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Biased Random Process of Randomly Moving Particles with Fixed Mean and Group Velocities

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Abstract: In a particle swarm with a fixed kinetic energy, the constituent particles move randomly and their velocity magnitudes follow a Maxwell distribution characterized by specific parameters. Within a certain time frame, particles in a sub-swarm may exhibit a bias in their movement direction. Thus, the particles may exhibit a special characteristic (biased random motion), where the group velocity in a particular direction remains constant u. For this biased particle swarm, composed of randomly moving particles with a fixed average speed c, particles have a higher probability of moving in a specific direction. Conversely, their likelihood of moving in all other directions is equally reduced. This study starts from the perspective of biased random walks and using Mathematica software proves that the diffusion rate of particles (in the reference frame $\mathcal{R}_u$ observed from $\mathcal{R}_0$) in all directions is slower than that of an unbiased particle swarm with the same average speed c. Specifically, the degree of reduction is determined by the Lorentz-like factor $\frac{\sqrt{c^2-u^2}}{c}$. Lastly, we present the Itô equation for this biased random motion and provide associated verification examples. This study aims to serve as a reference for comprehending the underlying principles of the special relativity effect.

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1. Introduction

Most random phenomena in nature exhibit characteristics of bias[1]. This includes random motion, as such phenomena tend not to occur in a strictly neutral or uniform manner. In a group composed of randomly moving particles, the motions of particles in its sub-groups tend to be biased. Among these biased random motions, several distinct motion processes warrant particular attention. These special motions include particles with a higher probability of movement in a specific direction, while maintaining an equivalent yet lower probability of movement in other directions[2]; particles with greater probabilities of movement toward a point, while maintaining an equivalent yet lower probability of movement in other directions[3]; and biased random processes of particles capable of generating rotation[4]. The study of such phenomena typically begins with the simplest biased random processes.

Biased random processes, akin to the first scenario previously described, have been the subject of extensive research[5]. The nature of the bias can shape the form of the expression for a random process. For example:

$$\mathrm{d}x(t) = uS_t\mathrm{d}t + \sigma S_t\mathrm{d}w(t), \tag{1}$$

where $w(t)$ denotes a Brownian motion process. In Eq. 1, when the drift (u) and diffusion (σ) terms grow synchronously according to S_t, geometric Brownian motion will be produced. There will be fractional Brownian motion when particles follow the motion law in Eq. 2.

$$\mathrm{d}x(t) = \theta[u - x(t)]\mathrm{d}t + \sigma\mathrm{d}w(t), \tag{2}$$

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where θ is the regression speed. Additionally, there is a simpler scenario where particles with the same jump speed have a higher probability of moving in a specific direction and equal probabilities of moving in other directions. This is a common biased phenomenon in physics. In a system with constant kinetic energy, it is generally challenging to alter the distribution of particle speeds, while the directions of motion can change more readily. When describing such systems using Eq. 3

$$\mathrm{d}x(t) = u\mathrm{d}t + \sigma \mathrm{d}w(t), \tag{3}$$

the system's kinetic energy will increase as the drift term u increases. It is obvious that such an equation does not reflect real scenarios. In contrast, when the drift term u increases, it may be a better solution to reduce the diffusion term σ accordingly. Unfortunately, there is currently no clear quantitative relationship available. This article will delve into this issue in detail and derive a quantitative relationship of Lorentz-like factor between the drift (u) and diffusion (σ) terms. This kind of problem is similar to the problem that the particle speed or time slows down in a moving frame in special relativity. Some researchers[6–9] demonstrated the diffusion process (random walk) with a special relativistic effect. However, it is the diffusion or random walk of particles when their speed is affected by the special relativity effect. The case involved in this study is that the particle velocity is determined, and the diffusion in the biased random particle swarm formed by them is restricted by the Lorentz-like factor. They are not the same kind of problem.

In my previous research[2], I demonstrated the special relativistic-like effect of biased random motion when particles move at a uniform speed of c. This article extends that work by considering particles whose speeds follow a Maxwell distribution with equal average speed c. I establish that such biased random motion retains its special relativistic-like effect. Additionally, I present the corresponding Itô equations and simulate the motion of related examples. My findings offer insights into the intricate relationship between biased random processes and the special relativity effect.

2. Results and Discussion

2.1. Biased Random Walk

In order to elucidate the relationship between the drift and diffusion terms, this study commences with an examination of random walks. Among various biased random walk scenarios, we specifically focus on a situation where the probability of a particle moving in one of the six possible directions in a 3-dimensional space is denoted as p, while the probability in the remaining five directions is given by $\frac{1-p}{5}$, subject to the condition $\frac{1}{6} < p < 1$. Although the case when $0 < p < \frac{1}{6}$ presents an intriguing scenario, it falls outside the purview of this study. Given a particle's jump speed as c, the collective or group velocity of all particles can be described by the equation:

$$u = \frac{6p-1}{5}c. \tag{4}$$

This can be expressed equivalently in the form $\{X_1(t), X_2(t), X_3(t), t \in T\}$, where $X_1(t), X_2(t)$ and $X_3(t)$ represent 1-dimensional random walks with a jump speed of $\frac{c}{\sqrt{3}}$. We introduce a 3-dimensional rectangular coordinate system, aligning its (1, 1, 1) direction parallel to the group velocity $\boldsymbol{u}$.

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Consequently, the 1-dimensional random walk model along the x_i-axis, which is one of the three equivalent coordinate axes, is given by:

$$P_{x,\text{bia}} = \begin{cases} \dfrac{1+u/c}{2}, & n>0, \\ \dfrac{1-u/c}{2}, & n<0, \\ 0, & \text{otherwise}, \end{cases} \tag{5}$$

where $n \in \mathbb{N}$. The corresponding unbiased 1-dimensional random walk model on the x_i-axis among the three equivalent coordinate axes is

$$P_{x,\text{unbia}} = \begin{cases} \dfrac{1}{2}, & n>0, \\ \dfrac{1}{2}, & n<0, \\ 0, & \text{otherwise}, \end{cases} \tag{6}$$

where $n \in \mathbb{N}$. The relative standard deviation of the two 1-dimensional random walk processes along the two equivalent x_i-axes is

$$\frac{\sigma_{\text{bia}}}{\sigma_{\text{unbia}}} = -\frac{12}{25}(p-1)(3p+2). \tag{7}$$

Consider the 3-dimensional random walk vectors, representing velocities, formed by $X_1(t)$, $X_2(t)$ and $X_3(t)$ over a unit time interval. Their norms follow the discrete Maxwell distribution, and the average norm is proportional to the standard deviation of any component vector along the equivalent axes[2]. Consequently, the average velocity magnitude of the biased 3-dimensional random walk is $\dfrac{\sqrt{c^2-u^2}}{c}$ times that of its unbiased counterpart, with u defined in Eq. 4 (see Part 1 of the Supplementary Information for the detailed Mathematica code). Evidently, the biased case represents a decelerative process. This also illuminates the relationship between the drift term and the attenuation rate of diffusion in a continuous process with constant energy from an alternative perspective. For a more rigorous depiction of this relationship in continuous time, further proofs are warranted. Yet, directly proving this for a continuous-time stochastic process poses challenges. We can transpose to the following strategy.

2.2. Continuous Biased Random Process

When particles in swarm $\mathcal{A}$ possess speeds governed by a Maxwell distribution with a mean of c, the associated scale parameter for this distribution is $\frac{1}{2}\sqrt{\frac{\pi}{2}}c$. Particles in $\mathcal{A}$ can be categorized into subgroups $\mathcal{A}_i$, based on their speeds, where each subgroup's velocity vectors terminate on a series of spheres of radius r_i $(i = 1, 2, 3, \cdots)$. The endpoints of these vectors for particles in $\mathcal{A}_i$ are uniformly distributed over the sphere with radius r_i. The number of particles in $\mathcal{A}$ distributed across these spheres follows a Maxwell distribution with respect to r, with the scale parameter $\frac{1}{2}\sqrt{\frac{\pi}{2}}c$ and mean radius $\bar{r} = c$. Hence, the scenario where particle speeds are uniformly c can be viewed as a specialized case $(r_i = c)$. The collective motion characteristics of particles whose velocity vectors terminate on spheres of radius r_i embody the motion traits of particles with speeds dictated by the Maxwell distribution. The particle counts in both $\mathcal{A}$ and $\mathcal{A}_i$ are substantial, with a uniform distribution. When a subgroup $\mathcal{A}$ of the parent particle swarm—where the magnitudes of the particle velocities follow a Maxwell distribution—moves along the z-axis (with a reference frame based on the parent particle swarm) at an average speed of u, it is equivalent to each sub-particle group $\mathcal{A}_i$ ($i = 1, 2, 3, \cdots$) moving along the z-axis with an average speed of $\dfrac{r_i\, u}{c}$.

Let R follow a Maxwell distribution with a scale parameter λ. Then aR will follow a Maxwell distribution with a scale parameter $a\lambda$. Specifically, along the z-axis, all the component speeds u_i in particle swarm $\mathcal{A}_i$, given by $u_i = \dfrac{r_i\, u}{c}$ (where $i = 1, 2, 3, \cdots$), follow a Maxwell distribution with a mean value of u. Moreover, the standard deviations $\sigma_i = \dfrac{r_i\sqrt{1 - u^2/c^2}}{\sqrt{3}}$ (where $i = 1, 2, 3, \cdots$) of the mixed distributions (we can regard the particles in $\mathcal{A}_i$ as a particle swarm possessing a mixed distribution with weight $w = \dfrac{c+u}{2c}$) across all spherical layers also conform to a Maxwell distribution. For the case along the x- or y-axis, the particles in $\mathcal{A}_i$ can also be viewed as a particle swarm possessing a mixed distribution with weight $w = \dfrac{c+u}{2c}$. Adopting a method analogous to that in my previous work[2], it can be demonstrated that the ratio of the standard deviation of the mixed distribution along the x- or y-axis to that of the unbiased moving particle on the same axis satisfies the relationship $\dfrac{\sqrt{c^2 - u^2}}{c}$, and the ratio of the standard deviation along the z-axis also satisfies this relationship (see Part 2 of the Supplementary Information for the detailed Mathematica code).

2.3. Itô Equation of Biased Random Processes

The above results indicate that for particles possessing a group velocity of u and with speeds following the Maxwell distribution, their speeds within $\mathcal{R}_u$ exhibit a deceleration characterized by a Lorentz-like factor. Despite this change in speed, their distributions still follow the Maxwell distribution within $\mathcal{R}_u$. This behavior permits their motion to be described by the following Itô equation when viewed from $\mathcal{R}_0$. It should be noted that the definitions of $\mathcal{R}_0$ and $\mathcal{R}_u$ align with those presented in my previous study[2].

$$\begin{cases} \mathrm{d}x_1(t) = u_1 \mathrm{d}t + \sigma \dfrac{\sqrt{c^2 - u^2}}{c}\, \mathrm{d}w_1(t), \\ \mathrm{d}x_2(t) = u_2 \mathrm{d}t + \sigma \dfrac{\sqrt{c^2 - u^2}}{c}\, \mathrm{d}w_2(t), \\ \mathrm{d}x_3(t) = u_3 \mathrm{d}t + \sigma \dfrac{\sqrt{c^2 - u^2}}{c}\, \mathrm{d}w_3(t), \end{cases} \tag{8}$$

where u_1, u_2 and u_3 are the drift speeds and $\sqrt{{u_1}^2 + {u_2}^2 + {u_3}^2} = u$; c is the average speed of particles; $w_1(t)$, $w_2(t)$ and $w_3(t)$ are Brownian motion processes; and σ is their standard deviation. Then, the slice distribution of Eq. 8 is

$$\frac{c^3\, \mathrm{e}^{-\frac{c^2 (x_i - t u_i)^2}{2\sigma^2 t (c^2 - v^2)}}}{2\sqrt{2}\pi^{\frac{3}{2}}\, \sigma^3 t^{\frac{3}{2}} (c^2 - u^2)^{\frac{3}{2}}}, \tag{9}$$

where i takes the values 1, 2 and 3, in accordance with the Einstein summation convention. Moreover, the forward equation of Eq. 8 is

$$\frac{\partial p(\boldsymbol{x}, t)}{\partial t} = \frac{\sigma^2 (c^2 - u^2)}{2c^2} \nabla^2 p(\boldsymbol{x}, t) - u_i \frac{\partial p(\boldsymbol{x}, t)}{\partial x_i}, \tag{10}$$

where $p(\boldsymbol{x}, t)$ is the probability of a particle at position $\boldsymbol{x}(x_1, x_2, x_3)$ and time t. This is also a drift-diffusion (or advection-diffusion) equation. Only the diffusion coefficient of this equation is limited by the advection term.

This result (3-dimensional biased Brownian motion) can be simulated by three 1-dimensional biased Brownian motions $\{X_1(t), X_2(t), X_3(t), t \in T\}$. Here, the trajectories of 300 particles randomly

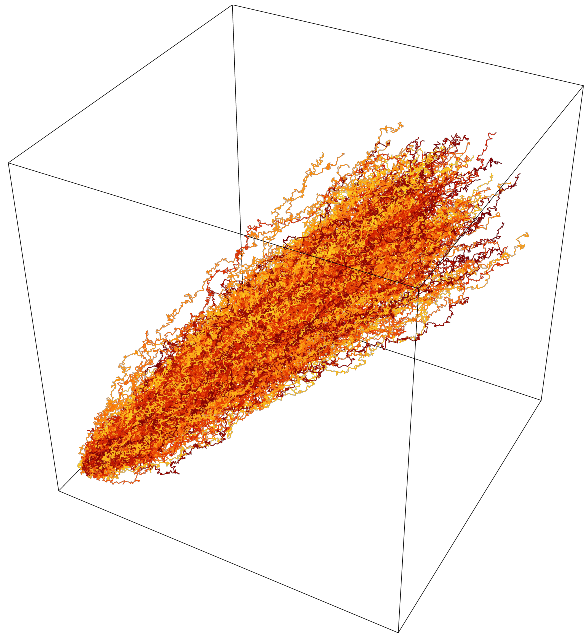

Figure 1. Simulation results of the trajectories of 300 particles randomly moving for 10 s ($c = 10$, $u = 6$ and $\sigma = 2$).

moving for 10 s with a time step of 0.01 s are presented in Fig. 1. There is no significant difference from the situation that is unbiased and not affected by the special relativistic-like effect.

3. Conclusions

For a particle swarm composed of particles with a fixed energy, assumed to be in reference frame $\mathcal{R}_0$, the magnitudes of the particle velocities follow a Maxwell distribution with parameter $\frac{1}{2}\sqrt{\frac{\pi}{2}}c$, yielding an average speed of c. When a sub-swarm of particles (assumed to be in reference frame $\mathcal{R}_u$), exhibits a higher probability of motion in a specific direction and equal yet lower probabilities in other directions (with the group velocity of this sub-swarm denoted as u), as observed from $\mathcal{R}_0$, the velocities of particles within this sub-swarm decelerate, adhering to a pattern determined by the Lorentz-like factor $\frac{\sqrt{c^2 - u^2}}{c}$. Consequently, the biased motion of such particles can be described by a set of Itô equations (Eq. 8). Both the time slice distribution and the forward equation for these biased particles differ from those in the reference frame $\mathcal{R}_0$.

Data Availability Statement: All data generated or analysed during this study are included in this published article and its supplementary information files.

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Supporting Information Available: Mathematica code for necessary calculation process and graphics. The following file is available free of charge.—Supporting Information.

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