In[ ]:=

Supplementary Information

(Mathematica v13.2.0 code of TraditionalForm)

In[ ]:=

Biased Random Process of Randomly Moving Particles with Constant Average Velocities

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NOTE:
1. The "Euclid Math One" regular and bold fonts are needed to display the contents correctly in this Notebook.
2. If there is no special case, the Mathematica code starts with gray "*In*[∘]:=" and is bold by default according to Mathematica's rules.

Part 1. Necessary Calculation Processes

In[∘]:= $$\textbf{Simplify}\left[\frac{\textbf{Variance}\left[\textbf{RandomWalkProcess}\left[\frac{1+\frac{6p-1}{5}}{2}, \frac{1-\frac{6p-1}{5}}{2}\right][t]\right]}{\textbf{Variance}\left[\textbf{RandomWalkProcess}\left[\frac{1}{2}\right][t]\right]}\right]$$

Out[∘]=

$$-\frac{12}{25}\left(3p^2-p-2\right)$$

In[∘]:= $$u=\frac{6p-1}{5}c;$$

$$\textbf{Simplify}\left[\left(\frac{\sqrt{c^2-u^2}}{c}\right)^2, \textbf{Assumptions}\to c>0\right]$$

Out[∘]=

$$-\frac{12}{25}\left(3p^2-p-2\right)$$

Part 2. Process of Obtaining the Lorentz Factor for Randomly Moving Particles with Speed Following Maxwell Distribution

In[∘]:= **PDF[TransformedDistribution[$a\,r$, {$r \approx$ MaxwellDistribution[λ]}], x]**

Out[∘]=

$$\begin{cases} \dfrac{\sqrt{\frac{2}{\pi}}\,x^2\,e^{-\frac{x^2}{2a^2\lambda^2}}}{a^3\lambda^3} & x>0 \\ 0 & \text{True} \end{cases}$$

In[∘]:= **PDF[MaxwellDistribution[$a\,\lambda$], x]**

Out[∘]=

$$\begin{cases} \dfrac{\sqrt{\frac{2}{\pi}}\,x^2\,e^{-\frac{x^2}{2a^2\lambda^2}}}{a^3\lambda^3} & x>0 \\ 0 & \text{True} \end{cases}$$

In[4]:= $$\textbf{Mean}\left[\textbf{MaxwellDistribution}\left[\frac{1}{2}\sqrt{\frac{\pi}{2}}\,c\right]\right]$$

Out[4]= c

For the case along the z-axis, we can regard the particles in $\mathcal{A}_i$ as a particle swarm possessing a mixed distribution with weight $w=\frac{c+u}{2c}$

In[ ]:=

```
Clear[c, u];
w = (c + u)/(2 c);
𝒟3 = TruncatedDistribution[{r u/c, r}, UniformDistribution[{-r, r}]];
𝒟4 = TruncatedDistribution[{-r, r u/c}, UniformDistribution[{-r, r}]];
𝒟34 = MixtureDistribution[{w, 1 - w}, {𝒟3, 𝒟4}];
Simplify[StandardDeviation[𝒟34], Assumptions → r > r u/c > 0]
```

Out[ ]=

$$\frac{r\sqrt{1-\frac{u^2}{c^2}}}{\sqrt{3}}$$

In[ ]:=

```
StandardDeviation[UniformDistribution[{-r, r}]]
```

Out[ ]=

$$\frac{r}{\sqrt{3}}$$

In[ ]:=

```
𝒟mz = TransformedDistribution[(r Sqrt[1 - u^2/c^2])/Sqrt[3], r ≈ MaxwellDistribution[1/2 Sqrt[π/2] c]];
𝒟z = TransformedDistribution[r/Sqrt[3], r ≈ MaxwellDistribution[1/2 Sqrt[π/2] c]];
Simplify[Mean[𝒟mz]/Mean[𝒟z], Assumptions → c > 0]
```

Out[ ]=

$$\sqrt{1-\frac{u^2}{c^2}}$$

For the case along the x- or y-axis, we can also regard the particles in $\mathcal{A}_i$ as a particle swarm possessing a mixed distribution with weight $w = \frac{c+u}{2\,c}$ (this code takes approximately 26 seconds).

```
In[∘]:= 𝒟 = TransformedDistribution[r Cos[θ] Sin[ArcCos[η]], {θ ≈ UniformDistribution[{-π, π}],
            η ≈ UniformDistribution[{-1, 1}], r ≈ MaxwellDistribution[1/2 Sqrt[π/2] c]}];
        𝒟1 = TransformedDistribution[r Cos[θ] Sin[ArcCos[η]], {θ ≈ UniformDistribution[{-π, π}],
            η ≈ UniformDistribution[{u/c, 1}], r ≈ MaxwellDistribution[1/2 Sqrt[π/2] c]}];
        𝒟2 = TransformedDistribution[r Cos[θ] Sin[ArcCos[η]], {θ ≈ UniformDistribution[{-π, π}],
            η ≈ UniformDistribution[{-1, u/c}], r ≈ MaxwellDistribution[1/2 Sqrt[π/2] c]}];
        w = (c + u)/(2 c);
        𝒟12 = MixtureDistribution[{w, 1 - w}, {𝒟1, 𝒟2}];
        σu = Simplify[StandardDeviation[𝒟12], Assumptions → 0 < u < c]
```

Out[∘]=

$$\frac{1}{2}\sqrt{\frac{\pi}{2}}\ \sqrt{c^2-u^2}$$

```
In[∘]:= Simplify[σu / StandardDeviation[𝒟], Assumptions → 0 < u < c]
```

Out[∘]=

$$\frac{\sqrt{c^2-u^2}}{c}$$

When $c = 10$ and $u = 6$, the distribution of $\mathcal{D}12$ along the x- or y-axes is like this (this code takes approximately 21 seconds):

```
In[∘]:= c = 10;
        u = 6;
        data = RandomVariate[𝒟12, 30 000 000];
        𝒟0 = SmoothKernelDistribution[data, {"Adaptive", Automatic, Automatic}];
        s2 = Plot[PDF[𝒟0, x], {x, -20, 20}, PlotRange → {{-21, 21}, {0, 0.091}},
           PlotStyle → {Blue, Thickness → 0.004}, AxesLabel → {HoldForm[Speed], HoldForm[Probability Density]},
           AxesStyle → Directive[Black, Thickness → 0.0018], TicksStyle → Directive[Black, Thickness → 0.0014],
           LabelStyle → Directive[Black, FontFamily → "Arial", FontSize → 15]]
```

Out[∘]=

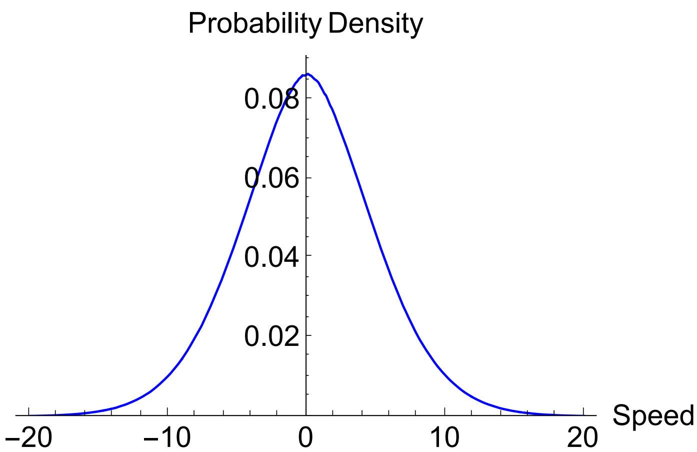

Figure S1 | Simulated probability density of the mixed distribution $\mathcal{D}12$ when $\bar{r} = c = 10$ and $u = 6$.

Part 3. Calculation Process regarding Ito Equation

In[∘]:=

```
Clear[c, u, σ];
proc = ItoProcess[dx[t] == u dt + Sqrt[c^2 - u^2]/c σ dw[t], x[t], {x, 0}, t, w ≈ WienerProcess[]];
Simplify[PDF[proc[t], x], Assumptions → c > 0 ∧ σ > 0]
Simplify[PDF[WienerProcess[u, Sqrt[c^2 - u^2]/c σ][t], x], Assumptions → c > 0 ∧ σ > 0]
SliceDistribution[WienerProcess[u1, Sqrt[c^2 - u^2]/c σ1], t]
```

Out[∘]=

$$\frac{c\, e^{-\frac{c^2 (x-t u)^2}{2 \sigma^2 t \left(c^2-u^2\right)}}}{\sqrt{2\pi}\, \sigma\, \sqrt{t\left(c^2-u^2\right)}}$$

Out[∘]=

$$\frac{c\, e^{-\frac{c^2 (x-t u)^2}{2 \sigma^2 t \left(c^2-u^2\right)}}}{\sqrt{2\pi}\, \sigma\, \sqrt{t}\, \sqrt{c^2-u^2}}$$

Out[∘]=

$$\mathrm{NormalDistribution}\left[t\,\mathrm{u1}, \frac{\sigma 1\, \sqrt{t}\, \sqrt{c^2-u^2}}{c}\right]$$

In[∘]:=

```
proc = ItoProcess[dx[t] == u dt + Sqrt[c^2 - u^2]/c σ dw[t], x[t], {x, 0}, t, w ≈ WienerProcess[]];
Activate[proc["KolmogorovForwardEquation"], Unevaluated[D]]
```

Out[∘]=

$$p^{(0,1)}(x, t) = \frac{\sigma^2 \left(c^2-u^2\right) p^{(2,0)}(x, t)}{2 c^2} - u\, p^{(1,0)}(x, t)$$

```
In[•]:= Clear[c, u, σ];
proc = ItoProcess[{dx1[t] == u1 dt + Sqrt[c^2 - u^2]/c σ dw1[t], dx2[t] == u2 dt + Sqrt[c^2 - u^2]/c σ dw2[t],
      dx3[t] == u3 dt + Sqrt[c^2 - u^2]/c σ dw3[t]}, {x1[t], x2[t], x3[t]}, {{x1, x2, x3}, {0, 0, 0}},
    t, {w1 ≈ WienerProcess[], w2 ≈ WienerProcess[], w3 ≈ WienerProcess[]}];
Simplify[PDF[proc[t], {x1, x2, x3}], Assumptions → c > 0 ∧ σ > 0]
Activate[proc["KolmogorovForwardEquation"], Unevaluated[D]]
```

Out[•]=

$$\frac{c^3 \exp\left(-\frac{c^2\left(t^2\left(\mathrm{u1}^2+\mathrm{u2}^2+\mathrm{u3}^2\right)-2t\left(\mathrm{u1\,x1}+\mathrm{u2\,x2}+\mathrm{u3\,x3}\right)+\mathrm{x1}^2+\mathrm{x2}^2+\mathrm{x3}^2\right)}{2\sigma^2 t\left(c^2-u^2\right)}\right)}{2\sqrt{2}\,\pi^{3/2}\sigma^3\sqrt{t^3\left(c^2-u^2\right)^3}}$$

Out[•]=

$$p^{(0,0,0,1)}(\mathrm{x1},\mathrm{x2},\mathrm{x3},t) = \frac{1}{2}\left(\frac{\sigma^2\left(c^2-u^2\right)p^{(0,0,2,0)}(\mathrm{x1},\mathrm{x2},\mathrm{x3},t)}{c^2}+\frac{\sigma^2\left(c^2-u^2\right)p^{(0,2,0,0)}(\mathrm{x1},\mathrm{x2},\mathrm{x3},t)}{c^2}+\frac{\sigma^2\left(c^2-u^2\right)p^{(2,0,0,0)}(\mathrm{x1},\mathrm{x2},\mathrm{x3},t)}{c^2}\right) - \mathrm{u1}\,p^{(1,0,0,0)}(\mathrm{x1},\mathrm{x2},\mathrm{x3},t) - \mathrm{u2}\,p^{(0,1,0,0)}(\mathrm{x1},\mathrm{x2},\mathrm{x3},t) - \mathrm{u3}\,p^{(0,0,1,0)}(\mathrm{x1},\mathrm{x2},\mathrm{x3},t)$$

Part 4. Figure Used in the Main Text

NOTE: To run these codes correctly, the contents in "**MyDirection = ∗∗**" in the next cell should be modified. It is similar to **MyDirection** = "**/Users/yourdirection/**". Then, run it (Shift+Enter) beforehand.

```
MyDirection = ∗∗;
Protect[MyDirection];
Off[General::wrsym];

## ## ## ## ## ## ## ## ## ## ## Figure1 ## ## ## ## ## ## ## ## ## ## ## ## ## ## ##

In[•]:= c = 10;
u = 6;
σ = 2;
proc = WienerProcess[Sqrt[u^2/3], Sqrt[c^2 - u^2]/c σ];
SeedRandom[123];
sample = Table[RandomFunction[proc, {0, 10, 0.01}, 3]["ValueList"]ᵀ, {300}];
figure1 = Graphics3D[Table[{ColorData["SolarColors"][RandomReal[]], Line@sample[[i]]}, {i, 300}],
    BoxStyle → Directive[Black, Thickness → 0.001]];
Export[MyDirection <> "figure1.png", figure1, Background → None, ImageResolution → 700];

## ## ## ## ## ## ## ## ## ## ## Figure1 ## ## ## ## ## ## ## ## ## ## ## ## ## ## ##
```