

In[]:=

Supplementary Information
(Mathematica v13.2.0 code of TraditionalForm)

In[]:=

**Biased Random Process of Randomly Moving Particles with Constant
Average Velocities**

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NOTE:

1. The "Euclid Math One" regular and bold fonts are needed to display the contents correctly in this Notebook.
2. If there is no special case, the Mathematica code starts with gray "**In[•]:=**" and is bold by default according to Mathematica's rules.

Part 1. Necessary Calculation Processes

$$\text{In[•]:= Simplify}\left[\frac{\text{Variance}\left[\text{RandomWalkProcess}\left[\frac{1+\frac{6p-1}{5}}{2}, \frac{1-\frac{6p-1}{5}}{2}\right][t]\right]}{\text{Variance}\left[\text{RandomWalkProcess}\left[\frac{1}{2}\right][t]\right]}\right]$$

Out[•]=

$$-\frac{12}{25}(3p^2 - p - 2)$$

$$\text{In[•]:= } u = \frac{6p-1}{5} c;$$

$$\text{Simplify}\left[\left(\frac{\sqrt{c^2 - u^2}}{c}\right)^2, \text{Assumptions} \rightarrow c > 0\right]$$

Out[•]=

$$-\frac{12}{25}(3p^2 - p - 2)$$

Part 2. Process of Obtaining the Lorentz-like Factor for Randomly Moving Particles with Speeds Following Maxwell Distribution

$$\text{In[•]:= PDF[TransformedDistribution[a r, {r \approx \text{MaxwellDistribution}[\lambda]}], x]$$

Out[•]=

$$\begin{cases} \frac{\sqrt{\frac{2}{\pi}} x^2 e^{-\frac{x^2}{2a^2\lambda^2}}}{a^3 \lambda^3} & x > 0 \\ 0 & \text{True} \end{cases}$$

$$\text{In[•]:= PDF[MaxwellDistribution[a \lambda], x]$$

Out[•]=

$$\begin{cases} \frac{\sqrt{\frac{2}{\pi}} x^2 e^{-\frac{x^2}{2a^2\lambda^2}}}{a^3 \lambda^3} & x > 0 \\ 0 & \text{True} \end{cases}$$

$$\text{In[•]:= Mean}\left[\text{MaxwellDistribution}\left[\frac{1}{2} \sqrt{\frac{\pi}{2}} c\right]\right]$$

Out[•]=

$$c$$

For the case along the z -axis, we can regard the particles in $\mathcal{A}_i$ as a particle swarm possessing a mixed distribution with weight $w = \frac{c+u}{2c}$

`In[*]:= Clear[c, u];`

$$w = \frac{c + u}{2c};$$

$$\mathcal{D}_3 = \text{TruncatedDistribution}\left[\left\{r \frac{u}{c}, r\right\}, \text{UniformDistribution}[\{-r, r\}]\right];$$

$$\mathcal{D}_4 = \text{TruncatedDistribution}\left[\left\{-r, r \frac{u}{c}\right\}, \text{UniformDistribution}[\{-r, r\}]\right];$$

$$\mathcal{D}_{34} = \text{MixtureDistribution}[\{w, 1 - w\}, \{\mathcal{D}_3, \mathcal{D}_4\}];$$

$$\text{Simplify}\left[\text{StandardDeviation}[\mathcal{D}_{34}], \text{Assumptions} \rightarrow r > r \frac{u}{c} > 0\right]$$

`Out[*]:=`

$$\frac{r \sqrt{1 - \frac{u^2}{c^2}}}{\sqrt{3}}$$

`In[*]:= StandardDeviation[UniformDistribution[\{-r, r\}]]`

`Out[*]:=`

$$\frac{r}{\sqrt{3}}$$

$$\text{In[*]:= } \mathcal{D}_{\text{mz}} = \text{TransformedDistribution}\left[\frac{r \sqrt{1 - \frac{u^2}{c^2}}}{\sqrt{3}}, r \approx \text{MaxwellDistribution}\left[\frac{1}{2} \sqrt{\frac{\pi}{2}} c\right]\right];$$

$$\mathcal{D}_z = \text{TransformedDistribution}\left[\frac{r}{\sqrt{3}}, r \approx \text{MaxwellDistribution}\left[\frac{1}{2} \sqrt{\frac{\pi}{2}} c\right]\right];$$

$$\text{Simplify}[\text{Mean}[\mathcal{D}_{\text{mz}}]/\text{Mean}[\mathcal{D}_z], \text{Assumptions} \rightarrow c > 0]$$

`Out[*]:=`

$$\sqrt{1 - \frac{u^2}{c^2}}$$

For the case along the x - or y -axis, we can also regard the particles in $\mathcal{A}_i$ as a particle swarm possessing a mixed distribution with weight $w = \frac{c + u}{2c}$ (this code takes approximately 26 seconds).

```

In[ ]:=  $\mathcal{D} = \text{TransformedDistribution}\left[r \cos[\theta] \sin[\text{ArcCos}[\eta]], \left\{ \theta \approx \text{UniformDistribution}[\{-\pi, \pi\}], \right. \right.$ 
 $\eta \approx \text{UniformDistribution}[\{-1, 1\}], r \approx \text{MaxwellDistribution}\left[\frac{1}{2} \sqrt{\frac{\pi}{2}} c\right]\left. \right\}$ ;
 $\mathcal{D}_1 = \text{TransformedDistribution}\left[r \cos[\theta] \sin[\text{ArcCos}[\eta]], \left\{ \theta \approx \text{UniformDistribution}[\{-\pi, \pi\}], \right. \right.$ 
 $\eta \approx \text{UniformDistribution}\left[\left\{\frac{u}{c}, 1\right\}\right], r \approx \text{MaxwellDistribution}\left[\frac{1}{2} \sqrt{\frac{\pi}{2}} c\right]\left. \right\}$ ;
 $\mathcal{D}_2 = \text{TransformedDistribution}\left[r \cos[\theta] \sin[\text{ArcCos}[\eta]], \left\{ \theta \approx \text{UniformDistribution}[\{-\pi, \pi\}], \right. \right.$ 
 $\eta \approx \text{UniformDistribution}\left[\left\{-1, \frac{u}{c}\right\}\right], r \approx \text{MaxwellDistribution}\left[\frac{1}{2} \sqrt{\frac{\pi}{2}} c\right]\left. \right\}$ ;
 $w = \frac{c + u}{2c}$ ;
 $\mathcal{D}_{12} = \text{MixtureDistribution}[\{w, 1 - w\}, \{\mathcal{D}_1, \mathcal{D}_2\}]$ ;
 $\sigma_u = \text{Simplify}[\text{StandardDeviation}[\mathcal{D}_{12}], \text{Assumptions} \rightarrow 0 < u < c]$ 

```

```

Out[ ]:=  $\frac{1}{2} \sqrt{\frac{\pi}{2}} \sqrt{c^2 - u^2}$ 

```

```

In[ ]:=  $\text{Simplify}[\sigma_u / \text{StandardDeviation}[\mathcal{D}], \text{Assumptions} \rightarrow 0 < u < c]$ 

```

```

Out[ ]:=  $\frac{\sqrt{c^2 - u^2}}{c}$ 

```

When $c = 10$ and $u = 6$, the distribution of $\mathcal{D}_{12}$ along the x - or y -axes is like this (this code takes approximately 21 seconds):

```

In[ ]:=  $c = 10$ ;
 $u = 6$ ;
data = RandomVariate[ $\mathcal{D}_{12}$ , 30 000 000];
 $\mathcal{D}_0 = \text{SmoothKernelDistribution}[\text{data}, \{\text{"Adaptive"}, \text{Automatic}, \text{Automatic}\}]$ ;
s2 = Plot[PDF[ $\mathcal{D}_0$ , x], {x, -20, 20}, PlotRange → {{-21, 21}, {0, 0.091}},
PlotStyle → {Blue, Thickness → 0.004}, AxesLabel → {HoldForm[Speed], HoldForm[Probability Density]},
AxesStyle → Directive[Black, Thickness → 0.0018], TicksStyle → Directive[Black, Thickness → 0.0014],
LabelStyle → Directive[Black, FontFamily → "Arial", FontSize → 15]]

```

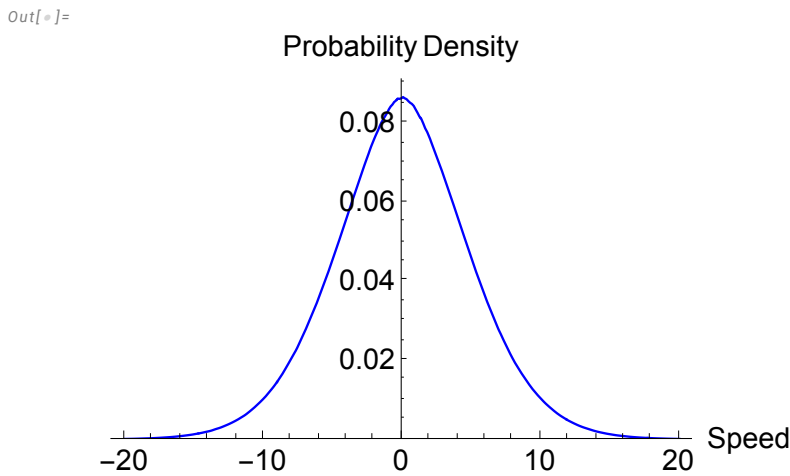


Figure S1 | Simulated probability density of the mixed distribution $\mathcal{D}_{12}$ when $\bar{r} = c = 10$ and $u = 6$.

Part 3. Calculation Process regarding Ito Equation

In[]:= Clear[c, u, σ];

```
proc = ItoProcess[ $\left\{dx1[t] == u1 dt + \frac{\sqrt{c^2 - u^2}}{c} \sigma dw1[t], dx2[t] == u2 dt + \frac{\sqrt{c^2 - u^2}}{c} \sigma dw2[t],\right.$ 
 $dx3[t] == u3 dt + \frac{\sqrt{c^2 - u^2}}{c} \sigma dw3[t]\}$ , {x1[t], x2[t], x3[t]}, {{x1, x2, x3}, {0, 0, 0}},
t, {w1 ≈ WienerProcess[], w2 ≈ WienerProcess[], w3 ≈ WienerProcess[]}]];
Simplify[PDF[proc[t], {x1, x2, x3}], Assumptions → c > 0 ∧ σ > 0]
Activate[proc["KolmogorovForwardEquation"], Unevaluated[D]]
```

Out[]:=

$$\frac{c^3 \exp\left(-\frac{c^2 (t^2 (u1^2 + u2^2 + u3^2) - 2t(u1 x1 + u2 x2 + u3 x3) + x1^2 + x2^2 + x3^2)}{2 \sigma^2 t (c^2 - u^2)}\right)}{2 \sqrt{2} \pi^{3/2} \sigma^3 \sqrt{t^3 (c^2 - u^2)^3}}$$

Out[]:=

$$p^{(0,0,0,1)}(x1, x2, x3, t) = \frac{1}{2} \left(\frac{\sigma^2 (c^2 - u^2) p^{(0,0,2,0)}(x1, x2, x3, t)}{c^2} + \frac{\sigma^2 (c^2 - u^2) p^{(0,2,0,0)}(x1, x2, x3, t)}{c^2} + \frac{\sigma^2 (c^2 - u^2) p^{(2,0,0,0)}(x1, x2, x3, t)}{c^2} \right) - u1 p^{(1,0,0,0)}(x1, x2, x3, t) - u2 p^{(0,1,0,0)}(x1, x2, x3, t) - u3 p^{(0,0,1,0)}(x1, x2, x3, t)$$

Part 4. Figure Used in the Main Text

NOTE: To run these codes correctly, the contents in "MyDirection = **" in the next cell should be modified. It is similar to MyDirection = "/Users/yourdirection/". Then, run it (Shift+Enter) beforehand.

```
MyDirection = **;
Protect[MyDirection];
Off[General::wrsym];
```

```
### ### ### ### ### ### ### ### ### ### Figure1 ### ### ### ### ### ### ### ### ### ### ### ###
```

In[93]:= c = 10;
u = 6;
σ = 2;

```
proc1 = WienerProcess[ $\sqrt{\frac{u^2}{3}}$ ,  $\frac{\sqrt{c^2 - u^2}}{c} \sigma$ ];
proc2 = WienerProcess[0, σ];
SeedRandom[123];
sample1 = Table[RandomFunction[proc1, {0, 10, 0.01}, 3][["ValueList"]], {300}];
sample2 = Table[RandomFunction[proc2, {0, 10, 0.01}, 3][["ValueList"]], {300}];
fg1 = Graphics3D[Table[{ColorData["Rainbow"][RandomReal[]], Line@sample1[[i]]}, {i, 300}],
BoxStyle → Directive[Black, Thickness → 0.001]];
fg2 = Graphics3D[Table[{ColorData["Rainbow"][RandomReal[]], Line@sample2[[i]]}, {i, 300}],
BoxStyle → Directive[Black, Thickness → 0.001]];
figure1 = GraphicsRow[{fg1, fg2}, ImageSize → 500, Spacings → Scaled[0.3],
Epilog → {Text[Style["a", 21, FontFamily → "Arial", Black, Bold], Scaled[{0.06, 0.91}]],
Text[Style["b", 21, FontFamily → "Arial", Black, Bold], Scaled[{0.56, 0.91}]]}];
Export[MyDirection <> "figure1.png", figure1, Background → None, ImageResolution → 700];
```

Figure1 ### ### ### ### ### ### ### ### ### ### ### ### ### ### ###