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# Magnified Attraction Due to Gravitationally Lensed Gravity

### A thought experiment proofs a largely disregarded effect of General Relativity

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**Abstract**

Can a gravitational lens magnify gravity ? The answer to this question is the subject of this article.

Light is deflected as it passes through curved spacetime caused by a mass distribution. This phenomenon, known as "gravitational lensing," has been extensively studied and is well understood. According to the general theory of relativity, the same deflection is expected for gravitational *waves*, because they also propagate at the speed of light. But this applies to gravitational *fields* as well. Thus, gravitational fields, which extend through curved spacetime, are also deflected correspondingly. This effect is referred to as "Gravitationally Lensed Gravity". It leads to the fact that the gravitational field strength originating from an object in the background of a gravitational lens is enhanced in the same way as its light intensity. If the background object is visible to an observer several times due to the gravitational lens, each of the images also exerts a corresponding gravitational attraction on her !

For the first time, a proof of the existence of this amazing effect is given. In a simple thought experiment the contraction of a spherically symmetric gravitational lens is discussed. The proof is essentially based on Birkhoff's theorem. According to this theorem, during the contraction additional null geodesics between the background object and the observer come into existence. To these null geodesics also contributions to the gravitational field are to be assigned. In the literature, similar descriptions can be found for the gravitational self-interaction of a particle in a curved spacetime background.

The article is organized as follows: In the introduction, the historical origin of the idea of the "Gravitationally Lensed Gravity" is briefly outlined. In the second chapter, the subsequently required basic principles and essential features of general relativity are explained. In the third chapter I analyze different arrangements of masses and describe the observations to be expected. The arrangements show step by step the transition from the classical "Newtonian limit" in a flat spacetime background to the "Gravitationally Lensed Gravity" in a curved spacetime background. Due to its clarity, the simple scenario may also serves as a suitable test case for the mathematical methods of general relativity. The concluding outlook provides references to current questions in astrophysics, especially the question of the existence of Dark Matter. Here, the general theory of relativity offers much simpler explanations with the "Gravitationally Lensed Gravity".

## Contents



## 1 Introduction

In 1977, the American physicist and later Nobel laureate Steven Weinberg wrote down in his popular science book "The First Three Minutes" [1] the following thoughts, which express his self-image as a theoretical physicist: "I do not believe that scientific progress is always best advanced by keeping an altogether open mind. It is often necessary to forget one's doubts and to follow the consequences of one's assumptions wherever they may lead." He emphasized: "Our mistake is not that we take our theories too seriously, but that we do not take them seriously enough." Still, the consequences of general relativity for the dynamics of galaxies are vastly underestimated. In particular, it has hardly been investigated so far how a large-scale curved spacetime background affects the gravitational interactions of further sources among each other.

Significant relativistic effects occur mainly in the vicinity of compact massive objects. Closely orbiting pairs of neutron stars or black holes emit energy in the form of gravitational waves and get closer and closer to each other. In the neighborhood of the supermassive black hole in the center of the Milky Way stars move on rosette-shaped trajectories around it [2] [3]. In contrast, relativistic effects on the motion of celestial bodies in the solar system are extremely small. The perihelion precession of the planet closest to the sun, Mercury, is 575 *arc seconds per century (!)*. Of these, 532" can be explained by interactions with the other planets of the solar system according to Newton's law of gravity.

For a long time, it was thought that the remaining difference of 43" is caused by an asteroid belt or another planet ("Vulcan") within Mercury's orbit [4]. But in 1915 Albert Einstein succeeded in making this undetectable matter completely obsolete by tracing the difference back to the deviation of Newton's law of gravity from the exact solution of his field equations [5].

For all other planets the deviations are negligibly small. From this experience results the preconception that weak gravitational interactions are generally described sufficiently well by Newton's law of gravity. That this is not accurate in every case is demonstrated in this article by a special example.

*Fig. 1: The curvature of spacetime around the massive foreground galaxy provides to us left and right two views of the spiral galaxy in the background. A third partially hidden view is located behind the foreground galaxy. This spectacular picture is the central detail of the picture "Lens Flair" of the Hubble Space Telescope showing the object with the designation SGAS J143845+145407. Credit: ESA/Hubble & NASA, J. Rigby [6]*

Somewhat different is the situation with the deflection of light by gravitational lenses. There we make the experience that significant effects become visible only far away from the mass accumulations causing them. Extremely strong gravitational fields are not required for their appearance. Large-scale weak curvatures extending over hundreds of thousands or even millions of light-years add up to a noticeable change in the direction of the light that finally reaches us. So we see the background objects in correspondingly altered directions, sometimes even multiple times. For example, Fig. 1 shows three views of the same spiral galaxy, provided to us by the strong gravitational lensing effect of the foreground galaxy. And for the so-called microlensing effect, even the gravitational field of a planet crossing the line of sight to a star in the background is sufficient to slightly deflect its light and focus it towards us. The associated transient brightness increase of the background star has actually led to the discovery of exoplanets [7].

The term "lens" coined by Einstein in his short article [8] in 1936 was an unfortunate choice. It implies a location-dependent "refractive index" in a Euclidean space – he had formulated it that way before in 1911 [9]. According to this view, the variation in the speed of light caused by this "refractive index" explains the curvature of the light rays. Since then, gravitational lenses are often represented as thin lenses in the framework of geometrical optics.

As Jürgen Ehlers et. al criticize [10] [11], this disregarding of the root cause – the curvature of spacetime – stands in the way of a more comprehensive understanding of gravitational lensing. Even today the visible phenomena resulting from gravitational lensing are sometimes called "mirage", which does not adequately reflect their true physical meaning.

Gravitational lensing is not just about light. Gravitational *waves* propagate at the same speed as light [12] and therefore follow the same paths through spacetime [13]. They should be focused and enhanced by gravitational lenses in the same way as light. In this respect, the curvature of spacetime inside spherical mass distributions achieves a much better focusing and correspondingly higher intensification than the curvature outside [14]. The possibility of observing focused gravitational waves (and neutrino beams) through the "transparent Sun" [15] was also analyzed by Robert J. Nemiroff [16]. He wondered if gravitational *fields* from distant sources should not be focused in the same way. In his 2005 paper "Can a gravitational lens magnify gravity?" [17] he put this question up for discussion and introduced the term "Gravitationally Lensed Gravity".

In more detail the historical development of this idea and related works of other authors is covered in [18]. There also the foundations and relevant models of general relativity are presented in a generally intelligible form. Here I restrict myself to outlining the essential relations by means of the concept of "null geodesics", the paths through spacetime available for light-fast effects.

# 2 Basic principles and essential features of general relativity

## 2.1 *Gravity propagates at the speed of light*

With the establishment of his special theory of relativity in 1905, Einstein abolished absolute time and absolute space. Observers moving relative to each other measure different spatial and temporal distances between the same two events. But all observers always measure the same speed of light (in vacuum). No information, no energy and no effect can propagate with greater speed. This ensures an absolute causal structure, i.e. the unambiguous sequence of causes and effects on which all observers agree.

Hermann Minkowski presented in 1908 [19] his four-dimensional spacetime as the stage for Einstein's special theory of relativity. Each event corresponds to a point in this spacetime with one temporal and three spatial coordinates. The spacetime interval $\Delta s^2$ between two considered events unifies temporal and spatial distances in one quantity which is the same for all observers:

$$\Delta s^2 = c^2\Delta t^2 - (\Delta x^2 + \Delta y^2 + \Delta z^2) = c^2\Delta t'^2 - (\Delta x'^2 + \Delta y'^2 + \Delta z'^2) \quad (1)$$

The Lorentz transformations between the coordinates of different observers can be derived from this absolute quantity. It also serves as an unambiguous criterion for possible causal relations. With the velocity $v$ of a signal between two events in a spatial distance $\Delta x$ holds:

$$\Delta s^2 = c^2\Delta t^2 - \Delta x^2 = c^2\Delta t^2 - v^2\Delta t^2 = (c^2 - v^2)\cdot\Delta t^2 \quad (2)$$

$\Delta s^2 > 0$ for $v^2 < c^2$: Positive spacetime intervals are called *time-like*. An object with rest mass (e.g. a clock) can move slower than light from the first event to the second.

$\Delta s^2 = 0$ for $v^2 = c^2$: Null-intervals are called *light-like*. The first event can only influence the second by light or an interaction propagating at the speed of light $c$.

Two events with time-like or light-like interval can be in a cause-and-effect relationship.

$\Delta s^2 < 0$ for $v^2 > c^2$: Negative spacetime intervals are called *space-like*. Such events cannot influence each other because this would require signals faster than light. Whether one event is measured as earlier, simultaneous or later with respect to the other depends on the observer. The only certainty is that the two events occur independently of each other at different locations.

A detailed description of the useful 2D ($x,t$) Minkowski diagrams and 3D ($x,y,t$) light cone representations of the causal structure can be found in [18].

Gravity is assumed to propagate in the same way and at the same speed as light. Quite analogously to the retarded Liénard-Wiechert potentials of electrodynamics, Minkowski described in his lecture [19] also gravity with past light cones. Sergei M. Kopeikin took up this way of representation again and coined the term "gravity null cone", which he applied to the gravitational field of the moving planet Jupiter [20] [21]. Also Dennis Rätzel et al. [22] used a past light cone to represent the retarded gravitational effect of a light pulse. All these depictions have in common that they show "regular" past light cones as they exist in the flat Minkowski spacetime. But only the curvature of spacetime and the associated folding of the past light cone [18] leads to the effects presented in the following.

Light *and gravity* propagate on the same paths through spacetime. These are called "**null-geodesics**" because all events on them have null-intervals to each other.

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### 2.2 *Instantaneous action despite propagation at the finite speed of light*

Newton's law of gravitation provided excellent service in celestial mechanics in the solar system region. But Kepler's laws of planetary motion are obtained correctly only if Newton's gravitation acts instantaneously: The current force acting on the planet from a distance is determined by the position of the sun at the same instant. This is based on the notion of absolute Euclidean space and absolute time.

From the special theory of relativity, it follows that no effect can propagate with a speed greater than that of light. This is also true for gravity. But a field strength which shows an *aberration*, i.e. does not act in the direction of the present position of the source, but in the direction of its former (retarded) position, does not allow stable orbits !

In general relativity, *velocity-dependent* components ensure that the current gravitational field strength points in the direction of the current position of the source, more precisely: in the direction of the current position *as extrapolated* from the former position, the former velocity and the former acceleration of the source. Thus, the speed limit of the speed of light is preserved: the current effect is based on the information transmitted at the speed of light from the former location of the source. But the (apparently) instantaneous effect is also preserved: the current field strength points in the direction of the (extrapolated) current position of the source. That it must be this way follows from the principle of relativity, which says that observers moving uniformly relative to each other must arrive at the same conclusions about physical processes [23] [18] [24]. For velocities that are small compared to the speed of light (and for small accelerations), the results of general relativity are virtually identical to Newton's law of gravity – at least throughout the solar system region.

### 2.3 *Schwarzschild's solution and the "Newtonian limit"*

The Universal Law of Gravitation, published by Newton in 1687, states that two point masses exert mutual forces of attraction of equal magnitude on each other, which are proportional to both masses and inversely proportional to the square of their distance. This can be reformulated into an expression for the field strength caused by a spherically symmetric mass distribution.

The field strength $g$ acting in the direction of the center of the mass distribution is proportional to its total mass $m$ and inversely proportional to the square of the distance $r$ from this center. The proportionality factor is the gravitational constant $G$:

$$g = G \cdot \frac{m}{r^2} \tag{3}$$

According to the Schwarzschild solution of Einstein's field equations, the locally measured field strength = gravitational acceleration $g$ results as (acc. to [4], here in SI units with the speed of light $c$):

$$g = G \cdot \frac{m}{r^2} \cdot \frac{1}{\sqrt{1 - \frac{G}{c^2} \cdot \frac{2m}{r}}} \tag{4}$$

The solution satisfies the correspondence principle: it converges to Newton's expression (3) in the case of *weak fields*, i.e. when the root approaches 1 for large distances or for small masses.

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### 2.4 *Birkhoff's theorem*

Birkhoff's theorem states:

The Schwarzschild solution is the only spherically symmetric vacuum solution of Einstein's field equations outside a spherically symmetric mass distribution.

Furthermore, it also applies:

The Minkowski metric is the only solution inside a massless void surrounded by a spherically symmetric mass distribution.

Here, the spherically symmetric mass- resp. energy-distributions do not have to be static. The outer gravitational field of a radially pulsating mass distribution remains unchanged, gravitational waves are not generated. An expanding spherically symmetric shell of mass or energy has no effect on the gravitational field in its interior (see e.g. [25]).

In [26] I consider a star which suffers a sudden mass loss by isotropic emission of energy throughout a light burst. According to Birkhoff's theorem, the gravitational field outside the expanding shell of photons remains unchanged, while inside it, the field is reduced and solely determined by the residual mass of the star in the center. This requires the propagation of gravity at the speed of light, because only this way the sudden decrease of the gravitational field strength can propagate together with the shell of photons. Such a monopole-like change of the gravitational field cannot propagate freely and remains bound to the propagating shell of the light burst's energy [27] [28]. Thus, Birkhoff's theorem requires and confirms the propagation of gravity at the speed of light.

Important for the understanding of the later following argumentation is the fact, that the outer gravitational field of every spherically symmetric distribution of equal total mass is identical. If such a mass distribution contracts, thus becomes more compact, its gravitational field – resp. the curvature of spacetime – remains unchanged in the region which was already outside the mass distribution before.

### 2.5 *Superposition principle and perturbation theory*

In the Newtonian theory of gravity, the gravitational effects of two or more celestial bodies are superposed. The total field strength results simply as the vectorial sum of the contributions of the individual celestial bodies. With this knowledge, the French mathematician Urbain Le Verrier was able to calculate from irregularities of the orbital motion of the planet Uranus the position of the planet Neptune, which was actually found there by the German astronomer Johann Gottfried Galle in 1846 [29]. This linear superposition of weak gravitational fields from several sources must of course be provided by Einstein's theory as well, otherwise it would not describe the astronomical observations correctly. For this purpose, one considers the contributions of the individual sources as small perturbations, which are added to the spacetime metric that exists without them. By this artificial separation into small disturbances ("gravitational fields") and an unchanged background metric ("spacetime"), a linearization of Einstein's field equations is achieved. This only serves the convenience and is not an expression of qualitative physical differences.

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### 2.6 *"Flat" spacetime and fluxes originating from a point source*

I now consider somewhat more fundamentally the relations between spacetime, null geodesics and location dependence of the light intensity and the gravitational field strength. The description is given from the point of view of an "outside" cartographer who is far away from the sources of gravity.

A "flat" – meaning not curved – spacetime is characterized by the following statements:

a) Time is universal: Resting clocks of the same construction tick at the same rate at every location. Once synchronized, they remain permanently synchronized.

b) The space is Euclidean: Each region of a cross-sectional plane of space can be mapped onto a two-dimensional Euclidean map with a constant scale valid for the entire map. Resting measuring rods of the same kind at any location and in any orientation within the considered plane are shown with the same length everywhere on the map.

If one depicts the propagation of light on such a Euclidean map, it is straight and equally fast at every location (homogeneity) and in every direction (isotropy). The light always advances the same distance on the map per time cycle of the cartographer's clock. Thus, the null geodesics in a flat spacetime are straight lines.

I now study a quantity, e.g. light energy, which propagates isotropically at the speed of light from a point source. The quantity emitted by the source per time uniformly radiates in all directions. From the source, the propagation follows the straight-line null geodesics. Discretizing the flow as a large number $N$ of straight lines starting from the point source and running isotropically in all directions, each of these straight lines represents a solid angle element $\Delta\Omega$:

$$\Delta\Omega = \frac{4\pi}{N} = 4\pi \cdot q \tag{5}$$

It corresponds to a fraction $q$ (quota) of the *source strength*, the total flux from the source. For the density $\sigma$ of the straight lines perpendicularly passing through a surface $\Delta A$ the following applies:

$$\sigma = \frac{N}{4\pi \cdot r^2} = \frac{1}{\Delta A} = \frac{1}{\Delta\Omega \cdot r^2} \tag{6}$$

Isotropy means a uniform distribution of straight lines over the surface of a sphere with radius $r$ around the center. The surface $\Delta A$ can be regarded as a section of this sphere surface.

This quantity $\sigma$ is illustrative, but it still contains the arbitrarily chosen number $N$. We therefore define a somewhat more abstract quantity, the distribution density $d$ as the fraction $q$ of flux per area $\Delta A$:

$$d = \frac{q}{\Delta A} = \frac{\Delta\Omega}{4\pi} \cdot \frac{1}{\Delta\Omega \cdot r^2} = \frac{1}{4\pi \cdot r^2} \tag{7}$$

The total flux – the "fraction" 1 – is distributed uniformly over a spherical surface around the center.

The straight-line course of the null geodesics from a point source is therefore the geometrical cause for the well-known inverse-square distance law. If *stationary* fluxes distribute evenly among the isotropic null geodesics, then the corresponding areal densities of the fluxes exhibit an inverse-square distance law.

In the case of light, energy flows along the straight-line null geodesics, the "light rays". The energy emitted by the source per time – the power $P$ – is divided onto the null geodesics. Then, the intensity $I$ is the areal density of these power fractions:

$$I(r) = P \cdot d(r) = P \cdot \frac{q}{\Delta A} = \frac{P}{4\pi} \cdot \frac{1}{r^2} \tag{8}$$

Consequently, for the energy flux density – the intensity $I$ – the inverse square distance law results.

In the case of the gravitational field, the straight-line null geodesics can be identified with the classical "field lines". The areal density of these field lines is the field strength $g$, for which also the inverse square distance law holds, as stated by Newton's law of gravity.

In flat spacetime, both the light intensity $I$ emitted by a star and its gravitational field strength $g$ decrease inversely with the square of the distance $r$:

$$I \sim \frac{1}{r^2} \quad \text{and} \quad g \sim \frac{1}{r^2} \tag{9}$$

So, this is not a coincidence, but is due to the common deeper cause: the geometry of the null geodesics in flat spacetime. It determines the spatial dependence of the local areal density of the flux fractions corresponding to the null geodesics, the distribution density $d$, to which these quantities are proportional:

$$I \sim d(r) \quad \text{and} \quad g \sim d(r) \tag{10}$$

Since both the gravitational field strength and the light intensity are proportional to the areal density of the null geodesics, the local change in light intensity can be used as an "indicator" of the local change in areal density of the null geodesics, and thus provide information about the local change in gravitational field strength [26]. The prerequisite for this is a constant light power of the source and a propagation of the light exclusively in vacuum, so that it can follow the null geodesics of spacetime undisturbed. Absorption, scattering or a refractive index of matter would alter the result and make it difficult to infer from the light intensity to the areal density of the null geodesics.

An observer does not notice the propagation speed of gravity as long as the mass of the source does not change. However, if the mass decreases due to a radiation burst, the corresponding gravitational field strength also decreases with the arrival of the radiation. Both arrive at the observer at the same time, delayed by the transit time $\Delta t_r$ of the propagation along the null geodesic at the speed of light. Here, the null geodesic is the radial line with length $r$ from the source to the observer:

$$\Delta t_r = \frac{r}{c} \tag{11}$$

Even if they are temporally constant, gravitational fields are permanently emerging from the source and propagating at the speed of light $c$. Therefore, they should not be denoted as "static" but as "*stationary*". The currently effective gravitational field strength of a star is thus determined by its *retarded* mass at the former time, when it also emitted the light, which the observer currently sees:

$$g(t,r) = G \cdot \frac{m(t - r/c)}{r^2} \tag{12}$$

The deeper into space we peer, the further back into the past we see. This relation between the spatial distance to a source and the temporal distance to its past state, whose light-fast effects we currently observe, can also be described with the notion of the past light cone of our observation event [18]. The paths along which these effects reach us are the same paths that the light takes, namely the null geodesics. In the flat Minkowski spacetime these are straight lines.

According to the above considerations we now can write for the gravitational field strength $g$:

$$g(t,r) = 4\pi \cdot G \cdot m(t - \Delta t_r) \cdot d(r) \tag{13}$$

This expression is not limited to flat spacetime; it retains its validity in curved spacetimes as well.

### 2.7 *"Curved" spacetimes*

A curved spacetime is characterized by the following statements:

a) Time is location-dependent: Resting clocks of the same construction do not tick at the same rate in every location. They cannot be synchronized everywhere. Processes closer to a large mass appear to run slower when viewed from a distance, as if in slow motion.

b) Space is non-Euclidean: Mapping a large region of a cross-sectional plane of space onto a two-dimensional Euclidean map requires a varying location- and direction-dependent scale, a "metric". Resting measuring rods of the same kind are shown with different lengths on the map, depending on their location and orientation within the considered plane.

Such location- and direction- dependent scales are familiar to us from world maps. They are unavoidable for a mapping of the extrinsically curved convexly domed surface of the Earth onto a plane map. The location- and direction-dependent length units varying across the map cause the shortest connections to appear curved on the map. If the position of an airplane on an intercontinental flight is marked on a world map at regular time intervals, its trajectory appears curved and its speed continuously varying, although it is flying straight ahead at a constant speed. Only by taking into account the location- and direction-dependent scale of the map one can retrieve the speed and direction of the airplane, which an observer on the ground would determine.

Of course, a cross-sectional *plane* through space is not domed, thus it does not have extrinsic curvature. According to general relativity, however, a mass causes an intrinsic curvature of space, which is reflected in the map's location- and direction-dependent scale. Locally straight constant motions result in globally curved trajectories on the map similar to those of an airplane on the world map. In addition, the "curvature of time" must be taken into account here, meaning the location-dependent tick rate of the local clocks.

The slowing down of the time cycle of the local clocks and the shortening of the representation of the local measuring rods on the map lead to an apparent reduction of the speed of light. From the map, it seems as if Euclidean space is equipped with a refractive index that depends on location and direction. However, a local observer at rest near the mass, who has brought a clock and a measuring rod with him from the cartographer, still measures an unchanged speed of light.

So, in general relativity, the mass-induced intrinsic curvature of spacetime leads to globally curved trajectories of light, which locally propagates straight-line uniformly. These curved paths are the null geodesics of the curved spacetime. Gravity, which also locally propagates in a straight line and uniformly at the speed of light, follows the globally curved null geodesics as well.

### 2.8 *Limits of the validity of the "Newtonian limit" in curved spacetimes*

As reasoned in 2.2 and 2.3, Newton's law of gravity is considered to be a *sufficiently good approximation to general relativity* for the mostly applicable case of *small velocities and weak fields*. To the present day it is still used to calculate the motion of whole galaxies !

Tacitly an additional assumption is made, namely that of a *negligible curvature of spacetime everywhere between source and observer*. As described in 2.2, even for moving objects no aberration of gravity occurs despite its finite propagation speed. Therefore, it is prematurely concluded that the speed of propagation of gravity has no effect at all. In a curved spacetime, however, a propagation at the speed of light is inevitably associated with a deflection. What consequences this has for large-scale gravitational fields extending through a curved spacetime background will be shown in the further course of this article.

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## 3 Deflection of the gravitational field of a distant object in a curved spacetime background

### 3.1 *Arrangement A: The "Newtonian limit" in flat regions of spacetime*

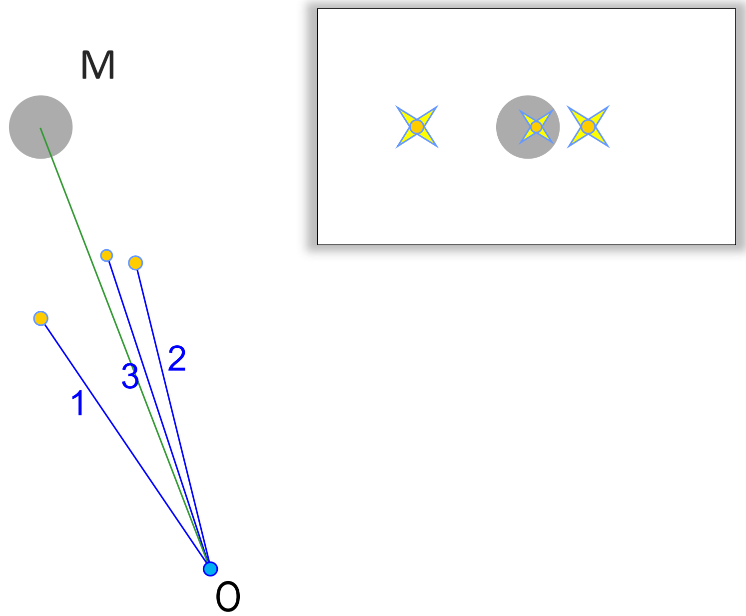


*Fig. 2: Effect of three stars and a larger mass distribution M on an observer O. Here the more distant mass M has no influence on the course of the paths of light and gravity from the three closer stars to the observer. The frame on the right shows the observer's view of the three stars and the mass distribution M in the background.*

Fig. 2 on the left shows the arrangement of three stars, a larger spherically symmetric mass distribution M and an observer O. The centers of the stars and the mass distribution lie in one plane with the observer. How the observer sees this arrangement is shown in the frame to the right.

The mass distribution M is located at a great distance from the observer. For its field strength contribution, the "Newtonian limit" applies. Also, the contributions of the closer stars can be calculated according to Newton, as their masses are sufficiently small. It is assumed that the stars and the mass distribution M are at rest relative to the observer. The observer is in a nearly curvature-free region of spacetime, the light of the stars reaches her on straight paths. At the observer's location, the gravitational field strength contribution of each mass acts in the direction of the line of sight from her to the center of the respective mass. Since these are weak fields, the superposition principle applies. The total field strength is simply the vectorial sum of the individual contributions. All masses are also considered to be *invariant in time*.

Thus, the prerequisites are given for the applicability of the "Newtonian limit". For this arrangement, Newton's law of gravitation leads to the same result as the general theory of relativity.

### 3.2 *Arrangement B: Straight null geodesic with "Newtonian limit"*

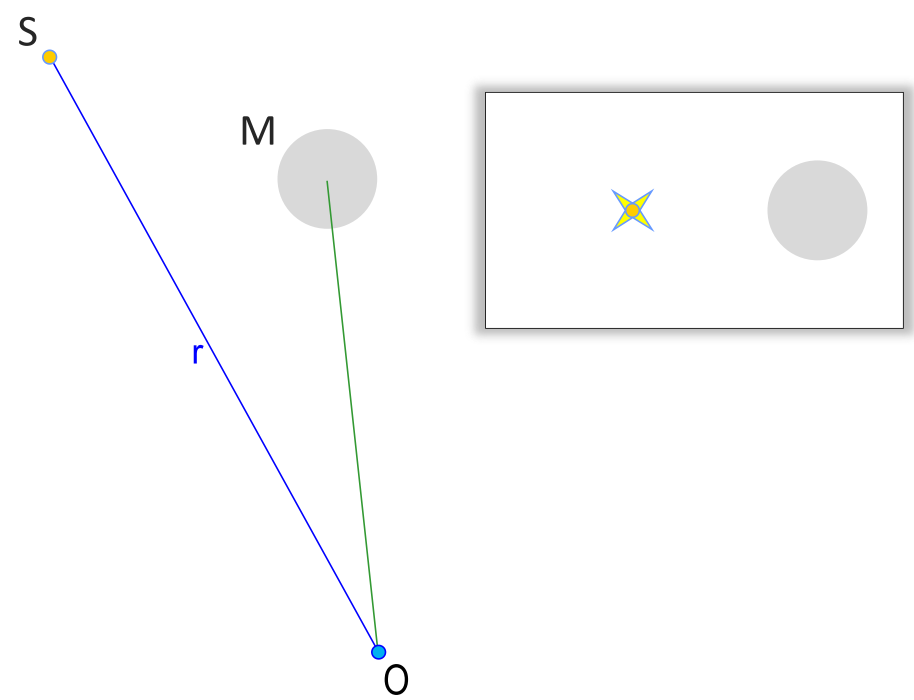


*Fig. 3: With sufficient lateral distance from the straight connecting line r, the mass distribution M has virtually no influence on the propagation of light and gravity from the star S along the straight line r to the observer O.*

Fig. 3 shows a larger spherically symmetric mass distribution M and a more distant star S. The straight connecting line r between the star and the observer O runs at such a large distance from the mass M that in this region spacetime is practically not curved. Thus, along this connecting line, the light from the star reaches the observer on a straight path. The star is at rest relative to the observer – as well as the mass distribution M. Their gravitational field strength contributions act in the directions in which the observer sees them, and are superposed.

The null geodesics, starting from the star S and running close to the connecting line r, are the straight lines of a flat spacetime. Correspondingly, the inverse-square distance law for the light intensity and for the gravitational field strength (Newton's law) still applies in the direction of the connecting line r. There, the distribution density d is according to (7):

$$d(r) = \frac{q}{\Delta A} = \frac{1}{4\pi \cdot r^2}$$

The currently effective gravitational field strength of the star is determined by its *retarded* mass at the time earlier by the transit time $\Delta t_r$, when also the light has left it, which the observer currently sees:

$$\Delta t_r = \frac{r}{c}$$

According to (13), for the gravitational field strength $g$ generally applies:

$$g(t, r) = 4\pi \cdot G \cdot m(t - \Delta t_r) \cdot d(r)$$

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Inserting the above terms leads to the result expected for a flat region of spacetime (12):

$$g(t,r) = G \cdot \frac{m(t - r/c)}{r^2}$$

For a constant mass $m$, Newton's law of gravitation here still gives the same result as general relativity.

### 3.3 *Arrangement C: Curved null geodesic, deflected gravity*

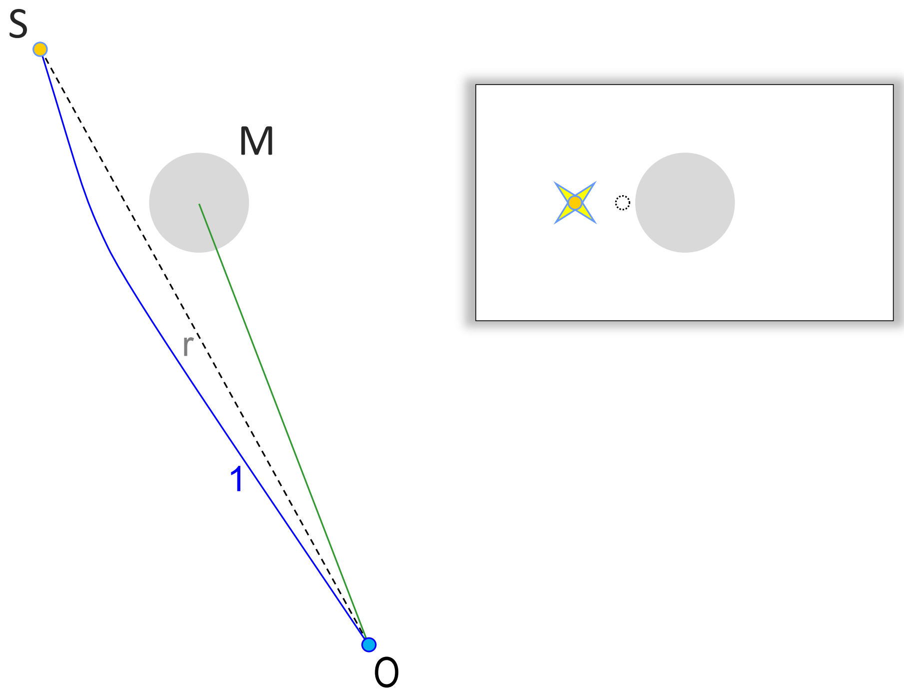


*Fig. 4: At a smaller distance from the straight connecting line r (dashed), the curved spacetime in the vicinity of the mass distribution M acts as a gravitational lens: it bends the course of path 1 of light and gravity from the star S to the observer O. The observer sees the star in the direction from which its light reaches her. It now appears shifted away from the mass M compared to its position before the close approach of the mass M (dotted circle in the frame on the right).*

How does the curved spacetime in the vicinity of the spherically symmetric mass distribution M affect the gravitational effect of the more distant star S passing through it?

Fig. 4 on the left side shows *a map* of the arrangement **C**. The observer O is located at a great distance from the mass distribution M. In her nearer surroundings the spacetime is flat in good approximation, there the Euclidean geometry is valid. This is also true for the surroundings of the star S. In the vicinity of the large mass distribution M, however, spacetime is curved. By using *Schwarzschild coordinates*, this area can also be represented on the map, but with a location- and direction-dependent scale (a metric). Also, temporal durations of processes there, which the cartographer has read on his clock, have to be converted to different local temporal durations.

The closer the mass M approaches the straight connecting line r on the map, the more the null geodesic 1, the path of light from the star to the observer, is bent. Although the star is shown at the same position on the map as in arrangement **B**, the transit time of the light has now become longer. This is due, on the one

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hand, to the greater length of the curved path 1 and, on the other hand, to the apparent slower tick rates of clocks nearer to the mass M, which together amounts to an apparent reduction of the speed of light.

The observer sees the star in the direction from which its light arrives at her. This direction does not correspond to the straight connecting line r from the star to her, which is shown dashed on the map. The light and the gravity would follow this straight line, if the mass M would not exist or would have a larger distance to it, similarly as in the arrangement **B**. The observer would then see the star in the sky at the position of the dotted circle. Now it appears to her shifted away from the mass M.

Which consequences has the curvature of the null geodesic 1 for the gravitational field strength reaching the observer from the star ? To explain this, I answer the following questions in accordance with the "Gravitationally Lensed Gravity":

**Q1:** In which direction does the gravitational field strength originating from the star act at the observer's location ?

**A1:** Since the star is at rest relative to the observer, no velocity- and acceleration-dependent components occur. Its gravitational field strength acts simply in the direction in which she sees the star. Gravity propagates along the same paths as light does, so both arrive at the observer from the same direction. The dashed straight connecting line r marks only the connecting line with the *shortest coordinate distance on the Euclidean map*. It has no meaning for the physics in the curved spacetime geometry.

**Q2:** How does the curved spacetime geometry determine the magnitude of the gravitational field strength of the star at the observer's location ?

**A2:** Since the dashed straight connecting line r has no physical meaning in curved spacetime any further, it cannot determine the magnitude of the gravitational field strength. In flat spacetime, we have seen that the inverse square distance law of the gravitational field strength is equivalent to the proportionality to the flux density of the null geodesics. This flux density is a local quantity that can be used also in curved spacetimes, that is, in the case of curved null geodesics. It determines as *geometric distribution density d* the intensity of the light from the star arriving at the observer, and likewise its gravitational field strength contribution effective at her. Furthermore, in the case that the light of the star shows a redshift, its intensity is decreased and correspondingly also its field strength contribution [26]. However, this fact does not play a role in our example here.

**Q3:** Which retarded mass of the star caused its currently observed gravitational field strength ?

**A3:** Besides the distribution density *d*, which represents the *transfer* from the source to the observer, the *source strength* is the second determining factor for the flux density. In the case of the gravitational field, this source strength is proportional to the mass, more precisely: to the retarded mass, "*the effect of which was fed in at the other end of the null geodesics at the time of emission*". The currently effective field strength contribution of the star contains the information about its mass at the past time when also the light that the observer is currently seeing left the star. This point in time is earlier than the current time $t$ of observation by the travel time $\Delta t_1$ of the light along the null geodesic 1. The effective retarded mass of the star is thus $m(t - \Delta t_1)$.

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### 3.4 *Arrangement D: Additional null geodesics, additional images, additional gravity !*

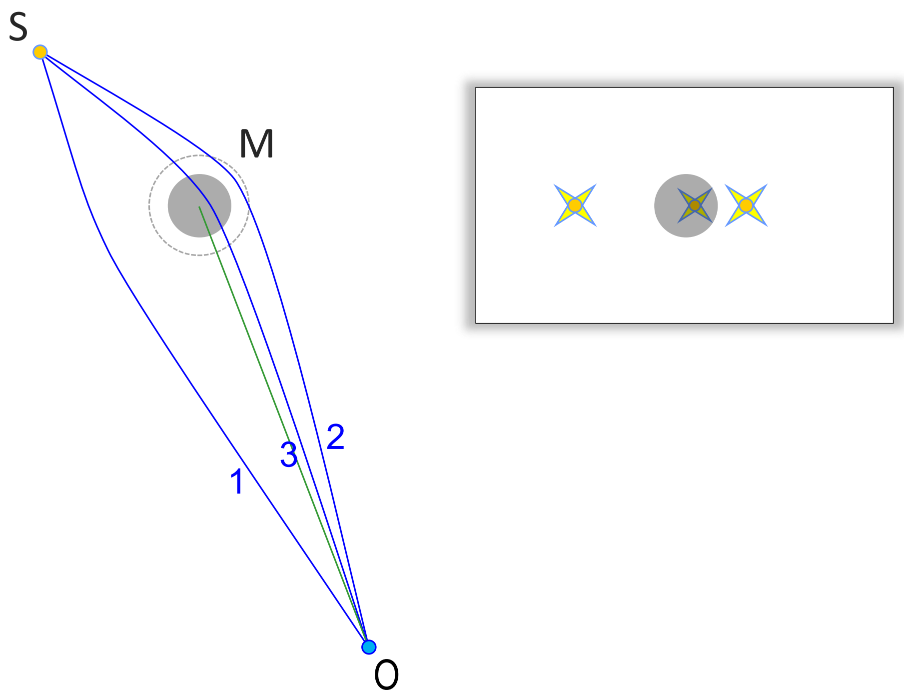


*Fig. 5: The contraction of the mass distribution M does not change its own field strength contribution at the observer O. Also, the null geodesic 1 is not affected by this. But now, the null geodesics 2 and 3 are newly coming into play and are as well transmitting light and gravitational effect from the star S to the observer.*

We now consider the consequences of a contraction of the spherically symmetric mass distribution M. According to Birkhoff's theorem, its gravitational field resp. the spacetime curvature caused by it does not change in the region which was already outside the mass distribution M before. That means, at the star S, at the observer O and in the whole area through which the null geodesic 1 runs, everything remains the same.

The circle shown dashed on the left in Fig. 5 marks the area from which the mass has retreated, it now also belongs to the outer field. There, the gravitational field has become stronger, respectively the curvature of the spacetime has become larger and therefore also the curvature of the null geodesics running through this area. If the mass distribution M has contracted sufficiently, then the now more strongly deflected null geodesic 2 arrives at the observer as well. Like 1, also 2 runs completely in vacuum through the outer gravitational field of the mass distribution M. Via it an *additional* connection from the star to the observer is established.

Null geodesics crossing the interior of the mass distribution M are deflected less the closer to the center they pass – if they pass right through the center, they are not deflected at all, for symmetry reasons. Therefore, such a null geodesic 3 must exist, which runs between 2 and the center, and whose smaller deflection is just large enough to reach the observer O as well.

The light travel time along the null geodesic 2 is larger than for the null geodesic 1, but it is largest for the null geodesic 3, which crosses the interior of the mass distribution M. In more detail, these relations are described by means of folded wavefronts of the light emanating from the source in [30] [31] and by means of folded past light cones of the observation event in [10] [18].

Concerning arrangement **D**, the decisive questions arise:

**Q4**: Are the effects transmitted along null geodesic 1 altered when the two other null geodesics additionally come into play ?

**A4**: No. According to Birkhoff's theorem, the gravitational field resp. the spacetime is not altered in the concerned area. Correspondingly, the course of the null geodesic 1 does not change. The direction, magnitude, and retardation of the gravitational field strength contribution it transmits are identical to those in the arrangement **C**.

**Q5**: Do the newly added null geodesics differ qualitatively from the null geodesic 1 ?

**A5**: No. Naturally, the course of the null geodesics differs, they arrive at the observer from different directions, show different distribution densities and retardations. But their physical functionality is the same. In particular for the null geodesic 2, which like the null geodesic 1 runs completely through the external gravitational field in vacuum, no principal physical difference can be reasoned. If besides the light a contribution of gravitational field strength from the star is transmitted via the null geodesic 1, then this is also the case for the null geodesic 2.

**Q6**: What do the additional null geodesics mean for the effects arriving at the observer ?

**A6**: Like the null geodesic 1, the newly added null geodesics 2 and 3 transmit light and gravitational effects from the star to the observer. There, these effects superpose in exactly the same way, as if they were coming from different objects. The observer now sees two additional "images" of the star, thus receives altogether more light from the star. Correspondingly, two additional contributions to the gravitational field strength have an effect on her, the star now attracts her in three ways and therefore more strongly overall !

**Q7**: Why do the effects via the null geodesics have to be regarded as being independent of each other ?

**A7**: Due to the longer transit time along 2 and the even longer one along 3, light and gravitational effects must have left the star correspondingly earlier to arrive at the observer simultaneously with the light and gravitational effect along 1. Thus, the effects left the star at different times, testifying possibly different states of the star, for example before and after a supernova explosion. The location of the observer determines the combination of the null geodesics reaching her and thus the combination of retarded states of the star, whose effects superpose at her.

In generalized form according to (13), the total contribution of the star is now written as:

$$g(t) = 4\pi \cdot G \cdot \sum_{n=1}^{3} m(t - \Delta t_n) \cdot d_n \tag{14}$$

$\Delta t_n$ is the travel time of the light along the null geodesic $n$ and $d_n$ is the corresponding distribution density, into which the course of the null geodesic $n$ from the star to the observer gets involved.

The arrangement **D** has the same effect on the observer as the arrangement **A**, suitable masses, luminous powers and distances of the three stars provided. In **D**, effects of the one star are guided by the gravitational lens along three curved paths to the observer. The arrival and superposition of the effects at her proceed in the same way as the effects in **A**, which reach her from the three stars along straight paths.

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### 3.5 *The proof of the existence of the "Gravitationally Lensed Gravity"*

Let us summarize the main findings from the previous considerations:

- The gravitational field is always locally propagating at the speed of light.
- It propagates the same way as light does, along null geodesics through spacetime.
- Each gravitational effect presently acting on the observer is retarded. It corresponds to the mass of the source at a time earlier by the transit time along the connecting null geodesic.
- All gravitational effects are superposed at the observer. The total field strength results as the vectorial sum of the individual gravitational field strength contributions.

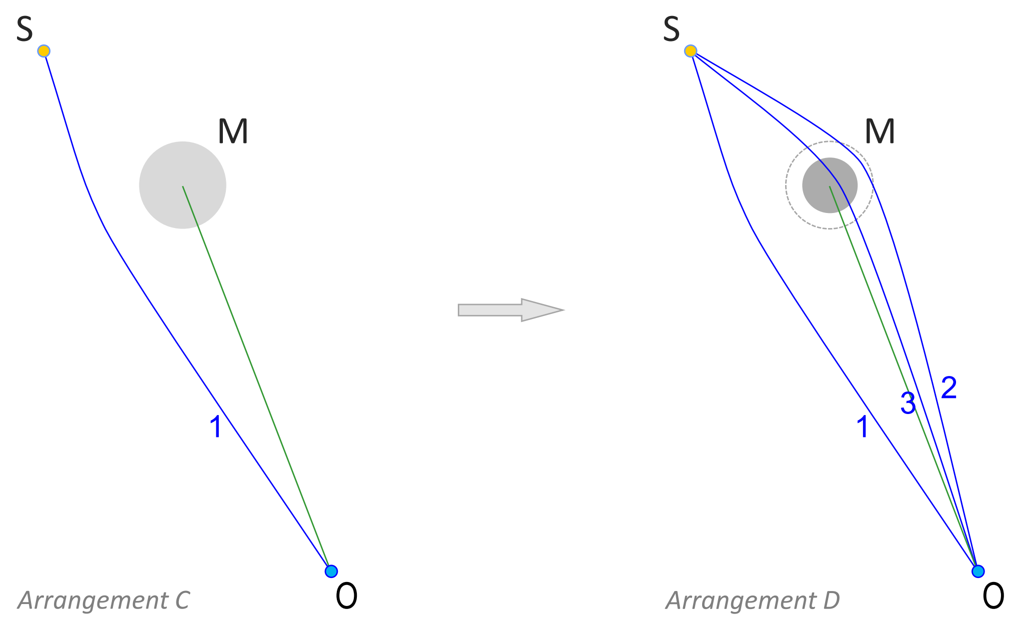


*Fig. 6: The transition from arrangement C to arrangement D.*

In our simple example, the "Gravitationally Lensed Gravity" most clearly appears at the transition from arrangement **C** to arrangement **D** by the mere contraction of the "lensing" mass distribution M:

- ➢ After the contraction of the spherically symmetric mass distribution M, the null geodesics 2 and 3 establish additional connections from the background star S to the observer O without affecting the pre-existing connection via null geodesic 1, as it is stated by Birkhoff's theorem.
- ➢ In particular for the null geodesic 2, which as well runs completely in vacuum through the external gravitational field around the mass distribution M, no qualitative physical difference to the null geodesic 1 can be reasoned. Therefore, it must also bring along a contribution of gravitational field strength from the background star S, which, however, corresponds to an earlier state of the star than the contribution coming along the null geodesic 1.
- ➢ Thus, the total gravitational field strength caused by the star at the observer increases by its contributions along the null geodesics 2 and 3.

So, here alone the contraction of the remote mass distribution M induces an increase in the gravitational field strength at the observer. This demonstrates the limits of the areas of validity of the "Newtonian limit". At the observer, according to Newton's law of gravity, such a contraction should show no effect at all !

Due to its clarity, the simple scenario provides a suitable test case for the mathematical methods of the general theory of relativity. These should produce the same result as the above reasoning, which is based only on the logical consistency of the principles of general relativity. So, they should confirm the existence of the additional contributions of gravitational field strength after the contraction of the spherically symmetric mass distribution M. It is also interesting to ask how the "Gravitationally Lensed Gravity" came to be overlooked for a century.

### 3.6 *From the literature: Curved null geodesics lead to gravitational self-force of a particle*

During the inspiral phase, two black holes orbit each other in ever tighter spirals at increasing speeds, emitting gravitational waves, until they finally merge. The exact sequences determine the temporal course of the frequency and amplitude of the radiated gravitational waves. In order to predict the expected courses ("templates") of the gravitational wave signals even for cases with extreme mass ratios of the merging partners, scenarios were considered, in which a small mass ("particle") orbits a much heavier black hole and slowly approaches it. During this process also a gravitational self-interaction of the small mass occurs.

Warren G. Anderson and Alan G. Wiseman [32] state that "in a curved background" "it is possible for the field, which 'leaves' the particle along a null geodesic, to re-intersect that particle, which follows a timelike geodesic, at a later time". During the calculation of the self-force, "when doing the integral over the Green's function, distributions (i.e. δ-functions and step functions) will be encountered when the source point and field point are null separated. In effect, the particle can 'feel' its own direct field 'sent' from points in the past."

Also, Marc Casals, Sam Dolan, Adrian C. Ottewill and Barry Wardell have studied similar scenarios in order "to shed more light on the physical nature of the self-force. For example, one may ask: what, if any, is the role of the null geodesics that begin and end on the worldline ?". Such null geodesics can start from the particle, loop around the black hole near the unstable photon orbit several times, and then end at the orbiting particle again. The authors confirm, that the retarded "times at which features appear in the partial self-force is intimately linked to the null geodesics intersecting the worldline" [33] [34] [35].

The scenarios of a particle orbiting in the vicinity of a black hole require taking into account any number of null geodesics that started at the particle's locations at distant times in its past, looped around the black hole many times, and end at the current location of the particle.

In contrast, the arrangements presented here are less extreme and should even be easier to calculate using similar methods.

## 4 Summary and outlook

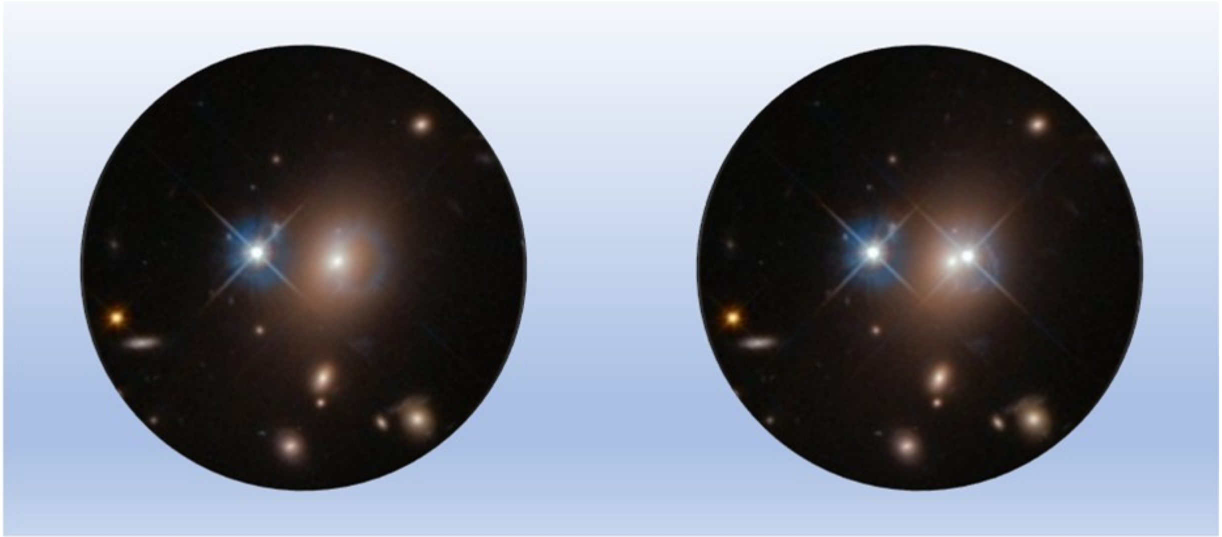

*Fig. 7: View of the observer through a telescope, on the left before and on the right after the contraction of a "lensing" spherical mass distribution and the associated appearance of the second image (the second "X") of the background object. The right picture shows the central detail of the original Hubble Space Telescope's picture of the "twin quasar" QSO 0957+561A/B. Credit: ESA/Hubble & NASA [36]. The left picture has been retouched.*

The gravitational field propagates as fast as the light along null geodesics through spacetime. In the range of a gravitational lens, spacetime is curved. Thus, also the null geodesics are curved and the gravitational field of an object in the background is deflected in the same way as its light is deflected. This effect follows from general relativity and is termed "Gravitationally Lensed Gravity". The gravitational field strength contribution caused by the background object at the location of an observer acts in the direction in which she sees it ("X") and superposes with the contributions of the other objects (Fig. 7 left). From Birkhoff's theorem it follows that the contraction of a spherically symmetric mass distribution acting as a gravitational lens can provide a second null geodesic between the background object and the observer without affecting the pre-existing first null geodesic. The appearance of a second image of the background object (the second "X") does not change anything about the first image (Fig. 7 right).

Naturally, with the first image also a gravitational effect from the background object reached the observer. Similar to the first one, the second null geodesic runs completely through vacuum around the mass distribution. No qualitative physical difference can be substantiated between these two. Consequently, besides the light, the second null geodesic must also provide a gravitational effect. This, however, originates from an earlier state of the background object than in the first image. In fact, the two images appear to the observer as if they showed different objects at different locations, whose light and gravity reached her on straight paths. She does not directly see the incoming light's curved paths and its common "true" place of origin. In the case of the "twin quasar" shown above, only after years of observations it could be confirmed that both images originate from the same source and have a transit time difference of 14 months with respect to each other [37] [38].

The propagation of the light and the gravitational field from a common source along the same null geodesics with the same speed in vacuum ensures without further ado the consistency of the locally observed effects everywhere. Suitable sensors provided, the observer detects a decrease in the gravitational field strength exactly at the moment when she sees the light of a supernova explosion. And the magnitude of the decrease corresponds to the energy density of the radiation passing her.

This is still true even if a gravitational lens produces multiple images of the star, in which the supernova light burst is visible at different times, meaning that the observer is passed by its radiation multiple times [26]. With the supernovae "Refsdal" [39] and "Requiem" [40] the Hubble Space Telescope has observed two such multiple lensed supernova events. Obviously, these arrangements can no longer be described with Newton's law of gravity. In view of Fig. 1, we now can state: When we *see* whole galaxies multiple times they also *attract* us multiple times !

### 4.1 *Does "Gravitationally Lensed Gravity" make Dark Matter obsolete ?*

A gravitational lens causes a curvature of the null geodesics. This results in a *redistribution of fluxes* from a source. The light intensity is greatly increased in some locations and slightly decreased in many others. No additional photons are generated. Likewise it works with the gravitational field strength. Described vividly, the field lines, which follow the course of the null geodesics, are bent and thus concentrated at some locations, there the field strength increases, and thinned out at other locations, there the field strength decreases. The effect is thus strongly anisotropic and, as we have seen, depends on the arrangement of the source, the gravitational lens, and the observer.

A simple model based on this idea was presented by Alexandre Deur in his 2021 paper "Relativistic corrections to the rotation curves of disk galaxies" [41]. He applied it to realistic mass distributions in the disks of galaxies and obtained the observed flat rotation curves – without adding Dark Matter ! Furthermore, he could show [42] that even the formation of large structures in the early universe results quite naturally from general relativity under consideration of "Gravitationally Lensed Gravity".

On the other hand, in spherically symmetric globular clusters, no net redistribution should result for symmetry reasons and thus no such effect should occur. This is exactly what is observed: "There is no evidence for dark matter within the globular clusters today." [43]

The consequent application of the general theory of relativity may help again – as already with the perihelion precession of Mercury – to make undetectable matter disappear forever.

## Acknowledgement

I would like to thank the engineers at NASA who succeeded in reactivating the Hubble Space Telescope in July 2021. Without them, the wonderful picture in Fig. 1 would not exist. And it was an earlier similar inspiring picture, also taken with the Hubble Space Telescope, that allowed me a glimpse into the very nature of gravitational lensing and "Gravitationally Lensed Gravity" in the first place.

Thanks are due to the scientists who have kindly answered my questions and strive to present their findings in a way that is understandable to an amateur. And not to forget the many experts and laymen who carefully create high-quality Wikipedia articles.

Many thanks to my family for their encouragement and especially to my wife for her support, understanding and patience.

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