, ,

EKELAND VARIATIONAL PRINCIPLE AND SOME OF ITS EQUIVALENTS ON A WEIGHTED GRAPH, COMPLETENESS AND THE OSC PROPERTY

ALI BASIT, ŞTEFAN COBZAŞ, AND MOKHWETHA DANIEL MABULA

Abstract. We prove a version of Ekeland Variational Principle (EkVP) in a weighted graph G and its equivalence to Caristi fixed point theorem and to Takahashi minimization principle. The usual completeness and topological notions are replaced with some weaker versions expressed in terms of the graph G. The main tool used in the proof is the OSC property for sequences in a graph. Converse results, meaning the completeness of graphs for which one of these principles holds is also considered.

1. Introduction

Ekeland Variational Principle (EkVP) [1] is one of the most important tools in nonlinear analysis that is used to minimize lower semicontinuous and bounded from below functions on a metric space. So far, it has been applied in various contexts, see [2], [3], [4], [5], [6], and the references therein. Due to its usefulness and applicability in mathematics and other related disciplines, EkVP has been studied in different contexts. Not so long ago, Alfuraidan, and Khamsi [7], following the approach taken by Jachymski [8], obtained a version of EkVP in metric spaces endowed with a graph, via the OSC property for sequences. On the other hand the "Completeness problem" (CP) in mathematics is an important problem which is to know under what circumstances the underlying space is complete. CP is linked with "End Problem" (EP). EP is an important problem in behavioral sciences which is to determine where and when human dynamics defined as succession of positions that starts from an initial position and follows transitions ends somewhere. For more on the relationship between CP and EP, we refer the interested reader to [9]. Notice that a thorough analysis of various situations when the validity of a result (variational principle, fixed point) on a metric space forces its completeness was given in [10].

We start by recalling Ekeland Variational Principle in its weak version and in its full one as well.

Theorem 1.1 (Ekeland Variational Principle - weak form (wEkVP)). *Let (X,d) be a complete metric space and $f : X \to \mathbb{R}\cup\{+\infty\}$ a lsc and bounded from below proper function. Then for every $\varepsilon > 0$ there exists an element $y_\varepsilon \in X$ such that*

$$f(y_\varepsilon) \le \inf f(X) + \varepsilon, \tag{1.1}$$

Date: January 25, 2023.

1991 *Mathematics Subject Classification.* Primary 58E30; Secondary 47H10, 05C22, 06F30, 54E50.

Key words and phrases. Ekeland variational principle, Takahashi minimization principle, Caristi fixed point theorem, weighted graph, partially ordered metric space, completeness, the OSC property.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 26 January 2023 doi:10.20944/preprints202301.0484.v1

and

$$f(y_\varepsilon) < f(y) + \varepsilon d(y, y_\varepsilon), \quad \forall y \in X \setminus \{y_\varepsilon\}. \tag{1.2}$$

The full version of EkVP is the following.

Theorem 1.2 (Ekeland Variational Principle – EkVP)**.** *Let (X,d) be a complete metric space and $f : X \to \mathbb{R} \cup \{+\infty\}$ a lsc bounded below proper function. Let $\varepsilon > 0$ and $x_0 \in \operatorname{dom} f$.*

Then given $\lambda > 0$ there exists $z = z_{\varepsilon,\lambda} \in X$ such that

$$\begin{aligned} &\text{(a)} \quad f(z) + \frac{\varepsilon}{\lambda} d(z, x_0) \le f(x_0); \\ &\text{(b)} \quad \forall x \in X,\ x \neq z,\ \ f(z) < f(x) + \frac{\varepsilon}{\lambda} d(z, x). \end{aligned} \tag{1.3}$$

If further, x_0 satisfies the condition

$$f(x_0) \le \inf f(X) + \varepsilon, \tag{1.4}$$

then

$$\text{(c)} \quad d(z, x_0) \le \lambda.$$

A function $f : X \to \mathbb{R} \cup \{\infty\}$ is called *proper* if $\operatorname{dom}(f) := \{x \in X : \varphi(x) \in \mathbb{R}\}$ is nonempty.

Remark 1.3. Notice that condition (a) in the above theorem implies $f(z) \le f(x_0)$. Sometimes EkVP is formulated with this condition instead of (a).

An important consequence of the full version of EkVP is obtained by taking $\lambda = \sqrt{\varepsilon}$ in Theorem 1.2.

Corollary 1.4. *Under the hypotheses of Theorem 1.2, for every $\varepsilon > 0$ and $x_0 \in X$ with $f(x_0) \le \inf f(X) + \varepsilon$ there exists $y_\varepsilon \in X$ such that*

$$\begin{aligned} &\text{(a)} \ \ f(y_\varepsilon) + \sqrt{\varepsilon} d(y_\varepsilon, x_0) \le f(x_0); \\ &\text{(b)} \ \ \forall x \in X,\ x \neq y_\varepsilon,\ \ f(y_\varepsilon) < f(x) + \sqrt{\varepsilon} d(y_\varepsilon, x); \\ &\text{(c)} \ \ d(y_\varepsilon, x_0) \le \sqrt{\varepsilon}. \end{aligned} \tag{1.5}$$

Taking further $\varepsilon_n = 1/n$, one obtains a sequence (y_n) approximating the minimum point of the function f.

Note that the validity of Ekeland Variational Principle (in its weak form) implies the completeness of the metric space X. This was discovered by Weston [11] in 1977 and rediscovered by Sullivan [12] in 1981 (see also the survey [13]).

More exactly, the following result holds.

Theorem 1.5. *For a metric space (X,d) the following are equivalent.*

1. *The metric space (X,d) is complete.*
2. *For every $\varepsilon > 0$ there exists an element $x_\varepsilon \in X$ such that*

$$f(x_\varepsilon) < f(x) + \varepsilon d(x, x_\varepsilon), \quad \forall x \in X \setminus \{x_\varepsilon\}. \tag{1.6}$$

Ekeland Variational Principle is equivalent to many results in fixed point theory, geometry of Banach spaces (the drop theorem) and optimization. We mention only two of these, namely Caristi fixed point theorem and Takahashi minimization principle.

We start with Caristi fixed point theorem [14] (see also [15]), presented both in single-valued and set-valued versions.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 26 January 2023 doi:10.20944/preprints202301.0484.v1

Theorem 1.6 (Caristi fixed point theorem)**.** *Let (X,d) be a complete metric space and $\varphi : X \to [0,\infty)$ be a lsc function.*

1. *If $T : X \to X$ is a mapping satisfying the condition*

$$\forall x \in X, \;\; d(T(x), x) \le \varphi(x) - \varphi(T(x)) \,, \tag{1.7}$$

then T has a fixed point, i.e. there exists $z \in X$ such that $z = T(z)$.

2. *If $T : X \rightrightarrows X$ is a set-valued mapping satisfying the condition*

$$\forall x \in X, \;\; \exists y \in T(x), \;\; d(y,x) \le \varphi(x) - \varphi(y) \,, \tag{1.8}$$

then T has a fixed point, i.e. there exists $z \in X$ such that $z \in T(z)$.

Another result is Takahashi minimization principle [16] (see also [17]).

Theorem 1.7 (Takahashi minimization principle)**.** *Let (X,d) be a complete metric space and let $\varphi : X \to [0,\infty)$ be a lsc function.*

If for every $x \in X$ satisfying the condition $\varphi(x) > \inf \varphi(X)$ there exists $y \in X \smallsetminus \{x\}$ such that

$$\varphi(y) + d(x,y) \le \varphi(x) \,, \tag{1.9}$$

then φ attains its infimum on X, i.e. there exists $z \in X$ such that $\varphi(z) = \inf \varphi(X)$.

Remark 1.8. Replacing φ with $\tilde{\varphi} = \varphi - \inf \varphi(X)$, the above results can be automatically extended to a lower bounded lsc function $\varphi : X \to \mathbb{R}$. Considering an extended proper lsc function $\varphi : X \to \mathbb{R} \cup \{\infty\}$, then the results hold on $\mathrm{dom}(\varphi) := \{x \in X : \varphi(x) \in \mathbb{R}\}$.

Recall that a function $\varphi : X \to \mathbb{R} \cup \{\infty\}$ is called proper if $\mathrm{dom}(\varphi) \neq \emptyset$.

2. Preliminaries

In this section we present some notions and results on partially ordered metric spaces and graph theory, needed for the main results of the paper.

A *partial order* on a nonempty set X is a reflexive, transitive and antisymmetric relation $\preceq$ on X. One also says that $(X,\preceq)$ is a *partially ordered set.* We consider orders $\preceq$ on a metric space (X,d) in which case we shall say that $(X,d,\preceq)$ is a partially ordered metric space. Initially, no relation between the order $\preceq$ and the metric d is supposed to hold but, in order to make the things to work, some connections are needed (see Definition 2.2).

We present now some preliminary notions from graph theory. A good introductory text, with many examples, is [18, Chapter 8] (see also [19] or [20]).

A *directed graph* (*digraph* for short) G is formed by a set $V(G)$, called the set of *vertices* of the graph G, and a set $E(G)$ of *edges* corresponding to ordered pairs $(u,v) \in V(G) \times V(G)$. The graph will be denoted by $G = (V(G), E(G))$. Two edges e_1, e_2 are called *parallel* if they correspond to the same pair (u,v). In this paper we shall always suppose that the graph G has no parallel edges, so that the set $E(G)$ can be viewed as a subset of $V(G) \times V(G)$. A *path* in the graph G connecting two points $u, v \in V(G)$ is a succession $(u,u_1), (u_1,u_2), \ldots, (u_n, v)$ of edges, where $n \ge 1$ and the points $u, u_1, \ldots, u_n, v$ are pairwise distinct. A path with $v = u$ is called a *cycle*. A graph without cycles is called *acyclic*.

A graph $G = (V(G), E(G))$ is called:

- *reflexive* if $(u,u) \in E(G)$ for all $u \in V(G)$;
- *transitive* if (u,v), $(v,w) \in E(G)$ implies $(u,w) \in E(G)$.

A *weight* on a graph $G = (V(G), E(G))$ is a function (usually non-negative) $d : E(G) \to \mathbb{R}$. We say that $G = (V(G), E(G), d)$ is a *weighted graph.* In this paper we shall suppose that the graph is contained in a metric space (X, d) and that the weight is the metric d.

The next remark emphasizes the tight connection between graphs and partial orders.

Remark 2.1. Let $(X, \preceq)$ be a partially ordered set. Put $V(G) = X$ and $E(G) = \{(x, y) \in X \times X : x \preceq y\}$. Then $G = (V(G), E(G))$ is a reflexive transitive acyclic directed graph.

Conversely, given a reflexive transitive acyclic directed graph $G = (V(G), E(G))$ put

$$x \le_G y \iff (x, y) \in E(G)\,. \tag{2.1}$$

Then $\le_G$ is a partial order on $V(G)$ and the graph $\tilde{G}$ associated to the order $\le_G$ in the way described above agrees with G.

Consequently, there is a one-to-one correspondence between reflexive transitive acyclic directed graphs and partial orders.

Only the relation between acyclicity and antisymmetry needs some explanation, the others (reflexivity, transitivity) being obvious.

Suppose that G is the graph corresponding to a partial order $\preceq$ in the way described in Remark 2.1. If $u, u_1, \ldots, u_n, u$ is a loop in G, then, by the definition of $E(G)$, $u \preceq u_1 \preceq u$ with $u_1 \ne u$, in contradiction to the antisymmetry of $\preceq$. This shows that the graph G is acyclic.

Conversely, let $\le_G$ be the partial order associated to a graph G (having the properties mentioned in Remark 2.1). If $u \le_G v$ and $v \le_G u$, for some $u \ne v$ in $V(G)$, then (u, v), $(v, u) \in E(G)$, i.e. u, v, u is a loop in G, in contradiction to the acyclicity of G.

We introduce now some notions in weighted graphs and their analogs in partially ordered metric spaces.

Definition 2.2.

- A weighted digraph $G = (V(G), E(G), d)$, where d is a metric on $V(G)$, is said to satisfy the OSC property if $(x, x_n) \in E(G)$ for all $n \in \mathbb{N}$, for every sequence (x_n) in $V(G)$ such that $x_n \xrightarrow{d} x$ and $(x_{n+1}, x_n) \in E(G)$ for all $n \in \mathbb{N}$.
- A partially ordered metric space $(X, d, \preceq)$ is said to satisfy the OSC condition if $x \preceq x_n$ for all $n \in \mathbb{N}$, for every sequence (x_n) in X such that $x_n \xrightarrow{d} x$ and $x_{n+1} \preceq x_n$ for all $n \in \mathbb{N}$.

Remark 2.3. In [7] the OSC property for graphs is defined with the supplementary condition that $(y, x_n) \in E(G)$, $n \in \mathbb{N}$, implies $(y, x) \in E(G)$. In a partially ordered metric space this corresponds to the condition that $x = \inf_n x_n$.

The condition OSC, as given in Definition 2.2, is also used by Jachymski [8].

For partially ordered metric spaces the OSC condition was introduced in [21] in the study of fixed points for contractions on ordered metric spaces. Some authors consider a weakened version of the OSC, where one asks that the conclusion holds only for a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of (x_n). As remarked Jachymski [8], the transitivity of $\preceq$ implies the equivalence of these conditions. Also, if $\preceq$ is a reflexive relation on the metric space (X, d) satisfying the OSC property, then $\preceq$ is transitive.

 doi:10.20944/preprints202301.0484.v1

We introduce now definitions of completeness and of some topological notions expressed in terms of the graph and of the partial order.

Definition 2.4. Let $G = (V(G), E(G), d)$ be a weighted digraph, where d is a metric on $V(G)$.

- A sequence (x_n) in $V(G)$ is called a G-*sequence* if $(x_{n+1}, x_n) \in E(G)$ for all $n \in \mathbb{N}$.
- A G-*Cauchy sequence* is a G-sequence that is Cauchy with respect to d.
- A subset Y of $V(G)$ is called G-*closed* if $y \in Y$ for every G-sequence (y_n) in Y, d-convergent to $y \in V(G)$.
- A function $f : V(G) \to \mathbb{R}$ is called G-*continuous* (G-lsc) if $\lim_n f(x_n) = f(x)$ (resp. $f(x) \le \liminf_n f(x_n)$) for every G-sequence (x_n) in $V(G)$ d-convergent to x.

In a partially ordered metric space $(X, d, \preceq)$ the notions of *decreasingly Cauchy sequence*, *decreasingly closed set*, *decreasingly continuous* (or lsc) function, can be defined in a similar way by replacing the condition $(x_{n+1}, x_n) \in E(G)$ with $x_{n+1} \preceq x_n$.

Let (X, d) be a metric space and $\varphi : X \to [0, \infty)$ be a function. We define a partial order $\le_\varphi$ on X by

$$\begin{aligned} x \le_\varphi y \iff & d(x, y) \le \varphi(y) - \varphi(x) \\ \iff & \varphi(x) + d(x, y) \le \varphi(y). \end{aligned} \tag{2.2}$$

Remark 2.5. The relation (2.2) is a partial order on X, called by some authors the Brønsted order (see [22], [23]). It is related to the Bishop-Phelps theorem on the denseness of the support functionals of a closed bounded convex subset of a Banach space, see [24], [25].

The following proposition contains some simple remarks about $\le_\varphi$.

Proposition 2.6. *Let* (X, d) *be a metric space,* $\varphi : X \to [0, \infty)$ *and let* $\le_\varphi$ *be defined by* (2.2)*.*

1. *The relation* $\le_\varphi$ *is a partial order on* X*.*
2. *Every* $\le_\varphi$*-decreasing sequence in* X *is Cauchy.*
3. *If* φ *is lsc, then* $y \le_\varphi x$ *for every sequence* (y_n) *in* X *satisfying* $y_n \le_\varphi x$ *and* $y_n \xrightarrow{d} y$*. Also, the partial order* $\le_\varphi$ *satisfies OSC.*

Proof. The proof of 1 is a straightforward verification.

2. Let (x_n) be a sequence in X such that $x_{n+1} \le x_n$ for all $n \in \mathbb{N}$. The inequality

$$0 \le d(x_n, x_{n+1}) \le \varphi(x_n) - \varphi(x_{n+1})$$

shows that $(\varphi(x_n))_{n\in\mathbb{N}}$ is a decreasing sequence in $\mathbb{R}_+$, so convergent and, hence, Cauchy. The transitivity of $\le_\varphi$ implies $x_{n+k} \le_\varphi x_n$, that is,

$$d(x_n, x_{n+k}) \le \varphi(x_n) - \varphi(x_{n+k}),$$

an inequality which shows that (x_n) is d-Cauchy.

3. We have

$$y_n \le_\varphi x \iff \varphi(y_n) + d(y_n, x) \le \varphi(x).$$

Taking into account the lsc of the function φ, one obtains

$$\varphi(y) + d(y, x) \le \liminf_n [\varphi(y_n) + d(y_n, x)] \le \varphi(x),$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 26 January 2023 doi:10.20944/preprints202301.0484.v1

that is, $y \le_\varphi x$.

Let now (y_n) be a sequence in X such that $y_{n+1} \le_\varphi y_n$, $n \in \mathbb{N}$, and $y_n \xrightarrow{d} y$. By transitivity $y_{n+k} \le_\varphi y_n$, $n, k \in \mathbb{N}$, which, fixing n and letting $k \to \infty$, yields $y \le_\varphi y_n$ for all $n \in \mathbb{N}$. □

Remark 2.7. Supposing φ only decreasingly lsc, then 3 holds for convergent decreasing sequences only. A similar result holds for G-sequences in a weighted graph G.

3. Ekeland Variational Principle in metric spaces endowed with a graph

The following theorem is an extension of a result proved in [7, Theorem 3.3] (see also [26]). The main modification consists in the replacement of topological and completeness conditions with their G-versions. A property $\mathcal{P}$ on a set X can be interpreted as a function $\mathcal{P} : X \to \{0, 1\}$, where $\mathcal{P}(x) = 1$ means that x has property $\mathcal{P}$, while $\mathcal{P}(x) = 0$ means that x does not have it. Then $\mathrm{Dom}(\mathcal{P}) = \{x \in X : \mathcal{P}(x) = 1\}$. If X is a topological space, then we say that $\mathcal{P}$ is closed if $\mathrm{Dom}(\mathcal{P})$ is closed.

The following example will be used to prove the equivalence of EkVP in weighted graphs to that in ordered metric spaces (see Theorem 3.5).

Example 3.1. Let $G = (V(G), E(G), d)$ be a weighted digraph, where d is a metric on $V(G)$ and let $\varphi : X \to [0, \infty)$ be a function. For $\varepsilon > 0$ we define a property $\mathcal{P}$ on X by

$$\mathcal{P}(x) \text{ holds for } x \in V(G) \iff \varphi(x) \le \inf \varphi(X) + \varepsilon\,.$$

Then

$$\mathrm{Dom}(\mathcal{P}) = \{x \in X : \varphi(x) \le \inf \varphi(X) + \varepsilon\}\,.$$

The set $\mathrm{Dom}(\mathcal{P})$ is closed (G-closed) provided the function φ is lsc (G-lsc).

The closedness property of $\mathrm{Dom}(\mathcal{P})$ follows from the fact that a function φ is lsc (G-lsc) if and only if the set $\{x \in X : \varphi(x) \le \alpha\}$ is closed (G-closed) for every $\alpha \in \mathbb{R}$.

Theorem 3.2. *Let $G = (V(G), E(G), d)$ be a reflexive transitive acyclic weighted digraph, where d is a G-complete metric on $V(G)$ having the OSC property for G-sequences. Consider a G-closed property $\mathcal{P}$ on $V(G)$ such that $\mathrm{Dom}(\mathcal{P})$ is nonempty and a G-lower semi-continuous function $\varphi : V(G) \to [0, \infty)$. For any given $\varepsilon > 0$ and $\lambda > 0$ let $x_0 \in \mathrm{Dom}(\mathcal{P})$ be such that*

$$\varphi(x_0) \le \inf \varphi(\mathrm{Dom}(\mathcal{P})) + \varepsilon \tag{3.1}$$

Then there exists $z \in \mathrm{Dom}(\mathcal{P})$ such that

$$\begin{aligned} &\text{(i)} \quad \varphi(z) + \lambda^{-1}\varepsilon\, d(x_0, z), \le \varphi(x_0),\\ &\text{(ii)} \quad d(x_0, z) \le \lambda,\\ &\text{(iii)} \quad \varphi(z) < \varphi(x) + \lambda^{-1}\varepsilon\, d(x, z), \end{aligned} \tag{3.2}$$

for all $x \in \mathrm{Dom}(\mathcal{P})$ such that $x \neq z$ and $(x, z) \in E(G)$.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 26 January 2023 doi:10.20944/preprints202301.0484.v1

Proof. Put, for convenience, $Y = \mathrm{Dom}(\mathcal{P})$. Let also $\gamma = \varepsilon/\lambda$ and $d_\gamma = \gamma \cdot d$. Then d_γ is a metric on $V(G)$, Lipschitz equivalent to d, so that all the properties holding for d holds for d_γ too. Define the partial order $\le_\gamma$ by (2.2) with d_γ instead of d. For $x \in Y$ let

$$S(x) = \{y \in Y : y \le_\gamma x \text{ and } (y,x) \in E(G)\}\,. \tag{3.3}$$

Claim I. *The sets $S(x)$ have the following properties*:

$$\begin{aligned} &\text{(i)} \quad y \in S(x) \textit{ implies } S(y) \subseteq S(x) \quad \textit{and} \quad \varphi(y) + d_\gamma(y,x) \le \varphi(x); \\ &\text{(ii)} \quad S(x) \textit{ is } G\textit{-closed.} \end{aligned} \tag{3.4}$$

Indeed, let $y \in S(x)$ and $z \in S(y)$. Then

$$\begin{aligned} &y \le_\gamma x \quad \text{and} \quad (y,x) \in E(G), \\ &z \le_\gamma y \quad \text{and} \quad (y,z) \in E(G)\,, \end{aligned}$$

so that, by the transitivity of $\le_\gamma$ and of $E(G)$,

$$z \le_\gamma x \text{ and } (z,x) \in E(G)\,,$$

that is, $z \in S(x)$.

Let (y_n) be a G-sequence in $S(x)$, d-convergent to some $y \in Y$. Then $y_n \le_\gamma x$ for all $n \in \mathbb{N}$, so that, by the G-lsc of φ, $y \le_\gamma x$ (see Proposition 2.6 and Remark 2.7). Also, $(y_n, x) \in E(G)$ and, by the OSC, $(y, y_n) \in E(G)$, so that, by transitivity, $(y,x) \in E(G)$.

Consequently, $y \in S(x)$, showing that $S(x)$ is G-closed.

Let $J(x) = \inf \varphi(S(x))$. We define now inductively a sequence of sets in Y.

Choose $x_1 \in S(x_0)$ such that

$$\varphi(x_1) \le J(x_0) + \frac{1}{2}\,,$$

and let $x_{n+1} \in S(x_n)$ be such that

$$\varphi(x_{n+1}) \le J(x_n) + \frac{1}{2^{n+1}},\ n \in \mathbb{N}.$$

Then (x_n) satisfies

$$x_{n+1} \le_\gamma x_n \quad \text{and} \quad (x_{n+1}, x_n) \in E(G),$$

for all $n \ge 0$, so that, By Proposition 2.6, it is a G-Cauchy sequence. Since Y is a G-closed subset of $V(g)$, it follows that it is also G-complete, so that the sequence (x_n) is d-convergent to some $z \in Y$.

Since $x_{n+k} \in S(x_{n+k}) \subseteq S(x_n)$ and $S(x_n)$ is G-closed, it follows $z \in S(x_n)$ for all $n \ge 0$. From $z \in S(x_0)$ and Claim I, $\varphi(z) + \lambda^{-1}\varepsilon\, d(x_0, z) \le \varphi(x_0)$, i.e. the inequality (i) from (3.2) holds true.

Let us show that

$$S(z) = \{z\}\,. \tag{3.5}$$

Let $x \in S(z)$. We have $S(z) \subseteq S(x_n)$ for all $n \ge 0$, so that, taking into account the choice of the elements x_n, one obtains

$$\varphi(x_{n+1}) - \frac{1}{2^{n+1}} \le J(x_n) \le \varphi(x)\,,$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 26 January 2023 doi:10.20944/preprints202301.0484.v1

implying $\varphi(x_{n+1})-\varphi(x)\le\frac{1}{2^{n+1}}$. Since $x\le_\gamma x_{n+1}$, we have

$$d_\gamma(x,x_{n+1})\le\varphi(x_{n+1})-\varphi(x)\le\frac{1}{2^{n+1}},$$

for all $n\ge0$. Letting $n\to\infty$, one obtains $d_\gamma(x,z)=0$ and so $x=z$.

Suppose now that $x\in Y$ is such that $x\ne z$ and $(x,z)\in E(G)$. Then, by (3.5), $x\notin S(z)$, so that the inequality $x\le_\gamma z$ fails, that is

$$\gamma d(x,z)>\varphi(z)-\varphi(x)\,,$$

which is equivalent to the inequality (iii) in (3.2).

Since $z\in S(x_0)$ we have $z\le_\gamma x_0$, so that, by (3.1),

$$\frac{\varepsilon}{\lambda}\,d(z,x_0)\le\varphi(x_0)-\varphi(z)\le\varphi(x_0)-\inf\varphi(Y)\le\varepsilon\,.$$

Hence $d(z,x_0)\le\lambda$. □

Taking $\mathcal{P}$ on X with $\operatorname{Dom}(\mathcal{P})=X$ one obtains the following weak form of EkVP in weighted graphs.

Corollary 3.3. *Let $G=(V(G),E(G),d)$ be a weighted graph satisfying the hypotheses of theorem 3.2. Then for every $\varepsilon,\lambda>o$ there exists $z\in X$ such that*

$$\varphi(z)<\varphi(x)+\lambda^{-1}\varepsilon\,d(x,z), \tag{3.6}$$

for all $x\in X\setminus\{z\}$ with $(x,z)\in E(G)$.

Remark 3.4. Similar results for equilibrium versions of Ekeland Variational Principle were obtained by Alfuraidan and Khamsi [2].

Theorem 3.5. *Consider the following statements.*

1. *For every reflexive transitive acyclic weighted digraph $G=(V(G),E(G),d)$, where d is a G-complete metric on $V(G)$ such that the OSC property for G-sequences is satisfied, the following property holds.*

   (A$_1$) *For any G-closed property $\mathcal{P}$ on $V(G)$ such that $\operatorname{Dom}(\mathcal{P})$ is nonempty, every G-lower semi-continuous function $\varphi:V(G)\to[0,\infty)$, any given $\varepsilon>0$ and $\lambda>0$ and $x_0\in\operatorname{Dom}(\mathcal{P})$ such that*

   $$\varphi(x_0)\le\inf\varphi(\operatorname{Dom}(\mathcal{P}))+\varepsilon\,,$$

   there exists $z\in\operatorname{Dom}(\mathcal{P})$ such that

   $$\begin{aligned} &\text{(i)} && \varphi(z)+\lambda^{-1}\varepsilon\,d(x_0,z)\le\varphi(x_0),\\ &\text{(ii)} && d(z,x_0)\le\lambda,\\ &\text{(iii)} && \varphi(z)<\varphi(x)+\lambda^{-1}\varepsilon\,d(x,z), \end{aligned} \tag{3.7}$$

   for all $x\in\operatorname{Dom}(\mathcal{P})$ with $x\ne z$ and $(x,z)\in E(G)$.

2. *For every partially ordered decreasingly complete metric space $(X,d,\preceq)$ with the OSC property for decreasing sequences, the following property holds.*

   (A$_2$) *For any decreasingly lower semi-continuous function $\varphi:X\to[0,\infty)$, any $\varepsilon>0$, any $\lambda>0$, and any $x_0\in X$ satisfying*

   $$\varphi(x_0)\le\inf\varphi(X)+\varepsilon$$

 doi:10.20944/preprints202301.0484.v1

there exists $z \in X$ such that

$$\begin{aligned} &\text{(i)} \quad \varphi(z) + \lambda^{-1}\varepsilon\, d(x_0, z) \le \varphi(x_0), \\ &\text{(ii)} \quad d(z, x_0) \le \lambda, \\ &\text{(iii)} \quad \varphi(z) < \varphi(x) + \varepsilon\lambda^{-1} d(x, z), \end{aligned} \tag{3.8}$$

for all $x \in X$ with $x \neq z$ and $x \preceq z$.

3. *For every complete metric space (X, d) the following property holds.*

   (A_3) *For any lsc function $\varphi : X \to [0, \infty)$, any $\varepsilon > 0$ and any $x_0 \in X$ satisfying*

   $$\varphi(x_0) \le \inf \varphi(X) + \varepsilon$$

   there exists $x_\varepsilon \in X$ such that

   $$\begin{aligned} &\text{(i)} \quad \varphi(x_\varepsilon) + \varepsilon\, d(x_\varepsilon, z) \le \varphi(x_0), \\ &\text{(ii)} \quad d(x_\varepsilon, x_0) \le 1, \\ &\text{(iii)} \quad \varphi(x_\varepsilon) < \varphi(x) + \varepsilon\, d(x, x_\varepsilon), \end{aligned} \tag{3.9}$$

   for all $x \in X$ with $x \neq x_\varepsilon$.

4. *For every complete metric space (X, d) the following property holds.*

   (A_4) *For any lsc function $\phi : X \to [0, \infty)$ and for any $\varepsilon > 0$ there exists $x_\varepsilon \in X$ such that*

   $$\phi(x_\varepsilon) \le \inf \phi(X) + \varepsilon$$

   and

   $$\varphi(x_\varepsilon) < \varphi(x) + \varepsilon\, d(x, x_\varepsilon),$$

   for all $x \in X$ with $x \neq x_\varepsilon$.

Then

$$1 \iff 2 \Rightarrow 3 \Rightarrow 4\,.$$

Proof. $1 \Rightarrow 2$. Take $V(G) = X$ and

$$E(G) = \{(x, y) \in X \times X : x \preceq y\}.$$

Then $G = (V(G), E(G))$ is a reflexive transitive acyclic digraph (see Remark 2.1). The completeness of $(X, d, \preceq)$ for decreasing Cauchy sequences implies the G-completeness of $(V(G), d)$.

Also, by the definition of the graph G, the OSC property for decreasing sequences in $(X, d, \preceq)$ implies the OSC property for G-sequences of the graph G.

Define now $\mathcal{P}$ as in Example 3.1, i.e. $\mathcal{P}(x)$ holds if and only if $\varphi(x) \le \inf \varphi(X) + \varepsilon$. Then $\mathrm{Dom}(\mathcal{P})$ is nonempty as φ is bounded below (by 0) and decreasingly closed (because φ is decreasingly lsc), hence $\mathrm{Dom}(\mathcal{P})$ is G-closed in G.

Now, a direct application of 1 yields the first two inequalities in (3.8) as well as the third one, but only for $x \in \mathrm{Dom}(\mathcal{P})$ with $x \neq z$ and $x \preceq z$. If $x \notin \mathrm{Dom}(\mathcal{P})$, then

$$\varphi(x) > \inf \varphi(X) + \varepsilon \ge \varphi(x_0) \ge \varphi(z),$$

so that

$$\varphi(z) < \varphi(x) < \varphi(x) + \varepsilon\lambda^{-1} d(x, z),$$

(no matter x satisfies $x \preceq z$ or not).

Consequently, the third inequality holds for all $x \in X$ with $x \neq z$ and $x \preceq z$.

$2 \Rightarrow 1$. Suppose that we are given a weighted graph $G(V(G), E(G), d)$, a function $\varphi : V(G) \to [0, \infty)$ and a property $\mathcal{P}$ on $V(G)$ such that the hypotheses from 1 hold for these data.

Define $X = \text{Dom}(\mathcal{P})$ and the partial order $\preceq$ by

$$x \preceq y \iff (x, y) \in E(G),$$

for $x, y \in X$. Again, the G-closedness of X and the G-completeness of G, imply the G-completeness of (X, d), and so the completeness of (X, d) for decreasing Cauchy sequences. The fact that φ is decreasingly lsc is a direct consequence of G-lower smicontinuity of φ.

Now, the conclusions from (A_1) follows from those in (A_2).

$2 \Rightarrow 3$. Define an order on X by

$$x \preceq y \iff d(x, y) \le \varphi(x) - \varphi(y).$$

Then $\preceq$ is a partial order on X which, by Proposition 2.6, satisfies OSC. The result follows now from 2 with $\lambda = 1$.

Indeed, by 2, the third inequality in (3.11) holds for all $x \in X \setminus \{x_\varepsilon\}$ with $x \preceq x_\varepsilon$ and, by the definition of the order $\preceq$, it is automatically satisfied for all $x \in X$ for which the inequality $x \preceq x_\varepsilon$ fails.

$3 \Rightarrow 4$. Notice that 4 is a particular case of 3. □

Theorem 3.6. *The following properties hold.*

1. *Let $G = (V(G), E(G), d)$ be a reflexive transitive acyclic digraph, where d is a metric on G such that the OSC property holds on G. The following are equivalent:*
   (i) *The metric space $(V(G), d)$ is complete.*
   (ii) *For any lower semi-continuous function $\varphi : V(G) \to [0, \infty)$ and any $\varepsilon > 0$ there exists $x_\varepsilon \in V(G)$ such that*

$$\varphi(x_\varepsilon) < \varphi(x) + \varepsilon\, d(x, x_\varepsilon), \tag{3.10}$$

   for all $x \in V(G)$ with $x \ne x_\varepsilon$ and $(x, x_\varepsilon) \in E(G)$.
2. *Let now (X, d) be a metric space. The following are equivalent:*
   (i) *The metric space (X, d) is complete.*
   (ii) *For any partial order $\preceq$ on X satisfying the OSC property, any continuous function $\varphi : X \to [0, \infty)$ and any $\varepsilon > 0$ there exists $x_\varepsilon \in X$ such that*

$$\varphi(x_\varepsilon) < \varphi(x) + \varepsilon\, d(x, x_\varepsilon), \tag{3.11}$$

   for all $x \in X$ with $x \ne x_\varepsilon$ and $x \preceq x_\varepsilon$.
3. *For a metric space (X, d) the following are equivalent:*
   (i) *The metric space (X, d) is complete.*
   (ii) *For any continuous function $\varphi : X \to [0, \infty)$, any $\varepsilon > 0$, $\lambda > 0$ and any $x_0 \in X$ satisfying*

$$\varphi(x_0) \le \inf \varphi(X) + \varepsilon$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 26 January 2023 doi:10.20944/preprints202301.0484.v1

there exists $z \in X$ such that

$$\begin{aligned} &\varphi(z) + \lambda^{-1}\varepsilon \le \varphi(x_0), \\ &d(x_\varepsilon, x_0) \le \lambda, \\ &\varphi(z) < \varphi(x) + \lambda^{-1}\varepsilon d(x,z), \end{aligned} \tag{3.12}$$

for all $x \in X$ with $x \neq z$.

(iii) *For any continuous function $\varphi : X \to [0,\infty)$ and for any $\varepsilon > 0$ there exists $x_\varepsilon \in X$ such that*

$$\varphi(x_\varepsilon) < \varphi(x) + \varepsilon d(x, x_\varepsilon)\,,$$

for all $x \in X$ with $x \neq x_\varepsilon$.

Proof. 3. (i) $\Rightarrow$ (ii) is Ekeland Variational principle, while (ii) $\Rightarrow$ (iii) is trivial. The implication (iii) $\Rightarrow$ (i) is Sullivan's result [12] (see Theorem 1.5).

2. We suppose that (ii) from 2 holds for the metric space (X,d) and show that, in this case, statement (iii) from 3 holds, which will imply the completeness of (X,d).

Let $\varphi : X \to [0,\infty)$ be a continuous function and $\varepsilon > 0$.

On the metric space (X,d) consider the order

$$x \preceq y \iff d(x,y) \le \varphi(y) - \varphi(x)\,.$$

Since φ is continuous, the OSC property holds in $(X,d,\preceq)$ (by Proposition 2.6). It follows that there exists $x_\varepsilon \in X$ such that the inequality (3.11) holds. Since $d(x,x_\varepsilon) > \varphi(x_\varepsilon) - \varphi(x)$ for all $x \in X$ for which $x \preceq x_\varepsilon$ fails, it follows that the inequality (3.11) holds for all $x \in X \setminus \{x_\varepsilon\}$ with $x \neq x_\varepsilon$. Consequently, the statement (ii) from 3 holds, implying the completeness of (X,d).

1. The relation

$$x \preceq y \iff (x,y) \in E(G)$$

establishes a one-to-one correspondence between reflexive transitive acyclic weighted graphs and partially ordered metric spaces (see Remark 2.1) as well as the equivalence between the properties expressed in terms of the graph and those expressed in terms of the order. Consequently, 1 is a rephrasing of 2 in terms of graphs. □

4. Caristi fixed point theorem and Takahashi minimization principle on weighted graphs

In this section we present versions of Caristi fixed point theorem (Theorem 1.6) and Takahashi minimization principle (Theorem 1.7) on weighted graphs.

We start with Caristi fixed point theorem.

Theorem 4.1. *Let $G = (V(G), E(G), d)$ be a reflexive transitive acyclic digraph, where d is a G-complete metric on $V(G)$ such that the OSC property for G-sequences holds on G and let $\varphi : X \to [0,\infty)$ be a G-lsc function.*

1. *If $T : X \to X$ is a mapping satisfying the condition*

$$\forall x \in X,\ \ d(T(x),x) \le \varphi(x) - \varphi(T(x))\,, \tag{4.1}$$

then T has a fixed point, i.e. there exists $z \in X$ such that $z = T(z)$.

2. *If $T : X \rightrightarrows X$ is a set-valued mapping such that for every $x \in X$ there exists $y \in T(x)$ satisfying the condition*

$$(y,x) \in E(G) \ \text{ and } \ d(y,x) \le \varphi(x) - \varphi(y)\,, \tag{4.2}$$

then T has a fixed point, i.e. there exists $z \in X$ such that $z \in T(z)$.

Proof. Consider again the order $\le_\varphi$ given by

$$x \le_\varphi y \iff d(x,y) \le \varphi(y) - \varphi(x)\,,$$

and for $x \in V(G)$ let

$$S(x) = \{y \in V(G) : y \le_\varphi x \ \text{ and } \ (y,x) \in E(G)\}\,.$$

Since a single-valued mapping $T : X \to X$ can be viewed as a set-valued one $x \mapsto \{T(x)\}$, $x \in X$, conditions (4.1) and (4.2) can be expressed as

$$\forall x \in X,\ \ S(x) \cap T(x) \ne \emptyset\,,$$

implying that 1 is a particular case of 2.

To prove 2 we appeal to Corollary 3.3. Observe that condition (3.6) for $\lambda = \varepsilon = 1$ can be expressed as: there exists $z \in X$ such that

$$S(z) = \{z\}\,.$$

But then the condition $S(z) \cap T(z) \ne \emptyset$ means that $z \in T(z)$, i.e. z is a fixed point for T. □

Remark 4.2. There exists a stronger version of Caristi fixed point theorem for set-valued mappings, namely by asking that, for all $x \in X$, $T(x) \ne \emptyset$ and (4.1) holds for all $y \in T(x)$. In this case the conclusion is that there exists $z \in X$ such that $T(z) = \{z\}$.

Indeed, in terms of the set $S(x)$ condition (4.1) means in this case that $\emptyset \ne T(x) \subseteq S(x)$. Then the existence of $z \in X$ such that $S(z) = \{z\}$ implies $T(z) = \{z\}$.

Theorem 4.3. *Let $G = (V(G), E(G), d)$ be a reflexive transitive acyclic digraph, where d is a G-complete metric on $V(G)$ such that the OSC property for G-sequences holds on G and let $\varphi : X \to [0,\infty)$ be a G-lsc function.*

If for every $x \in V(G)$ satisfying the condition $\varphi(x) > \inf \varphi(X)$ there exists $y \in V(G) \setminus \{x\}$ such that

$$(y,x) \in E(G) \ \text{ and } \ \varphi(y) + d(x,y) \le \varphi(x)\,, \tag{4.3}$$

then φ attains its infimum on $V(G)$, i.e. there exists $z \in V(G)$ such that $\varphi(z) = \inf \varphi(V(G))$.

Proof. Considering again the order $\le_\varphi$ and the sets $S(x)$ as given in the proof of Theorem 4.1, condition (4.3) can be expressed as

$$S(x) \setminus \{x\} \ne \emptyset\,,$$

for every $x \in V(G)$ with $\varphi(x) > \inf \varphi(V(G))$. By Corollary 3.3 there exists $z \in V(G)$ with $S(z) = \{z\}$. It follows that this z must satisfy

$$\varphi(z) = \inf \varphi(V(G)).$$

□

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 26 January 2023 doi:10.20944/preprints202301.0484.v1

5. The equivalence of principles

We prove in this section the equivalence of Ekekand, Caristi and Takahashi principles. We formulate them in terms of the order $\leq_\phi$ and the sets $S(x)$.

Theorem 5.1. *Let $G = (V(G), E(G), d)$ be a reflexive transitive acyclic digraph, where d is a metric on $V(G)$ and let $\varphi : X \to [0, \infty)$ be a function. Let $\leq_\varphi$ be the partial order on $V(G)$ given for $x, y \in V(G)$ by*

$$y \leq_\varphi x \iff d(x, y) \leq \varphi(x) - \varphi(y)\,,$$

and, for $x \in V(G)$, put

$$S(x) = \{y \in X : y \leq_\varphi x \;\; and \;\; (y, x) \in E(G)\}\,.$$

Then the following statements are equivalent.

(wEk) *The following holds*

$$\exists z \in V(G) \;\; such\ that \;\; S(z) = \{z\}\,. \tag{5.1}$$

(Car) *Any mapping $T : X \to X$ satisfying*

$$T(x) \in S(x) \;\; for\ all \;\; x \in V(G), \tag{5.2}$$

has a fixed point, i.e. there exists $z \in V(G)$ such that $T(z) = z$.

(Tak) *If*

$$S(x) \setminus \{x\} \neq \emptyset\,, \tag{5.3}$$

for every $x \in V(G)$ with $\varphi(x) > \inf \varphi(V(G))$, then there exists $z \in V(G)$ such that $\varphi(z) = \inf \varphi(V(G))$.

Proof. The implication (wEk) $\Rightarrow$ (Car) is contained in the proof of Theorem 4.1.

(Car) $\Rightarrow$ (wEk).

We prove the equivalent implication $\neg$(wEk) $\Rightarrow$ $\neg$(Car).

Observe that $\neg$(wEk) means that

$$S(x) \setminus \{x\} \neq \emptyset \;\; \text{for all} \;\; x \in V(G)\,. \tag{5.4}$$

Let $T : V(G) \to V(G)$ be defined for $x \in V(G)$ by $T(x) = y_x$, where $y_x \in S(x) \setminus \{x\}$. Then T satisfies (5.6), but, by the choice of y_x, $T(x) \neq x$ for all $x \in V(G)$, i.e. (Car) fails.

(Tak) $\iff$ (wEk).

We prove the equivalent assertion $\neg$(Tak) $\iff$ $\neg$(wEk).

Observe that (Tak) can be formally written as

$$\begin{aligned} &\big[\forall x \in V(G), \quad \varphi(x) > \inf \varphi(X) \;\Rightarrow\; S(x) \setminus \{x\} \neq \emptyset\big] \\ &\Rightarrow \big[\exists z \in V(G), \quad \varphi(z) = \inf \varphi(V(G))\big]\,, \end{aligned}$$

so that, its negation $\neg$(Tak) is given by

$$\begin{aligned} &\neg(\text{Tak}) \\ &\iff \big[\forall x \in V(G), \quad \varphi(x) > \inf \varphi(X) \;\Rightarrow\; S(x) \setminus \{x\} \neq \emptyset\big] \\ &\wedge \big[\forall z \in V(G), \quad \varphi(z) > \inf \varphi(V(G))\big] \\ &\iff \forall x \in V(G), \quad S(x) \setminus \{x\} \neq \emptyset \\ &\iff \neg(\text{wEk})\,. \end{aligned}$$

(The last equivalence from above follows from (5.4)). □

 doi:10.20944/preprints202301.0484.v1

These equivalences and Theorem 3.6 show that the completeness of $(V(G), d)$ is also equivalent to the fulfillment of each of these principles.

Corollary 5.2. *Let $G = (V(G), E(G), d)$ be a reflexive transitive acyclic digraph, where d is a metric on $V(G)$ such that the OSC property for G-sequences holds on G. Then the following statements are equivalent.*

1. *The metric space $(V(G), d)$ is G-complete.*
2. (wEk) *For every G-lsc function $\varphi : V(G) \to [0, \infty)$ there exists $z \in V(G)$ such that*

$$S(z) = \{z\}\,. \tag{5.5}$$

2. (Car) *For every G-lsc function $\varphi : V(G) \to [0, \infty)$ and any mapping $T : V(G) \to V(G)$ satisfying*

$$T(x) \in S(x) \;\; \textit{for all} \;\; x \in V(G), \tag{5.6}$$

there exists $z \in V(G)$ such that $T(z) = z$.

3. (Tak) *For every G-lsc function $\varphi : V(G) \to [0, \infty)$ such that*

$$S(x) \setminus \{x\} \neq \emptyset\,, \tag{5.7}$$

for all $x \in V(G)$ with $\varphi(x) > \inf \varphi(V(G))$, there exists $z \in V(G)$ with $\varphi(z) = \inf \varphi(V(G))$.

Remark 5.3. In the proof of Theorem 5.1 we have used some rules from Mathematical Logic (calculus of propositions and calculus of predicates). The sign $\neg$ stands for negation, $\vee$ is for "or", while $\wedge$ is for "and".

$$\begin{aligned}
&\neg(\neg p) \leftrightarrow p, && \neg(p \vee q) \leftrightarrow (\neg p \wedge \neg q), && \neg(p \wedge q) \leftrightarrow (\neg p \vee \neg q),\\
&(p \to q) \leftrightarrow (\neg p \vee q), && \neg(p \to q) \leftrightarrow (p \wedge \neg q), && (p \to q) \leftrightarrow (\neg q \to \neg p),\\
&\neg(\forall x,\ P(x)) \leftrightarrow (\exists x,\ \neg P(x)),\\
&\neg(\exists x,\ P(x)) \leftrightarrow (\forall x,\ \neg P(x))\,.
\end{aligned}$$

6. Conclusions

There are many extensions of Ekeland Variational Principles and its equivalences obtained either relaxing the conditions on the function φ (e.g., by considering functions with values in an ordered vector space) or considering spaces more general than the metric ones (uniform spaces, quasi-metric spaces, w-metric spaces, partial-metric spaces, etc.), or both.

In the present paper such an extension is given within the framework of metric spaces endowed with a graph and for a function φ satisfying a weaker notion of lower semi-continuity expressed in terms of the graph, completing in this way the results obtained by Alfuraidan and Khamsi [2] and Jachymski [8].

Author Contributions: The authors contributed equally to this work.

Funding: This researchers received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

Data Availability Statement: Not applicable.

References

[1] Ekeland, I. On the variational principle. *J. Math. Anal. Appl.* **1974**, 47, 324–353. 1
[2] Alfuraidan, M.R.; Khamsi, M. Graphical Ekeland's principle for equilibrium problems. *Proc. Amer. Math. Soc., Ser. B*, **2022**, 9, 33–40. 1, 8, 14
[3] Al-Homidan, S.; Ansari, Q.H.; Yao, J.-C. Some generalizations of Ekeland-type variational principle with applications to equilibrium problems and fixed point theory. *Nonlinear Anal.* **2008**, 69 , 26–139. 1
[4] de Figueiredo, D.G. *Lectures on the Ekeland variational principle with applications and detours.* Tata Institute of Fundamental Research Lectures on Mathematics and Physics, 81. Published for the Tata Institute of Fundamental Research, Bombay; by Springer-Verlag, Berlin, 1989. 1
[5] Ekeland, I. Nonconvex minimization problems. *Bull. Amer. Math. Soc.* (N.S.) **1979**, 1, 443–474. 1
[6] Qiu, J-H. An equilibrium version of vectorial Ekeland variational principle and its applications to equilibrium problems. *Nonlinear Anal. Real World Appl.* **2016**, 27, 26–42. 1
[7] Alfuraidan, M.R.; Khamsi, M. Ekeland variational principle on weighted graphs. *Proc. Amer. Math. Soc.*, **2019**, 147, 5313–5321. 1, 4, 6
[8] Jachymski, J. The contraction principle for mappings on a metric space with a graph. *Proc. Amer. Math. Soc.* **2008**, 136, 1359–1373. 1, 4, 14
[9] Bao, T.Q.; Cobzaş, S.; Soubeyran, A. Variational principles, completeness and the existence of traps in behavioral sciences. *Ann. Oper. Res.* **2018**, 269, 53–79. 1
[10] Cobzaş, S. Fixed points and completeness in metric and generalized metric spaces (Russian). *Fundam. Prikl. Mat.* **2018**, 22, 127–215; (English translation in *J. Math. Sci.* (N.Y.) **2020**, 250, 475–535). 1
[11] Weston, J.D. A characterization of metric completeness. *Proc. Amer. Math. Soc.* **1977**, 64, 186–188. 2
[12] Sullivan, F. A characterization of complete metric spaces. *Proc. Amer. Math. Soc.* **1981**, 83, 345–346. 2, 11
[13] Sullivan, F. Ordering and completeness of metric spaces. *Nieuw Arch. Wisk.* (3) **1981**, 29, 178–193. 2
[14] Caristi, J. Fixed point theorems for mappings satisfying inwardness conditions. *Trans. Amer. Math. Soc.* **215** (1976), 241–251. 2
[15] Caristi, J; Kirk, W.A. Geometric fixed point theory and inwardness conditions. in Geometry of Metric and Linear Spaces, Lect. Notes Math. 490, pp. 74–83, Springer-Verlag 1975. 2
[16] Takahashi, W. Existence theorems generalizing fixed point theorems for multivalued mappings. in Fixed point theory and applications (Marseille, 1989), Pitman Res. Notes Math. Ser., vol. 252, Longman Sci. Tech., Harlow, 1991, pp. 397–406. 3
[17] Takahashi, W. *Nonlinear functional analysis. Fixed point theory and its applications.* Yokohama Publishers, Yokohama, 2000. 3
[18] Johnsonbaugh, R. *Discrete mathematics.* 7th ed., Upper Saddle River, NJ: Prentice Hall, 2009. 3
[19] Bollobás, B. *Graph theory. An introductory course.* Corr. 3rd printing, Springer-Verlag, 1990. 3
[20] Wallis, W.D. *A beginner's guide to graph theory.* 2nd ed., Birkhäuser, Boston, MA, 2007. 3
[21] Nieto J.J.; Rodríguez-López, R. Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations. *Order* **2005**, 22, 223–239. 4
[22] Brøndsted, A. On a lemma of Bishop and Phelps. *Pacific J. Math.* **1974**, 55, 335–341. 5
[23] Brøndsted, A. Fixed points and partial orders. *Proc. Amer. Math. Soc.* **1976**, 60, 365–366. 5
[24] Bishop, E.; Phelps, R.R. The support functionals of a convex set. Proc. Sympos. Pure Math., Vol. VII pp. 27–35, Amer. Math. Soc. 1963. 5
[25] R. R. Phelps, *Convex functions, monotone operators and differentiability.* 2nd ed. Lect. Notes Math., vol. 1364. Springer-Verlag, Berlin, 1993. 5
[26] Alfuraidan, M.R.; Khamsi, M.A. Caristi fixed point theorem in metric spaces with a graph. *Abstr. Appl. Anal.* **2014**, Art. ID 303484, 5 pp. 6

doi:10.20944/preprints202301.0484.v1

Department of Mathematics, School of Sciences, University of Management and Technology, C-II, Johar Town, Lahore, 54770, Pakistan
E-mail address: basit.ali@umt.edu.pk

Babeş-Bolyai University, Faculty of Mathematics and Computer Science,, 400 084 Cluj-Napoca, Romania,
E-mail address: scobzas@math.ubbcluj.ro

Department of Mathematics, University of Pretoria, South Africa
E-mail address: Mokwetha.Mabula@up.ac.za