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Not peer-reviewed version

A Solution to the Collatz Conjecture Problem

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Posted Date: 6 November 2024

doi: 10.20944/preprints202301.0541.v18

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A Solution to the Collatz Conjecture Problem

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Abstract: Research Collatz odd sequence, change $(\times 3+1) \div 2^k$ operation in Collatz Conjecture to $(\times 3+2^m-1) \div 2^k$ operation. Expand loop Collatz odd sequence (if exists) in $(\times 3+2^m-1) \div 2^k$ odd sequence to become ∞-steps non-loop sequence. Build a $(\times 3+2^m-1) \div 2^k$ odd tree model and transform position model for odds in tree. Via comparing actual and virtual positions, prove if a $(\times 3+2^m-1) \div 2^k$ odd sequence can not converge after ∞ steps of $(\times 3+2^m-1) \div 2^k$ operation, the sequence must walk out of the boundary of the tree.

Keywords: Collatz Conjecture; Collatz odd sequence; $(\times 3+2^m-1) \div 2^k$ odd sequence; $(\times 3+2^m-1) \div 2^k$ odd tree; transform position

Terms: $(\times 3+1) \div 2^k$: $\forall b \in O\{1,3,5,7,9...\}$, $O\{...\}$ is an odd set, do $b \times 3+1$, then do k times $\div 2$ repeatedly until get an odd. At this point, these odds are called Collatz odds, and k is called the step property of b or this step. $(\times 3+2^m-1) \div 2^k$: $\forall a \in O\{3,5,7,9...\}$, highest binary bit is 2^{m-1}, do $a \times 3+2^m-1$, then do k times $\div 2$ repeatedly until get an odd. k is called the step property of a or this step.

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§1 Introduction About the Collatz Conjecture

The Collatz Conjecture is a famous math conjecture, named after mathematician Lothar Collatz, who introduced the idea in 1937. It is also known as the 3x+1 conjecture, the Ulam conjecture[1]. Many mathematicians have tried to prove it true or false and have expanded it to a more digit scale. But until today, it has not yet been proved.

The Collatz Conjecture concerns sequences of positive integers in which each term is obtained from the previous one as follows: if the previous integer is even, the next integer is the previous integer divided by 2. If the previous integer is odd, the next term is the previous integer multiply 3 and plus 1. The conjecture is that these sequences always reach 1, no matter which positive integer is chosen to start the sequence[1].

Here is an example for a typical integer $x=27$, takes up to 111 steps, increasing or decreasing step by step, climbing as high as 9232 before descending to 1[1].

27, 82, 41, 124, 62, 31, 94, 47, 142, 71, 214, 107, 322, 161, 484, 242, 121, 364, 182, 91, 274, 137, 412, 206, 103, 310, 155, 466, 233, 700, 350, 175, 526, 263, 790, 395, 1186, 593, 1780, 890, 445, 1336, 668, 334, 167, 502, 251, 754, 377, 1132, 566, 283, 850, 425, 1276, 638, 319, 958, 479, 1438, 719, 2158, 1079, 3238, 1619, 4858, 2429, 7288, 3644, 1822, 911, 2734, 1367, 4102, 2051, 6154, 3077, 9232, 4616, 2308, 1154, 577, 1732, 866, 433, 1300, 650, 325, 976, 488, 244, 122, 61, 184, 92, 46, 23, 70, 35, 106, 53, 160, 80, 40, 20, 10, 5, 16, 8, 4, 2, 1.

If the conjecture is false, there should exist some starting number which gives rise to a sequence that does not contain 1. Such a sequence would either enter a repeating cycle that excludes 1, or increase without bound[1]. No such sequence has been found by humans or computers after verifying a lot of numbers can reach 1. It is very difficult to prove these two cases exist or not.

This paper will try to prove the conjecture true from a special view. Because any even can become odd via $\div 2^k$ operation, this paper will research only odd characters in the conjecture sequence. The equivalence conjecture becomes: with random starting odd x, do $(\times 3+1) \div 2^k$ operation repeatedly, it always converges to 1. The above sequence can be written as follows, in which numbers on arrows are step property k in each step:

$27 \xrightarrow{1} 41 \xrightarrow{2} 31 \xrightarrow{1} 47 \xrightarrow{1} 71 \xrightarrow{1} 107 \xrightarrow{1} 161 \xrightarrow{2} 121 \xrightarrow{2} 91 \xrightarrow{1} 137 \xrightarrow{2} 103 \xrightarrow{1} 155 \xrightarrow{1} 233 \xrightarrow{2} 175 \xrightarrow{1} 263 \xrightarrow{1} 395 \xrightarrow{1} 593 \xrightarrow{2} 445 \xrightarrow{3} 167 \xrightarrow{1} 251 \xrightarrow{1} 377 \xrightarrow{2} 283 \xrightarrow{1} 425 \xrightarrow{2} 319 \xrightarrow{1} 479 \xrightarrow{1} 719 \xrightarrow{1} 1079 \xrightarrow{1} 1619 \xrightarrow{1} 2429 \xrightarrow{3} 911 \xrightarrow{1} 1367 \xrightarrow{1} 2051 \xrightarrow{1} 3077 \xrightarrow{4} 577 \xrightarrow{2} 433 \xrightarrow{2} 325 \xrightarrow{4} 61 \xrightarrow{3} 23 \xrightarrow{1} 35 \xrightarrow{1} 53 \xrightarrow{5} 5 \xrightarrow{4} 1$

§2 Equivalence of $(\times 3+1) \div 2^k$ and $(\times 3+2^m-1) \div 2^k$ Operation

Lemma 1 : $\forall a \in O\{3,5,7,9...\}$, highest binary bit is $2^{m-1}(m \geq 2)$, do $(\times 3+2^m-1) \div 2^k$ operation gets odd b, with complement odd 2^m-a do $(\times 3+1) \div 2^k$ operation gets odd c, then the step property of a and 2^m-a is same, $b+c=2^j$, and $b>c$.

Prove: First, $2^m > a > 2^{m-1} > 2^m - a$

$3 \times a + 2^m - 1 = b \times 2^{p_1}$

$3 \times (2^m - a) + 1 = c \times 2^{p_2}$

$3 \times (2^m - a) + 1 + 3 \times a + 2^m - 1 = 2^{m+2} = b \times 2^{p_1} + c \times 2^{p_2}$

Because both b and c are odds, then

$2^{p_1} = 2^{p_2}$, and $b + c = 2^{m+2-p_1} = 2^j$

$3 \times a + 2^m - 1 - (3 \times (2^m - a) + 1) = 6 \times a - 2^{m+1} - 4 > 6 \times 2^{m-1} - 2^{m+1} - 4 = 2^m - 4 \geq 0$

Then $b > c$.

For example, $(9 \times 3 + 16 - 1) \div 2 = 21$, $(7 \times 3 + 1) \div 2 = 11$, $21 + 11 = 2^5$, $21 > 11$

This states, with any Collatz odd $2^m - a(a > 2^m - a)$ do $(\times 3+1) \div 2^k$ operation, we can do $(\times 3+2^m-1) \div 2^k$ operation with its complement odd a instead, they are equivalent.

For example, use 7 as the starting Collatz odd, the Collatz odd sequence is: $7, 11, 17, 13, 5, 1$. Use 9 as corresponding starting complement odd, $(\times 3+2^m-1) \div 2^k$ sequence is: $9, 21, 47, 51, 27, 7$.

Lemma 2 : $\forall a = 2^m - 1 (m \geq 2)$, do $(\times 3+2^m-1) \div 2^k$ operation get odd b, $b = a$.

Prove: $a \times 3 + 2^m - 1 = (2^m - 1) \times 3 + 2^m - 1 = (2^m - 1) \times 2^2$

$b = (2^m - 1) \times 2^2 \div 2^2 = 2^m - 1 = a$

We call odds of form $2^m - 1$ is the convergence state for any odds(> 3), if they can reach it after doing $(\times 3+2^m-1) \div 2^k$ operation repeatedly.

Lemma 3: $\forall a \in O\{3,5,7,9...\}$, highest binary bit is $2^{m-1}(m \geq 2)$, do $\times 3 + 2^m - 1$ operation gets an even, result must grow 2 binary bits, with odd part of result do $\times 3 + 2^m - 1$ operation repeatedly, m is the highest binary bit number plus 1 of the odd part produced in each step, then count of successive binary bit 1 in the most front head part of odd produced in each step must remain unchanged or increased, and must increase 1 within 3 steps, until reach convergence state.

Prove: $a = 2^{m-1} + a_1 \times 2^{m-2} + ...a_{m-2} \times 2 + 1, (a_1...a_{m-2} = 0 \;\; or \;\; 1)$

$a \times 3 + 2^m - 1 = 2^{m+1} + 2^{m-1} + 3 \times (a_1 \times 2^{m-2} + ...a_{m-2} \times 2) + 2$, tail part $3 \times (a_1 \times 2^{m-2} + ...a_{m-2} \times 2) + 2$ is possible to carry 0, 1 or 2 bits to the head part. If carries 0 bit, head part changes nothing; if carries 1 bit, head part $+2^{m-1}$; if carries 2 bits, head part $+2^m + 0 \times 2^{m-1}$. All cases, highest binary bit is 2^{m+1}, grows 2 binary bits.

If the most front head of the first odd or odd part of the result has only one successive binary bit 1, has the following 4 cases:

Case 1: the odd part is in binary form 1000*...1. The head part becomes 101000 after doing $\times 3 + 2^m$. Tail part *...1 is possible to carry 0, 1 or 2 bits to the head part after doing $\times 3 - 1$. If carries 0 bit, head part changes nothing; if carries 1 bit, head part +1, becomes 101001; if carries 2 bits, head part +10, becomes 101010. All cases form of head part is the same with case 3, count of successive binary bit 1 in the most front head remains unchanged.

Case 2: the odd part is in binary form 1001*...1. The head part becomes 101011 after doing $\times 3 + 2^m$. Tail part *...1 is possible to carry 0, 1 or 2 bits to the head part after doing $\times 3 - 1$. If carries 0 bit, head part changes nothing, its form is the same with case 3; if carries 1 bit, head part +1, becomes 101100, its form is the same with case 4; if carries 2 bits, head part +10, becomes 101101, its form is the same with case 4. All cases count of successive binary bit 1 remains unchanged.

Case 3: the odd part is in binary form 1010*...1. The head part becomes 101110 after doing $\times 3 + 2^m$. Tail part *...1 is possible to carry 0, 1 or 2 bits to the head part after doing $\times 3 - 1$. If carries 0 bit, head part changes nothing, its form is the same with case 4, count of successive binary bit 1 remains unchanged; if carries 1 bit, head part +1, becomes 101111, its form is the same with case 4, count of successive binary bit 1 remains unchanged; if carries 2 bits, head part +10, becomes 110000, count of successive binary bit 1 increases 1.

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Case 4: the odd part is in binary form 1011*...1. The head part becomes 110001 after doing $\times 3 + 2^m$. Tail part *...1 is possible to carry 0, 1 or 2 bits to the head part after doing $\times 3 - 1$. All cases count of successive binary bit 1 increases 1.

Hence, within 3 steps, count of successive binary bit 1 in the most front head part of odd increases 1 until reach convergence state. In the convergence state, the count of successive binary bit 1 always remains unchanged.

Similar to cases where the most front head of the first odd or odd part of the result produced in each step has more than one successive binary bit 1.

For example, $1110001 + 11100010 + 2^7 - 1 = 111010010$, $11101001 + 111010010 + 2^8 - 1 = 1110111010$, $111011101 + 1110111010 + 2^9 - 1 = 11110010110$.

Corollary 1: If exists a loop Collatz odd sequence $b_1, b_2, ...b_i, b_1, b_2, ...b_i (i > 3)$, expand in corresponding $(\times 3 + 2^m - 1) \div 2^k$ odd sequence, get an odd sequence $a_1, a_2, ...a_i, a_{i+1}, a_{i+2}, ...a_{2i}$, then $a_{i+1} \neq a_1, a_{i+2} \neq a_2, ...a_{2i} \neq a_i$.

§3 $(\times 3 + 2^m - 1) \div 2^k$ Odd Tree

Use all odds in order in set $O\{3, 5, 7, 9...\}$ to build a "tree":

...

L6: 129(321.1) 131(81.3) 133(327.1) 135(165.2) 137(333.1) 139(21.5) 141(339.1) 143(171.2) 145(345.1) 147(87.3) 149(351.1) 151(177.2) 153(357.1) 155(45.4) 157(363.1) 159(183.2) 161(369.1) 163(93.3) 165(375.1) 167(189.2) 169(381.1) 171(3.8) 173(387.1) 175(195.2) 177(393.1) 179(99.3) 181(399.1) 183(201.2) 185(405.1) 187(51.4) 189(411.1) 191(207.2) 193(417.1) 195(105.3) 197(423.1) 199(213.2) 201(429.1) 203(27.5) 205(435.1) 207(219.2) 209(441.1) 211(111.3) 213(447.1) 215(225.2) 217(453.1) 219(57.4) 221(459.1) 223(231.2) 225(465.1) 227(117.3) 229(471.1) 231(237.2) 233(477.1) 235(15.6) 237(483.1) 239(243.2) 241(489.1) 243(123.3) 245(495.1) 247(249.2) 249(501.1) 251(63.4) 253(507.1) 255

L5: 65(161.1) 67(41.3) 69(167.1) 71(85.2) 73(173.1) 75(11.5) 77(179.1) 79(91.2) 81(185.1) 83(47.3) 85(191.1) 87(97.2) 89(197.1) 91(25.4) 93(203.1) 95(103.2) 97(209.1) 99(53.3) 101(215.1) 103(109.2) 105(221.1) 107(7.6) 109(227.1) 111(115.2) 113(233.1) 115(59.3) 117(239.1) 119(121.2) 121(245.1) 123(31.4) 125(251.1) 127

L4: 33(81.1) 35(21.3) 37(87.1) 39(45.2) 41(93.1) 43(3.6) 45(99.1) 47(51.2) 49(105.1) 51(27.3) 53(111.1) 55(57.2) 57(117.1) 59(15.4) 61(123.1) 63

L3: 17(41.1) 19(11.3) 21(47.1) 23(25.2) 25(53.1) 27(7.4) 29(59.1) 31

L2: 9(21.1) 11(3.4) 13(27.1) 15

L1: 5(11.1) 7

L0: 3

Graph 1 $(\times 3 + 2^m - 1) \div 2^k$ odd tree

All odds in the tree do $(\times 3 + 2^m - 1) \div 2^k$ operation, $a.b$ in () means result is $a \times 2^b$ after front odd doing $(\times 3 + 2^m - 1) \div 2^k$ operation, b is step property. Layer i has 2^i elements, the first element is $2^{i+1} + 1$, and the last element is $2^{i+2} - 1$, which is the convergence state.

In this tree, because the element count of each layer is 2 times that of the downward layer, we can transform all element positions in original layers to one specific layer, layer $i - 2$ transform to layer $i - 1$ do $\times 2$, layer i transform to layer $i - 1$ do $\div 2$..., then all transformed positions (to layer $i - 1$) can not exceed 2^{i-1}.

Lemma 4: Suppose a is any odd before convergence state in layer $m - 1$, $a \times 3 + 2^{m+1} - 1 = b \times 2^{p_1}$, b is odd, then b is in layer $m + 1 - p_1$.

Prove: According to Lemma 3, the highest binary bit of $b \times 2^{p_1}$ is 2^{m+2}, then the highest binary bit of b is 2^{m+2-p_1}, which is in layer $m + 1 - p_1$.

Lemma 5: Suppose a is any odd before convergence state in layer $m - 1$. The highest binary bit is 2^m, $2^{m+1} - a > 3$, do $(\times 3 + 2^m - 1) \div 2^k$ get odd b, b do $(\times 3 + 2^m - 1) \div 2^k$ get odd c. Then transform

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position (to layer $m-1$) of b is bigger than which of a. If step property of a is $p_1=2$, step property of b is $p_2=2$, the transform position increment ratio from a to c is $\frac{3}{4}$.

Prove: Position of a in layer $m-1$ is: $\frac{a-2^m+1}{2}$,

$3\times a+2^{m+1}-1=b\times 2^{p_1}$, b is in layer $m-p_1+1$

Position of b in layer $m-p_1+1$ is: $\frac{b-2^{m-p_1+2}+1}{2}$,

Position of b in layer $m-1$ is: $\frac{b-2^{m-p_1+2}+1}{2^{3-p_1}}$

$3^2\times a+3\times 2^{m+1}-3+2^{m+3}-2^{p_1}=c\times 2^{p_1+p_2}$, c is in layer $m+3-p_1-p_2$

Position of c in layer $m+3-p_1-p_2$ is: $\frac{c-2^{m+4-p_1-p_2}+1}{2}$

Position of c in layer $m-1$ is: $\frac{c-2^{m+4-p_1-p_2}+1}{2^{5-p_1-p_2}}$

Transform position increment from odd a to b is:

$$\delta=\frac{b-2^{m-p_1+2}+1}{2^{3-p_1}}-\frac{a-2^m+1}{2}=\frac{b+1-2^{2-p_1}\times a-2^{2-p_1}}{2^{3-p_1}}$$
$$=\frac{b\times 2^{p_1}+2^{p_1}-2^2\times a-2^2}{2^3}=\frac{3\times a+2^{m+1}-1+2^{p_1}-2^2\times a-2^2}{2^3}$$
$$=\frac{2^{m+1}-a+2^{p_1}-5}{2^3}$$

Only when $2^{m+1}-a=1$ and $p_1=2$ or $2^{m+1}-a=3$ and $p_1=1$, $\delta=0$, these two cases are convergence state or quasi convergence state of Collatz odd sequence. Other cases $\delta>0$, even in expanding loop sequence. When $2^{m+1}-a>3$, the bigger p_1 is, the bigger δ will be.

Continue:

$$\frac{b-2^{m-p_1+2}+1}{2^{3-p_1}}-\frac{a-2^m+1}{2}=\frac{b+1-2^{2-p_1}\times a-2^{2-p_1}}{2^{3-p_1}}$$

$\frac{c-2^{m+4-p_1-p_2}+1}{2^{5-p_1-p_2}}-\frac{b-2^{m-p_1+2}+1}{2^{3-p_1}}=\frac{c+1-2^{2-p_2}\times b-2^{2-p_2}}{2^{5-p_1-p_2}}$, transform position increment ratio from a to c is:

$$r=\frac{c+1-2^{2-p_2}\times b-2^{2-p_2}}{2^{5-p_1-p_2}}\times\frac{2^{3-p_1}}{b+1-2^{2-p_1}\times a-2^{2-p_1}}$$
$$=\frac{c\times 2^{p_2}+2^{p_2}-2^2\times b-2^2}{2^2}\times\frac{2^{p_1}}{b\times 2^{p_1}+2^{p_1}-2^2\times a-2^2}$$
$$=\frac{c\times 2^{p_1+p_2}+2^{p_1+p_2}-2^{2+p_1}\times b-2^{2+p_1}}{2^2}\times\frac{1}{3\times a+2^{m+1}-1+2^{p_1}-2^2\times a-2^2}$$
$$=\frac{3^2\times a+3\times 2^{m+1}-3+2^{m+3}-2^{p_1}+2^{p_1+p_2}-2^{2+p_1}\times b-2^{2+p_1}}{2^2}\times\frac{1}{2^{m+1}+2^{p_1}-a-5}$$
$$=\frac{3^2\times a+3\times 2^{m+1}-3+2^{m+3}-2^{p_1}+2^{p_1+p_2}-2^2\times(3\times a+2^{m+1}-1)-2^{2+p_1}}{2^2}\times\frac{1}{2^{m+1}+2^{p_1}-a-5}$$
$$=\frac{3\times 2^{m+1}-3\times a-5\times 2^{p_1}+2^{p_1+p_2}+1}{2^2}\times\frac{1}{2^{m+1}+2^{p_1}-a-5}$$
$$=\frac{3\times(2^{m+1}+2^{p_1}-a-5)+2^{p_1+p_2}-8\times 2^{p_1}+16}{2^2\times(2^{m+1}+2^{p_1}-a-5)}$$
$$=\frac{3}{4}+\frac{2^{p_1+p_2}-8\times 2^{p_1}+16}{2^2\times(2^{m+1}+2^{p_1}-a-5)}$$
$$=\frac{3}{4}+\frac{2^{p_1}\times(2^{p_2}-8)+16}{2^2\times(2^{m+1}+2^{p_1}-a-5)}$$

Because $2^{m+1}-a>3$, $2^{m+1}+2^{p_1}-a-5>0$. Then if $p_2=1$ and $p_1\geq 2$ or $p_2=2$ and $p_1\geq 3$, $r<\frac{3}{4}$; if $p_1=p_2=2$, $r=\frac{3}{4}$; other cases $r>\frac{3}{4}$.

Because position in all layers can be transformed to layer $m-1$ only by $\times 2^k$(k is 0, negative or positive integer) operation, the transform position increment regularity is also suitable for all subsequent steps, as long as in each step corresponding Collatz odd is bigger than 3.

§4 Convergence of Transform Position for Odds in $(\times 3+2^m-1)\div 2^k$ Odd Tree

Suppose starting odd a is any odd before convergence state in layer $m-1$, and s_0 is its position in layer $m-1$, $s_i(i\geq 1)$ is transform position (to layer $m-1$) of odd produced in step i, deduce common transform position s_i:

$$s_0=\frac{a-2^m+1}{2}=2^{m-1}+\frac{1-(2^{m+1}-a)}{2}$$
$$s_1=\frac{3\times a+2^{m+1}-1-2^{m+2}+2^{p_1}}{2^3}=\frac{3\times a-1+2^{p_1}+2^{m+1}-2^{m+2}}{2^3}=2^{m-1}+\frac{2^{p_1}-3^1\times(2^{m+1}-a)-1}{2^{2\times 1+1}}$$
$$s_2=\frac{3^2\times a+3\times 2^{m+1}-3+2^{m+3}-2^{p_1}-2^{m+4}+2^{p_1+p_2}}{2^{2\times 2+1}}=\frac{3^2\times a-3-2^{p_1}+2^{p_1+p_2}+3\times 2^{m+1}+2^{m+3}-2^{m+4}}{2^{2\times 2+1}}=2^{m-1}+\frac{2^{p_1+p_2}-3^2\times(2^{m+1}-a)-3-2^{p_1}}{2^{2\times 2+1}}$$

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$$
\begin{aligned}
s_3 &= \frac{3^3 \times a + 3^2 \times 2^{m+1} - 3^2 + 3 \times 2^{m+3} - 3 \times 2^{p_1} + 2^{m+5} - 2^{p_1+p_2} - 2^{m+6} + 2^{p_1+p_2+p_3}}{2^{2\times 3+1}} \\
&= \frac{3^3 \times a - 3^2 - 3 \times 2^{p_1} - 2^{p_1+p_2} + 2^{p_1+p_2+p_3} + 3^2 \times 2^{m+1} + 3 \times 2^{m+3} + 2^{m+5} - 2^{m+6}}{2^{2\times 3+1}} \\
&= \frac{3^3 \times a - 3^2 - 3 \times 2^{p_1} - 2^{p_1+p_2} + 2^{p_1+p_2+p_3} + 2^{m+1} \times (3^2 + 3 \times 2^2 + 2^4) - 2^{m+1} \times 2^5}{2^{2\times 3+1}} \\
&= \frac{3^3 \times a - 3^2 - 3 \times 2^{p_1} - 2^{p_1+p_2} + 2^{p_1+p_2+p_3} + 2^{m+1} \times (2^6 - 3^3) - 2^{m+1} \times 2^5}{2^{2\times 3+1}} \\
&= 2^{m-1} + \frac{2^{p_1+p_2+p_3} - 3^3 \times (2^{m+1} - a) - 3^2 - 3 \times 2^{p_1} - 2^{p_1+p_2}}{2^{2\times 3+1}}
\end{aligned}
$$

...

Then $\forall i \in N(1,2,3,4,5...)$

$$
\begin{aligned}
s_i &= \frac{3^i a - 3^{i-1} - 3^{i-2} \times 2^{p_1} - ...2^{p_1+...p_{i-1}} + 2^{p_1+...p_i} + 3^{i-1} \times 2^{m+1} + 3^{i-2} \times 2^{m+3} + ...2^{m+1+2(i-1)} - 2^{m+1+2(i-1)+1}}{2^{2i+1}} \\
&= \frac{3^i a - 3^{i-1} - 3^{i-2} \times 2^{p_1} - ...2^{p_1+...p_{i-1}} + 2^{p_1+...p_i} + 2^{m+1} \times (2^{2i} - 3^i) - 2^{m+1} \times 2^{2(i-1)+1}}{2^{2i+1}} \\
&= 2^{m-1} + \frac{2^{p_1+p_2+...p_i} - 3^i \times (2^{m+1} - a) - 3^{i-1} - 3^{i-2} \times 2^{p_1} - ...2^{p_1+p_2+...p_{i-1}}}{2^{2i+1}}.
\end{aligned}
$$

Further, use $s_{(p_1,p_2...p_i)}$ to represent the transform position (to layer $m-1$) of odd produced in step i from starting odd a in layer $m-1$. We can change the value of step property $p_k(1 \leq k \leq i)$ to different positive integers or delete middle steps in order to compare two transform positions. At this point, the modified transform position is called virtual transform position. Use the same common formula to calculate two kinds of transform positions.

Lemma 6: a is a positive rational number; $\forall a, 2^{m+1} > a > 2^{m+1} - a > 1 (m > 1)$, the highest binary bit of its integer part is 2^m, do an operation similar to $(\times 3 + 2^m - 1) \div 2^k$: $a \times 3 + 2^{m+1} - 1 = b \times 2^2$, $3 \times b \times 2^2 + 2^{m+3} - 2^2 = c \times 2^4$. Then $2^{m+1} > b > 2^{m+1} - b > 1$, $2^{m+1} > c > 2^{m+1} - c > 1$, the virtual transform position (to layer $m-1$) of b is bigger than which of a, and virtual transform position increment ratio from a to c is $\frac{3}{4}$.

Prove: $b = \frac{3}{4} \times a + 2^{m-1} - \frac{1}{4} < \frac{3}{4} \times 2^{m+1} + 2^{m-1} - \frac{1}{4} = 2^{m+1} - \frac{1}{4} < 2^{m+1}$

Because $a > 2^{m+1} - a$, $a > 2^m$,

$b = \frac{3}{4} \times a + 2^{m-1} - \frac{1}{4} > \frac{3}{4} \times 2^m + 2^{m-1} - \frac{1}{4} = 2^m + 2^{m-2} - \frac{1}{4} > 2^m$. Then $b > 2^{m+1} - b$.

Because $2^{m+1} - a > 1$, $a < 2^{m+1} - 1$,

$2^{m+1} - b = 2^{m+1} - \frac{3}{4} \times a - 2^{m-1} + \frac{1}{4} > 2^{m+1} - \frac{3}{4} \times (2^{m+1} - 1) - 2^{m-1} + \frac{1}{4} = 1$.

Then $2^{m+1} > b > 2^{m+1} - b > 1$.

Because $c = \frac{3}{4} \times b + 2^{m-1} - \frac{1}{4}$, use the same way to prove $2^{m+1} > c > 2^{m+1} - c > 1$.

Because two kinds of transform position formulas are the same, virtual transform position increment from odd a to b is:

$\delta = s_1 - s_0 = \frac{2^{m+1} - a + 2^2 - 5}{2^3}$. When $2^{m+1} - a > 1$, $\delta > 0$.

The virtual transform position increment ratio from a to c is:

$r = \frac{3}{4} + \frac{2^2 \times (2^2 - 8) + 16}{2^2 \times (2^{m+1} + 2^2 - a - 5)} = \frac{3}{4}$

Obviously, the virtual transform position increment regularity is also suitable for all subsequent steps, as long as the step property of each step is 2.

Lemma 7: For any non-convergence starting odd a in layer $m-1$, if $2^{m+1} - a > 3$, $s_{(1,2,2,2...)} > s_{(2,2,2...)}$.

Prove:

$s_{(1,2)} - s_{(2)} = \frac{2^3 - 3^2 \times (2^{m+1} - a) - 3 - 2}{2^5} - \frac{2^2 - 3 \times (2^{m+1} - a) - 1}{2^3} = \frac{3}{4} \times \frac{2^{m+1} - a - 3}{2^3} > 0$

$s_{(1,2,2)} - s_{(2,2)} = \frac{2^5 - 3^3 \times (2^{m+1} - a) - 3^2 - 3^1 \times 2 - 2^3}{2^7} - \frac{2^4 - 3^2 \times (2^{m+1} - a) - 3 - 2^2}{2^5}$

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$= \frac{2^5-(3^3\times(2^{m+1}-a)+3^2+3^1\times2+2^3)-4\times2^4+4\times(3^2\times(2^{m+1}-a)+3+2^2)}{2^7}$
$= (\frac{3}{4})^2 \times \frac{2^{m+1}-a-3}{2^3} > 0$
$s_{(1,2,2,2)} - s_{(2,2,2)} = \frac{2^{5+2}-3\times(3^3\times(2^{m+1}-a)+3^2+3^1\times2+2^3)-2^5}{2^{7+2}} - \frac{2^{4+2}-3\times(3^2\times(2^{m+1}-a)+3+2^2)-2^4}{2^{5+2}}$
$= \frac{3\times2^5-3\times(3^3\times(2^{m+1}-a)+3^2+3^1\times2+2^3)}{2^{7+2}} - \frac{3\times2^4-3\times(3^2\times(2^{m+1}-a)+3+2^2)}{2^{5+2}}$
$= \frac{3}{4} \times (s_{(1,2,2)} - s_{(2,2)})$
$= (\frac{3}{4})^3 \times \frac{2^{m+1}-a-3}{2^3} > 0$
If all $p_i = 2(1 \le i < \infty)$, use mathematical induction to get:
$s_{(1,p_1,p_2...p_i)} - s_{(p_1,p_2...p_i)} = (\frac{3}{4})^i \times \frac{2^{m+1}-a-3}{2^3} > 0$, then $s_{(1,2,2,2...)} > s_{(2,2,2...)}$.

Lemma 8: For any non-convergence starting odd a in layer $m-1$, if $2^{m+1}-a > 3$, $s_{(1,1,2,2,2...)} > s_{(2,2,2...)}$.

Prove:
$s_{(1,1,2)} - s_{(2)} = \frac{2^4-3^3\times(2^{m+1}-a)-3^2-3^1\times2-2^2}{2^7} - \frac{2^2-3\times(2^{m+1}-a)-1}{2^3} = \frac{3}{4} \times \frac{7\times(2^{m+1}-a)-17}{2^5} > 0$
$s_{(1,1,2,2)} - s_{(2,2)} = \frac{2^6-3^4\times(2^{m+1}-a)-3^3-3^2\times2-3\times2^2-2^4}{2^9} - \frac{2^4-3^2\times(2^{m+1}-a)-3-2^2}{2^5}$
$= (\frac{3}{4})^2 \times \frac{7\times(2^{m+1}-a)-17}{2^5} > 0$
$s_{(1,1,2,2,2)} - s_{(2,2,2)} = (\frac{3}{4})^3 \times \frac{7\times(2^{m+1}-a)-17}{2^5} > 0$
If all $p_i = 2(1 \le i < \infty)$, use mathematical induction to get:
$s_{(1,1,p_1,p_2...p_i)} - s_{(p_1,p_2...p_i)} = (\frac{3}{4})^i \times \frac{7\times(2^{m+1}-a)-17}{2^5} > 0$, then $s_{(1,1,2,2,2...)} > s_{(2,2,2...)}$.

Lemma 9: For any non-convergence starting odd a in layer $m-1$, if after doing $(\times3+2^m-1)\div2^k$ operation, corresponding Collatz odd in each step is bigger than 3, then $s_{(p_1,p_2...,p_i,1...,1,2^+,1...,1,2^+)} > s_{(p_1,p_2...,p_i,2,2)}$, where $2^+ \ge 2$.

Prove:
$s_{(2^+,2)} - s_{(2,2)} = \frac{2^{2^++2}-3^2\times(2^{m+1}-a)-3-2^{2^+}}{2^5} - \frac{2^{2+2}-3^2\times(2^{m+1}-a)-3-2^2}{2^5} = \frac{3\times2^{2^+}-3\times2^2}{2^5} \ge 0$
According to Lemma 7 and Lemma 8.
$s_{(p_1,p_2...,p_i,1...,1,2^+)} \ge s_{(p_1,p_2...,p_i,1...,1,2)} > s_{(p_1,p_2...,p_i,2)}$,
$s_{(p_1,p_2...,p_i,1...,1,2^+,1...,1,2^+)} > s_{(p_1,p_2...,p_i,1...,1,2^+,2)} \ge s_{(p_1,p_2...,p_i,1...1,2,2)} > s_{(p_1,p_2...,p_i,2,2)}$.
The result can be extended to more cases, such as $s_{(p_1,p_2...,p_i,1...,1,2^+,1...,1,2^+,1...,1,2^+)}$...

As in the previous example, with starting Collatz odd 27, choose 37 as corresponding starting odd in $(\times3+2^m-1)\div2^k$ odd tree, which is in layer 4, $s_0 = 3$, original transform positions (to layer 4) in subsequent steps are (choose part Collatz odds: $27, 31, 121, 91, 103, 175...$, corresponding odds in $(\times3+2^m-1)\div2^k$ sequence are: $37, 97, 1927, 1957, 3993, 16209...$):

$s_{(1,2)} = 8.5$, $s_{(1,2,1,1,1,1,2)} = 14.125$, $s_{(1,2,1,1,1,1,2,2)} = 14.59375$, $s_{(1,2,1,1,1,1,2,2,1,2)} = 15.203125$, $s_{(1,2,1,1,1,1,2,2,1,2,1,1,2)} = 15.66015625...$

Corresponding virtual transform positions (to layer 4) are:
$s_{(2)} = 6.25$, $s_{(2,2)} = 8.6875$, $s_{(2,2,2)} = 10.515625$, $s_{(2,2,2,2)} = 11.88671875$, $s_{(2,2,2,2,2)} = 12.9150390625...$
$s_{(1,2)} > s_{(2)}$, $s_{(1,2,1,1,1,1,2)} > s_{(2,2)}$, $s_{(1,2,1,1,1,1,2,2)} > s_{(2,2,2)}$,
$s_{(1,2,1,1,1,1,2,2,1,2)} > s_{(2,2,2,2)}$, $s_{(1,2,1,1,1,1,2,2,1,2,1,1,2)} > s_{(2,2,2,2,2)}$...
It can be verified that before Collatz odd in this sequence reaches 1, these inequalities hold true.

This is to say, if we delete all (1) steps in a long sequence and change all (2^+) steps to (2) steps, the final virtual transform position is smaller than original, if corresponding original Collatz odd in each step is bigger than 3.

In Lemma 7 and Lemma 8, when $i \to \infty$ and all $p_i = 2$, both $s_{(1,p_1,p_2...p_i)} - s_{(p_1,p_2...p_i)}$ and $s_{(1,1,p_1,p_2...p_i)} - s_{(p_1,p_2...p_i)}$ are $\to 0$. But in real Collatz odd sequence, it does not exist ∞ successive (2) steps (see Lemma 11). In above deduction procedure, before $i \to \infty$, when we delete (1) steps or change (2^+) steps to (2) steps, actual transform position for original sequence already becomes bigger than modified virtual transform position. Hence, this case does not influence the conclusion in Lemma 9.

Lemma 10: If there exists any loop Collatz odd sequence, the step count must be bigger than 2.

Prove: For any Collatz odd $a > 1$, suppose $3 \times a + 1 = a \times 2^{p_1}$.

Then $(2^{p_1} - 3) \times a = 1$, there exists no odd solution.

Suppose $3 \times a + 1 = b \times 2^{p_1}, 3 \times b + 1 = a \times 2^{p_2}$, where odd $b > 1$.

Then $9 \times a + 3 = 3 \times b \times 2^{p_1}, 3 \times b \times 2^{p_1} + 2^{p_1} = a \times 2^{p_1+p_2}$.

Get $2^{p_1} + 3 = (2^{p_1+p_2} - 9) \times a > 7 \times a$, and $2^{p_1} = \frac{3 \times a+1}{b} < 3 \times a + 1$, then $3 \times a + 1 + 3 > 2^{p_1} + 3 > 7 \times a$. $a < 1$, it is contradictory.

Hence, if there exists any loop Collatz odd sequence, the step count must be bigger than 2. This way, according to Corollary 1. We can expand loop Collatz odd sequence (if exists), get a ∞-steps $(\times 3 + 2^m - 1) \div 2^k$ odd sequence, and it is no longer loop sequence.

Lemma 11: If there exists a ∞-steps non-convergence Collatz odd sequence, the step property of any tail part of the sequence is not possible to always be 1 (or 2).

Prove: Suppose Collatz odd $a \neq 1$. Does i times $(\times 3 + 1) \div 2$ operation get odd $b \neq 1$, then

$b = \frac{3^i \times a + 3^{i-1} + 3^{i-2} \times 2 + ... 3 \times 2^{i-2} + 2^{i-1}}{2^i} = \frac{3^i \times a + 3^i - 2^i}{2^i} = \frac{3^i \times (a+1)}{2^i} - 1$

b is a positive integer, hence

$(a + 1) \mod 2^i \equiv 0$

With definite odd a, this equation does not hold true when $i \to \infty$.

Same way to prove if there exists a ∞-steps non-convergence Collatz odd sequence, the step property of any tail part of the sequence is not possible to always be 2.

Lemma 12: There is not possible to exist a loop Collatz odd sequence or ∞-steps non-convergence Collatz odd sequence.

Prove: If there exists, all odds in the sequence must be bigger than 3, because 3 is very close to 1 in Collatz odd sequence. Change the sequence to $(\times 3 + 2^m - 1) \div 2^k$ odd sequence, expand loop Collatz odd sequence, and then in both cases we get a ∞-steps $(\times 3 + 2^m - 1) \div 2^k$ odd sequence; there are many (1) and (2^+) steps in it.

Select a part sequence from the original $(\times 3 + 2^m - 1) \div 2^k$ odd sequence, odd a in layer $m - 1$ as starting odd, and the last step is (2^+); there are many (1) and (2^+) steps in the middle. Use the common transform position formula to produce a transform position (to layer $m - 1$) sequence. Delete all (1) steps before the last step and change all (2^+) steps to (2) steps. Use the common transform position formula to produce a new z-steps virtual transform position sequence.

According to Lemma 6, the transform position increment ratio of the new transform position sequence is always $\frac{3}{4}$. According to Lemma 9, the final transform position of the original sequence is (can also be gotten from the common transform position formula):

$s_{original} > s_{new} = \frac{a-2^m+1}{2} + (\frac{2^{m+1}-a+2^2-5}{2^3}) \times (1 + \frac{3}{4} + (\frac{3}{4})^2 + ...(\frac{3}{4})^{z-1})$

According to Lemma 11, the original Collatz odd sequence must appear (2^+) steps continuously after 0 or more (1) steps each time; the count of (2^+) steps must be infinite.

When $z \to \infty$,

$s_{original} > s_{new} = \frac{a-2^m+1}{2} + (\frac{2^{m+1}-a-1}{2^3}) \times 4 = 2^{m-1}$

Walk out of the boundary of the $(\times 3 + 2^m - 1) \div 2^k$ odd tree; it is not possible in the real world.

§5 Conclusion

This way, we have proved that the Collatz Conjecture is true. If there exists a loop or ∞-steps non-convergence Collatz odd sequence, change it to $(\times 3 + 2^m - 1) \div 2^k$ odd sequence. Both cases exist ∞ steps, and will finally walk out of the boundary of the $(\times 3 + 2^m - 1) \div 2^k$ odd tree.

References

1. Wikipedia. *Collatz Conjecture* TheFreeDictionary.com mirror. Ia.thefreedictionary.com.
2. Jeffrey C. Lagarias. *The 3x + 1 Problem and Its Generalizations*, American Mathematical Monthly; Vol. 92, No. 1, pp 3-23.(1985).
3. Jeffrey C. Lagarias. *The 3x+1 Problem: An Overview*, arXiv:2111.02635.
4. Jishe Feng. *Solve the 3x+1 problem by the multiplication and division of binary numbers*, doi:10.20944/preprints202301.0163.v3.

 doi:10.20944/preprints202301.0541.v18

5. Baoyuan Duan. *A proof for 3x+1 guess*, vixra:2110.0116V5.
6. Terence Tao. *Almost all orbits of the Collatz map attain almost bounded values*, arVix:1909.03562.