_D-q)k_1}||\omega_{2,d_1}(\tau,m,\epsilon)||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \leq \frac{C_1}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}}|\epsilon|^{(\delta_D-q)k_1}\varpi_2 \quad (136)$$

for $1 \leq q \leq \delta_D - 1$ in a row with

$$||\frac{\tau^{k_1}}{P_m(\tau)}\int_0^{\tau^{k_1}}(\tau^{k_1}-s)^{\frac{d_{D,q}+k_1(q-p)}{k_1}-1}s^p R_D(\sqrt{-1}m)\omega_{2,d_1}(s^{1/k_1},m,\epsilon)\frac{ds}{s}||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \leq \frac{C_1}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}}|\epsilon|^{(\delta_D-p)k_1}||\omega_{2,d_1}(\tau,m,\epsilon)||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \leq \frac{C_1}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}}|\epsilon|^{(\delta_D-p)k_1}\varpi_2 \quad (137)$$

for $1 \leq p \leq q-1$ with $1 \leq q \leq \delta_D - 1$.

The estimates for the following two components of $\mathcal{G}_\epsilon$ simply recast the ones obtained in (104) and (108). Indeed,

$$||\frac{\tau^{k_1}}{P_m(\tau)}\int_0^{\tau^{k_1}}(\tau^{k_1}-s)^{\frac{d_{l,q}}{k_1}-1}\int_{-\infty}^{+\infty}A_l(m-m_1,\epsilon)s^q \times R_l(\sqrt{-1}m_1)\omega(s^{1/k_1},m_1)\frac{ds}{s}dm_1||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \leq \frac{1}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}}C_2\mathbf{A}_{l,\epsilon_0}C_1M_{k_1,\delta_D}|\epsilon|^{d_{l,q}}||\omega(\tau,m)||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \quad (138)$$

for $1 \leq q \leq \delta_l$ and $1 \leq l \leq D-1$ in parallel with

$$
\begin{aligned}
&|| \frac{\tau^{k_1}}{P_m(\tau)} \int_0^{\tau^{k_1}} (\tau^{k_1}-s)^{\frac{d_{l,q}+k_1(q-p)}{k_1}-1} \\
&\qquad \times \int_{-\infty}^{+\infty} A_l(m-m_1,\epsilon) s^p R_l(\sqrt{-1}m_1) \omega(s^{1/k_1},m_1) \frac{ds}{s} dm_1 ||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \\
&\qquad\qquad \leq \frac{1}{C_P (r_{Q,R_D})^{\frac{1}{k_1 \delta_D}}} C_2 \boldsymbol{A}_{l,\epsilon_0} C_1 M_{k_1,\delta_D} |\epsilon|^{d_{l,q}+k_1(q-p)} ||\omega(\tau,m)||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}
\end{aligned} \tag{139}
$$

for $1 \leq p \leq q-1$ and $1 \leq q \leq \delta_l$ with $1 \leq l \leq D-1$. Furthermore, the two ensuing constituents of $\mathcal{G}_\epsilon$ mirror the one reached in (104) and (108) and draw on the estimates (128) from Proposition 5. Namely,

$$
\begin{aligned}
&|| \frac{\tau^{k_1}}{P_m(\tau)} \int_0^{\tau^{k_1}} (\tau^{k_1}-s)^{\frac{d_{l,q}}{k_1}-1} \int_{-\infty}^{+\infty} A_l(m-m_1,\epsilon) s^q \\
&\qquad \times R_l(\sqrt{-1}m_1) \omega_{2,d_1}(s^{1/k_1},m_1,\epsilon) \frac{ds}{s} dm_1 ||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \\
&\qquad \leq \frac{1}{C_P (r_{Q,R_D})^{\frac{1}{k_1 \delta_D}}} C_2 \boldsymbol{A}_{l,\epsilon_0} C_1 M_{k_1,\delta_D} |\epsilon|^{d_{l,q}} ||\omega_{2,d_1}(\tau,m,\epsilon)||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \\
&\qquad\qquad \leq \frac{1}{C_P (r_{Q,R_D})^{\frac{1}{k_1 \delta_D}}} C_2 \boldsymbol{A}_{l,\epsilon_0} C_1 M_{k_1,\delta_D} |\epsilon|^{d_{l,q}} \varpi_2
\end{aligned} \tag{140}
$$

for $1 \leq q \leq \delta_l - 1$ and $1 \leq l \leq D-1$ in tandem with

$$
\begin{aligned}
&|| \frac{\tau^{k_1}}{P_m(\tau)} \int_0^{\tau^{k_1}} (\tau^{k_1}-s)^{\frac{d_{l,q}+k_1(q-p)}{k_1}-1} \\
&\qquad \times \int_{-\infty}^{+\infty} A_l(m-m_1,\epsilon) s^p R_l(\sqrt{-1}m_1) \omega_{2,d_1}(s^{1/k_1},m_1,\epsilon) \frac{ds}{s} dm_1 ||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \\
&\qquad \leq \frac{1}{C_P (r_{Q,R_D})^{\frac{1}{k_1 \delta_D}}} C_2 \boldsymbol{A}_{l,\epsilon_0} C_1 M_{k_1,\delta_D} |\epsilon|^{d_{l,q}+k_1(q-p)} ||\omega_{2,d_1}(\tau,m,\epsilon)||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \\
&\qquad\qquad \leq \frac{1}{C_P (r_{Q,R_D})^{\frac{1}{k_1 \delta_D}}} C_2 \boldsymbol{A}_{l,\epsilon_0} C_1 M_{k_1,\delta_D} |\epsilon|^{d_{l,q}+k_1(q-p)} \varpi_2
\end{aligned} \tag{141}
$$

provided that $1 \leq p \leq q-1$ and $1 \leq q \leq \delta_l - 1$ with $1 \leq l \leq D-1$.

The next element of $\mathcal{G}_\epsilon$ we pay regard is $\mathcal{F}_1(\tau,m,\epsilon)/P_m(\tau)$ and is displayed in (131). Its bounds are obtained in a similar way as the ones reached in (111). Indeed,

$$
\begin{aligned}
&|| \frac{\mathcal{F}_1(\tau,m,\epsilon)}{P_m(\tau)} ||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \\
&\qquad \leq \frac{1}{C_P (r_{Q,R_D})^{\frac{1}{k_1 \delta_D}}} \max_{m \in \mathbb{R}} \left| \frac{1}{R_D(\sqrt{-1}m)} \right| \epsilon_0 \times \sup_{x \geq 0} e^{-\nu x^{k_1}} (1+x^{2k_1}) \sum_{j_1 \in J_1} \boldsymbol{F}_{1,j_1,\epsilon_0} \epsilon_0^{j_1-1} x^{j_1-1}
\end{aligned} \tag{142}
$$

which can be subsided close to 0 provided that $\epsilon_0 > 0$ is tiny enough since $0 \notin J_1$.

We handle the second and third pieces of $\mathcal{A}_{\mathcal{G}_\epsilon}(\tau,m,\epsilon)$. Paying heed to (109) and the bounds (27), Proposition 2 kindles

$$
\begin{aligned}
&||\frac{1}{P_m(\tau)}\int_{-\infty}^{+\infty} B_1(m-m_1,\epsilon)\omega(\tau,m_1)dm_1||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \\
&\quad \leq \frac{1}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}}\max_{m\in\mathbb{R}}\left|\frac{1}{R_D(\sqrt{-1}m)}\right|C_2||B_1(m,\epsilon)||_{(\beta,\mu)}||\omega(\tau,m)||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \\
&\quad\quad \leq \frac{1}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}}\max_{m\in\mathbb{R}}\left|\frac{1}{R_D(\sqrt{-1}m)}\right|C_2\boldsymbol{B}_{1,\epsilon_0}||\omega(\tau,m)||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}. \qquad (143)
\end{aligned}
$$

and bearing in mind the estimates (128) from Proposition 5,

$$
\begin{aligned}
&||\frac{1}{P_m(\tau)}\int_{-\infty}^{+\infty} B_2(m-m_1,\epsilon)\omega_{2,d_1}(\tau,m_1,\epsilon)dm_1||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \\
&\quad \leq \frac{1}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}}\max_{m\in\mathbb{R}}\left|\frac{1}{R_D(\sqrt{-1}m)}\right|C_2||B_2(m,\epsilon)||_{(\beta,\mu)}||\omega_{2,d_1}(\tau,m,\epsilon)||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \\
&\quad\quad \leq \frac{1}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}}\max_{m\in\mathbb{R}}\left|\frac{1}{R_D(\sqrt{-1}m)}\right|C_2\boldsymbol{B}_{2,\epsilon_0}\varpi_2. \qquad (144)
\end{aligned}
$$

ensues.

Thanks to the factorization (113) with (114) and the bounds (128) from Proposition 5, we can apply Proposition 3 in order to address the last three terms of $\mathcal{A}_{\mathcal{G}_\epsilon}(\tau,m,\epsilon)$. Namely,

$$
\begin{aligned}
&||\frac{1}{P_m(\tau)}\int_{-\infty}^{+\infty}\tau^{k_1}\int_0^{\tau^{k_1}} P_1(\sqrt{-1}(m-m_1))\omega((\tau^{k_1}-s)^{1/k_1},m-m_1) \\
&\quad \times P_2(\sqrt{-1}m_1)\omega_{2,d_1}(s^{1/k_1},m_1,\epsilon)\frac{1}{(\tau^{k_1}-s)s}dsdm_1||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \\
&\quad \leq \frac{C_3}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}}||\omega(\tau,m)||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}||\omega_{2,d_1}(\tau,m,\epsilon)||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \\
&\quad\quad \leq \frac{C_3}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}}||\omega(\tau,m)||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}\varpi_2 \qquad (145)
\end{aligned}
$$

together with

$$
\begin{aligned}
&||\frac{1}{P_m(\tau)}\int_{-\infty}^{+\infty}\tau^{k_1}\int_0^{\tau^{k_1}} P_3(\sqrt{-1}(m-m_1))\omega((\tau^{k_1}-s)^{1/k_1},m-m_1) \\
&\quad \times P_4(\sqrt{-1}m_1)\omega(s^{1/k_1},m_1)\frac{1}{(\tau^{k_1}-s)s}dsdm_1||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \\
&\quad\quad \leq \frac{C_3}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}}||\omega(\tau,m)||^2_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \qquad (146)
\end{aligned}
$$

as well as

$$||\frac{1}{P_m(\tau)}\int_{-\infty}^{+\infty}\tau^{k_1}\int_0^{\tau^{k_1}}P_5(\sqrt{-1}(m-m_1))\omega_{2,d_1}((\tau^{k_1}-s)^{1/k_1},m-m_1,\epsilon)$$
$$\times P_6(\sqrt{-1}m_1)\omega_{2,d_1}(s^{1/k_1},m_1,\epsilon)\frac{1}{(\tau^{k_1}-s)s}dsdm_1||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$$
$$\leq \frac{C_3}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}}||\omega_{2,d_1}(\tau,m,\epsilon)||^2_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \leq \frac{C_3}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}}\varpi_2^2. \quad (147)$$

We pin down the constants $\epsilon_0 > 0$ and $\boldsymbol{B}_{j,\epsilon_0} > 0$, $c_{P_k,P_{k+1}} \in \mathbb{C}^*$, for $j = 1,2$ and $k = 1,3,5$ proximate to 0 together with a suitable radius $\varpi_1 > 0$ in a way that the next inequality

$$\Big(\sum_{q=1}^{\delta_D-1}|a_{q,\delta_D}|\Big[\frac{1}{\Gamma(\frac{d_{D,q}}{k_1})}k_1^q\frac{C_1}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}}\epsilon_0^{(\delta_D-q)k_1}\varpi_1$$
$$+\sum_{1\leq p\leq q-1}|A_{q,p}|\frac{1}{\Gamma(\frac{d_{D,q}+k_1(q-p)}{k_1})}k_1^p\frac{C_1}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}}\epsilon_0^{(\delta_D-p)k_1}\varpi_1\Big]\Big)$$
$$+\sum_{1\leq p\leq \delta_D-1}|A_{\delta_D,p}|\frac{k_1^p}{\Gamma(\delta_D-p)}\frac{C_1}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}}\epsilon_0^{(\delta_D-p)k_1}\varpi_1$$
$$+\Big(\delta_D\sum_{q=1}^{\delta_D-1}|a_{q,\delta_D-1}|\Big[\frac{1}{\Gamma(\frac{d_{D,q}}{k_1})}k_1^q\frac{C_1}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}}\epsilon_0^{(\delta_D-q)k_1}\varpi_2$$
$$+\sum_{1\leq p\leq q-1}|A_{q,p}|\frac{1}{\Gamma(\frac{d_{D,q}+k_1(q-p)}{k_1})}k_1^p\frac{C_1}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}}\epsilon_0^{(\delta_D-p)k_1}\varpi_2\Big]\Big)$$
$$+\sum_{l=1}^{D-1}\epsilon_0^{\Delta_l-d_l}\Big[\sum_{q=1}^{\delta_l}|a_{q,\delta_l}|\Big[\frac{k_1^q}{\Gamma(\frac{d_{l,q}}{k_1})}\frac{1}{(2\pi)^{1/2}}\frac{1}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}}C_2\boldsymbol{A}_{l,\epsilon_0}C_1M_{k_1,\delta_D}\epsilon_0^{d_{l,q}}\varpi_1$$
$$+\sum_{1\leq p\leq q-1}|A_{q,p}|\frac{k_1^p}{\Gamma(\frac{d_{l,q}+k_1(q-p)}{k_1})}\frac{1}{(2\pi)^{1/2}}\frac{1}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}}C_2\boldsymbol{A}_{l,\epsilon_0}C_1M_{k_1,\delta_D}\epsilon_0^{d_{l,q}+k_1(q-p)}\varpi_1\Big]$$
$$+\delta_l\sum_{q=1}^{\delta_l-1}a_{q,\delta_l-1}\Big[\frac{k_1^q}{\Gamma(\frac{d_{l,q}}{k_1})}\frac{1}{(2\pi)^{1/2}}\frac{1}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}}C_2\boldsymbol{A}_{l,\epsilon_0}C_1M_{k_1,\delta_D}\epsilon_0^{d_{l,q}}\varpi_2$$
$$+\sum_{1\leq p\leq q-1}|A_{q,p}|\frac{k_1^p}{\Gamma(\frac{d_{l,q}+k_1(q-p)}{k_1})}\frac{1}{(2\pi)^{1/2}}\frac{1}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}}C_2\boldsymbol{A}_{l,\epsilon_0}C_1M_{k_1,\delta_D}\epsilon_0^{d_{l,q}+k_1(q-p)}\varpi_2\Big]\Big]$$
$$+\mathbb{A}_{\mathcal{G}} \leq \varpi_1 \quad (148)$$

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holds where

$$
\begin{aligned}
\mathbb{A}_{\mathcal{G}} = \frac{1}{C_{P}(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}} \max_{m\in\mathbb{R}} \left|\frac{1}{R_D(\sqrt{-1}m)}\right| \epsilon_0 \\
\times \sup_{x\geq 0} e^{-\nu x^{k_1}}(1+x^{2k_1}) \sum_{j_1\in J_1} \boldsymbol{F}_{1,j_1,\epsilon_0}\epsilon_0^{j_1-1}x^{j_1-1} \\
+ \frac{1}{(2\pi)^{1/2}} \frac{1}{C_{P}(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}} \max_{m\in\mathbb{R}} \left|\frac{1}{R_D(\sqrt{-1}m)}\right| C_2 \boldsymbol{B}_{1,\epsilon_0}\varpi_1 \\
+ \frac{1}{(2\pi)^{1/2}} \frac{1}{C_{P}(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}} \max_{m\in\mathbb{R}} \left|\frac{1}{R_D(\sqrt{-1}m)}\right| C_2 \boldsymbol{B}_{2,\epsilon_0}\varpi_2 \\
+ |c_{P_1,P_2}| \frac{1}{(2\pi)^{1/2}} \frac{C_3}{C_{P}(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}} \varpi_1\varpi_2 + |c_{P_3,P_4}| \frac{1}{(2\pi)^{1/2}} \frac{C_3}{C_{P}(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}} \varpi_1^2 \\
+ |c_{P_5,P_6}| \frac{1}{(2\pi)^{1/2}} \frac{C_3}{C_{P}(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}} \varpi_2^2 \quad (149)
\end{aligned}
$$

We check that all the terms on the left handside of (148) except $\mathbb{A}_{\mathcal{G}}$ can be tapered off since they contain positive powers of $\epsilon_0 > 0$ in particular due to the constraint (18). Besides, the constant $\mathbb{A}_{\mathcal{G}}$ can be lessen provided that the constants ϵ_0 and $\boldsymbol{B}_{j,\epsilon_0}$, $c_{P_k,P_{k+1}}$, for $j = 1,2$ and $k = 1,3,5$ are taken in the vicinity of 0.

At last, stacking up all the above bounds (133), (134), (135), (136), (137), (138), (139), (140) (141), (142), (143), (144), (145), (146), (147) under the contingency (148) yield that $\mathcal{G}_\epsilon$ maps $\bar{B}_{\varpi_1}$ into itself.

In the second part of the proof, we address the second item of Proposition 6. Let ω_1, ω_2 be elements of the ball $\bar{B}_{\varpi_1}$ of the space $F^{d_1}_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$ with radius $\varpi_1 > 0$ chosen as in the first part of the proof.

We provide norm estimates for each part of the difference $\mathcal{G}_\epsilon(\omega_1) - \mathcal{G}_\epsilon(\omega_2)$. The bounds for the foremost five blocks of the difference are barely the ones found in (117), (118), (119), (120), (121). Namely,

$$
\begin{aligned}
||\frac{\tau^{k_1}}{P_m(\tau)} \int_0^{\tau^{k_1}} (\tau^{k_1}-s)^{\frac{d_{D,q}}{k_1}-1} s^q R_D(\sqrt{-1}m)\big(\omega_1(s^{1/k_1},m) - \omega_2(s^{1/k_1},m)\big)\frac{ds}{s}||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \\
\leq \frac{C_1}{C_{P}(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}} |\epsilon|^{(\delta_D-q)k_1} ||\omega_1(\tau,m) - \omega_2(\tau,m)||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \quad (150)
\end{aligned}
$$

for $1 \leq q \leq \delta_D - 1$ along with

$$
\begin{aligned}
||\frac{\tau^{k_1}}{P_m(\tau)} \int_0^{\tau^{k_1}} (\tau^{k_1}-s)^{\frac{d_{D,q}+k_1(q-p)}{k_1}-1} s^p R_D(\sqrt{-1}m) \\
\times \big(\omega_1(s^{1/k_1},m) - \omega_2(s^{1/k_1},m)\big)\frac{ds}{s}||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \\
\leq \frac{C_1}{C_{P}(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}} |\epsilon|^{(\delta_D-p)k_1} ||\omega_1(\tau,m) - \omega_2(\tau,m)||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \quad (151)
\end{aligned}
$$

for $1 \leq p \leq q-1$ with $1 \leq q \leq \delta_D - 1$ and

$$
\begin{aligned}
||\frac{\tau^{k_1}}{P_m(\tau)} \int_0^{\tau^{k_1}} (\tau^{k_1}-s)^{\delta_D-p-1} s^p R_D(\sqrt{-1}m)\big(\omega_1(s^{1/k_1},m) - \omega_2(s^{1/k_1},m)\big)\frac{ds}{s}||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \\
\leq \frac{C_1}{C_{P}(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}} |\epsilon|^{(\delta_D-p)k_1} ||\omega_1(\tau,m) - \omega_2(\tau,m)||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \quad (152)
\end{aligned}
$$

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as long as $1 \leq p \leq \delta_D - 1$. Furthermore,

$$||\frac{\tau^{k_1}}{P_m(\tau)}\int_0^{\tau^{k_1}}(\tau^{k_1}-s)^{\frac{d_{l,q}}{k_1}-1}\int_{-\infty}^{+\infty}A_l(m-m_1,\epsilon)s^q R_l(\sqrt{-1}m_1)$$
$$\times\left(\omega_1(s^{1/k_1},m_1)-\omega_2(s^{1/k_1},m_1)\right)\frac{ds}{s}dm_1||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$$
$$\leq \frac{1}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}}C_2\boldsymbol{A}_{l,\epsilon_0}C_1M_{k_1,\delta_D}|\epsilon|^{d_{l,q}}||\omega_1(\tau,m)-\omega_2(\tau,m)||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}. \quad (153)$$

holds for $1 \leq l \leq D-1$ and $1 \leq q \leq \delta_l$ together with

$$||\frac{\tau^{k_1}}{P_m(\tau)}\int_0^{\tau^{k_1}}(\tau^{k_1}-s)^{\frac{d_{l,q}+k_1(q-p)}{k_1}-1}\int_{-\infty}^{+\infty}A_l(m-m_1,\epsilon)s^p R_l(\sqrt{-1}m_1)$$
$$\times\left(\omega_1(s^{1/k_1},m_1)-\omega_2(s^{1/k_1},m_1)\right)\frac{ds}{s}dm_1||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$$
$$\leq \frac{1}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}}C_2\boldsymbol{A}_{l,\epsilon_0}C_1M_{k_1,\delta_D}|\epsilon|^{d_{l,q}+k_1(q-p)}||\omega_1(\tau,m)-\omega_2(\tau,m)||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}. \quad (154)$$

for $1 \leq l \leq D-1$, $1 \leq q \leq \delta_l$ and $1 \leq p \leq q-1$. Besides, bounds for the sixth piece of $\mathcal{G}_\epsilon(\omega_1) - \mathcal{G}_\epsilon(\omega_2)$ result from (143) and are written

$$||\frac{1}{P_m(\tau)}\int_{-\infty}^{+\infty}B_1(m-m_1,\epsilon)(\omega_1(\tau,m_1)-\omega_2(\tau,m_1))dm_1||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$$
$$\leq \frac{1}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}}\max_{m\in\mathbb{R}}\left|\frac{1}{R_D(\sqrt{-1}m)}\right|C_2\boldsymbol{B}_{1,\epsilon_0}||\omega_1(\tau,m)-\omega_2(\tau,m)||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}. \quad (155)$$

The treatment of the seventh piece of $\mathcal{G}_\epsilon(\omega_1) - \mathcal{G}_\epsilon(\omega_2)$ springs from (145). Indeed,

$$||\frac{1}{P_m(\tau)}\int_{-\infty}^{+\infty}\tau^{k_1}\int_0^{\tau^{k_1}}P_1(\sqrt{-1}(m-m_1))$$
$$\times\left(\omega_1((\tau^{k_1}-s)^{1/k_1},m-m_1)-\omega_2((\tau^{k_1}-s)^{1/k_1},m-m_1)\right)$$
$$\times P_2(\sqrt{-1}m_1)\omega_{2,d_1}(s^{1/k_1},m_1,\epsilon)\frac{1}{(\tau^{k_1}-s)s}dsdm_1||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$$
$$\leq \frac{C_3}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}}||\omega_1(\tau,m)-\omega_2(\tau,m)||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}||\omega_{2,d_1}(\tau,m,\epsilon)||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$$
$$\leq \frac{C_3}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}}||\omega_1(\tau,m)-\omega_2(\tau,m)||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}\varpi_2 \quad (156)$$

The hindmost term of the difference $\mathcal{G}_{\epsilon}(\omega_1) - \mathcal{G}_{\epsilon}(\omega_2)$ can be processed in a similar way as for the difference (123) given by (126). Namely,

$$
\begin{aligned}
&||\frac{1}{P_m(\tau)}\int_{-\infty}^{+\infty}\tau^{k_1}\int_0^{\tau^{k_1}} P_3(\sqrt{-1}(m-m_1))\omega_1((\tau^{k_1}-s)^{1/k_1},m-m_1)\\
&\qquad\times P_4(\sqrt{-1}m_1)\omega_1(s^{1/k_1},m_1)\frac{1}{(\tau^{k_1}-s)s}dsdm_1\\
&\quad-\frac{1}{P_m(\tau)}\int_{-\infty}^{+\infty}\tau^{k_1}\int_0^{\tau^{k_1}} P_3(\sqrt{-1}(m-m_1))\omega_2((\tau^{k_1}-s)^{1/k_1},m-m_1)\\
&\qquad\times P_4(\sqrt{-1}m_1)\omega_2(s^{1/k_1},m_1)\frac{1}{(\tau^{k_1}-s)s}dsdm_1||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}\\
&\quad\leq \frac{1}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}}C_3(||\omega_1(\tau,m)||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}+||\omega_2(\tau,m)||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)})\\
&\qquad\times||\omega_1(\tau,m)-\omega_2(\tau,m)||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}\\
&\qquad\leq \frac{1}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}}C_3 2\varpi_1||\omega_1(\tau,m)-\omega_2(\tau,m)||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}. \qquad (157)
\end{aligned}
$$

We skirt the constants $\epsilon_0 > 0$, $\boldsymbol{B}_{1,\epsilon_0} > 0$ and $c_{P_1P_2} \in \mathbb{C}^*$, $c_{P_3P_4} \in \mathbb{C}^*$ nearby the origin in a manner that the next inequality

$$
\begin{aligned}
&\Big(\sum_{q=1}^{\delta_D-1}|a_{q,\delta_D}|\Big[\frac{1}{\Gamma(\frac{d_{D,q}}{k_1})}k_1^q\frac{C_1}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}}\epsilon_0^{(\delta_D-q)k_1}\\
&\quad+\sum_{1\leq p\leq q-1}|A_{q,p}|\frac{1}{\Gamma(\frac{d_{D,q}+k_1(q-p)}{k_1})}k_1^p\frac{C_1}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}}\epsilon_0^{(\delta_D-p)k_1}\Big]\Big)\\
&\quad+\sum_{1\leq p\leq \delta_D-1}|A_{\delta_D,p}|\frac{k_1^p}{\Gamma(\delta_D-p)}\frac{C_1}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}}\epsilon_0^{(\delta_D-p)k_1}\\
&\quad+\sum_{l=1}^{D-1}\epsilon_0^{\Delta_l-d_l}\Big[\sum_{q=1}^{\delta_l}|a_{q,\delta_l}|\Big[\frac{k_1^q}{\Gamma(\frac{d_{l,q}}{k_1})}\frac{1}{(2\pi)^{1/2}}\frac{1}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}}C_2\boldsymbol{A}_{l,\epsilon_0}C_1M_{k_1,\delta_D}\epsilon_0^{d_{l,q}}\\
&\quad+\sum_{1\leq p\leq q-1}|A_{q,p}|\frac{k_1^p}{\Gamma(\frac{d_{l,q}+k_1(q-p)}{k_1})}\frac{1}{(2\pi)^{1/2}}\frac{1}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}}C_2\boldsymbol{A}_{l,\epsilon_0}C_1M_{k_1,\delta_D}\epsilon_0^{d_{l,q}+k_1(q-p)}\Big]\Big]\\
&\quad+\mathbb{S}_{\mathcal{G}}\leq\frac{1}{2} \qquad (158)
\end{aligned}
$$

holds where

$$
\begin{aligned}
\mathbb{S}_{\mathcal{G}} &= \frac{1}{(2\pi)^{1/2}}\frac{1}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}}\max_{m\in\mathbb{R}}\Big|\frac{1}{R_D(\sqrt{-1}m)}\Big|C_2\boldsymbol{B}_{1,\epsilon_0}\\
&\quad+|c_{P_1,P_2}|\frac{1}{(2\pi)^{1/2}}\frac{C_3}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}}\varpi_2+|c_{P_3,P_4}|\frac{1}{(2\pi)^{1/2}}\frac{C_3}{C_P(r_{Q,R_D})^{\frac{1}{k_1\delta_D}}}2\varpi_1. \qquad (159)
\end{aligned}
$$

We notice that all the terms appearing in the left handside of (158) excluding $\mathbb{S}_{\mathcal{G}}$ can be dwindled since they involve positive powers of ϵ_0 according to the constraints (18). Furthermore, the term $\mathbb{S}_{\mathcal{G}}$ can be depleted whenever the constants $\boldsymbol{B}_{1,\epsilon_0} > 0$ and $c_{P_1P_2} \in \mathbb{C}^*$, $c_{P_3P_4} \in \mathbb{C}^*$ are taken close to 0.

In the end, the coupling of all the above bounds (150), (151), (152), (153), (154), (155), (156), (157) under the condition (158) triggers the shrinking feature (132) for the map $\mathcal{G}_{\epsilon}$.

In conclusion, we select the radius $\varpi_1 > 0$ and pinpoint the constants $\epsilon_0 > 0$, $\boldsymbol{B}_{j,\epsilon_0} > 0$, for $j = 1,2$, along with $c_{P_k,P_{k+1}} \in \mathbb{C}^*$, for $k = 1,3,5$ nearby the origin, in a way they obey both (148) and (158). These values taken for granted, the map $\mathcal{G}_{\epsilon}$ fulfills both inclusion and shrinking properties described in the items of Proposition 6. □

The oncoming proposition provides a solution to the second convolution equation (50) with (51) crafted in the Banach spaces displayed in Definition 6.

Proposition 7. *Consider an appropriate inner radius $r_{Q,R_D} > 0$ and aperture $\eta_{Q,R_D} > 0$ of the sector S_{Q,R_D} together with an unbounded sector S_{d_1} and radius $\rho > 0$ that respect the requirements of Lemma 5. Then, a radius $\epsilon_0 > 0$ along with constants $\boldsymbol{B}_{j,\epsilon_0} > 0$, for $j = 1,2$ and $c_{P_k,P_{k+1}} \in \mathbb{C}^*$, for $k = 1,3,5$ can be pinned down nearby 0 together with an appropriate radius $\varpi_1 > 0$ in a way that for all $\epsilon \in D_{\epsilon_0} \setminus \{0\}$, a unique solution $\omega_{1,d_1}(\tau,m,\epsilon)$ to (50), (51) exists that is favoured with the next features*

- *the map $(\tau,m) \mapsto \omega_{1,d_1}(\tau,m,\epsilon)$ belongs to $F^{d_1}_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$ under the restriction*

$$\sup_{\epsilon \in D_{\epsilon_0} \setminus \{0\}} ||\omega_{1,d_1}(\tau,m,\epsilon)||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \leq \varpi_1. \tag{160}$$

- *the partial map $\epsilon \mapsto \omega_{1,d_1}(\tau,m,\epsilon)$ stands for an analytic map from $D_{\epsilon_0} \setminus \{0\}$ into $\mathbb{C}$, for any prescribed $\tau \in S_{d_1} \cup D_\rho$ and $m \in \mathbb{R}$.*

Proof. Let the constants $\epsilon_0 > 0$, $\boldsymbol{B}_{j,\epsilon_0} > 0$, for $j = 1,2$ and $c_{P_k,P_{k+1}} \in \mathbb{C}^*$, for $k = 1,3,5$ together with $\varpi_1 > 0$ be fixed as in Proposition 6. The proposition 6 asserts that $\mathcal{G}_{\epsilon}$ induces a contractive map from the closed ball and complete space $\bar{B}_{\varpi_1}$ into itself for the distance $d(x,y) = ||x-y||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$ inherited from the norm on the Banach space $F^{d_1}_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$.

The classical Banach fixed point theorem then claims that $\mathcal{G}_{\epsilon}$ boasts a unique fixed point $\omega_{1,d_1}(\tau,m,\epsilon)$ inside the ball $\bar{B}_{\varpi_1}$, for all $\epsilon \in D_{\epsilon_0} \setminus \{0\}$. In other words,

$$\mathcal{G}_{\epsilon}(\omega_{1,d_1}(\tau,m,\epsilon)) = \omega_{1,d_1}(\tau,m,\epsilon) \tag{161}$$

holds. Furthermore, the map $\omega_{1,d_1}(\tau,m,\epsilon)$ depends analytically on ϵ since $\mathcal{G}_{\epsilon}$ itself does on the domain $D_{\epsilon_0} \setminus \{0\}$. On the other hand, we observe that the convolution equation (50) can be reorganized as the equation (161) by moving the term

$$(k_1\tau^{k_1})^{\delta_D} R_D(\sqrt{-1}m)\omega_{1,d_1}(\tau,m,\epsilon)$$

from the right to the left handside of (50) and dividing by the resulting equation by the map $P_m(\tau)$ given by (86). As a result, the unique fixed point $\omega_{1,d_1}(\tau,m,\epsilon)$ of $\mathcal{G}_{\epsilon}$ penned in $\bar{B}_{\varpi_1}$ precisely solves (50), (51). The result ensues. □

7. Building up a finite set of holomorphic solutions to the coupling of partial differential equations (36), (37)

7.1. Fourier-Laplace transforms solutions to the pairing (36), (37)

In this section, we exhibit genuine analytic solutions expressed by means of Fourier-Laplace transforms to the coupling (36), (37) reached at the end of Subsection 3.1.

Proposition 8. *For all unbounded sectors S_{d_1} with bisecting direction $d_1 \in \mathbb{R}$ and disc D_ρ that obey the demands of Lemma 5, we introduce the two partial maps*

$$(u_1, z) \mapsto U_{j,d_1}(u_1, z, \epsilon) = \frac{k_1}{(2\pi)^{1/2}} \int_{L_{d_{1,u_1}}} \int_{-\infty}^{+\infty} \omega_{j,d_1}(\tau, m, \epsilon) \exp\left(-\left(\frac{\tau}{u_1}\right)^{k_1}\right) e^{\sqrt{-1}zm} \frac{d\tau}{\tau} dm \quad (162)$$

for $j = 1, 2$, for all $\epsilon \in D_{\epsilon_0} \setminus \{0\}$ where the Borel map ω_{2,d_1} is manufactured in Proposition 5 and solves (49), the Borel map ω_{1,d_1} is crafted in Proposition 7 and fulfills (50), (51) and the radius $\epsilon_0 > 0$ is taken in agreement with Proposition 5 and Proposition 7 and $L_{d_{1,u_1}} = [0, +\infty)e^{\sqrt{-1}d_{1,u_1}}$ stands for a halfline in a direction $d_{1,u_1} \in \mathbb{R}$ suitably chosen and described below.

The maps $U_{j,d_1}(u_1, z, \epsilon)$, $j = 1, 2$, are endowed with the next two properties.

- *They define holomorphic functions that are bounded by a constant not relying on ϵ on a product $U_{1,d_1} \times H_{\beta'}$ where U_{1,d_1} represents a bounded open sector centered at 0 with bisecting direction d_1, for any given $0 < \beta' < \beta$.*
- *The map $U_{2,d_1}(u_1, z, \epsilon)$ solves the equation (36) for prescribed initial data $U_{2,d_1}(0, z, \epsilon) \equiv 0$. The map $U_{1,d_1}(u_1, z, \epsilon)$ is subjected to the equation (37) for vanishing data $U_{1,d_1}(0, z, \epsilon) \equiv 0$*

The sector U_{1,d_1} is submitted to the next technical constraints:

1. *A positive real number $\Delta_1 > 0$ can be singled out with the next property: for all $u_1 \in U_{1,d_1}$, a direction $d_{1,u_1} \in \mathbb{R}$ (that might rely on u_1) can be favoured with*

$$e^{\sqrt{-1}d_{1,u_1}} \in S_{d_1} \ , \ \cos(k_1(d_{1,u_1} - \arg(u_1))) > \Delta_1. \quad (163)$$

2. *The radius $r_{U_{1,d_1}} > 0$ of U_{1,d_1} withstands the next upper bounds*

$$0 < r_{U_{1,d_1}} < \Delta_1^{1/k_1} \frac{|\epsilon|}{(\nu + \tilde{\Delta}_1)^{1/k_1}} \quad (164)$$

for some positive real number $\tilde{\Delta}_1 > 0$, where $\Delta_1 > 0$ is defined in the above item.

Proof. We discuss the first item of the proposition. We mind the maps ω_{2,d_1} and ω_{1,d_1} constructed in Propositions 5 and 7 and we select a bounded sector U_{1,d_1} that matches the above prerequisite (163) and (164). We set $u_1 \in U_{1,d_1}$ and take

$$\tau = re^{\sqrt{-1}d_{1,u_1}} \in L_{d_{1,u_1}}$$

for given real number $r \geq 0$ where d_{1,u_1} is the direction chosen above. Then, then next two inequalities for the Borel maps hold.

- A constant $\varpi_2 > 0$ can be found for which the next bounds

$$\begin{aligned} &|\omega_{2,d_1}(\tau, m, \epsilon)| |\exp\left(-\left(\frac{\tau}{u_1}\right)^{k_1}\right)| |e^{\sqrt{-1}zm}| |\frac{1}{\tau}| \\ &\leq \varpi_2 (1 + |m|)^{-\mu} e^{-\beta|m|} \frac{1}{|\epsilon|} \exp\left(\nu\left(\frac{r}{|\epsilon|}\right)^{k_1}\right) \exp\left(-\left(\frac{r}{|u_1|}\right)^{k_1} \cos(k_1(d_{1,u_1} - \arg(u_1)))\right) e^{-m\mathrm{Im}(z)} \\ &\qquad \leq \varpi_2 (1 + |m|)^{-\mu} e^{-(\beta-\beta')|m|} \frac{1}{|\epsilon|} \exp\left(\nu\left(\frac{r}{|\epsilon|}\right)^{k_1}\right) \exp\left(-\left(\frac{r}{|u_1|}\right)^{k_1} \Delta_1\right) \\ &\qquad\qquad \leq \varpi_2 (1 + |m|)^{-\mu} e^{-(\beta-\beta')|m|} \frac{1}{|\epsilon|} \exp\left(-\left(\frac{\tilde{\Delta}_1}{|\epsilon|^{k_1}}\right) r^{k_1}\right) \end{aligned} \quad (165)$$

hold for all $r \geq 0$, all $m \in \mathbb{R}$.

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- Similarly, a constant $\varpi_1 > 0$ can be singled out with the bounds

$$|\omega_{1,d_1}(\tau,m,\epsilon)||\exp\left(-(\frac{\tau}{u_1})^{k_1}\right)||e^{\sqrt{-1}zm}||\frac{1}{\tau}| \leq \varpi_1(1+|m|)^{-\mu}e^{-(\beta-\beta')|m|}\frac{1}{|\epsilon|}\exp\left(-(\frac{\tilde{\Delta}_1}{|\epsilon|^{k_1}})r^{k_1}\right) \quad (166)$$

provided that $r \geq 0$ and $m \in \mathbb{R}$.

As a result, we reach the next two upper bounds for the maps U_{j,d_1}, $j = 1,2$. Namely,

$$|U_{2,d_1}(u_1,z,\epsilon)| \leq \frac{k_1\varpi_2}{(2\pi)^{1/2}}\int_0^{+\infty}\frac{1}{|\epsilon|}\exp\left(-(\frac{\tilde{\Delta}_1}{|\epsilon|^{k_1}})r^{k_1}\right)dr\int_{-\infty}^{+\infty}e^{-(\beta-\beta')|m|}dm \leq \frac{k_1\varpi_2}{(2\pi)^{1/2}}\int_0^{+\infty}\exp(-\tilde{\Delta}_1 r_1^{k_1})dr_1\int_{-\infty}^{+\infty}e^{-(\beta-\beta')|m|}dm \quad (167)$$

by means of the change of variable $r = |\epsilon|r_1$ in the integral together with

$$|U_{1,d_1}(u_1,z,\epsilon)| \leq \frac{k_1\varpi_1}{(2\pi)^{1/2}}\int_0^{+\infty}\exp(-\tilde{\Delta}_1 r_1^{k_1})dr_1\int_{-\infty}^{+\infty}e^{-(\beta-\beta')|m|}dm \quad (168)$$

for all $u_1 \in U_{1,d_1}$, $z \in H_{\beta'}$ and all $\epsilon \in D_{\epsilon_0} \setminus \{0\}$. We observe that the right handside of both (167) and (168) are unconstrained constants relatively to ϵ on $D_{\epsilon_0} \setminus \{0\}$. The first item ensues.

Concerning the second item, we remind from Proposition 5 (resp. Proposition 7) that the Borel map $\omega_{2,d_1}(\tau,m,\epsilon)$ (resp. $\omega_{1,d_1}(\tau,m,\epsilon)$) is shown to solve the associated convolution equation (49) (resp. (50), (51)). By tracking reversedly the computations made in Subsection 3.2, we deduce that for all $\epsilon \in D_{\epsilon_0} \setminus \{0\}$, the next properties hold.

- The holomorphic map $U_{2,d_1}(u_1,z,\epsilon)$ given by the expression (162) for $j = 2$ obeys the equation (47), then fulfills (43) and finally solves (36) on the domain $U_{1,d_1} \times H_{\beta'}$, for prescribed initial data $U_{2,d_1}(0,z,\epsilon) \equiv 0$.
- The holomorphic map $U_{1,d_1}(u_1,z,\epsilon)$ expressed in the form (162) for $j = 1$ conforms to the equation (48), then satisfies (44) and finally is subjected (37) on the domain $U_{1,d_1} \times H_{\beta'}$, for vanishing initial data $U_{1,d_1}(0,z,\epsilon) \equiv 0$.

The second item of Proposition 8 follows. □

7.2. Construction of a finite family of genuine solutions to the coupling (36), (37) and sharp bounds for the neighboring differences of related maps

We need to refer to the usual definition of a good covering in $\mathbb{C}^*$ given in the textbook [23].

Definition 7. *Let $\varsigma \geq 2$ be an integer. We consider a set $\underline{\mathcal{E}} = \{\mathcal{E}_p\}_{0 \leq p \leq \varsigma-1}$ of open bounded sectors $\mathcal{E}_p$ centered at 0 endowed with the next three properties*

1. *The intersection of two neighboring sectors $\mathcal{E}_p$ and $\mathcal{E}_{p+1}$ is not empty for any $0 \leq p \leq \varsigma - 1$, where the convention $\mathcal{E}_\varsigma = \mathcal{E}_0$ is chosen.*
2. *The intersection of any three sectors $\mathcal{E}_p$, $\mathcal{E}_q$ and $\mathcal{E}_r$ for distinct integers $p,q,r \in \{0,\dots,\varsigma-1\}$ is empty.*
3. *The union of all the sectors $\mathcal{E}_p$ is subjected to*

$$\bigcup_{p=0}^{\varsigma-1}\mathcal{E}_p = U \setminus \{0\}$$

for some neighborhood U of 0 in $\mathbb{C}$.

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Such a set $\underline{\mathcal{E}}$ is designated as a good covering in $\mathbb{C}^*$.

The next definition displays some domains in $\mathbb{C}$ which are crucially involved in the set up of genuine solutions.

Definition 8. *We consider two finite sets of bounded open sectors centered at 0,*

$$\underline{\mathcal{U}}_1 = \{U_{1,d_p}\}_{0 \leq p \leq \varsigma-1} \ , \ \underline{\mathcal{E}} = \{\mathcal{E}_p\}_{0 \leq p \leq \varsigma-1}$$

and a bounded sector $\mathcal{T}$ centered at 0, for which the next list of constraints is required.

1. *For each $0 \leq p \leq \varsigma - 1$ and fixed $\epsilon \in D_{\epsilon_0} \setminus \{0\}$, for some given radius $\epsilon_0 > 0$, the sector U_{1,d_p} has bisecting direction $d_p \in \mathbb{R}$ and obeys the next three rules*
   - *For each $0 \leq p \leq \varsigma - 1$, one can single out an unbounded sector S_{d_p} centered at 0 with bisecting direction d_p that is subjected to the requirements of Lemma 5 (namely for which the lower bounds (89) and (90) hold).*
   - *For each $0 \leq p \leq \varsigma - 1$, a positive real number $\Delta_p > 0$ can be selected in a way that for all $u_1 \in U_{1,d_p}$, a direction d_{p,u_1} (that might depend on u_1) can be found with*
   $$e^{\sqrt{-1}d_{p,u_1}} \in S_{d_p} \ , \ \cos(k_1(d_{p,u_1} - \arg(u_1))) > \Delta_p. \tag{169}$$
   - *The radius $r_{U_{1,d_p}} > 0$ of U_{1,d_p} is constrained to the next upper bounds*
   $$0 < r_{U_{1,d_p}} < \Delta_p^{1/k_1} \frac{|\epsilon|}{(\nu + \tilde{\Delta}_p)^{1/k_1}} \tag{170}$$
   for some positive real number $\tilde{\Delta}_p > 0$, where $\Delta_p > 0$ is determined in the above item.
2. *The radius $r_{\mathcal{T}} > 0$ of the sector $\mathcal{T}$ satisfies the restriction*
   $$r_{\mathcal{T}} < \frac{\Delta_p^{1/k_1}}{(\nu + \tilde{\Delta}_p)^{1/k_1}}$$
   where Δ_p, $\tilde{\Delta}_p$ are specified in 1. for $0 \leq p \leq \varsigma - 1$. Besides, the sectors $\mathcal{E}_p$ share a common radius given by ϵ_0, for $0 \leq p \leq \varsigma - 1$.
3. *For all $0 \leq p \leq \varsigma - 1$, the sectors $\mathcal{E}_p$ and $\mathcal{T}$ stick to the feature*
   $$\epsilon t \in U_{1,d_p}$$
   provided that $\epsilon \in \mathcal{E}_p$ and $t \in \mathcal{T}$.
4. *The set $\underline{\mathcal{E}}$ stands for a good covering in $\mathbb{C}^*$. Furthermore, the aperture of the sector $\mathcal{T}$ is taken nearby 0 in a way that the set*
   $$I_1 = \{p \in \{0, \ldots, \varsigma - 1\} / \epsilon t \notin (-\infty, 0], \ \text{for all} \ \ \epsilon \in \mathcal{E}_p, \text{all} \ \ t \in \mathcal{T}\}$$
   is not empty.

These sets $\underline{\mathcal{U}}_1$, $\underline{\mathcal{E}}$ and the sector $\mathcal{T}$ form a so-called fitting collection of sectors.

In the next proposition, we shape a finite family of analytic solutions to the coupling of auxiliary problems (36), (37).

Proposition 9. *We consider a fitting collection of sectors $\underline{\mathcal{U}}_1$, $\underline{\mathcal{E}}$ and $\mathcal{T}$ in the sense of Definition 8. The solutions to (36), (37) are cooked up as follows.*

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– *The equation (36) possesses a finite set of holomorphic solutions* $(u_1, z) \mapsto U_{2,d_p}(u_1, z, \epsilon)$*, for* $0 \leq p \leq \varsigma - 1$*, on the domain* $U_{1,d_p} \times H_{\beta'}$*, for all* $\epsilon \in D_{\epsilon_0} \setminus \{0\}$*, where* ϵ_0 *is proximate to 0, for any* $0 < \beta' < \beta$*, that fulfills the initial condition* $U_{2,d_p}(0, z, \epsilon) \equiv 0$*. These maps enjoy the next two qualities: for each* $0 \leq p \leq \varsigma - 1$*,*

1. *the map* $(u_1, z) \mapsto U_{2,d_p}(u_1, z, \epsilon)$ *is bounded by a constant unconstrained to* ϵ *in* $D_{\epsilon_0} \setminus \{0\}$*, on the product* $U_{1,d_p} \times H_{\beta'}$*.*
2. *the map* $U_{2,d_p}(u_1, z, \epsilon)$ *is represented as Fourier inverse and Laplace transforms,*

$$U_{2,d_p}(u_1, z, \epsilon) = \frac{k_1}{(2\pi)^{1/2}} \int_{L_{d_p,u_1}} \int_{-\infty}^{+\infty} \omega_{2,d_p}(\tau, m, \epsilon) \exp\left(-(\frac{\tau}{u_1})^{k_1}\right) e^{\sqrt{-1}zm} \frac{d\tau}{\tau} dm \quad (171)$$

where the Borel maps $(\tau, m) \mapsto \omega_{2,d_p}(\tau, m, \epsilon)$ *appertain to the Banach space* $F^{d_p}_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$ *and are subjected to*

$$\sup_{\epsilon \in D_{\epsilon_0} \setminus \{0\}} ||\omega_{2,d_p}(\tau, m, \epsilon)||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \leq \varpi_2 \quad (172)$$

for suitable constants $\varpi_2 > 0$ *and radius* $\rho > 0$*, for all* $\epsilon \in D_{\epsilon_0} \setminus \{0\}$*.*

– *The equation (37) (where the expression* $U_2(u_1, z, \epsilon)$ *needs to be replaced by* $U_{2,d_p}(u_1, z, \epsilon)$*) owns a finite set of holomorphic solutions* $(u_1, z) \mapsto U_{1,d_p}(u_1, z, \epsilon)$*, for* $0 \leq p \leq \varsigma - 1$*, on the domain* $U_{1,d_p} \times H_{\beta'}$*, for all* $\epsilon \in D_{\epsilon_0} \setminus \{0\}$*, where* ϵ_0 *is closed to 0, for any* $0 < \beta' < \beta$*, with the initial condition* $U_{1,d_p}(0, z, \epsilon) \equiv 0$*. These maps are endowed with the next two features: for each* $0 \leq p \leq \varsigma - 1$*,*

1. *the map* $(u_1, z) \mapsto U_{1,d_p}(u_1, z, \epsilon)$ *is bounded on the product* $U_{1,d_p} \times H_{\beta'}$ *by a constant not relying to* ϵ *in* $D_{\epsilon_0} \setminus \{0\}$*.*
2. *the map* $U_{1,d_p}(u_1, z, \epsilon)$ *is expressed by means of a Fourier inverse and Laplace transforms,*

$$U_{1,d_p}(u_1, z, \epsilon) = \frac{k_1}{(2\pi)^{1/2}} \int_{L_{d_p,u_1}} \int_{-\infty}^{+\infty} \omega_{1,d_p}(\tau, m, \epsilon) \exp\left(-(\frac{\tau}{u_1})^{k_1}\right) e^{\sqrt{-1}zm} \frac{d\tau}{\tau} dm \quad (173)$$

where the Borel maps $(\tau, m) \mapsto \omega_{1,d_p}(\tau, m, \epsilon)$ *are crafted in the Banach space* $F^{d_p}_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$ *with bounds*

$$\sup_{\epsilon \in D_{\epsilon_0} \setminus \{0\}} ||\omega_{1,d_p}(\tau, m, \epsilon)||_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \leq \varpi_1 \quad (174)$$

for appropriate constants $\varpi_1 > 0$ *and radius* $\rho > 0$*, for all* $\epsilon \in D_{\epsilon_0} \setminus \{0\}$*.*

Proof. The proposition 9 is a downright consequence of Proposition 8 and of the very definition of fitting collections of sectors depicted in Definition 8. □

In the next proposition we examine a finite set of maps related to the analytic solutions of the coupling (36), (37). In particular, we obtain a control on their consecutive differences which turns out to be an essential information in the study of their parametric asymptotic expansions.

Proposition 10. *Let us prescribe a fitting collection of sectors* $\underline{\mathcal{U}}_1$*,* $\underline{\mathcal{E}}$ *and* $\mathcal{T}$ *in accordance with Definition 8. For each* $0 \leq p \leq \varsigma - 1$*, we set up the maps*

$$u_{j,p}(t, z, \epsilon) = U_{j,d_p}(\epsilon t, z, \epsilon) \quad (175)$$

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for $j = 1,2$, where U_{j,d_p} are described in Proposition 9. The next attributes hold: for all $0 \leq p \leq \varsigma - 1$,

- *the maps $u_{j,p}(t,z,\epsilon)$, $j = 1,2$, are bounded holomorphic on the product $\mathcal{T} \times H_{\beta'} \times \mathcal{E}_p$ and satisfy $u_{j,p}(0,z,\epsilon) \equiv 0$,*
- *one can exhibit constants $M_{p,j} > 0$ and $K_{p,j} > 0$ such that*

$$|u_{j,p+1}(t,z,\epsilon) - u_{j,p}(t,z,\epsilon)| \leq M_{p,j} \exp\big(-\frac{K_{p,j}}{|\epsilon|^{k_1}}\big) \tag{176}$$

for all $t \in \mathcal{T}$, all $\epsilon \in \mathcal{E}_{p+1} \cap \mathcal{E}_p$, all $z \in H_{\beta'}$, for $j = 1,2$, where we adopt the convention $u_{j,\varsigma} = u_{j,0}$.

Proof. The first item is a direct outcome of the properties of the maps U_{j,d_p}, $j = 1,2$, stated in Proposition 9 and from the characteristics 2. and 3. of the sectors $\mathcal{E}_p$ and $\mathcal{T}$ listed in Definition 8.

The second item follows from a path deformation argument. Indeed, let us take $p \in \{0,\dots,\varsigma-1\}$ and $j \in \{1,2\}$. For any given $m \in \mathbb{R}$ and fixed $\epsilon \in D_{\epsilon_0} \setminus \{0\}$, the partial maps $\tau \mapsto \omega_{j,d_k}(\tau,m,\epsilon)$, $k = p, p+1$, represent analytic continuation on the sector S_{d_k} of a common analytic map we denote $\tau \mapsto \omega_j(\tau,m,\epsilon)$ on the disc D_ρ.

For any prescribed $\epsilon \in \mathcal{E}_{p+1} \cap \mathcal{E}_p$ and $t \in \mathcal{T}$, we deform the oriented path $L_{d_{p+1,\epsilon t}} - L_{d_{p,\epsilon t}}$ into the union of three oriented curves

- Two halflines

$$L_{d_{p+1,\epsilon t};\rho/2} = [\rho/2,+\infty)e^{\sqrt{-1}d_{p+1,\epsilon t}} \ , \ -L_{d_{p,\epsilon t};\rho/2} = -[\rho/2,+\infty)e^{\sqrt{-1}d_{p,\epsilon t}}.$$

- An arc of circle

$$C_{p,p+1,\epsilon t;\rho/2} = \{\frac{\rho}{2}e^{\sqrt{-1}\theta}/\theta \in (d_{p,\epsilon t}, d_{p+1,\epsilon t})\}$$

  centered at 0 with radius $\rho/2$ that connects the above two halflines.

Then, the classical Cauchy's theorem enables us to reshape the next difference into a sum of three contributions

$$\begin{aligned} u_{j,p+1}&(t,z,\epsilon) - u_{j,p}(t,z,\epsilon) \\ &= \frac{k_1}{(2\pi)^{1/2}} \int_{L_{d_{p+1,\epsilon t};\rho/2}} \int_{-\infty}^{+\infty} \omega_{j,d_{p+1}}(\tau,m,\epsilon) \exp\big(-(\frac{\tau}{\epsilon t})^{k_1}\big) e^{\sqrt{-1}zm} \frac{d\tau}{\tau} dm \\ &\quad - \frac{k_1}{(2\pi)^{1/2}} \int_{L_{d_{p,\epsilon t};\rho/2}} \int_{-\infty}^{+\infty} \omega_{j,d_p}(\tau,m,\epsilon) \exp\big(-(\frac{\tau}{\epsilon t})^{k_1}\big) e^{\sqrt{-1}zm} \frac{d\tau}{\tau} dm \\ &\quad + \frac{k_1}{(2\pi)^{1/2}} \int_{C_{p,p+1,\epsilon t;\rho/2}} \int_{-\infty}^{+\infty} \omega_j(\tau,m,\epsilon) \exp\big(-(\frac{\tau}{\epsilon t})^{k_1}\big) e^{\sqrt{-1}zm} \frac{d\tau}{\tau} dm. \end{aligned} \tag{177}$$

We provide upper bounds for the first piece of (177)

$$I_1 = \Big|\frac{k_1}{(2\pi)^{1/2}} \int_{L_{d_{p+1,\epsilon t};\rho/2}} \int_{-\infty}^{+\infty} \omega_{j,d_{p+1}}(\tau,m,\epsilon) \exp\big(-(\frac{\tau}{\epsilon t})^{k_1}\big) e^{\sqrt{-1}zm} \frac{d\tau}{\tau} dm\Big|.$$

Based on the bounds (165), (166) and (172), (174) together with the requirements asked in Definition 8, we arrive at

$$I_1 \leq \frac{\varpi_j k_1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} e^{-(\beta-\beta')|m|} dm \times \int_{\rho/2}^{+\infty} \frac{1}{|\epsilon|} \exp\Big(- \frac{\tilde{\Delta}_{p+1}}{|\epsilon|^{k_1}} r^{k_1} \Big) dr$$
$$\leq \frac{2\varpi_j k_1}{(2\pi)^{1/2}} \int_{0}^{+\infty} e^{-(\beta-\beta')m} dm \times \int_{\rho/2}^{+\infty} \frac{1}{|\epsilon|} \Big\{ \frac{|\epsilon|^{k_1}}{\tilde{\Delta}_{p+1}} \frac{1}{k_1 r^{k_1-1}} \Big\} \Big\{ \frac{\tilde{\Delta}_{p+1}}{|\epsilon|^{k_1}} k_1 r^{k_1-1} \exp\Big(- \frac{\tilde{\Delta}_{p+1}}{|\epsilon|^{k_1}} r^{k_1} \Big) \Big\} dr$$
$$\leq \frac{2\varpi_j k_1}{(2\pi)^{1/2}} \frac{1}{\beta-\beta'} \frac{|\epsilon|^{k_1-1}}{\tilde{\Delta}_{p+1}} \frac{1}{k_1(\rho/2)^{k_1-1}} \exp\Big(- \frac{\tilde{\Delta}_{p+1}}{|\epsilon|^{k_1}} (\rho/2)^{k_1} \Big) \quad (178)$$

provided that $\epsilon \in \mathcal{E}_{p+1} \cap \mathcal{E}_p$, $t \in \mathcal{T}$ and $z \in H_{\beta'}$.

In the same vein, we can get upper bounds for the second piece

$$I_2 = \Big| \frac{k_1}{(2\pi)^{1/2}} \int_{L_{d_{p,\epsilon t};\rho/2}} \int_{-\infty}^{+\infty} \omega_{j,d_p}(\tau,m,\epsilon) \exp\Big(- (\frac{\tau}{\epsilon t})^{k_1} \Big) e^{\sqrt{-1}zm} \frac{d\tau}{\tau} dm \Big|$$

of (177). Namely,

$$|I_2| \leq \frac{2\varpi_j k_1}{(2\pi)^{1/2}} \frac{1}{\beta-\beta'} \frac{|\epsilon|^{k_1-1}}{\tilde{\Delta}_p} \frac{1}{k_1(\rho/2)^{k_1-1}} \exp\Big(- \frac{\tilde{\Delta}_p}{|\epsilon|^{k_1}} (\rho/2)^{k_1} \Big) \quad (179)$$

for all $\epsilon \in \mathcal{E}_{p+1} \cap \mathcal{E}_p$, $t \in \mathcal{T}$ and $z \in H_{\beta'}$.

At last, we handle the integral along the arc of circle closing (177),

$$I_3 = \Big| \frac{k_1}{(2\pi)^{1/2}} \int_{C_{p,p+1,\epsilon t;\rho/2}} \int_{-\infty}^{+\infty} \omega_j(\tau,m,\epsilon) \exp\Big(- (\frac{\tau}{\epsilon t})^{k_1} \Big) e^{\sqrt{-1}zm} \frac{d\tau}{\tau} dm \Big|.$$

Owing to the bounds (172) and (174), we observe that

$$|\omega_j(\tau,m,\epsilon)| \leq \varpi_j (1+|m|)^{-\mu} e^{-\beta|m|} \frac{\rho/2}{|\epsilon|} \exp\Big(\nu \frac{(\rho/2)^{k_1}}{|\epsilon|^{k_1}} \Big) \quad (180)$$

as long as $\tau \in C_{p,p+1,\epsilon t;\rho/2}$, $m \in \mathbb{R}$ and $\epsilon \in \mathcal{E}_{p+1} \cap \mathcal{E}_p$. Besides, in view of the restrictions discussed in Definition 8 1. it follows that

$$\cos(k_1(\theta - \arg(\epsilon t))) > \Delta_{p,p+1} = \min(\Delta_p, \Delta_{p+1}) \quad (181)$$

for all $t \in \mathcal{T}$, $\epsilon \in \mathcal{E}_{p+1} \cap \mathcal{E}_p$, granted that the angle θ belongs to $(d_{p,\epsilon t}, d_{p+1,\epsilon t})$ or $(d_{p+1,\epsilon t}, d_{p,\epsilon t})$. By virtue of (180) and (181), we come up with some constant $\tilde{\Delta}_{p,p+1} > 0$ with

$$I_3 \leq \frac{k_1 \varpi_j}{(2\pi)^{1/2}} \Big(\int_{-\infty}^{+\infty} e^{-(\beta-\beta')|m|} dm \Big)$$
$$\times \Big| \int_{d_{p,\epsilon t}}^{d_{p+1,\epsilon t}} \frac{1}{|\epsilon|} \exp\Big(\nu \frac{(\rho/2)^{k_1}}{|\epsilon|^{k_1}} \Big) \exp\Big(- \frac{(\rho/2)^{k_1}}{|\epsilon t|^{k_1}} \Delta_{p,p+1} \Big) \frac{\rho}{2} d\theta \Big|$$
$$\leq \frac{2k_1 \varpi_j}{(2\pi)^{1/2}(\beta-\beta')} |d_{p+1,\epsilon t} - d_{p,\epsilon t}| \frac{1}{|\epsilon|} \exp\Big(- \frac{\tilde{\Delta}_{p,p+1}}{|\epsilon|^{k_1}} (\rho/2)^{k_1} \Big) \frac{\rho}{2} \quad (182)$$

 doi:10.20944/preprints202301.0582.v1

contingent upon $t \in \mathcal{T}$, $\epsilon \in \mathcal{E}_{p+1} \cap \mathcal{E}_p$ and $z \in H_{\beta'}$. Hence, we deduce that

$$I_3 \leq \frac{2k_1\varpi_j}{(2\pi)^{1/2}(\beta-\beta')}|d_{p+1,\epsilon t} - d_{p,\epsilon t}|\frac{\rho}{2}\frac{1}{|\epsilon|}\exp\big(-\frac{\tilde{\Delta}_{p,p+1}}{2|\epsilon|^{k_1}}(\rho/2)^{k_1}\big)\exp\big(-\frac{\tilde{\Delta}_{p,p+1}}{2|\epsilon|^{k_1}}(\rho/2)^{k_1}\big)$$
$$\leq \frac{2k_1\varpi_j}{(2\pi)^{1/2}(\beta-\beta')}|d_{p+1,\epsilon t} - d_{p,\epsilon t}|\frac{\rho}{2}\mathcal{C}_{k_1,\rho,\tilde{\Delta}_{p,p+1}}\exp\big(-\frac{\tilde{\Delta}_{p,p+1}}{2|\epsilon|^{k_1}}(\rho/2)^{k_1}\big) \quad (183)$$

holds, where

$$\mathcal{C}_{k_1,\rho,\tilde{\Delta}_{p,p+1}} = \sup_{x\geq 0} x\exp\big(-\frac{\tilde{\Delta}_{p,p+1}}{2}(\rho/2)^{k_1}x^{k_1}\big)$$

as long as $\epsilon \in \mathcal{E}_{p+1} \cap \mathcal{E}_p$, $t \in \mathcal{T}$ and $z \in H_{\beta'}$.

In summary, the splitting (177) along with the bounds (178), (179) and (183) beget the awaited estimates (176). □

8. Main statement of the paper. Construction of a finite set of holomorphic solutions to the leading problem (14). Description of their parametric asymptotic expansion

8.1. Parametric Gevrey asymptotic expansions of the associated maps (175)

We first call to mind a result known as the Ramis-Sibuya theorem, see Lemma XI-2-6 in [23].

Theorem (R.S.) *Let $\{\mathcal{E}_p\}_{0\leq p\leq \varsigma-1}$ be a good covering in $\mathbb{C}^*$ be fixed as described in Definition 7. We denote $(\mathbb{F}, ||.||_{\mathbb{F}})$ a Banach space over $\mathbb{C}$. For all $0 \leq p \leq \varsigma - 1$, we set $G_p : \mathcal{E}_p \to \mathbb{F}$ as holomorphic functions that obey the next requirements*

1. *The maps G_p are bounded on $\mathcal{E}_p$ for all $0 \leq p \leq \varsigma - 1$.*
2. *The difference $\Theta_p(\epsilon) = G_{p+1}(\epsilon) - G_p(\epsilon)$ defines a holomorphic map on the intersection $Z_p = \mathcal{E}_{p+1} \cap \mathcal{E}_p$ which is exponentially flat of order k_1, for some integer $k_1 \geq 1$, meaning that one can select two constants $C_p, A_p > 0$ for which*
$$||\Theta_p(\epsilon)||_{\mathbb{F}} \leq C_p \exp(-\frac{A_p}{|\epsilon|^{k_1}})$$
holds provided that $\epsilon \in Z_p$, for all $0 \leq p \leq \varsigma - 1$. By convention, we set $G_\varsigma = G_0$ and $\mathcal{E}_\varsigma = \mathcal{E}_0$.

Then, one can find a formal power series $\hat{G}(\epsilon) = \sum_{n\geq 0} G_n\epsilon^n$ with coefficients G_n belonging to $\mathbb{F}$, which is the common Gevrey asymptotic expansion of order $1/k_1$ relatively to ϵ on $\mathcal{E}_p$ for all the maps G_p, for $0 \leq p \leq \varsigma - 1$. It means that two constants $K_p, M_p > 0$ can be singled out with the error bounds

$$||G_p(\epsilon) - \sum_{n=0}^{N} G_n\epsilon^n||_{\mathbb{F}} \leq K_p M_p^{N+1}\Gamma(1+\frac{N+1}{k_1})|\epsilon|^{N+1} \quad (184)$$

for all integers $N \geq 0$, all $\epsilon \in \mathcal{E}_p$, all $0 \leq p \leq \varsigma - 1$.

In the next proposition we exhibit asymptotic expansions of Gevrey type for the two sets of related maps introduced in Proposition 10, $\{u_{j,p}(t,z,\epsilon)\}_{0\leq p\leq\varsigma-1}$, $j = 1,2$, relatively to the variable ϵ.

Proposition 11. *We denote $\mathbb{F}_{\beta',\mathcal{T}}$ the Banach space of bounded holomorphic functions on the product $\mathcal{T} \times H_{\beta'}$ which are $\mathbb{C}-$valued, equipped with the sup norm. Then, for $j = 1,2$, a formal power series*

$$\hat{\mathbb{G}}_j(\epsilon) = \sum_{n\geq 0}\mathbb{G}_{n,j}(t,z)\frac{\epsilon^n}{n!} \quad (185)$$

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with coefficients $\mathbb{G}_{n,j}(t,z)$, $n \geq 0$, in $\mathbb{F}_{\beta',\mathcal{T}}$ can be shaped that obey the next error bounds. For all $0 \leq p \leq \varsigma - 1$, two constants $K_{p,j} > 0$ and $M_{p,j} > 0$ can be chosen with

$$\sup_{\substack{t\in\mathcal{T} \\ z\in H_{\beta'}}} |u_{j,p}(t,z,\epsilon) - \sum_{n=0}^{N} \mathbb{G}_{n,j}(t,z)\frac{\epsilon^n}{n!}| \leq K_{p,j}(M_{p,j})^{N+1}\Gamma(1+\frac{N+1}{k_1})|\epsilon|^{N+1} \tag{186}$$

for all integers $N \geq 0$, all $\epsilon \in \mathcal{E}_p$.

Proof. Let $j = 1,2$. For all $0 \leq p \leq \varsigma - 1$, let us define the maps $G_{j,p} : \mathcal{E}_p \to \mathbb{F}_{\beta',\mathcal{T}}$ by the expression $G_{j,p}(\epsilon) := (t,z) \mapsto u_{j,p}(t,z,\epsilon)$. For $0 \leq p \leq \varsigma - 1$, these functions share the next two features:

- The maps $G_{j,p}$ are bounded holomorphic on the sector $\mathcal{E}_p$, according to the first item of Proposition 10.
- The differences $\Theta_{j,p}(\epsilon) := G_{j,p+1}(\epsilon) - G_{j,p}(\epsilon)$ are submitted to the bounds

$$||\Theta_{j,p}(\epsilon)||_{\mathbb{F}_{\beta',\mathcal{T}}} \leq M_{p,j} \exp\big(-\frac{K_{p,j}}{|\epsilon|^{k_1}}\big)$$

  for the constants $M_{p,j} > 0$ and $K_{p,j} > 0$ obtained in Proposition 10, whenever $\epsilon \in \mathcal{E}_{p+1} \cap \mathcal{E}_p$, where the convention $G_{j,\varsigma} = G_{j,0}$ and $\mathcal{E}_\varsigma = \mathcal{E}_0$ is in use.

As a result, the requirements 1. and 2. of the Theorem (R.S.) are matched for the sets of maps $\{G_{j,p}\}_{0\leq p\leq\varsigma-1}$, $j = 1,2$. We deduce the existence of formal series $\hat{\mathbb{G}}_j(\epsilon)$, $j = 1,2$, which are the Gevrey asymptotic expansion of order $1/k_1$ relatively to ϵ on $\mathcal{E}_p$ shared by all the maps $G_{j,p}$ for $0 \leq p \leq \varsigma - 1$. This is tantamount to the statement of Proposition 11 and the awaited bounds (186). □

8.2. Statement of the main result

The next statement stands for the pinnacle of our work.

Theorem 1. *Let us prescribe a fitting collection of sectors $\underline{\mathcal{U}}_1$, $\underline{\mathcal{E}}$ and $\mathcal{T}$ accordingly to Definition 8. We take for granted that all the conditions (15), (16), (17), (18), (19), (20), (22), (23), (24), (25), (26) and (27) enumerated in Subsection 2.3 are fulfilled.*

Then, provided that the constants $\epsilon_0 > 0$ and $\boldsymbol{C}_{1,\epsilon_0} > 0$, $\boldsymbol{B}_{j,\epsilon_0} > 0$, $j = 1,2$, along with $c_{Q_1Q_2} \in \mathbb{C}^*$ and $c_{P_jP_{j+1}} \in \mathbb{C}^*$, $j = 1,3,5$ are nonvanishing and taken proximate to 0, the main equation

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$$
\begin{aligned}
Q(\partial_z)u(t,z,\epsilon) &= (\epsilon t)^{d_D}(t\partial_t)^{\delta_D}R_D(\partial_z)u(t,z,\epsilon) + \sum_{l=1}^{D-1}\epsilon^{\Delta_l}t^{d_l}a_l(z,\epsilon)(t\partial_t)^{\delta_l}R_l(\partial_z)u(t,z,\epsilon)\\
&+ f(t,z,\epsilon) + c_1(z,\epsilon)\big[\frac{1}{2\sqrt{-1}\pi}(\gamma_\epsilon^* - \mathrm{id})u(t,z,\epsilon)\big]\log(\epsilon t)\\
&+ b_1(z,\epsilon)\Big[u(t,z,\epsilon) - \big[\frac{1}{2\sqrt{-1}\pi}(\gamma_\epsilon^* - \mathrm{id})u(t,z,\epsilon)\big]\log(\epsilon t)\Big] + b_2(z,\epsilon)\frac{1}{2\sqrt{-1}\pi}(\gamma_\epsilon^* - \mathrm{id})u(t,z,\epsilon)\\
&+ c_{Q_1Q_2}Q_1(\partial_z)\big[\frac{1}{2\sqrt{-1}\pi}(\gamma_\epsilon^* - \mathrm{id})u(t,z,\epsilon)\big] \times Q_2(\partial_z)\big[\frac{1}{2\sqrt{-1}\pi}(\gamma_\epsilon^* - \mathrm{id})u(t,z,\epsilon)\big] \times \log(\epsilon t)\\
&+ c_{P_1P_2}P_1(\partial_z)\Big[u(t,z,\epsilon) - \big[\frac{1}{2\sqrt{-1}\pi}(\gamma_\epsilon^* - \mathrm{id})u(t,z,\epsilon)\big]\log(\epsilon t)\Big]\\
&\times P_2(\partial_z)\big[\frac{1}{2\sqrt{-1}\pi}(\gamma_\epsilon^* - \mathrm{id})u(t,z,\epsilon)\big]\\
&+ c_{P_3P_4}P_3(\partial_z)\Big[u(t,z,\epsilon) - \big[\frac{1}{2\sqrt{-1}\pi}(\gamma_\epsilon^* - \mathrm{id})u(t,z,\epsilon)\big]\log(\epsilon t)\Big]\\
&\times P_4(\partial_z)\Big[u(t,z,\epsilon) - \big[\frac{1}{2\sqrt{-1}\pi}(\gamma_\epsilon^* - \mathrm{id})u(t,z,\epsilon)\big]\log(\epsilon t)\Big]\\
&+ c_{P_5P_6}P_5(\partial_z)\big[\frac{1}{2\sqrt{-1}\pi}(\gamma_\epsilon^* - \mathrm{id})u(t,z,\epsilon)\big] \times P_6(\partial_z)\big[\frac{1}{2\sqrt{-1}\pi}(\gamma_\epsilon^* - \mathrm{id})u(t,z,\epsilon)\big] \quad (187)
\end{aligned}
$$

with vanishing initial data

$$u(0,z,\epsilon) \equiv 0 \quad (188)$$

possesses a finite set of bounded holomorphic solutions $(t,z,\epsilon) \mapsto u_p(t,z,\epsilon)$, for all $p \in I_1$, where I_1 is the subset of $\{0,\ldots,\varsigma-1\}$ introduced in the item 4. of Definition 8, on the domain $\mathcal{T} \times H_{\beta'} \times \mathcal{E}_p$. In the equation (187), the formal monodromy operator around 0, γ_ϵ^* acts on the analytic map $\epsilon \mapsto u_p(t,z,\epsilon)$ through Definition 5 by use of (11). The next additional features hold.

- *For each $p \in I_1$, the solution u_p can be expressed by means of a Fourier/Laplace transform*

$$u_p(t,z,\epsilon) = u_{1,p}(t,z,\epsilon) + u_{2,p}(t,z,\epsilon)\log(\epsilon t) \quad (189)$$

where

$$u_{j,p}(t,z,\epsilon) = \frac{k_1}{(2\pi)^{1/2}}\int_{L_{d_p,\epsilon t;\rho/2}}\int_{-\infty}^{+\infty}\omega_{j,d_p}(\tau,m,\epsilon)\exp\big(-(\frac{\tau}{\epsilon t})^{k_1}\big)e^{\sqrt{-1}zm}\frac{d\tau}{\tau}dm \quad (190)$$

for Borel maps $(\tau,m) \mapsto \omega_{j,d_p}(\tau,m,\epsilon)$, $j=1,2$, that belong to the Banach space $F^{d_p}_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$ under the restrictions (172) and (174).
- *The two components $u_{j,p}(t,z,\epsilon)$, $j=1,2$, of $u_p(t,z,\epsilon)$ are endowed with Gevrey asymptotic expansions $\hat{\mathbb{G}}_j(\epsilon)$ given by (185) of order $1/k_1$ relatively to ϵ on $\mathcal{E}_p$ displayed in (186).*
- *If one sets the formal expression*

$$\hat{\mathbb{G}}(\epsilon) = \hat{\mathbb{G}}_1(\epsilon) + \hat{\mathbb{G}}_2(\epsilon)\log(\epsilon t), \quad (191)$$

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then, $\hat{\mathbb{G}}(\epsilon)$ conforms to the next equation

$$
\begin{aligned}
Q(\partial_z)\hat{\mathbb{G}}(\epsilon) &= (\epsilon t)^{d_D}(t\partial_t)^{\delta_D}R_D(\partial_z)\hat{\mathbb{G}}(\epsilon) + \sum_{l=1}^{D-1}\epsilon^{\Delta_l}t^{d_l}a_l(z,\epsilon)(t\partial_t)^{\delta_l}R_l(\partial_z)\hat{\mathbb{G}}(\epsilon)\\
&+ f(t,z,\epsilon) + c_1(z,\epsilon)\big[\frac{1}{2\sqrt{-1}\pi}(\gamma_\epsilon^* - \mathrm{id})\hat{\mathbb{G}}(\epsilon)\big]\log(\epsilon t)\\
&+ b_1(z,\epsilon)\Big[\hat{\mathbb{G}}(\epsilon) - \big[\frac{1}{2\sqrt{-1}\pi}(\gamma_\epsilon^* - \mathrm{id})\hat{\mathbb{G}}(\epsilon)\big]\log(\epsilon t)\Big] + b_2(z,\epsilon)\frac{1}{2\sqrt{-1}\pi}(\gamma_\epsilon^* - \mathrm{id})\hat{\mathbb{G}}(\epsilon)\\
&+ c_{Q_1Q_2}Q_1(\partial_z)\big[\frac{1}{2\sqrt{-1}\pi}(\gamma_\epsilon^* - \mathrm{id})\hat{\mathbb{G}}(\epsilon)\big] \times Q_2(\partial_z)\big[\frac{1}{2\sqrt{-1}\pi}(\gamma_\epsilon^* - \mathrm{id})\hat{\mathbb{G}}(\epsilon)\big] \times \log(\epsilon t)\\
&+ c_{P_1P_2}P_1(\partial_z)\Big[\hat{\mathbb{G}}(\epsilon) - \big[\frac{1}{2\sqrt{-1}\pi}(\gamma_\epsilon^* - \mathrm{id})\hat{\mathbb{G}}(\epsilon)\big]\log(\epsilon t)\Big]\\
&\times P_2(\partial_z)\big[\frac{1}{2\sqrt{-1}\pi}(\gamma_\epsilon^* - \mathrm{id})\hat{\mathbb{G}}(\epsilon)\big]\\
&+ c_{P_3P_4}P_3(\partial_z)\Big[\hat{\mathbb{G}}(\epsilon) - \big[\frac{1}{2\sqrt{-1}\pi}(\gamma_\epsilon^* - \mathrm{id})\hat{\mathbb{G}}(\epsilon)\big]\log(\epsilon t)\Big]\\
&\times P_4(\partial_z)\Big[\hat{\mathbb{G}}(\epsilon) - \big[\frac{1}{2\sqrt{-1}\pi}(\gamma_\epsilon^* - \mathrm{id})\hat{\mathbb{G}}(\epsilon)\big]\log(\epsilon t)\Big]\\
&+ c_{P_5P_6}P_5(\partial_z)\big[\frac{1}{2\sqrt{-1}\pi}(\gamma_\epsilon^* - \mathrm{id})\hat{\mathbb{G}}(\epsilon)\big] \times P_6(\partial_z)\big[\frac{1}{2\sqrt{-1}\pi}(\gamma_\epsilon^* - \mathrm{id})\hat{\mathbb{G}}(\epsilon)\big] \quad (192)
\end{aligned}
$$

where the formal monodromy operator around 0, γ_ϵ^* acts on the formal expression $\epsilon \mapsto \hat{\mathbb{G}}(\epsilon)$ by means of the formula (9) from Definition 4.

Proof. For all $p \in I_1$, where I_1 is the set described in the item 4. of Definition 8, we define

$$u_p(t,z,\epsilon) = u_{1,p}(t,z,\epsilon) + u_{2,p}(t,z,\epsilon)\log(\epsilon t)$$

where the maps $u_{j,p}$ are introduced in (175) of Proposition 10.

As a result of the definition of I_1 together with the first item of Proposition 10 and the classical limit $\lim_{x\to 0} x^\alpha \log(x) = 0$, for any natural number $\alpha \geq 1$, we check that the map $u_p(t,z,\epsilon)$ represents a bounded holomorphic function on the product $\mathcal{T} \times H_{\beta'} \times \mathcal{E}_p$ that vanishes at $t = 0$, meaning that $u_p(0,z,\epsilon) \equiv 0$ for all $z \in H_{\beta'}$ and $\epsilon \in \mathcal{E}_p$.

According to Proposition 9, we know that for each $\epsilon \in D_{\epsilon_0} \setminus \{0\}$,

- the map $(u_1,z) \mapsto U_{2,d_p}(u_1,z,\epsilon)$ stands for a solution of the equation (36) on the domain $U_{1,d_p} \times H_{\beta'}$,
- the map $(u_1,z) \mapsto U_{1,d_p}(u_1,z,\epsilon)$ embodies a solution of (37) where the expression $U_2(u_1,z,\epsilon)$ is asked to be replaced by $U_{2,d_p}(u_1,z,\epsilon)$ on the domain $U_{1,d_p} \times H_{\beta'}$.

Then, on the basis of the computations (35), (34) and (33) performed reversedly from Subsection 3.1, we deduce that $u_p(t,z,\epsilon)$ solves the main equation (14), rephrased as (187), on the domain $\mathcal{T} \times H_{\beta'} \times \mathcal{E}_p$, for all $p \in I_1$.

The first item of Theorem 1 follows from the Fourier/Laplace representation of the maps $U_{j,d_p}(u_1,z,\epsilon)$, $j = 1,2$, displayed in Proposition 9 that are used to define the components $u_{j,p}(t,z,\epsilon)$ in (175).

The second item of Theorem 1 merely restates the result obtained in Proposition 11.

We focus on the third item. We first need to disclose partial differential equations that the maps $u_{j,p}(t,z,\epsilon)$, $j = 1,2$ turn out to fulfill. Indeed, the usual chain rule enables the next computation

$$t\partial_t u_{j,p}(t,z,\epsilon) = (u_1\partial_{u_1}U_{j,d_p})(\epsilon t,z,\epsilon)$$

for all $0 \leq p \leq \varsigma - 1$, $j = 1,2$, provided that $t \in \mathcal{T}$, $\epsilon \in \mathcal{E}_p$ and $z \in H_{\beta'}$. According to the statement discussed in Proposition 9, that the partial map $(u_1, z) \mapsto U_{2,d_p}(u_1, z, \epsilon)$ matches the equation (36) on the domain $U_{1,d_p} \times H_{\beta'}$, whenever $\epsilon \in D_{\epsilon_0} \setminus \{0\}$, we observe that the map $u_{2,p}(t,z,\epsilon)$ satisfies the next equation

$$Q(\partial_z)u_{2,p}(t,z,\epsilon) = (\epsilon t)^{d_D}\left[(t\partial_t)^{\delta_D} R_D(\partial_z)u_{2,p}(t,z,\epsilon)\right] + \sum_{l=1}^{D-1} \epsilon^{\Delta_l} t^{d_l} a_l(z,\epsilon)(t\partial_t)^{\delta_l} R_l(\partial_z)u_{2,p}(t,z,\epsilon) + F_2(\epsilon t,z,\epsilon) + c_1(z,\epsilon)u_{2,p}(t,z,\epsilon) + c_{Q_1,Q_2}\left[Q_1(\partial_z)u_{2,p}(t,z,\epsilon))\right] \times \left[Q_2(\partial_z)u_{2,p}(t,z,\epsilon)\right] \quad (193)$$

as long as $t \in \mathcal{T}$, $z \in H_{\beta'}$ and $\epsilon \in \mathcal{E}_p$. On the other hand, since the partial map $(u_1, z) \mapsto U_{1,d_p}(u_1, z, \epsilon)$ obeys the equation (37) on the domain $U_{1,d_p} \times H_{\beta'}$, for $\epsilon \in D_{\epsilon_0} \setminus \{0\}$, it follows that the map $u_{1,p}(t,z,\epsilon)$ fulfills the next equation coupled to (193),

$$Q(\partial_z)u_{1,p}(t,z,\epsilon) = (\epsilon t)^{d_D}\left[(t\partial_t)^{\delta_D} R_D(\partial_z)u_{1,p}(t,z,\epsilon) + \delta_D (t\partial_t)^{\delta_D - 1} R_D(\partial_z)u_{2,p}(t,z,\epsilon)\right] + \sum_{l=1}^{D-1} \epsilon^{\Delta_l} t^{d_l} a_l(z,\epsilon)\left[(t\partial_t)^{\delta_l} R_l(\partial_z)u_{1,p}(t,z,\epsilon) + \delta_l (t\partial_t)^{\delta_l - 1} R_l(\partial_z)u_{2,p}(t,z,\epsilon)\right] + F_1(\epsilon t,z,\epsilon) + b_1(z,\epsilon)u_{1,p}(t,z,\epsilon) + b_2(z,\epsilon)u_{2,p}(t,z,\epsilon) + c_{P_1P_2}\left[P_1(\partial_z)u_{1,p}(t,z,\epsilon)\right] \times \left[P_2(\partial_z)u_{2,p}(t,z,\epsilon)\right] + c_{P_3P_4}\left[P_3(\partial_z)u_{1,p}(t,z,\epsilon)\right] \times \left[P_4(\partial_z)u_{1,p}(t,z,\epsilon)\right] + c_{P_5P_6}\left[P_5(\partial_z)u_{2,p}(t,z,\epsilon)\right] \times \left[P_6(\partial_z)u_{2,p}(t,z,\epsilon)\right] \quad (194)$$

provided that $t \in \mathcal{T}$, $z \in H_{\beta'}$ and $\epsilon \in \mathcal{E}_p$.

The next classical result (stated in Proposition 8 p. 66 from [7]) will be essential to deduce recursion relations for the coefficients $\mathbb{G}_{n,j}(t,z)$, $n \geq 0$ of $\hat{\mathbb{G}}_j(\epsilon)$ from the partial differential equations that govern the components $u_{j,p}(t,z,\epsilon)$, $j = 1,2$.

Proposition 12. *Let $f : G \to \mathbb{F}$ be a holomorphic map from a bounded open sector G centered at 0 in $\mathbb{C}^*$ into a complex Banach space $\mathbb{F}$ equipped with a norm $||.||_{\mathbb{F}}$. The next statements are equivalent*

– *There exists a formal power series $\hat{f}(\epsilon) = \sum_{n \geq 0} f_n \epsilon^n / n!$ in $\mathbb{F}[[\epsilon]]$ which is the asymptotic expansion of f on G, meaning that for all closed sector S of G centered at 0, one can associate a sequence $(c(N,S))_{N \geq 0}$ of positive real numbers such that*

$$||f(\epsilon) - \sum_{n=0}^{N-1} f_n \epsilon^n / n!||_{\mathbb{F}} \leq c(N,S)|\epsilon|^N$$

for all $\epsilon \in S$, all integers $N \geq 1$.

– *All $n-$th derivatives of f denoted $f^{(n)}(\epsilon)$ are continuous at 0 and satisfy*

$$\lim_{\substack{\epsilon \to 0 \\ \epsilon \in G}} ||f^{(n)}(\epsilon) - f_n||_{\mathbb{F}} = 0$$

for all integers $n \geq 0$.

We first derive some recursion relations for the coefficients $\mathbb{G}_{m,2}(t,z)$, $m \geq 0$. To that aim we take the derivative of order $m \geq 0$ on the left and right handside of (193) relatively to ϵ for any integer $m \geq 0$. Indeed, owing to the Leibniz rule, we deduce

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$$
\begin{aligned}
Q(\partial_z)\partial_\epsilon^m u_{2,p}(t,z,\epsilon) &= \sum_{m_1+m_2=m} \frac{m!}{m_1!m_2!}(\partial_\epsilon^{m_1}\epsilon^{d_D})t^{d_D}(t\partial_t)^{\delta_D}R_D(\partial_z)\left[\partial_\epsilon^{m_2}u_{2,p}(t,z,\epsilon)\right] \\
&+ \sum_{l=1}^{D-1}\sum_{m_1+m_2+m_3=m}\frac{m!}{m_1!m_2!m_3!}(\partial_\epsilon^{m_1}\epsilon^{\Delta_l})t^{d_l} \times \left[(\partial_\epsilon^{m_2}a_l(z,\epsilon)\right] \times (t\partial_t)^{\delta_l}R_l(\partial_z)\left[\partial_\epsilon^{m_3}u_{2,p}(t,z,\epsilon)\right] \\
&+ \partial_\epsilon^m F_2(\epsilon t,z,\epsilon) + \sum_{m_1+m_2=m}\frac{m!}{m_1!m_2!}\left[\partial_\epsilon^{m_1}c_1(z,\epsilon)\right] \times \left[\partial_\epsilon^{m_2}u_{2,p}(t,z,\epsilon)\right] \\
&+ c_{Q_1,Q_2}\sum_{m_1+m_2=m}\frac{m!}{m_1!m_2!}\left[Q_1(\partial_z)\partial_\epsilon^{m_1}u_{2,p}(t,z,\epsilon)\right] \times \left[Q_2(\partial_z)\partial_\epsilon^{m_2}u_{2,p}(t,z,\epsilon)\right] \quad (195)
\end{aligned}
$$

for all $m \geq 0$, all $t \in \mathcal{T}$, $z \in H_{\beta'}$ and $\epsilon \in \mathcal{E}_p$. Owing to the asymptotic expansion (186) for $j = 2$, the application of Proposition 12 yields the next limits

$$
\lim_{\substack{\epsilon \to 0 \\ \epsilon \in \mathcal{E}_p}} \sup_{\substack{t \in \mathcal{T} \\ z \in H_{\beta'}}} |\partial_\epsilon^m u_{2,p}(t,z,\epsilon) - \mathbb{G}_{m,2}(t,z)| = 0 \quad (196)
$$

for all integers $m \geq 0$ and any given $0 \leq p \leq \varsigma - 1$. We let ϵ tend to 0 on the sector $\mathcal{E}_p$ in the above equality (195) and with the help of (196) combined with the observation that both maps $u_{2,p}(t,z,\epsilon)$ and $\mathbb{G}_{m,2}(t,z)$ are holomorphic with respect to (t,z) on the product $\mathcal{T} \times H_{\beta'}$, we reach the next relation for the coefficients $\mathbb{G}_{m,2}(t,z)$, $m \geq 0$,

$$
\begin{aligned}
Q(\partial_z)\mathbb{G}_{m,2}(t,z) &= \sum_{m_1+m_2=m}\frac{m!}{m_1!m_2!}(\partial_\epsilon^{m_1}\epsilon^{d_D})(0)t^{d_D}(t\partial_t)^{\delta_D}R_D(\partial_z)\mathbb{G}_{m_2,2}(t,z) \\
&+ \sum_{l=1}^{D-1}\sum_{m_1+m_2+m_3=m}\frac{m!}{m_1!m_2!m_3!}(\partial_\epsilon^{m_1}\epsilon^{\Delta_l})(0)t^{d_l} \times \left[(\partial_\epsilon^{m_2}a_l)(z,0)\right] \times (t\partial_t)^{\delta_l}R_l(\partial_z)\mathbb{G}_{m_3,2}(t,z) \\
&+ \partial_\epsilon^m F_2(\epsilon t,z,\epsilon)_{|\epsilon=0} + \sum_{m_1+m_2=m}\frac{m!}{m_1!m_2!}\left[(\partial_\epsilon^{m_1}c_1)(z,0)\right] \times \mathbb{G}_{m_2,2}(t,z) \\
&+ c_{Q_1,Q_2}\sum_{m_1+m_2=m}\frac{m!}{m_1!m_2!}\left[Q_1(\partial_z)\mathbb{G}_{m_1,2}(t,z)\right] \times \left[Q_2(\partial_z)\mathbb{G}_{m_2,2}(t,z)\right] \quad (197)
\end{aligned}
$$

for all $m \geq 0$, provided that $t \in \mathcal{T}$, $z \in H_{\beta'}$.

This enables us to display some partial differential equation fulfilled by the formal expansion $\hat{\mathbb{G}}_2(\epsilon)$. Namely, we know that the maps $\epsilon \mapsto \epsilon^{d_D}$, $\epsilon \mapsto \epsilon^{\Delta_l}$, $\epsilon \mapsto a_l(z,\epsilon)$ together with $\epsilon \mapsto F_2(\epsilon t,z,\epsilon)$ are analytic on the disc D_{ϵ_0}. Their convergent Taylor series are expressed as

$$
\begin{aligned}
\epsilon^{d_D} = \sum_{m\geq 0}\frac{(\partial_\epsilon^m\epsilon^{d_D})(0)}{m!}\epsilon^m \, , \, \epsilon^{\Delta_l} = \sum_{m\geq 0}\frac{(\partial_\epsilon^m\epsilon^{\Delta_l})(0)}{m!}\epsilon^m \, , \, a_l(z,\epsilon) = \sum_{m\geq 0}\frac{(\partial_\epsilon^m a_l)(z,0)}{m!}\epsilon^m, \\
c_1(z,\epsilon) = \sum_{m\geq 0}\frac{(\partial_\epsilon^m c_1)(z,0)}{m!}\epsilon^m \, , \, F_2(\epsilon t,z,\epsilon) = \sum_{m\geq 0}\frac{\partial_\epsilon^m F_2(\epsilon t,z,\epsilon)_{|\epsilon=0}}{m!}\epsilon^m \quad (198)
\end{aligned}
$$

for all $\epsilon \in D_{\epsilon_0}$. Then, departing from (185), we get the formal Taylor expansion of the next pieces that involve $\hat{\mathbb{G}}_2(\epsilon)$. Namely,

$$
\begin{aligned}
&(\epsilon t)^{d_D}\left[(t\partial_t)^{\delta_D}R_D(\partial_z)\hat{\mathbb{G}}_2(\epsilon)\right] \\
&\quad = t^{d_D}\sum_{m\geq 0}\left[\sum_{m_1+m_2=m}\frac{(\partial_\epsilon^{m_1}\epsilon^{d_D})(0)}{m_1!}(t\partial_t)^{\delta_D}R_D(\partial_z)\frac{\mathbb{G}_{m_2,2}(t,z)}{m_2!}\right]\epsilon^m \quad (199)
\end{aligned}
$$

and

$$\epsilon^{\Delta_l} t^{d_l} a_l(z,\epsilon)(t\partial_t)^{\delta_l} R_l(\partial_z)\hat{\mathbb{G}}_2(\epsilon) = t^{d_l} \sum_{m\geq 0} \Big[\sum_{m_1+m_2+m_3=m} \frac{(\partial_\epsilon^{m_1}\epsilon^{\Delta_l})(0)}{m_1!} \times \Big[\frac{(\partial_\epsilon^{m_2}a_l)(z,0)}{m_2!}\Big] \times (t\partial_t)^{\delta_l} R_l(\partial_z) \frac{\mathbb{G}_{m_3,2}(t,z)}{m_3!}\Big]\epsilon^m \quad (200)$$

along with

$$c_1(z,\epsilon)\hat{\mathbb{G}}_2(\epsilon) = \sum_{m\geq 0} \Big[\sum_{m_1+m_2=m} \Big[\frac{(\partial_\epsilon^{m_1}c_1)(z,0)}{m_1!}\Big] \times \frac{\mathbb{G}_{m_2,2}(t,z)}{m_2!}\Big]\epsilon^m \quad (201)$$

and

$$[Q_1(\partial_z)\hat{\mathbb{G}}_2(\epsilon)] \times [Q_2(\partial_z)\hat{\mathbb{G}}_2(\epsilon)] = \sum_{m\geq 0} \Big[\sum_{m_1+m_2=m} \Big[\frac{Q_1(\partial_z)\mathbb{G}_{m_1,2}(t,z)}{m_1!}\Big] \times \Big[\frac{Q_2(\partial_z)\mathbb{G}_{m_2,2}(t,z)}{m_2!}\Big]\Big]\epsilon^m. \quad (202)$$

As a result, the relation (197) and the above formal expansions prompt the next partial differential equation satisfied by $\hat{\mathbb{G}}_2(\epsilon)$,

$$Q(\partial_z)\hat{\mathbb{G}}_2(\epsilon) = (\epsilon t)^{d_D}[(t\partial_t)^{\delta_D} R_D(\partial_z)\hat{\mathbb{G}}_2(\epsilon)] + \sum_{l=1}^{D-1} \epsilon^{\Delta_l} t^{d_l} a_l(z,\epsilon)(t\partial_t)^{\delta_l} R_l(\partial_z)\hat{\mathbb{G}}_2(\epsilon) + F_2(\epsilon t,z,\epsilon) + c_1(z,\epsilon)\hat{\mathbb{G}}_2(\epsilon) + c_{Q_1,Q_2}[Q_1(\partial_z)\hat{\mathbb{G}}_2(\epsilon)] \times [Q_2(\partial_z)\hat{\mathbb{G}}_2(\epsilon)]. \quad (203)$$

In the next part of the proof, we exhibit recursion relations for the coefficients $G_{m,1}(t,z)$, $m \geq 0$. We proceed by taking the $m-$th derivative of both handsides of (194) with respect to ϵ for any given integer $m \geq 0$. Indeed, the Leibniz rule yields

$$\begin{aligned}
Q(\partial_z)\partial_\epsilon^m u_{1,p}(t,z,\epsilon) = &\sum_{m=m_1+m_2} \frac{m!}{m_1!m_2!}[\partial_\epsilon^{m_1}\epsilon^{d_D}]t^{d_D} \times \Big[(t\partial_t)^{\delta_D} R_D(\partial_z)[\partial_\epsilon^{m_2}u_{1,p}(t,z,\epsilon)] \\
&+ \delta_D (t\partial_t)^{\delta_D-1} R_D(\partial_z)[\partial_\epsilon^{m_2}u_{2,p}(t,z,\epsilon)]\Big] + \sum_{l=1}^{D-1} \sum_{m_1+m_2+m_3=m} \frac{m!}{m_1!m_2!m_3!} \\
&\times [\partial_\epsilon^{m_1}\epsilon^{\Delta_l}]t^{d_l} \times [(\partial_\epsilon^{m_2}a_l)(z,\epsilon)] \times \Big[(t\partial_t)^{\delta_l} R_l(\partial_z)[(\partial_\epsilon^{m_3}u_{1,p})(t,z,\epsilon)] \\
&+ \delta_l (t\partial_t)^{\delta_l-1} R_l(\partial_z)[(\partial_\epsilon^{m_3}u_{2,p})(t,z,\epsilon)]\Big] + \partial_\epsilon^m(F_1(\epsilon t,z,\epsilon)) \\
&+ \sum_{m=m_1+m_2} \frac{m!}{m_1!m_2!}[(\partial_\epsilon^{m_1}b_1)(z,\epsilon)] \times [(\partial_\epsilon^{m_2}u_{1,p})(t,z,\epsilon)] \\
&+ \sum_{m=m_1+m_2} \frac{m!}{m_1!m_2!}[(\partial_\epsilon^{m_1}b_2)(z,\epsilon)] \times [(\partial_\epsilon^{m_2}u_{2,p})(t,z,\epsilon)] \\
&+ c_{P_1P_2} \sum_{m=m_1+m_2} \frac{m!}{m_1!m_2!}[P_1(\partial_z)(\partial_\epsilon^{m_1}u_{1,p})(t,z,\epsilon)] \times [P_2(\partial_z)(\partial_\epsilon^{m_2}u_{2,p})(t,z,\epsilon)] \\
&+ c_{P_3P_4} \sum_{m=m_1+m_2} \frac{m!}{m_1!m_2!}[P_3(\partial_z)(\partial_\epsilon^{m_1}u_{1,p})(t,z,\epsilon)] \times [P_4(\partial_z)(\partial_\epsilon^{m_2}u_{1,p})(t,z,\epsilon)] \\
&+ c_{P_5P_6} \sum_{m=m_1+m_2} \frac{m!}{m_1!m_2!}[P_5(\partial_z)(\partial_\epsilon^{m_1}u_{2,p})(t,z,\epsilon)] \times [P_6(\partial_z)(\partial_\epsilon^{m_2}u_{2,p})(t,z,\epsilon)] \quad (204)
\end{aligned}$$

for all $m \geq 0$, all $t \in \mathcal{T}$, $z \in H_{\beta'}$ and $\epsilon \in \mathcal{E}_p$. Besides, the asymptotic expansion (186) for $j = 1$ warrants the application of Proposition 12 in order to reach the limits

$$\lim_{\substack{\epsilon \to 0 \\ \epsilon \in \mathcal{E}_p}} \sup_{\substack{t \in \mathcal{T} \\ z \in H_{\beta'}}} |\partial_\epsilon^m u_{1,p}(t,z,\epsilon) - \mathbb{G}_{m,1}(t,z)| = 0 \tag{205}$$

for all integers $m \geq 0$ and any prescribed $0 \leq p \leq \varsigma - 1$. We allow the parameter ϵ to get close to 0 in the relation (204). Based on the above limits (205) combined with (196) and the fact that the maps $u_{j,p}(t,z,\epsilon)$ and $\mathbb{G}_{m,j}(t,z)$, $j = 1,2$ rely holomorphically in the variable (t,z) on the product $\mathcal{T} \times H_{\beta'}$, we obtain the next relation for the coefficients $\mathbb{G}_{m,1}(t,z)$, $m \geq 0$,

$$\begin{aligned}
Q(\partial_z)\mathbb{G}_{m,1}(t,z) = & \sum_{m=m_1+m_2} \frac{m!}{m_1!m_2!} [(\partial_\epsilon^{m_1} \epsilon^{d_D})(0)] t^{d_D} \times \Big[(t\partial_t)^{\delta_D} R_D(\partial_z) \mathbb{G}_{m_2,1}(t,z) \\
& + \delta_D (t\partial_t)^{\delta_D - 1} R_D(\partial_z) \mathbb{G}_{m_2,2}(t,z) \Big] + \sum_{l=1}^{D-1} \sum_{m_1+m_2+m_3=m} \frac{m!}{m_1!m_2!m_3!} \\
& \times [(\partial_\epsilon^{m_1} \epsilon^{\Delta_l})(0)] t^{d_l} \times [(\partial_\epsilon^{m_2} a_l)(z,0)] \times \Big[(t\partial_t)^{\delta_l} R_l(\partial_z) \mathbb{G}_{m_3,1}(t,z) \\
& + \delta_l (t\partial_t)^{\delta_l - 1} R_l(\partial_z) \mathbb{G}_{m_3,2}(t,z) \Big] + \partial_\epsilon^m (F_1(\epsilon t, z, \epsilon))_{|\epsilon=0} \\
& + \sum_{m=m_1+m_2} \frac{m!}{m_1!m_2!} [(\partial_\epsilon^{m_1} b_1)(z,0)] \times \mathbb{G}_{m_2,1}(t,z) \\
& + \sum_{m=m_1+m_2} \frac{m!}{m_1!m_2!} [(\partial_\epsilon^{m_1} b_2)(z,0)] \times \mathbb{G}_{m_2,2}(t,z) \\
& + c_{P_1P_2} \sum_{m=m_1+m_2} \frac{m!}{m_1!m_2!} [P_1(\partial_z)\mathbb{G}_{m_1,1}(t,z)] \times [P_2(\partial_z)\mathbb{G}_{m_2,2}(t,z)] \\
& + c_{P_3P_4} \sum_{m=m_1+m_2} \frac{m!}{m_1!m_2!} [P_3(\partial_z)\mathbb{G}_{m_1,1}(t,z)] \times [P_4(\partial_z)\mathbb{G}_{m_2,1}(t,z)] \\
& + c_{P_5P_6} \sum_{m=m_1+m_2} \frac{m!}{m_1!m_2!} [P_5(\partial_z)\mathbb{G}_{m_1,2}(t,z)] \times [P_6(\partial_z)\mathbb{G}_{m_2,2}(t,z)]
\end{aligned} \tag{206}$$

for all $m \geq 0$, whenever $t \in \mathcal{T}$ and $z \in H_{\beta'}$.

This latter recusion relation leads to some partial differential equation governing the formal expression $\hat{\mathbb{G}}_1(\epsilon)$ given by (185). In the process, we use the convergent Taylor expansions (198) together with

$$F_1(\epsilon t, z, \epsilon) = \sum_{m \geq 0} \frac{\partial_\epsilon^m F_1(\epsilon t, z, \epsilon)_{|\epsilon=0}}{m!} \epsilon^m \ , \ b_j(z,\epsilon) = \sum_{m \geq 0} \frac{(\partial_\epsilon^m b_j)(z,0)}{m!} \epsilon^m \tag{207}$$

for $j = 1,2$ which are valid for all $\epsilon \in D_{\epsilon_0}$ and from which the next list of computations are deduced

$$\begin{aligned}
& (\epsilon t)^{d_D} [(t\partial_t)^{\delta_D} R_D(\partial_z) \hat{\mathbb{G}}_1(\epsilon) + \delta_D (t\partial_t)^{\delta_D - 1} R_D(\partial_z) \hat{\mathbb{G}}_2(\epsilon)] \\
& = t^{d_D} \sum_{m \geq 0} \Big[\sum_{m=m_1+m_2} \frac{(\partial_\epsilon^{m_1} \epsilon^{d_D})(0)}{m_1!} [(t\partial_t)^{\delta_D} R_D(\partial_z) \frac{\mathbb{G}_{m_2,1}(t,z)}{m_2!} \\
& \qquad + \delta_D (t\partial_t)^{\delta_D - 1} R_D(\partial_z) \frac{\mathbb{G}_{m_2,2}(t,z)}{m_2!}] \Big] \epsilon^m
\end{aligned} \tag{208}$$

and

$$\epsilon^{\Delta_l} t^{d_l} a_l(z,\epsilon)\left[(t\partial_t)^{\delta_l} R_l(\partial_z)\hat{\mathbb{G}}_1(\epsilon) + \delta_l (t\partial_t)^{\delta_l - 1} R_l(\partial_z)\hat{\mathbb{G}}_2(\epsilon)\right]$$
$$= t^{d_l} \sum_{m\geq 0} \Big[\sum_{m_1+m_2+m_3=m} \frac{(\partial_\epsilon^{m_1}\epsilon^{\Delta_l})(0)}{m_1!} \times \frac{(\partial_\epsilon^{m_2} a_l)(z,0)}{m_2!} \times \Big[(t\partial_t)^{\delta_l} R_l(\partial_z)\frac{\mathbb{G}_{m_3,1}(t,z)}{m_3!}$$
$$+ \delta_l (t\partial_t)^{\delta_l - 1} R_l(\partial_z)\frac{\mathbb{G}_{m_3,2}(t,z)}{m_3!}\Big]\Big]\epsilon^m \quad (209)$$

along with

$$b_j(z,\epsilon)\hat{\mathbb{G}}_j(\epsilon) = \sum_{m\geq 0}\Big[\sum_{m=m_1+m_2} \frac{(\partial_\epsilon^{m_1} b_j)(z,0)}{m_1!} \times \frac{\mathbb{G}_{m_2,j}(t,z)}{m_2!}\Big]\epsilon^m \quad (210)$$

for $j = 1,2$. Futhermore, the next identities hold

$$[P_1(\partial_z)\hat{\mathbb{G}}_1(\epsilon)] \times [P_2(\partial_z)\hat{\mathbb{G}}_2(\epsilon)]$$
$$= \sum_{m\geq 0}\Big[\sum_{m=m_1+m_2} [P_1(\partial_z)\frac{\mathbb{G}_{m_1,1}(t,z)}{m_1!}] \times [P_2(\partial_z)\frac{\mathbb{G}_{m_2,2}(t,z)}{m_2!}]\Big]\epsilon^m \quad (211)$$

with

$$[P_3(\partial_z)\hat{\mathbb{G}}_1(\epsilon)] \times [P_4(\partial_z)\hat{\mathbb{G}}_1(\epsilon)]$$
$$= \sum_{m\geq 0}\Big[\sum_{m=m_1+m_2} [P_3(\partial_z)\frac{\mathbb{G}_{m_1,1}(t,z)}{m_1!}] \times [P_4(\partial_z)\frac{\mathbb{G}_{m_2,1}(t,z)}{m_2!}]\Big]\epsilon^m \quad (212)$$

and

$$[P_5(\partial_z)\hat{\mathbb{G}}_2(\epsilon)] \times [P_6(\partial_z)\hat{\mathbb{G}}_2(\epsilon)]$$
$$= \sum_{m\geq 0}\Big[\sum_{m=m_1+m_2} [P_5(\partial_z)\frac{\mathbb{G}_{m_1,2}(t,z)}{m_1!}] \times [P_6(\partial_z)\frac{\mathbb{G}_{m_2,2}(t,z)}{m_2!}]\Big]\epsilon^m \quad (213)$$

As a consequence of the above computations, the relation (206) triggers the next partial differential equation fulfilled by $\hat{\mathbb{G}}_1(\epsilon)$ and coupled with (203),

$$Q(\partial_z)\hat{\mathbb{G}}_1(\epsilon) = (\epsilon t)^{d_D}\left[(t\partial_t)^{\delta_D} R_D(\partial_z)\hat{\mathbb{G}}_1(\epsilon)\right.$$
$$\left.+ \delta_D (t\partial_t)^{\delta_D - 1} R_D(\partial_z)\hat{\mathbb{G}}_2(\epsilon)\right] + \sum_{l=1}^{D-1} \epsilon^{\Delta_l} t^{d_l} a_l(z,\epsilon)\left[(t\partial_t)^{\delta_l} R_l(\partial_z)\hat{\mathbb{G}}_1(\epsilon)\right.$$
$$\left.+ \delta_l (t\partial_t)^{\delta_l - 1} R_l(\partial_z)\hat{\mathbb{G}}_2(\epsilon)\right] + F_1(\epsilon t, z, \epsilon) + b_1(z,\epsilon)\hat{\mathbb{G}}_1(\epsilon) + b_2(z,\epsilon)\hat{\mathbb{G}}_2(\epsilon)$$
$$+ c_{P_1P_2}[P_1(\partial_z)\hat{\mathbb{G}}_1(\epsilon)] \times [P_2(\partial_z)\hat{\mathbb{G}}_2(\epsilon)] + c_{P_3P_4}[P_3(\partial_z)\hat{\mathbb{G}}_1(\epsilon)] \times [P_4(\partial_z)\hat{\mathbb{G}}_1(\epsilon)]$$
$$+ c_{P_5P_6}[P_5(\partial_z)\hat{\mathbb{G}}_2(\epsilon)] \times [P_6(\partial_z)\hat{\mathbb{G}}_2(\epsilon)]. \quad (214)$$

In conclusion, we have checked by means of (203) that the power series $\hat{\mathbb{G}}_2(\epsilon)$ formally solves the same partial differential equations as the function $u_{2,p}(t,z,\epsilon)$ stated in (193). In addition, through (214) and (194) we observe that the formal power series $\hat{\mathbb{G}}_1(\epsilon)$ and the map $u_{1,p}(t,z,\epsilon)$ obey identical coupled partial differential equations. Then, drew on the computations (35), (34) and (33) performed reversedly from Subsection 3.1, we deduce that the formal expression $\hat{\mathbb{G}}(\epsilon)$ stated in (191) conforms the same equation as the analytic map $u_p(t,z,\epsilon)$ given in (187) and recast as (192) where the formal monodromy operator around 0 given by γ_ϵ^* acts on the formal expression $\hat{\mathbb{G}}(\epsilon)$ by dint of the formula (9) in Definition 4. This completes the proof of the third item of Theorem 1. □

 doi:10.20944/preprints202301.0582.v1

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