Article

Quickening Data-Aware Conformance Checking through Temporal Algebras

Giacomo Bergami, Samuel Appleby, and Graham Morgan

Newcastle University, United Kingdom; {Giacomo.Bergami,S.Appleby,Graham.Morgan}@newcastle.ac.uk
* Correspondence: Giacomo.Bergami@newcastle.ac.uk
† This paper is an extended version of our paper published in IDEAS'22: Proceedings of the 26th International Database Engineered Applications Symposium.

Abstract: This paper extends our seminal paper on KnoBAB for efficient Conformance Checking computations performed on top of a customised relational model. After defining our proposed temporal algebra for temporal queries ($\texttt{xt}\mathrm{LTL}_f$), we show that this can express existing temporal languages over finite and non-empty traces such as LTL_f. This paper also proposes a parallelisation strategy for such queries thus reducing conformance checking into an embarrassingly parallel problem leading to super-linear speed up. This paper also presents how a single $\texttt{xt}\mathrm{LTL}_f$ operator (or even entire sub-expressions) might be efficiently implemented via different algorithms thus paving the way to future algorithmic improvements. Finally, our benchmarks remark that our proposed implementation of $\texttt{xt}\mathrm{LTL}_f$ (KnoBAB) outperforms state-of-the-art conformance checking software running on LTL_f logic, be it data or dateless.

Keywords: Logical Artificial Intelligence; Knowledge Bases; Query Plan; Temporal Logic; Conformance Checking; Temporal Data Mining; Intraquery Parallelism

Citation: Bergami, G.; Appleby, S.; Morgan, G. Quickening Data-Aware Conformance Checking through Temporal Algebras. *Information* **2022**, *1*, 0. https://doi.org/

Received:
Accepted:
Published:

1. Introduction

(Temporal) conformance checking is increasingly at the heart of ARTIFICIAL INTELLIGENCE activities: due to its logical foundation, assessing whether a sequence of distinguishable events (i.e., a *trace*) does not conform to the expected process behaviour (*process model*) reduces to the identification of the specific unfulfilled temporal patterns, represented as logical *clauses*. This leads to Explainable AI, as the process model's violation becomes apparent. Clauses are the instantiation of a specific behavioural pattern (i.e., *template*) that expresses temporal correlation between actions being carried out (*activations*) and their expected results (*targets*). These, therefore, differ from traditional association rules [1], as they can also describe complex temporal requirements: e.g., whether the target should immediately follow (ChainResponse) or precede (ChainPrecedence) the activation, if the former might happen any time in the future (Response), or if the target should have never happened in the past (Precedence). These temporal constraints can be fully expressed in a LINEAR TEMPORAL LOGIC OVER FINITE TRACES (LTL_f) and its extensions; this logic is referred to as linear as it assumes that, in a given sequence of events of interest, it exists only one possible future event immediately following a given one. This differs from stochastic process modelling, where each event is associated with a probabilistic distribution of possibly following events [2,3]. Such a formal language can be extended to express correlations between activations and targets through binary predicates correlating data payloads. Events are also associated with either an action or a piece of state information represented as an *activity label*. Collections of traces are usually referred to as *log*.

Despite its theoretical foundations, state-of-the-art conformance checking techniques for entire logs expose sub-optimal run-time behaviour [4]. The reasons are the following: while performing conformance checking over relational databases requires computing costly aggregation conditions [5], tailored solutions do not exploit efficient query planning

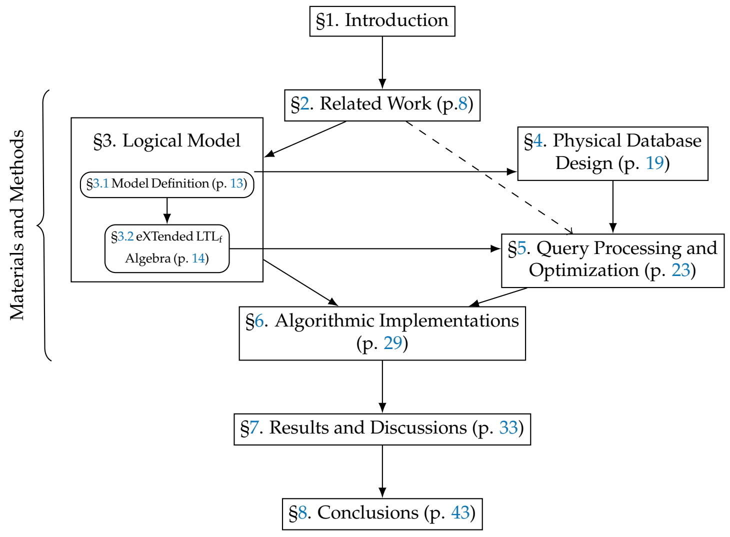

§1. Introduction
§2. Related Work (p.8)
§3. Logical Model
§3.1 Model Definition (p. 13)
§3.2 eXTended LTLf Algebra (p. 14)
§4. Physical Database Design (p. 19)
§5. Query Processing and Optimization (p. 23)
§6. Algorithmic Implementations (p. 29)
§7. Results and Discussions (p. 33)
§8. Conclusions (p. 43)
Materials and Methods

Figure 1. Table of Contents

and data access minimization, thus requiring scanning the traces multiple times [6]. Efficiency becomes of the uttermost importance after observing that conformance checking's run-time enhancement has a strong impact on a whole wide range of practical use case scenarios (§1.1). To make conformance checking computations efficient, we synthesise temporal data derived from a system (be it digital or real) to a simplified representation in the Relational Database model. In this instance, we use xtLTL$_f$, our proposed extension of LTL$_f$, to represent process models. While the original LTL$_f$ merely asserts whether a trace is *conformant* to the model, our proposed algebra returns all of the traces satisfying the temporal behaviour, as well as activated and targeted events. As a temporal representation in the declarative model provides a point-of-relativity in the context of correctness (i.e., time itself may dictate if traces maintain correctness throughout the logical declarations expressed by the model), the considerations of such temporal issues significantly increase the checking requirement. This is due to a need to visit logical declarations for correctness in the context of each temporal instance.

This paper extends our previous work [4], where we clearly showed the disruptiveness of the relational model for efficiently running temporal queries outperforming state-of-the-art model checking systems. While our original work [4] provided just a brief rationale behind the success of KnoBAB[1], this paper wants to deep-dive into each possible contribution leading to our implementation success.

1. We formalise (§3.1) and characterise (§4) the relational model so to faithfully represent our log. This will prelude the full formalisation of the xtLTL$_f$ algebra.
2. We show that the xtLTL$_f$ algebra (§3.2) can not only express declarative languages such as Declare [7] as in our previous work but can express the semantics of LTL$_f$ formula by returning any non-empty finite trace satisfying the latter if loaded in our relational representation (see Appendix). We also show for the first time a formalisation for data correlation conditions associated with binary temporal operators.
3. We propose different possible algorithms for some xtLTL$_f$ operators (§6), and we then discuss both theoretically (Supplement IV.2) and empirically (§7.1) which might be

[1] The acronym stands for KNOWLEDGE BASE FOR ALIGNMENTS AND BUSINESS PROCESS MODELLING. The Business Process Mining literature often uses the term KNOwledge Base differently from customary database literature: while in the former, the intended meaning is a customary relational representation for trace data, in the latter, we often require that such representation provides a machine-readable representation of data so to infer novel facts or to detect inconsistencies.

preferred under different trace length ϵ or log size $|\mathcal{L}|$ conditions. This leads to the definition of hybrid algorithms [8].

4. Our benchmarks demonstrate that our implementation still outperforms conformance checking techniques running on both relational databases (§7.2) and on tailored solutions (§7.5) when customary algorithms are chose for implementing xtLTL$_f$ operators.
5. We show (§7.3) how the query plan's execution might be parallelised, thus further improving with super-linear speed-up our previous running time results.
6. We discuss (§7.4) how different data accessing strategies achievable through query rewriting might affect the query's running time.

Figure 1 provides a graphical depiction of this paper's table of contents, with the mutual dependencies between its sections. Appendices and Supplements start from p. 44.

1.1. Case Studies

The present section shows a broad-ranging set of real case studies requiring efficient conformance checking computations in LTL$_f$*. This, therefore, motivates the need for our proposed approach in a practical sense.*

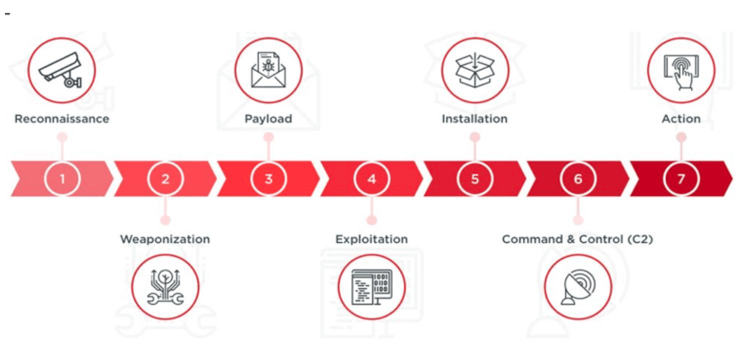
1.1.1. Cyber Security

Intrusion detection for cyber security aims at auditing an environment for identifying potential flaws that can be later on remedied and fixed. While *anomaly-based* approaches raise an alarm if the observed behaviour differs significantly from the expected one, *signature-based* approaches check whether attack patterns might be recognised from the environment. The latter are often used to mitigate the high false-alarm rates of the former [9]. Expected behaviour might be encoded as process models expressed in LTL$_f$, which, when violated, lead to the detection of an attack: such a language can be directly exploited to represent several different kinds of attacks, such as Denial Of Service, Buffer Overflows, and Password Guessing [9]. In his dissertation [10], Ray shows how malware can be detected by determining LTL$_f$ formulae discriminating between system-calls patterns generated by malicious software from expected run-time behaviour. Recent developments [11,12] showed that it is possible to perform prediction (and therefore reasoning) on potentially infinite sequences by analysing a finite subsequence of the overall behaviour within a sliding window; Buschjäger et al. [11] predict future events not covered by the sliding window by correlating them to the patterns observed in such a window. By associating a positive label to each finite subsequence preceding or containing an attack, and a negative one otherwise, we can also extract temporal models detecting subsequences containing attacks [13]. This entails that real-time verification boils down, to some extent, to offline monitoring, as we guarantee that it is sufficient to analyse currently-observed behaviours to predict and detect an attack. The learned model, once validated, can be exploited in the aforementioned real-time verification systems [9].

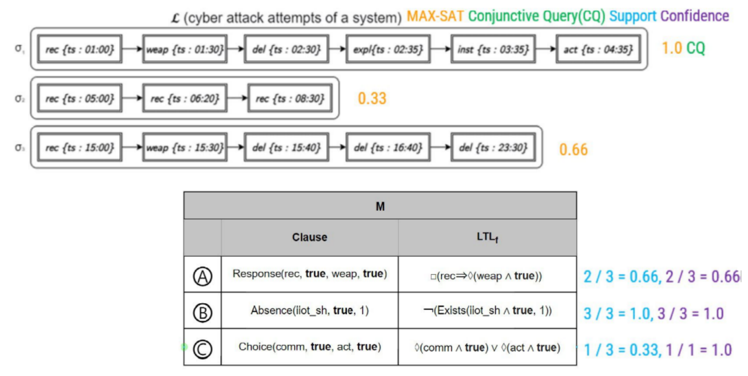
Example 1. *The Cyber Kill Chain® framework*[2] *allows the identification and prevention of intrusion activities on computer systems. This framework is based on a military tactic simply known as a* ***kill chain***[3]*, which breaks down the attack into the following phases: target identification, marshalling and organizing forces towards the target, starting an attack, and target neutralization. Lockheed Martin reformulated these steps to be transferred to the IT world and redirected the attack against a virtual target. These phases were reformulated as follows:*

Reconnaissance (*rec*): *an attacker observes the situation from the outside in order to identify targets and tactics. As the attacker mainly collects information regarding the system's vulnerabilities, this is the hardest part to detect.*

[2] https://www.lockheedmartin.com/en-us/capabilities/cyber/cyber-kill-chain.html
[3] https://en.wikipedia.org/wiki/Kill_chain



(a) A possible formulation of the Cyber Kill Chain® [https://www.bulletproof.co.uk/blog/what-is-the-cyber-kill-chain].



M			
	Clause	LTL_f	
Ⓐ	Response(rec, **true**, weap, **true**)	$\square(rec \Rightarrow \Diamond(weap \wedge \textbf{true}))$	2 / 3 = 0.66, 2 / 3 = 0.66
Ⓑ	Absence(iiot_sh, **true**, 1)	$\neg(Exists(iiot\_sh \wedge \textbf{true}, 1))$	3 / 3 = 1.0, 3 / 3 = 1.0
Ⓒ	Choice(comm, **true**, act, **true**)	$\Diamond(comm \wedge \textbf{true}) \vee \Diamond(act \wedge \textbf{true})$	1 / 3 = 0.33, 1 / 1 = 1.0

(b) A high-level use-case of an intrusion attack on a software system.

Weaponization (*weap*): *after gathering the information, the cybercriminal implements his strategy through a software artefact. This detection will have greater chances of success in the future after post-mortem analysis, when either a temporal model is mined over the collected attack data or the strategy is directly inferred from available artefacts (e.g., malaware).*

Payload or Delivery (*del*): *the cybercriminal devises a way to infiltrate the host system that hides the previously produced artefact (e.g., a trojan). This must sound as harmless as possible to fool the system.*

Exploitation (*expl*): *the cybercriminal exploits the system's vulnerabilities and infiltrates it through the previous "cover". At this stage, the defensive system should raise the alarm if any kind of unusual behaviour is detected while increasing the security level.*

Installation (*inst*): *the weapon escapes the payload and gets installed into the host computer system. At this point, any kind of suspected behaviour might be detected by malicious system calls.*

Command & Control (*comm*): *the weapon establishes a communication with the cybercriminal for receiving orders from the attacker. The system should detect any kind of suspicious network communication and should attempt to break the communication channel.*

Action (*act*): *the intruder starts the attack on the system. At this stage, the attack should be more evident, and the* ***Industrial IoT Shields*** *(iiot_sh), such as network devices protection, should be activated.*

Figure 2a describes the actions (and therefore activity labels) of interest. Having defined the actions that should be monitored, records of activities can be stored as traces within a log. This is represented by the top image in Figure 2b, where we define three distinct attacks as distinct traces ($\sigma_1, \sigma_2, \sigma_3$). Each trace contains the event information leading up to the completion of an attack attempt (which may be (un)successful). Data payload information is also considered, and here this is *provided as the unique timestamp (**ts**) associated to each event. Trace payload information is not simulated here, but is described and applied in Example 2.*

A temporal model might describe a completely successful attack. The occurrence of the aforementioned phases can be described through a temporal declarative language Declare [7], where each constraint is an instantiated Declare clause (see Table 1). The satisfaction of a model requires the satisfaction of all constituent clauses. The model described as the bottom table in Figure 2b is only one example of a myriad of possible solutions, which can either be manually defined (as here), or generated through mining/learning techniques. Our model consists of three clauses: Ⓐ all reconnaissance events should be followed by a weaponization, Ⓑ there should be no IoT shields in place, and Ⓒ either command and control or action should occur. As we will see in Example 2, these clauses might also contain data payload conditions setting constraints over the data.

We can either compute the degree to which the model is satisfied, or per trace, each requiring different metrics. An example of a trace-wise metric is Max-SAT while Support and Confidence values can be computed per clause. Definitions as well as details of these calculations can be found in §5.2.3. By providing the trace metrics, we are able to analyse the scenarios with respect to the model, and therefore help provide insight into the exhibits of any backdoors in the software. On the contrary, providing model metrics allows us to establish the suitability of a model and its constituent clauses; for example, clauses with low Support but high Confidence may indicate a correlation between events. Last, a conjunctive query will return all the traces satisfying all the model clauses. From Figure 2a it is evident that the only trace where a successful attack occurred is σ_1, as returned by the Conjunctive Query, providing the grounds that we have a suitable model.

On *blockchains*, each trace event represents a proper blockchain event, thus including function or event invocations issued by one or more smart contracts. In particular, *smart contracts* are sets of conditions specified in self-executing programs [14], which include protocols within which the parties will fulfil some promises [15]. Given that smart contracts can also be seen as postconditions activated upon the occurrence of specified pre-conditions [16], they are also exploited as security measures reducing malicious and accidental exceptions [14]. As per previous considerations, we can directly encode the smart contract premises in LTL_f, as well as represent the whole smart contract as a whole LTL_f formula under the assumption that the blockchain guarantees its execution [16]. Therefore, we can perform post-mortem analysis checking whether a given run-time abides by the rules imposed by the system.

1.1.2. Industry 4.0

Smart factories enable the collection and analysis of data through advanced sensors and embedded software for better decision-making. These enable monitoring each phase of the entire production process in both real-time and domain-specific applications where the safety of both autonomous cyber-physical systems as well as human workers is at stake [17]. This is of the uttermost importance, as both humans and machines cooperate in the same environment where a minimal violation of safety requirements might damage the overall production process, thus reflecting in maintenance costs. This calls for logical-based formal methods providing correctness guarantees [18]. Run-time verification [18] and prediction [12] have started gaining momentum against customary static analysis tools: in fact, *real* complex systems such as factories are often hard to predict and analyse before execution. As run-time verification can be deployed as a permanent testing condition on the environment, Mao et al. [18] show that this approach is complete, thus reducing the complicated model-checking problem into a simpler conformance checking one. PROGRAMMABLE LOGIC CONTROLLERS (PLC) are at the heart of this mechanism, where controllers can take decisions over previously-observed events. PLC work similarly to smart contracts in the previous scenario: at each “scan cycle”, the controllers perceive through sensors the status change of the environment (e.g., variations of temperature and pressure). This information is then fed to the internal logic, which, on the other hand, might decide to intervene directly in the environment by sending signals to some actuators (e.g., controlling the pressure and temperature on the system). Due to the similarity of PLC to smart contracts, these might also exploit $\mathrm{LTL_f}$[4] for determining security requirements: when a safety condition is violated, the PLC might activate an alarm while ensuring the system works within safe operation ranges [18].

In some other industrial scenarios, we might be interested in detecting unexpected variations in time series reflecting the fluctuation of some perceived variables (e.g., variations in temperature and pressure). Latest developments [12] showed that (industrial) time series could also be represented as traces: we might assign to each event an activity label ϖ if the current event has a data payload which values upper bound the ones from immediately preceding event's payload, and $\neg\varpi$ otherwise. Consequently, we can encode disparate data variation patterns in $\mathrm{LTL_f}$ reflecting different types of data volatility or steep increases/decreases [12]. This shows how $\mathrm{LTL_f}$ can also represent *anomaly-based* problems by reducing those to the identification of anomaly patterns [19].

1.1.3. Healthcare

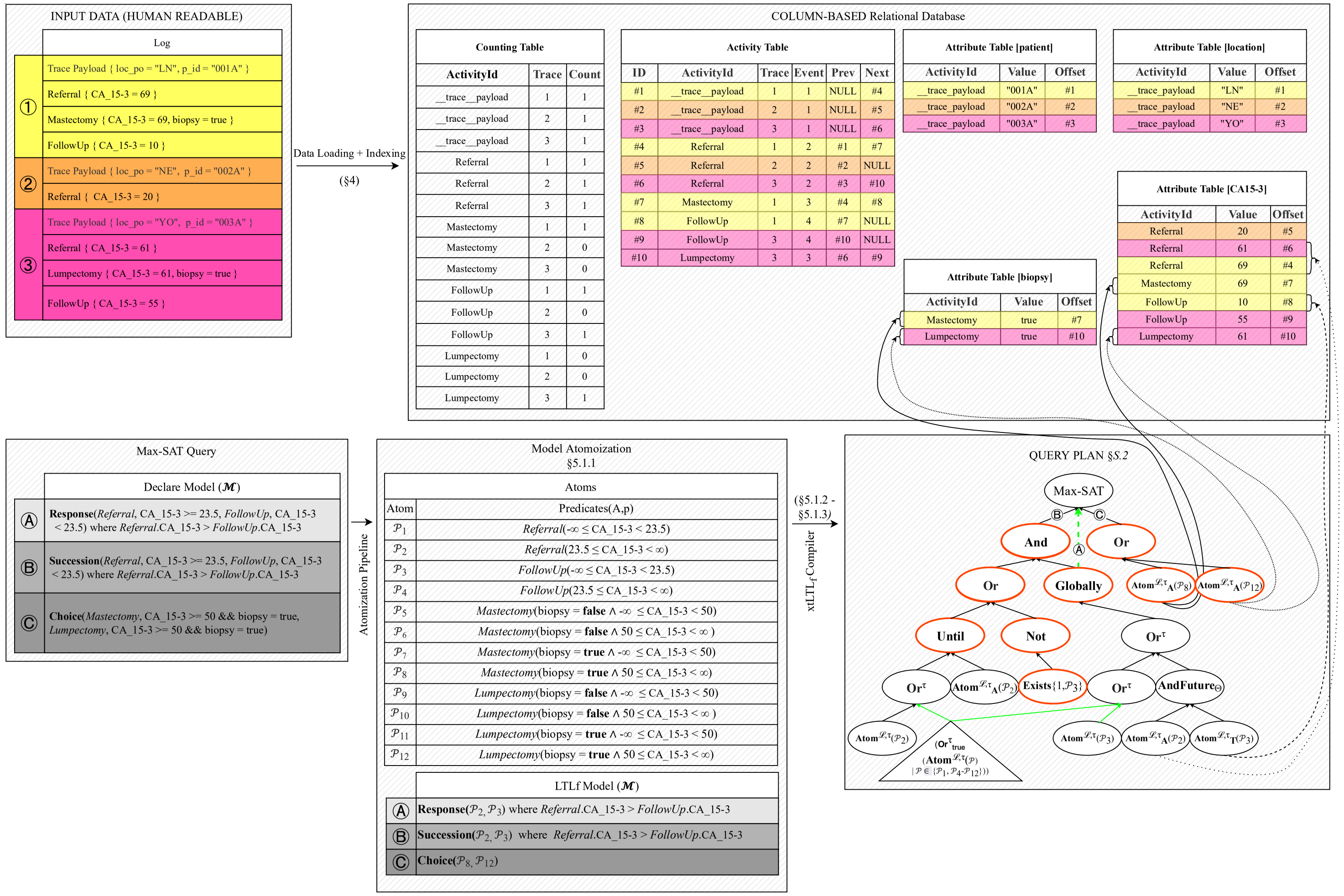
A medical process describes clinical-related procedures as well as organizational management ones (e.g., registration, admission, and discharge) [20]. The renowned openEHR[5] standard distinguishes the former in four main archetypes: an observation, recording patients' clinical symptoms (e.g., body temperature, blood pressure); an evaluation, providing preliminary diagnosis and assessing the patient's health based on the former results; and an instruction, the execution of the treatment plan proposed by a physician (e.g., prescribing, examine, and testing). An action describes the way to intervene or treat medical patients according to the treatment plan (e.g., drug administration, blood matching). Once encoded as such, each process representing an instantiation of a medical process, i.e., a patient's clinical course, can be then collected and represented in a log. As such, each action is going to be represented as a distinct activity label of a given event [21] which might contain relevant payload information recording the outcome of the clinical procedures, as well as demographical information related to the patient [20] for future socio-clinical analyses [22].

Declarative temporal languages such as Declare can then be exploited to provide a descriptive approach specifying temporal constraints among activities without strictly enforcing their order of completion, thus restricting the order of application of a specific set of activities [20]. As these models come with temporal semantics expressed in $\mathrm{LTL_f}$, these are for all intents and purposes process models. As such, those might be applied to detect discrepancies between clinical guidelines, expressed by the aforementioned model, and the actual process executions collected in a log. This is of the utmost concern as often deviations represent errors compromising the patient recovery [21] which, if efficiently and identified in advance, lead to an increased patient satisfaction as well a reduction of healthcare costs (e.g., due to mismanagement) [20].

Example 2. *To minimise costs and unrequired procedures, only ill patients should receive treatment. So, sufferers not receiving treatment (false negatives) and non-sufferers receiving treatment (false positives) need to be minimised. Figure 3 proposes a simplified scenario where we consider 2 event payload keys:* `CA 15-3` *(cancer antigen concentration in a patients blood), and* `biopsy` *(biopsies should be taken before any procedure is acted upon). Our model targets only breast cancer patients with successful therapies that describe a medical protocol and the desired patients' health condition at each step.* Ⓒ *states two possible surgical operations for breast tumours are mastectomy or lumpectomy if the biopsy is positive and the CA-13.5 is way above (≥ 50) the guard level being 23.5 units per ml, and* Ⓐ*-*Ⓑ *any successful treatment should decrease the CA-13.5 levels, which should be below the guard level; such correlation data condition is expressed via a Θ condition (introduced*

[4] ptLTL as per [18], a version of LTL allowing reasoning on past events so as to avoid semi-decidable computations for traces of infinite length, might be still represented through an equivalent $\mathrm{LTL_f}$ formula evaluated over a finite sliding window [12] bounded by the first and the latest event. Please observe that the difference between LTL and $\mathrm{LTL_f}$ is that only the latter considers traces of finite length.

[5] https://www.openehr.org/



	Log
①	Trace Payload { loc_po = "LN", p_id = "001A" }
	Referral { CA_15-3 = 69 }
	Mastectomy { CA_15-3 = 69, biopsy = true }
	FollowUp { CA_15-3 = 10 }
②	Trace Payload { loc_po = "NE", p_id = "002A" }
	Referral { CA_15-3 = 20 }
③	Trace Payload { loc_po = "YO", p_id = "003A" }
	Referral { CA_15-3 = 61 }
	Lumpectomy { CA_15-3 = 61, biopsy = true }
	FollowUp { CA_15-3 = 55 }

Counting Table

ActivityId	Trace	Count
__trace__payload	1	1
__trace__payload	2	1
__trace__payload	3	1
Referral	1	1
Referral	2	1
Referral	3	1
Mastectomy	1	1
Mastectomy	2	0
Mastectomy	3	0
FollowUp	1	1
FollowUp	2	0
FollowUp	3	1
Lumpectomy	1	0
Lumpectomy	2	0
Lumpectomy	3	1

Activity Table

ID	ActivityId	Trace	Event	Prev	Next
#1	__trace__payload	1	1	NULL	#4
#2	__trace__payload	2	1	NULL	#5
#3	__trace__payload	3	1	NULL	#6
#4	Referral	1	2	#1	#7
#5	Referral	2	2	#2	NULL
#6	Referral	3	2	#3	#10
#7	Mastectomy	1	3	#4	#8
#8	FollowUp	1	4	#7	NULL
#9	FollowUp	3	4	#10	NULL
#10	Lumpectomy	3	3	#6	#9

Attribute Table [patient]

ActivityId	Value	Offset
__trace_payload	"001A"	#1
__trace_payload	"002A"	#2
__trace_payload	"003A"	#3

Attribute Table [location]

ActivityId	Value	Offset
__trace_payload	"LN"	#1
__trace_payload	"NE"	#2
__trace_payload	"YO"	#3

Attribute Table [CA15-3]

ActivityId	Value	Offset
Referral	20	#5
Referral	61	#6
Referral	69	#4
Mastectomy	69	#7
FollowUp	10	#8
FollowUp	55	#9
Lumpectomy	61	#10

Attribute Table [biopsy]

ActivityId	Value	Offset
Mastectomy	true	#7
Lumpectomy	true	#10

	Declare Model ($\mathcal{M}$)
Ⓐ	**Response**(*Referral*, CA_15-3 >= 23.5, *FollowUp*, CA_15-3 < 23.5) where *Referral*.CA_15-3 > *FollowUp*.CA_15-3
Ⓑ	**Succession**(*Referral*, CA_15-3 >= 23.5, *FollowUp*, CA_15-3 < 23.5) where *Referral*.CA_15-3 > *FollowUp*.CA_15-3
Ⓒ	**Choice**(*Mastectomy*, CA_15-3 >= 50 && biopsy = true, *Lumpectomy*, CA_15-3 >= 50 && biopsy = true)

Atoms

Atom	Predicates(A,p)
$\mathcal{P}_1$	*Referral*($-\infty \leq$ CA_15-3 < 23.5)
$\mathcal{P}_2$	*Referral*($23.5 \leq$ CA_15-3 $< \infty$)
$\mathcal{P}_3$	*FollowUp*($-\infty \leq$ CA_15-3 < 23.5)
$\mathcal{P}_4$	*FollowUp*($23.5 \leq$ CA_15-3 $< \infty$)
$\mathcal{P}_5$	*Mastectomy*(biopsy = **false** $\wedge -\infty \leq$ CA_15-3 < 50)
$\mathcal{P}_6$	*Mastectomy*(biopsy = **false** $\wedge\ 50 \leq$ CA_15-3 $< \infty$)
$\mathcal{P}_7$	*Mastectomy*(biopsy = **true** $\wedge -\infty \leq$ CA_15-3 < 50)
$\mathcal{P}_8$	*Mastectomy*(biopsy = **true** $\wedge\ 50 \leq$ CA_15-3 $< \infty$)
$\mathcal{P}_9$	*Lumpectomy*(biopsy = **false** $\wedge -\infty \leq$ CA_15-3 < 50)
$\mathcal{P}_{10}$	*Lumpectomy*(biopsy = **false** $\wedge\ 50 \leq$ CA_15-3 $< \infty$)
$\mathcal{P}_{11}$	*Lumpectomy*(biopsy = **true** $\wedge -\infty \leq$ CA_15-3 < 50)
$\mathcal{P}_{12}$	*Lumpectomy*(biopsy = **true** $\wedge\ 50 \leq$ CA_15-3 $< \infty$)

	LTLf Model ($\mathcal{M}$)
Ⓐ	**Response**($\mathcal{P}_2, \mathcal{P}_3$) where *Referral*.CA_15-3 > *FollowUp*.CA_15-3
Ⓑ	**Succession**($\mathcal{P}_2, \mathcal{P}_3$) where *Referral*.CA_15-3 > *FollowUp*.CA_15-3
Ⓒ	**Choice**($\mathcal{P}_8, \mathcal{P}_{12}$)

Figure 3. KnoBAB Architecture for Breast Cancer patients. Each trace ①-③ represents one single patient's clinical history, are represented with unique colouring. Please observe that the atomization process does not consider data distribution but rather partitions the data space as described by the data activation and target conditions. In the query plan, green arrows remark access to shared sub-queries as in [23], and thick red ellipses remark which operators are untimed.

by a where). A twinned negative model (not in Figure) might better discriminate healthy patients from patients where the therapy was unsuccessful. Novel situations can be represented as a log. E.g., in Figure 3, we have three patients: ① a cancer patient with a successful mastectomy, ② a healthy patient, and ③ an unsuccessful lumpectomy, thus suggesting that the patient might still have some cancerous cells. Given the aforementioned model, patient ① will satisfy the model as the surgical operation was successful, ② will not satisfy the model because neither mastectomy nor lumpectomy was required ($\mathcal{M}$ is only fulfilled for successful procedures), and ③ will not satisfy the target condition, even though the correlation condition was met. Our model of interest should only return ① as an outcome of the conformance checking process.

2. Related Work

eXtensible Event Stream (XES). This paper relies on temporal data represented as a temporally ordered sequence of events (*trace* or *streams*), where events are associated to at most one action described by a single *activity label* [24]. In this paper, we formally characterize payloads as part of both events and traces while, in our previous work, we only considered payloads from events [25].

Given an arbitrarily ordered set of keys K and a set of values V, a **tuple** [26] is a finite function $p\colon K \to V$, where each key is either associated with a value in V or is undefined. After denoting $\perp$ as a `null` element missing from the set of values ($\perp \notin V$), we can express that κ is not associated to a value in p as $p(\kappa) = \perp$, thus $\kappa \notin \mathrm{dom}(p)$. An empty tuple ε has an empty domain.

(Data) *payloads* are tuples, where values can represent either categorical data or numerical data. An *event* σ^i_j is a pair $\langle \mathsf{a}, p\rangle \in \Sigma \times V^K$, where Σ is a finite set of activity labels and p is a finite function describing the data payload. A *trace* σ^i is a ordered sequence of distinct events $\sigma^i_1, \dots, \sigma^i_n$, which is distinguished from the other traces by a case id i; n represents the trace's length ($n = |\sigma^i|$). If a payload is also associated with the whole trace, this can be easily mimicked by adding an extra initial event containing such a payload with an associated label of `__trace_payload`. A *log* $\mathcal{L}$ is a finite set of traces $\{\, \sigma^1, \dots, \sigma^m \,\}$. In this paper, we further restrict our interest to the traces containing at least one event, as empty traces are meaningless as they are not describing any temporal behaviour of interest. Last, we denote as $\beta\colon \Sigma \leftrightarrow \{\, 1, \dots, |\Sigma| \,\}$ the bijection mapping each activity label occurring in the log to an unique id.

Example 3. *The log $\mathcal{L}$ in Figure 3 comprises three distinct traces $\mathcal{L} = \{\, \sigma^1, \sigma^2, \sigma^3 \,\}$. In particular, the second trace comprises two events $\sigma^2 = \sigma^2_1\sigma^2_2$, where the first event represents the trace payload, and therefore $\sigma^2_1 = \langle \texttt{\_\_trace\_payload}, p\rangle$ having $p(loc\_po) =$ "NE" and $p(p\_id) =$ "002A". The other event is $\sigma^2_2 = \langle \texttt{Referral}, \tilde{p}\rangle$, where payload $\tilde{p}$ is only associated to the CA-13.5 levels as $\tilde{p}(\text{CA-13.5}) = 20$. Similar considerations can be carried out for the other log traces.*

Linear Temporal Logic over finite traces (LTL$_\text{f}$). LTL$_\text{f}$ is a well-established extension of *modal logic* considering the possible worlds as finite traces, where each event represents a single relevant instant of time. The time is thereby linear, discrete, and future-oriented. The *syntax* of an well-formed LTL$_\text{f}$ formula φ is defined as follows:

$$\varphi ::= \mathsf{a} \mid \neg\varphi \mid \varphi \vee \varphi' \mid \varphi \wedge \varphi' \mid \bigcirc\varphi \mid \Box\varphi \mid \Diamond\varphi \mid \varphi\,\mathcal{U}\,\varphi' \qquad (1)$$

where $\mathsf{a} \in \Sigma$. It *semantics* it is usually defined in terms of First Order Logic [27] for a given trace σ^i at a current time j (e.g., for event σ^i_j) and, for the sake of completeness, this is provided in Supplement I. Other operators can be seen as syntactic sugar: **WeakUntil** is denoted as $\varphi\;\mathcal{W}\;\varphi' := \varphi\;\mathcal{U}\;\varphi' \vee \Box\varphi$, while the implication can be rewritten as $\varphi \Rightarrow \varphi' := (\neg\varphi) \vee (\varphi \wedge \varphi')$. Generally, binary operators bridge activation and target conditions appearing in two distinct sub-formulæ. The semantics associated with activity labels, consistently with the literature on business process execution traces [25], assumes that

Table 1. Declare templates illustrated as exemplifying clauses. $A \wedge p$ ($B \wedge q$) represents the *activation* (*target*) condition, A (B) denotes the activity label, and p (q) is the data payload condition.

Type	Exemplifying clause (c_l)	Natural Language Specification for Traces	LTL$_f$ Semantics ($[\![c_l]\!]$)
Simple	$\mathsf{Init}(A,p)$	The trace should start with an activation	$A \wedge p$
	$\mathsf{Exists}(A,p,n)$	Activations should occur at least n times	$\Diamond(A \wedge p \wedge \bigcirc([\![\mathsf{Exists}(A,p,n-1)]\!]))$
	$\mathsf{Absence}(A,p,n+1)$	Activations should occur at most n times	$\neg[\![\mathsf{Exists}(A,p,n+1)]\!]$
	$\mathsf{Precedence}(A,p,B,q)$	Events preceding the activations should not satisfy the target	$\neg(B \wedge p)\ \mathcal{W}\ (A \wedge p)$
(Mutual) Correlation	$\mathsf{ChainPrecedence}(A,p,B,q)$	The activation is immediately preceded by the target.	$\Box(\bigcirc(A \wedge p) \Rightarrow (B \wedge q))$
	$\mathsf{Choice}(A,p,A',p')$	One of the two activation conditions must appear.	$\Diamond(A \wedge p) \vee \Diamond(A' \wedge p')$
	$\mathsf{Response}(A,p,B,q)$	The activation is either followed by or simultaneous to the target.	$\Box((A \wedge p) \Rightarrow \Diamond(B \wedge q))$
	$\mathsf{ChainResponse}(A,p,B,q)$	The activation is immediately followed by the target.	$\Box((A \wedge p) \Rightarrow \bigcirc(B \wedge q))$
	$\mathsf{RespExistence}(A,p,B,q)$	The activation requires the existence of the target.	$\Diamond(A \wedge p) \Rightarrow \Diamond(B \wedge q)$
	$\mathsf{ExlChoice}(A,p,A',p')$	Only one activation condition must happen.	$[\![\mathsf{Choice}(\mathtt{A},p,\mathtt{A'},p')]\!] \wedge [\![\mathsf{NotCoExistence}(\mathtt{A},p,\mathtt{A'},p')]\!]$
	$\mathsf{CoExistence}(A,p,B,q)$	RespExistence, and vice versa.	$[\![\mathsf{RespExistence}(\mathtt{A},p,\mathtt{B},q)]\!] \wedge [\![\mathsf{RespExistence}(\mathtt{B},q,\mathtt{A},p)]\!]$
	$\mathsf{Succession}(A,p,B,q)$	The target should only follow the activation.	$[\![\mathsf{Precedence}(\mathtt{A},p,\mathtt{B},q)]\!] \wedge [\![\mathsf{Response}(\mathtt{A},p,\mathtt{B},q)]\!]$
	$\mathsf{ChainSuccession}(A,p,B,q)$	Activation immediately follows the target, and the target immediately preceeds the activation.	$\Box((A \wedge p) \Leftrightarrow \bigcirc(B \wedge q))$
	$\mathsf{AltResponse}(A,p,B,q)$	If an activation occurs, no other activations must happen until the target occurs.	$\Box((A \wedge p) \Rightarrow (\neg(A \wedge p)\ \mathcal{U}\ (B \wedge q)))$
	$\mathsf{AltPrecedence}(A,p,B,q)$	Every activation must be preceded by an target, without any other activation in between	$[\![\mathsf{Precedence}(\mathtt{A},p,\mathtt{B},q)]\!] \wedge \Box((A \wedge p) \Rightarrow \bigcirc(\neg(A \wedge p)\ \mathcal{W}\ (B \wedge q))$
Not.	$\mathsf{NotCoExistence}(A,p,B,q)$	The activation nand the target happen.	$\neg(\Diamond(A \wedge p) \wedge \Diamond(B \wedge q))$
	$\mathsf{NotSuccession}(A,p,B,q)$	The activation requires that no target condition should follow.	$\Box((A \wedge p) \Rightarrow \neg\Diamond(B \wedge q))$

in each point of the sequence, one and only one element from Σ holds. We state that a trace σ^i is *conformant* to an LTL$_f$ formula iff. it satisfies it starting from the first event: $\sigma^i \vDash \varphi \Leftrightarrow \sigma^i_1 \vDash \varphi$, and *deviant* otherwise [25]. The Declare language described in the next paragraph provides an application for such a logic. As relational algebra describes the semantics for SQL [28,29], LTL$_f$ is extensively applied [30] as a semantics for formally expressing temporal and human-readable declarative constraints such as Declare.

At the time of the writing, different authors proposed several extensions for representing data conditions in LTL$_f$. The simplest extension are *compound conditions* $\mathsf{a} \wedge q$, which are the conjunction of data predicate $q \in \mathbf{Prop}$ to the activity label a [25]. Nevertheless, this straightforward solution is not able to express correlation conditions in the data which might be relevant in business scenarios [31], as representing correlations as single atoms requires decomposing the former into disjunctions of formulae [32]. Despite prior attempts to define a temporal logic expressing correlation conditions, no explicit formal semantics on how this can be evaluated was provided [6]. This poses a problem to the current practitioner, as this hinders the process of both checking formally the equivalence among two languages expressing correlation conditions, as well as providing a correct implementation of such an operator. We, on the other hand, propose a relational representation of $\mathtt{xt}$LTL$_f$, where the semantics of all of the operators, thus including the ones requiring correlation conditions, is clearly laid out and implemented.

Declare. Temporal declarative languages pinpoint highly variable scenarios, where state machines provide complicated graph models that can be hardly understandable by the common business stake-holder [33]. Among all possible temporal declarative languages, we constrain our interest to **Declare**, originally proposed in [7]. Every single temporal pattern is expressed through *templates* (i.e., an abstract parameterized property: Table 1 column 2), which are then instantiated on a set of real activation, target, or correlation conditions. We can then categorize each Declare template from [30] through these conditions and the ability to express correlations between two temporally distant events happening in one trace: simple templates (Table 1, rows 1-3) only involving activation conditions; (mutual) correlation templates (rows from 4 to 15), which describe a dependency between activation and target conditions, thus including correlations between the two; and negative relation templates (last 2 rows), which describe a negative dependency between two events in correlation. Despite these templates may appear quite similar, they generate completely different finite state machines, thus suggesting that these conditions are not interchange-

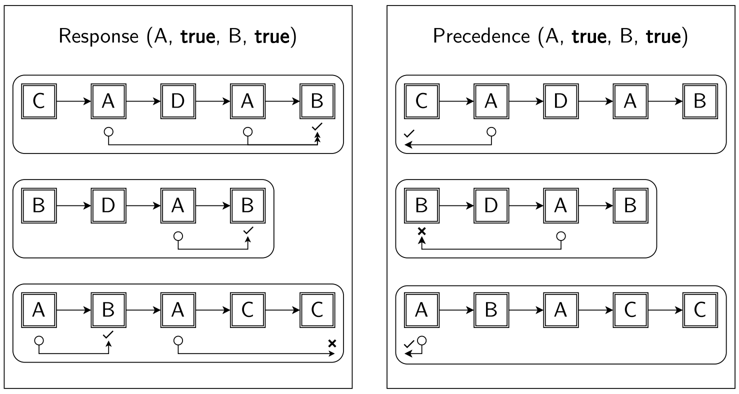

Response (A, **true**, B, **true**)
Precedence (A, **true**, B, **true**)
C
A
D
A
B
B
D
A
B
A
B
A
C
C

Figure 4. Two exemplifying clauses distinguishing Response and Precedence behaviours. Traces are represented as temporally ordered events associated with activity labels (boxed). Activation (or target) conditions are here circled (or ticked/crossed). Ticks (or crosses) indicate a (un)successful match of a target condition. For all activations there must be an un-failing target condition; for precedence, we shall consider at most one activation. These conditions require the usage of multiple join tests per trace.

able[6]. Figure 4 exemplifies the behavioural difference between two clauses differing only on the template of choice.

A Declare Model is composed of a set of clauses $\mathcal{M} = \{ c_l \}_{l \leq n, n \in \mathbb{N}}$ which have to be contemporarily satisfied in order to be true. Given the semantic representation in LTL_f $[\![c_l]\!]$ of each model clause c_l, we state that a trace σ^i is conformant to a model $\mathcal{M}$ iff. such a trace is conformant to each LTL_f formula $[\![c_l]\!]$ associated to the model clause $c_l \in \mathcal{M}$. Consequently, a Declare model can be represented as a finitary conjunction of the LTL_f representation of each of its clauses, $[\![\mathcal{M}]\!] := \bigwedge_{c_l \in \mathcal{M}} [\![c_l]\!]$: for this, the MAXIMUM-SATISFIABILITY PROBLEM (Max-SAT) for each trace counts the ratio between the satisfied clauses over the whole model size.

This consideration can be later on extended to also data predicates through predicate atomization [25], as discussed in the next paragraph.

Common Subquery Problem. Query caching mechanisms [34] are customary solutions for improving query runtime by holding partially-computed results in temporary tables referred to as materialised views, under the assumption that the queries sharing common data are pipelined [35]. Recently, Kechar et al. [23] proposed a novel approach that can also be run when queries are run contemporarily: it is sufficient to find the shared subqueries before actually running those so that when those are run, are stored into materialised views while guaranteeing that those are computed at most once.

Example 4. *Figure 3 shows how this idea might be transferred to our use case scenario requiring running multiple declarative clauses:* RESPONSE *is both a subquery of* SUCCESSION *as well as a distinct declarative clause of interest. Green arrows remark operators' output shared among operators expressed in our proposed* $\mathtt{xt}\,\text{LTL}_f$ *extension of* $\mathtt{xt}\,\text{LTL}_f$. *Please also observe that operators with the same name and arguments but marked either with activation, target, or no specification are considered different as they provide different results, and therefore are not merged together. This includes distinctions between timed and untimed operators, which will be discussed in greater detail in §3.2.*

To further minimize tables' access times, it is possible to take this reasoning to its extreme by minimising the data access per data predicate so as to avoid accessing the same table multiple times. In order to do so, we need to partition the data space according to the queries at our disposal as in our previous work [25] and described in greater detail in

[6] http://ltlf2dfa.diag.uniroma1.it/

Table 2. Definition of the atoms from Figure 3 in terms of partitioning over the elementary intervals.

Referral	$CA\text{-}15.3 < 23.5$	$CA\text{-}15.3 \geq 23.5$
	p_1	p_2

Mastectomy	$CA\text{-}15.3 < 50$	$CA\text{-}15.3 \geq 50$
$biopsy =$ **false**	p_5	p_6
$biopsy =$ **true**	p_7	p_8

FollowUp	$CA\text{-}15.3 < 23.5$	$CA\text{-}15.3 \geq 23.5$
	p_3	p_4

Lumpectomy	$CA\text{-}15.3 < 50$	$CA\text{-}15.3 \geq 50$
$biopsy =$ **false**	p_9	p_{10}
$biopsy =$ **true**	p_{11}	p_{12}

Table 3. Intermediate steps to generate distinct atoms for the Referral data predicates from Example 7.

(a) Interval decomposition in *basic intervals* $\mu(\text{Mastectomy}, \cdot)$.

$\mu(\text{Mastectomy}, CA\text{-}15.3)$		
$CA\text{-}15.3 < 0$	$CA15.3 < 0$	
$CA\text{-}15.3 < 50$	$CA15.3 < 0$	$0 \leq CA\text{-}15.3 < 50$
$CA\text{-}15.3 \geq 50$	$50 \leq CA15.3 \leq 1000$	$CA\text{-}15.3 > 1000$
$CA\text{-}15.3 > 1000$	$CA15.3 > 1000$	

$\mu(\text{Mastectomy}, biopsy)$				
$biopsy = 0$	$biopsy = 0$			
$biopsy = 1$	$biopsy = 1$			
$biopsy \neq 0$	$biopsy < 0$	$0 < biopsy < 1$	$biopsy = 1$	$biopsy > 1$
$biopsy \neq 0$	$biopsy < 0$	$biopsy = 0$	$0 < biopsy < 1$	$biopsy > 1$

(b) Atom generation by partitioning the data space $\bigtimes_{\kappa \in K} \mu(\text{Mastectomy}, \kappa)$.`elementaryIntervals()` with $K = \{\, biopsy, CA\text{-}15.3 \,\}$.

	$biopsy < 0$	$biopsy = 0$	$0 < biopsy =< 1$	$biopsy = 1$	$biopsy > 1$
$CA15.3 < 0$	p_1	p_2	p_3	p_4	p_5
$0 \leq CA15.3 < 50$	p_6	p_7	p_8	p_9	p_{10}
$50 \leq CA15.3 \leq 1000$	p_{11}	p_{12}	p_{13}	p_{14}	p_{15}
$CA15.3 > 1000$	p_{16}	p_{17}	p_{18}	p_{19}	p_{20}

Supplement II. This process can be eased if we assume that each payload condition p and p' for the declarative clauses within a model $\mathcal{M}$ is represented in Disjunctive Normal Form (DNF) [36]: in this scenario, data predicates q are in DNF if they are a disjunction of one or more conjunctions of one or more data intervals referring to just one payload key.

Example 5. *The model illustrated in Figure 2a and discussed in former Example 1 comes with data conditions associated to neither activation nor target conditions. Therefore, no atomisation process is performed. So, each event in a log might just be distinguished by its activity label [25].*

Example 6. *Each distinct payload conditions associated to either activation or target conditions in Figure 3 can be expressed as one single atom, as there are no overlapping data conditions associated to the same activity label and each data condition can be mapped into one single elementary interval associated to an activity label. The next example will provide another use case example and a different model on the same dataset leading to a decomposition of payload conditions into a disjunction of several atoms. Table 2 shows the partitioning of the data payloads associated to each activity label in the log by the elementary interval of interest.*

Example 7. *Let us suppose to return all the false negative and false positive* Mastectomy *cases related to the that are not caused by data imputation errors. For this, we want to get all of the negative biopsies having CA15.3 levels greater than the guard level of 50 and positive biopsies having CA15.3 below the same threshold. Under the assumption that biopsy values were imputed through numerical numbers thus leading to more imputation errors, we are ignoring cases where both CA15.3 and biopsy values are out of scale, that is, we want to ignore the data where CA15.3 levels are negative or above 1000, and where the biopsy values are neither true* (1.0) *nor false* (0.0)*. For this, we can outline the following model:*

Table 4. Table of Notation

Symbol	Definition	Type	Comments
		Set Theory	
$\varnothing$			An empty set contains no items.
$(\preceq, S)$			A partially ordered set (*poset*) is a relational structure for which $\preceq$ is a partial ordering over S [36]. $\preceq$ over S might be represented as a lattice, referred to as the Hasse diagram.
$\top_S$	$\forall a \in S.a \preceq \top_S$	$\top_S \in S$	Given a poset $(\preceq, S)$, $\top_S$ is the unique greatest element of S.
$\complement C$	$\mathcal{U} \backslash C$		*Complement* set: given an universe $\mathcal{U}$, the complement returns all of the elements that do not belong to C.
$\bigtimes_{\kappa \in K} f(\kappa)$	$f(\kappa_1) \times \dots \times f(\kappa_n)$		*Generalised cross product* for ordered sets K where $\kappa_1 \prec \dots \prec \kappa_n$
$\lvert C \rvert$	$\sum_{c \in C} 1$		The *cardinality* of a finite set indicates the number of contained items.
$\wp(C)$	$\{T \mid T \subseteq C\}$		The *powerset* of C is the set whose elements are all of the subsets of C.
		XES Model & LTL_f	
Σ			Finite set of activity labels
K			Finite set of ordered (payload) keys, κ
V			Finite set of (payload) values
p	$[\kappa_1 \mapsto v_1, \dots]$	$K \to V$	Tuple (or finite function) mapping keys $\kappa_1 \in K$ to values in $v_1 \in V$
$\perp$	$\perp \notin V$		`NULL` value
σ^i_j	$\langle p, a \rangle$	$\Sigma \times V^K$	Event
σ^i	$\sigma^i_1, \dots, \sigma^i_n$		Trace, sequence of temporarily ordered events.
$\mathcal{L}$	$\{\sigma^1, \dots, \sigma^m\}$		Log, set of traces.
β		$\Sigma \leftrightarrow \{1, \dots, \lvert\Sigma\rvert\}$	Bijection mapping each activity label to its unique identifier.
φ	Equation 1	Expression	An LTL_f expression.
$\vDash$			$\Gamma \vDash \varphi$ denotes that φ is *satisfied* for the world/environment Γ.
		$\texttt{xtLTL}_\text{f}$	
ψ	§3.2	Expression	eXTended LTL_f Algebra expression.
$A(k)/T(k)/M(h,k)$		ω	*Marks* associated to activation/target/matching conditions.
ρ	$\{\langle i, j, L \rangle, \dots\}$	$\Omega = \{\wp(\mathbb{N} \times \mathbb{N} \times S) \mid S \in \wp(\omega)\}$	*Intermediate representation* returned by each $\texttt{xtLTL}_\text{f}$ operator
$T[i]$		$T[i] \in T$	Accessing the *i*-th record of a sequence T.
$\Theta(x,y)$		Binary Predicate	*Correlation condition* between activated and targeted events.
$\Theta^{-1}(y,x)$	$\Theta(x,y)$	Binary Predicate	*Inverted/Flipped* correlation condition.
True		Binary Predicate	Always-true binary predicate.
$E^i_\Theta(M_1, M_2)$	Equation S2	Algorithm 6	*Existential matching condition* for which there exists at least one event in M_1, M_2 providing a match.
$A^i_\Theta(M_1, M_2)$	Equation S3	Algorithm 7	*Universal matching condition* returning a non-empty set if each event expressed in the maps M_1, M_2 provides a match.
$\mathcal{T}^{F,i}_\Theta(M_1, M_2)$	Equation S4	$\mathcal{T}^{F,i}_\Theta(M_1, M_2) \in \wp(\omega) \cup \{\textbf{False}\}$	*Testing functor* returning **False** iff., despite the maps contain activated and targeted events, the matching condition $F^i_\Theta(M_1, M_2)$ is empty. It returns $F^i_\Theta(M_1, M_2)$ otherwise.
		Pseudocode	
$\uparrow$			Null pointer or terminated iterator.
Iterator(ρ)		POINTER	On ρ non-empty, it returns the iterator pointing to the first record in ρ
`current`(it)		DEREFERENCE	Element pointer by the pointer/iterator *it*.
LOWERBOUND(d,b,e,v)		BINARY SEARCH	Given a beginning b and end e iterator range within a sequential and sorted data structure by increasing order, LOWERBOUND[1] returns either the first location in this range pointing at a value greater or equal to v or e otherwise.
UPPERBOUND(d,b,e,v)		BINARY SEARCH	Given a beginning b and end e iterator range within a sequential and sorted data structure by increasing order, UPPERBOUND[2] returns either the first location in this range pointing to a value strictly less to v or e otherwise.
		Time Complexity	
ϵ			Maximum trace length.
ℓ			Maximum length of the third component of the intermediate representation.

[1] https://en.cppreference.com/w/cpp/algorithm/lower_bound
[2] https://en.cppreference.com/w/cpp/algorithm/upper_bound

$$\begin{aligned}\mathcal{M}' = \{ & \mathit{Choice}(\mathsf{Mastectomy}, \mathit{biopsy} = 0.0 \land \mathit{CA15.3} \geq 50, \mathsf{Mastectomy}, \mathit{biopsy} = 1.0 \land \mathit{CA15.3} < 50), \\ & \mathit{Absence}(\mathsf{Mastectomy}, \mathit{CA15.3} > 1000 \lor \mathit{CA15.3} < 0), \\ & \mathit{Absence}(\mathsf{Mastectomy}, \mathit{biopsy} \neq 1.0 \lor \mathit{biopsy} \neq 0.0)\}\end{aligned} \tag{2}$$

This implies that we are interested to decompose the intervals pertaining to both CA-15.3 and biopsy into elementary intervals: Table 3a shows that only CA-15.3 < 50 and CA-15.3 ≥ 50 are decomposed into two elementary intervals, as the former also includes the range CA-15.3 < 0 while the latter also includes CA-15.3 > 1000. Elementary intervals not occurring in the initially collected ones are not reported in this graphical representation. Table 3b shows the partitioning of the Mastectomy *data payload induced by the elementary intervals of interest; the former data conditions can be now rewritten after Equation S1 in the Supplement as follows:*

1. $\bigvee \mathsf{Atom}_{\mu,ad}(\mathsf{Mastectomy}, biopsy=0.0\wedge CA15.3\geq 50) = p_{12} \vee p_{17}$
2. $\bigvee \mathsf{Atom}_{\mu,ad}(\mathsf{Mastectomy}, biopsy=1.0\wedge CA15.3<50) = p_{4} \vee p_{9}$
3. $\bigvee \mathsf{Atom}_{\mu,ad}(\mathsf{Mastectomy}, CA15.3>1000\vee CA15.3<0) = p_{1} \vee \cdots \vee p_{5} \vee p_{16} \vee \cdots \vee p_{20}$
4. $\bigvee \mathsf{Atom}_{\mu,ad}(\mathsf{Mastectomy}, biopsy\neq 0.0\vee biopsy\neq 1.0) = p_1 \vee p_3 \vee p_5 \vee p_6 \vee p_8 \vee p_{10} \vee p_{11} \vee p_{13} \vee p_{15} \vee p_{16} \vee p_{18} \vee p_{20}$

where each atom is defined as a conjunction of compound conditions defined upon the previously collected elementary intervals. Some examples are then the following:

- $p_1 := biopsy < 0 \wedge CA\text{-}15.3 < 0$
- $p_2 := biopsy = 0 \wedge CA\text{-}15.3 < 0$

This decomposition will enable us to reduce the data access time while scanning the tables efficiently.

Further Notation. We represent relational tables as a sequence of records indexed by id as per the physical relational model: given a relational table T, $T[i]$ represents the i-th record in T counting from 1. We denote $f = [x \mapsto y, z \mapsto t]$ a finite function such that $f(x) = y$ and $f(z) = t$. Table 4 collects the notation used throughout the paper.

3. Logical Model

Differently from our previous work [4], we provide a full definition of the (logical) model, thus describing the relational schema and how such tables are instantiated so to fully represent the original log $\mathcal{L}$ (§3.1). This is a required preliminary step, as this will provide the required background to understand the definitions for the `xt`LTL$_f$ *operators (§3.2). These operators, differently from the* LTL$_f$ *ones, are defined over the aforementioned model and assess the satisfiability of multiple traces loaded in such a model.*

The discussion on how such tables are loaded and indexed is postponed when discussing the physical model (§4), as well as the different algorithms associated to the different operators (§6).

3.1. Model Definition

KnoBAB provides a tabular (i.e., relational) representation of the log $\mathcal{L}$, so to efficiently query it through tailored relational operators (xtLTL$_f$). If the log does not contain data payloads, the entire log can be represented into two relational tables, CountingTable$_{\mathcal{L}}$(`Activity`, `Trace`,`Count`) and ActivityTable$_{\mathcal{L}}$(`Activity`,`Trace`,`Event`,`Prev`,`Next`). While the former can efficiently assess how many events in the same given trace share the same activity label, the latter allows a faithful reconstruction of the activity label associated to the traces. In particular, we use the former to assess whether a trace contains a given activity label at all. Such tables are then defined as follows:

Definition 1 (CountingTable)**.** *Given a log $\mathcal{L}$, the CountingTable$_{\mathcal{L}}$(`Activity`,`Trace`,`Count`) counts for each trace in $\mathcal{L}$ how many times each activity label occurs. More formally:*

$$\mathit{CountingTable}_{\mathcal{L}} = \left[\langle \beta(\mathsf{a}), i, |\{\sigma^i_j \in \sigma^i | \sigma^i_j = \langle \mathsf{a}, p\rangle\}|\rangle \;\middle|\; \mathsf{a} \in \Sigma, \sigma^i \in \mathcal{L}\right]$$

A record $\langle \beta(\mathsf{a}), i, n\rangle$ states that the i-the trace from the log $\sigma^i \in \mathcal{L}$ contains n occurrences of a*-labelled events with id $\beta(\mathsf{a})$.*

Definition 2 (ActivityTable)**.** *Given a log $\mathcal{L}$, the ActivityTable(`Activity`,`Trace`,`Event`,`Prev`,`Next`) lists all of the possible events occurring in each log trace, where `Prev` (π) and `Next` (ϕ) are offsets pointing to the row representing the immediately preceding or following event in the trace if any. More formally:*

$$\begin{aligned}
&\mathit{ActivityTable}_{\mathcal{L}} = \\
&\left[\, \langle \beta(\mathsf{a}), i, j, \pi, \phi \rangle \;\middle|\; \mathsf{a} \in \Sigma, \sigma^i \in \mathcal{L}, \sigma^i_j \in \sigma^i, \sigma^i_j = \langle \mathsf{a}, p \rangle\,, \right.\\
&\qquad (j = 1 \wedge \pi = \bot) \vee (\exists h, \pi', \phi'.\, \langle h, i, j-1, \pi', \phi' \rangle \in \mathit{ActivityTable}_{\mathcal{L}}[\pi],\\
&\qquad \left.(j = |\sigma^i| \wedge \phi = \bot) \vee (\exists h, \pi', \phi'.\, \langle h, i, j+1, \pi', \phi' \rangle \in \mathit{ActivityTable}_{\mathcal{L}}[\phi] \right]
\end{aligned}$$

A record $\langle \beta(\mathsf{a}), i, j, \pi, \phi \rangle$ states that the j-th event of the i-th log trace ($\sigma^i_j \in \sigma^i$, $\sigma^i \in \mathcal{L}$) has an activity label a *and that its preceding and following event (if any) are respectively located on the π-th and ϕ-th record of the same table.*

Please observe that `Prev` and `Next` are computed after bulk inserting while loading and indexing the data (see LOADINGANDINDEXING from Algorithm 1). If a log is associated to either trace or event payloads, we must store for each payload the values associated to keys k in an $\mathsf{AttributeTable}^k_{\mathcal{L}}$(`Activity`,`Value`,`Offset`), where `Offset` points to the event described in the $\mathsf{ActivityTable}_{\mathcal{L}}$.

Definition 3 (AttributeTable)**.** *Given a log $\mathcal{L}$, for each attribute $\kappa \in K$ associated to at least one value in a payload, we define a table $\mathit{AttributeTable}^{\kappa}_{\mathcal{L}}$(`Activity`,`Value`,`Offset`) associating each value to the pertaining event's payload as follows:*

$$\mathit{AttributeTable}^{\kappa}_{\mathcal{L}} = \left[\, \langle \beta(\mathsf{a}), p(\kappa), \pi \rangle \;\middle|\; \sigma^i \in \mathcal{L}, \sigma^i_j \in \sigma^i, \sigma^i_j = \langle \mathsf{a}, p \rangle\,, p(\kappa) \neq \bot \right]$$

A record $\langle \beta(a), v, \pi \rangle$ states that the event $\sigma^i_j = \langle \mathsf{a}, p \rangle$ stored in the ActivityTable associated to the π-th offset contains a payload p associating κ to a value v ($p(\kappa) = v$).

Please observe that, similarly to the former table, the offset π is also computed while loading and indexing the data: this is discussed in greater detail in §4.2.2.

Example 8. *Figure 3 provides a graphical depiction of the tables storing our data. The records are also sorted by ascending order induced by the first three cells of each record, as required by our Physical Database Design (§4). For representation purposes, the first cell of each row shows the activity label* a *rather than its associated unique id $\beta(\mathsf{a})$.*

3.2. *eXTended* LTL$_f$ *Algebra* (`xt`LTL$_f$)

We extend the operators provided in our previous work [4] into more minimal ones, thus better describing the data access on the relational model. Furthermore, we provide a full formal characterization for each of these operators via their access to the aforementioned relational tables. Please observe that, similarly to the relational algebra, each `xt`LTL$_f$ *operator might come with different possible algorithms [37], which are discussed in §6.*

Our operators come in two flavours: timed and untimed. While the former are marked by a τ and return all of the traces' events for which a given condition holds, the latter guarantee that such a condition will hold any time from the beginning of the trace. Furthermore, these operators assess the satisfiability of all the log traces simultaneously and not only one trace at a time as per LTL$_\mathsf{f}$.

Each `xt`LTL$_\mathsf{f}$ operator returns an ordered set of triplets $\langle i, j, L \rangle$, where each triplet states that an event σ^i_j from trace σ^i satisfies a condition specified by the returning operator. If L is not empty, we might also observe an activation ($A(k) \in L, k \geq j$), a target ($T(k) \in L, k \geq j$), or a matching/correlation condition ($M(h,k) \in L, k, h \geq j$) associated to the current event σ^i_j. Those three are stored in L, a vector ordered by *mark* type and referenced event id. The set of triplets, named *intermediate result* ρ, is implemented as a vector and sorted by increasing trace and event id, as sorted vectors guarantee efficient intersection and union operations, as well as efficient event counting within the same trace through linear scanning. Binary operators associated with a non-**True** binary predicate Θ return matching/correlation conditions $M(h,k) \in L$ if at least one activation and one target condition were matched, depending on the definition of the operator. As we are going to see next, if the output comes from a base operator, as defined in the next section, L might contain a single activation or target corresponding to the immediately returned event.

3.2.1. Base Operators

First, we discuss the base operators directly accessing the tables. These might have an associated marker specifying whether the event of interest is considered an activation (A) or a target (T) condition; if none is required, the mark can be omitted from the operator. The $\mathsf{Activity}^{\tau}(\mathsf{a})^{\mathcal{L}}_{A/T}$ operator lists all of the events associated to an activation label a. As the $\mathsf{ActivityTable}_{\mathcal{L}}$ directly provides this information, this operator is defined as follows:

$$\mathsf{Activity}^{\mathcal{L},\tau}_{A/T}(\mathsf{a}) = \{\, \langle i, j, \{A/T(j)\}\rangle \mid \exists \pi, \phi.\, \langle \beta(\mathsf{a}), i, j, \pi, \phi\rangle \in \mathsf{ActivityTable}_{\mathcal{L}} \,\}$$

We can also make similar considerations for single *elementary interval* representable as an LTL_f *compound condition* $\mathsf{a} \wedge \mathit{lower} \leq \kappa \leq \mathit{upper}$, which can be run as a single range query over an $\mathsf{AttributeTable}^{\kappa}_{\mathcal{L}}$. As each of its records has an offset π to the $\mathsf{ActivityTable}_{\mathcal{L}}$, this resolves the trace id and event id information required for the intermediate result. This operator can therefore be formalised as follows:

$$\mathsf{Compound}^{\mathcal{L},\tau}_{A/T}(\mathsf{a}, \kappa, [\mathit{lower}, \mathit{upper}]) = \left\{ \langle i, j, \{A/T(j)\}\rangle \;\middle|\; \begin{array}{l} \exists \pi, \pi', \phi, v.\; \mathit{lower} \leq v \leq \mathit{upper}, \langle \beta(\mathsf{a}), v, \pi\rangle \in \mathsf{AttributeTable}^{\kappa}_{\mathcal{L}}, \\ \mathsf{ActivityTable}_{\mathcal{L}}[\pi] = \langle \beta(a), i, j, \pi', \phi\rangle \end{array} \right\}$$

If we want to list all of the initial (or terminal) events of a trace, we can directly access the ActivityTable and provide a linear scan over the number of the possible traces through its associated secondary index. If we are not interested in whether the trace starts with a specific activity label, then we can define the $\mathsf{First}^{\mathcal{L},\tau}_{A}$ (and $\mathsf{Last}^{\mathcal{L},\tau}_{A}$) operators as follows:

$$\mathsf{First}^{\mathcal{L},\tau}_{A} = \{\, \langle i, 1, \{A(1)\}\rangle \mid \exists \mathsf{a}, \phi.\, \langle \beta(\mathsf{a}), i, 1, \bot, \phi\rangle \in \mathsf{ActivityTable}_{\mathcal{L}} \,\}$$
$$\mathsf{Last}^{\mathcal{L},\tau}_{A} = \left\{ \langle i, |\sigma^i|, \{A(|\sigma^i|)\}\rangle \;\middle|\; \exists \mathsf{a}, \pi.\, \langle \beta(\mathsf{a}), i, |\sigma^i|, \pi, \bot\rangle \in \mathsf{ActivityTable}_{\mathcal{L}} \right\}$$

On the other hand, Init (and Ends) are the specific refinements of the former operators if we are also interested in retrieving events with a specific activity label. These can be defined as follows:

$$\mathsf{Init}^{\mathcal{L}}_{A}(\mathsf{a}) = \{\, \langle i, 1, \{A(1)\}\rangle \mid \exists \phi.\, \langle \beta(\mathsf{a}), i, 1, \bot, \phi\rangle \in \mathsf{ActivityTable}_{\mathcal{L}} \,\}$$

$$\mathsf{Ends}^{\mathcal{L}}_{A}(\mathsf{a}) = \left\{ \langle i, 1, \{A(|\sigma^i|)\}\rangle \;\middle|\; \exists \pi.\, \langle \beta(\mathsf{a}), i, |\sigma^i|, \pi, \bot\rangle \in \mathsf{ActivityTable}_{\mathcal{L}} \right\}$$

Given a natural number n, $\mathsf{Exists}(\mathsf{a}, n)^{\mathcal{L}}_{A}$ lists the traces containing at least n events with an activity label a. As $\mathsf{Absence}(\mathsf{a}, n)^{\mathcal{L}}_{A}$ is the substantial negation of the former, this lists the traces containing at most $n-1$ events with an activity label a. Please observe that these operators directly provide the formal semantics for the homonym Declare template. As the CountingTable precisely contains the counting information required to solve this query, these operators can be formalised as follows:

$$\mathsf{Exists}^{\mathcal{L}}_{A}(\mathsf{a}, n) = \{\, \langle i, 1, \{A(1)\}\rangle \mid \exists m \geq n.\, \langle \beta(\mathsf{a}), i, m\rangle \in \mathsf{CountTable}_{\mathcal{L}} \,\}$$
$$\mathsf{Absence}^{\mathcal{L}}_{A}(\mathsf{a}, n) = \{\, \langle i, 1, \{A(1)\}\rangle \mid \exists m < n.\, \langle \beta(\mathsf{a}), i, m\rangle \in \mathsf{CountTable}_{\mathcal{L}} \,\}$$

The **Unary Operator** paragraph is going to show how these last two operators can be generalised for counting the salient event information returned by any sub-expression returning an operand ρ.

3.2.2. Unary Operators

The unary xtLTL$_f$ operators come in two flavours: the first ones extend some of the former operators for compound conditions or atoms not necessarily associated with activity labels, while the second ones directly extend the unary operators from LTL$_f$.

Base Operators' generalizations. We extend the definition of Init/Ends or Exists/Absence for any possible set of events of interest listed in an intermediate result ρ, not necessarily associated to the same activity label. We first define Exists and Absence operator as such: instead of exploiting the counting table, we now actually need to count the events returned in ρ for each trace and return an intermediate result triplet iff. they satisfy the counting condition. These can be then defined as follows:

$$\mathsf{Exists}_n(\rho) = \left\{ \langle i, 1, \cup_{\langle i,j,L_j\rangle\in\rho} L_j \rangle \;\middle|\; n \leq |\{\langle i, j, L'\rangle \in \rho\}| \right\}$$

$$\mathsf{Absence}_n(\rho) = \left\{ \langle i, 1, \cup_{\langle i,j,L_j\rangle\in\rho} L_j \rangle \;\middle|\; n > |\{\langle i, j, L'\rangle \in \rho\}| \right\}$$

Similarly, while the operators accessing the CountingTable (Exists/Absence) return the result by linearly scanning such a table, their generalised counterparts require to scan their operand ρ as returned from a subexpression of choice, and then creaming them off depending on how many events per trace were in ρ. As we might observe, we might exploit the previously provided operators when we want to evaluate conditions only associated with activity labels, while we might need to exploit the former if we are interested in results associated with compound conditions whose evaluation is returned in ρ.

Last, we refine Init and Ends for a given operand ρ, to keep only the events at the beginning or end of a given trace:

$$\mathsf{Init}(\rho) = \{ \langle i, 1, L\rangle \in \rho \mid j = 1 \}$$

$$\mathsf{Ends}(\rho) = \left\{ \langle i, 1, L\rangle \;\middle|\; \langle i, |\sigma^i|, L\rangle \in \rho \right\}$$

LTL$_f$ extensions. The unary xtLTL$_f$ operators work differently from the corresponding ones in LTL$_f$: while the latter compute the semantics from the first occurring operator appearing in the formula towards the leaves, the former assume to receive intermediate results from the leaves.

This structural difference also imposes an explicit distinction between timed and untimed operators. This is required as each operator is completely agnostic from the semantics associated with the upstream operator, and therefore the downstream operator has to combine the incoming intermediate results appropriately. This motivates why LTL$_f$ operators do not have to provide such an explicit distinction from their syntactical standpoint.

Such a premise motivates the counter-intuitive definition of the timed Next$^\tau$ operator if compared to the homonym in LTL$_f$: as this needs to return the events for which desired temporal constraints happens immediately after them, it needs to assume that the desired forthcoming temporal behaviour is the one received as an input ρ, for which all the events preceding the ones listed in ρ are the ones of interest. As per the previous statement, it also follows that this operator shall never possess an equivalent untimed flavour. From these considerations, Next$^\tau$ is formally defined as follows:

$$\mathsf{Next}^\tau(\rho) = \{ \langle i, j-1, L\rangle \mid \langle i, j, L\rangle \in \rho, j > 1 \}$$

We now discuss the definition of "globally". As per previous considerations, checking that all of the events in a trace satisfy a given condition corresponds to retrieving all of the events satisfying such a condition, for then counting if the length of the returned events corresponds to the trace length. Similarly, the timed version of the same operator shall test the same condition for each possible event and return the points in the trace after which the desired condition always happens in the future. These operators are therefore defined as follows:

$$\mathsf{Globally}^{\tau}(\rho) = \left\{ \langle i, j, \cup_{\substack{j \leq k \leq |\sigma^i| \\ \langle i,k,L_k\rangle \in \rho}} L_k \rangle \;\middle|\; \langle i, j, L_j\rangle \in \rho,\ |\sigma^i| - j + 1 = \left|\{\langle i,k,L_k\rangle \in \rho | j \leq k \leq |\sigma^i|\}\right| \right\}$$

$$\mathsf{Globally}(\rho) = \left\{ \langle i, 1, \cup_{\langle i,j,L_k\rangle \in \rho} L_k \rangle \;\middle|\; |\sigma^i| = \left|\{\langle i,k,L_k\rangle \in \rho\}\right| \right\}$$

The operators expressing the eventuality that a condition shall happen in the future undergo similar considerations, with the only difference that these do not require to test that all of the trace events from a given point in time will satisfy a given condition, as it suffices that at least one event will satisfy it. The Future operator with its timed counterpart are then formally defined as follows:

$$\mathsf{Future}^{\tau}(\rho) = \left\{ \langle i, j, \cup_{\substack{j \leq k \leq |\sigma^i| \\ \langle i,k,L_k\rangle \in \rho}} L_k \rangle \;\middle|\; \exists h \geq j, L.\, \langle i, h, L\rangle \in \rho \right\}$$

$$\mathsf{Future}(\rho) = \left\{ \langle i, 1, \cup_{\langle i,k,L_k\rangle \in \rho} L_k \rangle \;\middle|\; \exists j, L.\, \langle i, j, L\rangle \in \rho \right\}$$

Timed and untimed negations are implemented dissimilarly by design. While the timed negation returns all of the events that are in the log but which were not returned in the previous computation ρ, the untimed version returns the traces containing no events associated to the provided input. These operators are therefore defined as follows:

$$\mathsf{Not}^{\tau}(\rho) = \{ \langle i, j, \varnothing\rangle \mid (\nexists L.\, \langle i,j,L\rangle \in \rho) \land \exists \alpha, \pi, \phi.\, \langle \alpha, i, j, \pi, \phi\rangle \in \mathsf{ActivityTable}_{\mathcal{L}} \}$$

$$\mathsf{Not}(\rho) = \{ \langle i, 1, \varnothing\rangle \mid (\nexists j, L.\, \langle i,j,L\rangle \in \rho) \land \exists \alpha, j, \pi, \phi.\, \langle \alpha, i, j, \pi, \phi\rangle \in \mathsf{ActivityTable}_{\mathcal{L}} \}$$

3.2.3. Binary Operators.

Differently from the LTL$_\mathrm{f}$ binary operators, the xtLTL$_\mathrm{f}$ binary operators are specifically tailored to express data correlation conditions Θ between activation and target payloads. This requires that one of the two operands, either ρ or ρ', returns activated events while the other provides targeted ones. Supplement III discusses the formal definition of predicates assessing whether an event $\langle i, j, L\rangle \in \rho$ matches with another event $\langle i, j', L'\rangle \in \rho'$ on the basis of their matched and activated events in L and L'. After this, we have the definition of our required binary operators.

The until operators work similarly to the other LTL$_\mathrm{f}$-derived unary operators. The timed until returns all of the events within the trace satisfying the until condition, expressed by returning all of the "activated" events σ^i_j listed in the right operand (as they trivially satisfy the until condition) alongside all of the "targeted' events σ^i_j from the left operand with $k < j$ at a distance $j - k + 1$ from the second operand's event while guaranteeing that all the events in $\sigma^i_k, \dots, \sigma^i_{j-1}$ appear in the first operand while satisfying the matching condition within this temporal window. The untimed version of this operator performs such considerations only from the beginning of the trace. Those are defined as follows:

$$\mathsf{Until}^{\tau}_{\Theta}(\rho_1,\rho_2) = \rho_2 \cup \Big\{ \langle i,k,\tau\rangle \,\Big|\, \exists j>k.\, \langle i,j,L\rangle \in \rho_2, (\forall k \le h < j.\, \langle i,h,L\rangle \in \rho_1), \tau := \mathcal{T}^{A,i}_{\Theta}([k\mapsto L]_{k\le h<j}, [h \mapsto L_h]_{k\le h<j,\langle i,h,L_h\rangle\in\rho_1}), \tau \neq \mathbf{False} \Big\}$$

$$\mathsf{Until}_{\Theta}(\rho_1,\rho_2) = \{\, \langle i,j,L\rangle \in \rho_2 \mid j=1 \,\} \cup \Big\{ \langle i,1,\tau\rangle \,\Big|\, \exists j>1, L.\, \langle i,j,L\rangle \in \rho_2, (\forall 1 \le k < j.\, \langle i,k,L_k\rangle \in \rho_1), \tau := \mathcal{T}^{A,i}_{\Theta}([k\mapsto L]_{i\le k<j}, [k \mapsto L_k]_{i\le k<j,\langle i,k,L_k\rangle\in\rho_1}), \tau \neq \mathbf{False} \Big\}$$

where $\mathcal{T}^{A,i}_{\Theta}$ performs[7] the correlation tests and returns the set of the matches if any and, if no match was successful, it returns **False**. Differently from $\mathsf{Until}^{\tau}_{\Theta}$ and Until_{Θ}, the rest of the binary operators assume to receive "activated" (or "targeted") events from the left (right) operand. The timed conjunction states that a join condition effectively happens in a given event σ^i_j if both operands return such an event and their associated activation and target conditions match. So, we only care for activation and target conditions at the same event σ^i_j. For its untimed counterpart, we state that a trace satisfies the conjunction of events if it exists at least one activation condition from the left operand matching with a target from the right operand if any; this corresponds to coalescing the activations and target conditions on the first event while requiring that at least one of them occurs. These two operators can then be defined as follows:

$$\mathsf{And}^{\tau}_{\Theta}(\rho_1,\rho_2) = \Big\{ \langle i,j,\tau\rangle \,\Big|\, \exists L_1, L_2.\, \langle i,j,L_1\rangle \in \rho_1, \langle i,j,L_2\rangle \in \rho_2, \tau := \mathcal{T}^{E,i}_{\Theta}([j\mapsto L_1],[j\mapsto L_2]), \tau \neq \mathbf{False} \Big\}$$

$$\mathsf{And}_{\Theta}(\rho_1,\rho_2) = \Big\{ \langle i,1,\tau\rangle \,\Big|\, \exists j,j',L,L'.(\langle i,j,L\rangle \in \rho_1 \wedge \langle i,j',L'\rangle \in \rho_2), \tau := \mathcal{T}^{E,i}_{\Theta}([1\mapsto \cup\{L_j|\, \langle i,j,L_j\rangle \in \rho_1\}],[1\mapsto \cup\{L_j|\, \langle i,j,L_j\rangle \in \rho_2\}]), \tau \neq \mathbf{False} \Big\}$$

The disjunctive version of the timed conjunctive operator returns either the result of the conjunctive operator or the events that did not temporally match from each respective operator. The only difference with its untimed version is that the latter merges all potential activation or target conditions from either of the two operands.

$$\mathsf{Or}^{\tau}_{\Theta}(\rho_1,\rho_2) = \mathsf{And}^{\tau}_{\Theta}(\rho_1,\rho_2) \cup \{\, \langle i,j,L\rangle \in \rho_1 \mid \nexists L'.\, \langle i,j,L'\rangle \in \rho_2 \,\} \cup \{\, \langle i,j,L\rangle \in \rho_2 \mid \nexists L'.\, \langle i,j,L'\rangle \in \rho_1 \,\}$$

$$\mathsf{Or}_{\Theta}(\rho_1,\rho_2) = \mathsf{And}_{\Theta}(\rho_1,\rho_2) \cup \{\, \langle i,1,\cup\{L|\exists j.\, \langle i,j,L\rangle \in \rho_1\}\rangle \mid \nexists j,L'.\, \langle i,j,L'\rangle \in \rho_2 \,\} \cup \{\, \langle i,1,\cup\{L|\exists j.\, \langle i,j,L\rangle \in \rho_2\}\rangle \mid \nexists j,L'.\, \langle i,j,L'\rangle \in \rho_1 \,\}$$

As we will see, the choice of characterizing Or with an E^i_{Θ} match while coalescing the activation and target conditions on the first trace event allows us to express the Choice template from Declare with one single operator while preserving its expected LTL$_f$ semantics.

3.2.4. Derived Operators.

Similarly to relational algebra, we can now compose some frequently occurring operators together for enhancing the overall time complexity associated with the execution of frequently appearing subqueries in declare. Appendix A.2 will show that computing

[7] Please see Supplement III for more details.

these operators is equivalent to computing their semantically equivalent xtLTL$_\mathrm{f}$ expression containing multiple operators.

$$\mathsf{AndFuture}^{\tau}_{\Theta}(\rho_1,\rho_2)=\Big\{\,\langle i,j,\tau\rangle\;\Big|\;\exists L.\,\langle i,j,L\rangle\in\rho_1,(\exists L',k\geq j.\,\langle i,k,L'\rangle\in\rho_2),\\ \tau:=\mathcal{T}^{E,i}_{\Theta}([j\mapsto L],[j\mapsto\cup_{h\geq j,\langle i,h,L_h\rangle\in\rho_2}L_h]),\tau\neq\mathbf{False}\Big\}$$

$$\mathsf{AndGlobally}^{\tau}_{\Theta}(\rho_1,\rho_2)=\Big\{\,\langle i,j,\tau\rangle\;\Big|\;\exists L.\,\langle i,j,L\rangle\in\rho_1,(\forall|\sigma^i|\geq k\geq j.\exists L'.\,\langle i,k,L'\rangle\in\rho_2),\\ \tau:=\mathcal{T}^{A,i}_{\Theta}([j\mapsto L],[j\mapsto\cup_{h\geq j,\langle i,h,L_h\rangle\in\rho_2}L_h]),\tau\neq\mathbf{False}\Big\}$$

$$\mathsf{AndNextGlobally}^{\tau}_{\Theta}(\rho_1,\rho_2)=\Big\{\,\langle i,j,\tau\rangle\;\Big|\;\exists L,L'.\,\langle i,j,L\rangle\in\rho_1,\langle i,j+1,L'\rangle\in\rho_2,\\ (\forall|\sigma^i|\geq k>j.\exists L''.\,\langle i,k,L''\rangle\in\rho_2),\\ \tau:=\mathcal{T}^{A,i}_{\Theta}([j\mapsto L],[j\mapsto\cup_{h>j,\langle i,h,L_h\rangle\in\rho_2}L_h]),\tau\neq\mathbf{False}\Big\}$$

For easing the pseudocode readability, we can also define an $\mathsf{Atom}^{\mathcal{L},\tau}_{A/T}(p_i)$ operator computing the conjunction of all of the compound conditions characterizing each atom:

$$\mathsf{Atom}^{\mathcal{L},\tau}_{A/T}(p_i)=\mathop{\mathsf{And}^{\tau}_{\mathbf{True}}}_{\kappa\in K}\mathsf{Compound}^{\mathcal{L},\tau}_{A/T}(\mathsf{a},\kappa,[low_\kappa,up_\kappa])\text{ s.t. }p_i:=\mathsf{a}\wedge\bigwedge_{\kappa\in K}low_\kappa\leq\kappa\leq up_\kappa$$

Properties of the xtLTL$_\mathrm{f}$ Algebra. We furnish the previous definitions with some formal proofs which, so as not to burden the reader, are postponed to the Appendix. We show that xtLTL$_\mathrm{f}$ is as expressive as traditional LTL$_\mathrm{f}$, as we can show that each LTL$_\mathrm{f}$ expression evaluated over a finite and non-empty trace σ corresponds to an xtLTL$_\mathrm{f}$ expression evaluated over the representation of such a trace within the proposed logical model; as the proofs of Lemma A1 and A2 in Appendix A.1 are constructive, they show the translation process from LTL$_\mathrm{f}$ formulæ to equivalent xtLTL$_\mathrm{f}$ expressions.

Next, we also show that the timed and untimed operators correspond to the intended semantics: that is, for each timed operator having a corresponding untimed operator if the former states that the timed formula is satisfied by the i-th trace starting from time j, it follows that the sub-trace of i starting from time j will satisfy the corresponding untimed formula. This shows the correctness of the untimed operators concerning their timed definitions (Lemma A3).

In Appendix A.2, we show that the Declare template Choice can be fully implemented by exploiting an untimed Or operator (Corollary A1) while the latter still abides to the rules of LTL$_\mathrm{f}$ semantics. We also motivate the need of the derived operators in terms of equivalence to the intended xtLTL$_\mathrm{f}$ expressions (Lemma A5-**??**) as well as in terms of improved computational complexity (§6.4) and run time (§7.1). The latter is discussed after describing the physical model in more detail alongside the algorithms associated with each operator, which is introduced in the following section.

4. Physical Database Design

This section shows how the defined model (§3.1) is represented in primary memory in terms of indices and data structures (§4.1). We also illustrate the algorithm loading a log in such representation of choice (§4.2).

4.1. Primary Memory Data Structures

At the time of the writing, KnoBAB is primarily an in-memory database. This is a common assumption in the conformance checking domain where most of the log datasets are quite compact and nicely fit in primary memory.

In order to be both memory and time efficient in our operations, the sub-record referring to the first three columns of both the $\mathsf{CountingTable}_{\mathcal{L}}$ and the $\mathsf{ActivityTable}_{\mathcal{L}}$ are fully stored in primary memory as an unsigned 64-bit unsigned integer, while the `Prev` and `Next` are more efficiently stored as pointers to the table records rather than being an offset. After sorting the $\mathsf{CountingTable}_{\mathcal{L}}$, we directly obtain the occurrence of each activity label a within the log by accessing the records in the range $[|\mathcal{L}| \cdot (\beta(\mathsf{a}) - 1) + 1,\ |\mathcal{L}| \cdot \beta(\mathsf{a})]$.

Indexing data structures, on the other hand, ease the access to the $\mathsf{ActivityTable}_{\mathcal{L}}$, as different traces might have different lengths and activity labels might be differently distributed among the traces. Therefore, we exploit a clustered and sparse primary index for determining which is the first event associated with a given activity label; as the traces in such a table are represented as a doubly linked list, its secondary index maps each trace-id to a block that, in turn, points to the head (first event of the trace) and the tail (last event of the trace) of such a doubly linked list.

The deduplication of trace and event payloads in distinct $\mathsf{AttributeTable}^{\kappa}_{\mathcal{L}}$ for each key κ follows the prescriptions of the query and memory-efficient representation of columnar-based storages [38]. In our implemnetation, such tables are sorted in ascending order by their three first columns. Each $\mathsf{AttributeTable}^{\kappa}_{\mathcal{L}}$ is also associated with two indices: the clustered and sparse primary index maps each activity label's id $\beta(\mathsf{a})$ to the records referring to values contained in a-labelled events, and a dense secondary index associates an $\mathsf{ActivityTable}_{\mathcal{L}}$ record offset to an $\mathsf{AttributeTable}^{\kappa}_{\mathcal{L}}$ record offset if and only if the event described in $\mathsf{ActivityTable}_{\mathcal{L}}$ has a payload containing a value associated to a key κ. While data range queries leverage the former, the latter is used for reconstructing the payload associated with a given event when identified by its offset in the $\mathsf{ActivityTable}_{\mathcal{L}}$. A relevant use case for doing so is the reconstruction of the event payload information while performing the Θ correlation condition, as well as reconstructing the original log leading to the loading of the internal database. RECONSTRUCTLOG function in Algorithm 1 shows the computation of the latter.

Example 9. *With reference to Figure 3, let us consider some events with activity label* $\mathsf{Mastectomy}$ *associated to an unique id* $\beta(\mathsf{Mastectomy}) = 3$. *The offsets for accessing the records in the* $\mathit{CountingTable}_{\mathcal{L}}$ *defining the number of events per trace with such a label is* $[3 \cdot (3-1) + 1, 3 \cdot 3] = [7, 9]$.

The $\mathit{ActivityTable}_{\mathcal{L}}$*'s primary index allows the access to the first record within the table recording a* $\mathsf{Mastectomy}$ *event, i.e.,* $\mathit{ActivityTable}_{\mathcal{L}}.\mathtt{primary\_index}[\beta(\mathsf{Mastectomy})] = 7$; *the index implicitly returns the last event associated to such an activity label by decreasing the offset to the following activity label by one, i.e.,* $\mathit{ActivityTable}_{\mathcal{L}}.\mathtt{primary\_index}[\beta(\mathsf{Mastectomy}) + 1] - 1 = 7$ [8]. *This indicates that there exists only one event throughout the whole log associated with such an activity label. We will exploit this mechanism for returning the events associated to* $\mathit{Activity}^{\mathcal{L},\tau}_{A/T}(\mathsf{Mastectomy})$. *As the seventh record of such a table refers to the third event of the first trace,* $\mathit{Activity}^{\mathcal{L},\tau}_{A}(\mathsf{Mastectomy})$ *will then return* $\{\ \langle 1, 3, \{A(3)\}\rangle\ \}$.

Last, we discuss how we can leverage $\mathit{AttributeTable}^{\kappa}_{\mathcal{L}}$*'s primary indices for returning results associated to a* $\mathit{Atom}^{\mathcal{L},\tau}_{A/T}$ *operator. Let us consider atom* p_{12}: *we can see that this is associated to the elementary intervals* $\mathsf{Lumpectomy} \wedge \mathit{biopsy} = \textbf{\textit{true}}$ *and* $\mathsf{Lumpectomy} \wedge 50 \leq \mathit{CA\_15.3} < +\infty$. *By definition of the operator of interest, we have then that:*

$$\begin{aligned} \mathit{Atom}^{\mathcal{L},\tau}_{A}(p_{12}) = \mathit{And}^{\tau}_{\textbf{\textit{True}}}(&\mathsf{Compound}^{\mathcal{L},\tau}_{A}(\mathsf{Lumpectomy} \wedge \mathit{biopsy} = \textbf{\textit{true}}), \\ &\mathsf{Compound}^{\mathcal{L},\tau}_{A}(\mathsf{Lumpectomy} \wedge \mathit{CA\_15.3} \geq 50)) \end{aligned}$$

The first $\mathsf{Compound}$ *operator will access the primary index from* $\mathit{AttributeTable}^{\mathit{biopsy}}_{\mathcal{L}}$ *while the second one will access the one from* $\mathit{AttributeTable}^{\mathit{CA\_15.3}}_{\mathcal{L}}$. *Then, the primary index of each ta-*

[8] Please remember that if the activity label is such that $\beta(\mathsf{a}) = |\Sigma|$, then the final offset to be considered corresponds to the $\mathsf{ActivityTable}_{\mathcal{L}}$ size.

*ble maps each activity to the offsets of the first and last record: $\mathsf{AttributeTable}_{\mathcal{L}}^{biopsy}$.`primary_index` $[\beta(\mathsf{Lumpectomy})] = \langle 2,2\rangle$ and $\mathsf{AttributeTable}_{\mathcal{L}}^{CA\_15.3}$.`primary_index`$[\beta(\mathsf{Lumpectomy})] = \langle 7,7\rangle$. Then, within these returned record offsets, we perform range queries respectively looking for records satisfying biopsy $=$ **true** and $50 \leq CA\_15.3 < +\infty$. All of the $\mathsf{ActivityTable}_{\mathcal{L}}\kappa$'s records satisfying these conditions point to the tenth record of the $\mathsf{ActivityTable}_{\mathcal{L}}$ referring to the third event of the third trace. Therefore, $\mathsf{Atom}_A^{\mathcal{L},\tau}(p_{12})$ returns $\{\ \langle 3,3,\{A(3)\}\rangle\ \}$.*

4.2. Populating the Database

We discuss two subsequent steps for loading a log in our proposed relational model: we preliminary sort the data by activity label id, event id, and values (§4.2.1) for then loading the sorted record in the tables while generating their primary and secondary indices (§4.2.2). These are computed in quasi-linear time with respect to the full log size.

4.2.1. Bulk Insertion

KnoBAB uses BULKINSERTION to pre-load the tables' data into an intermediate representation by pre-sorting it according to the ascending order induced by the first column of the tables of interest. Algorithm 1 shows the loading of the following three maps referring to the aforementioned tables. *(i)* `CountBulkMap` counts the occurrence of each activity label per track, implying that the absence of a trace identifier for a given $\beta(\mathsf{a})$ value presupposes the absence of a given activity label a within a trace; as the name suggests, we use this to later on populate the CountingTable. *(ii)* The `ActToEventBulkVector` prepares the insertion of sorted data in $\mathsf{ActivityTable}_{\mathcal{L}}$ by associating an activity label to each event and its associated trace containing it. *(iii)* Similarly to the `ActToEventBulkMap`, the `AttBulkMap`$_k$ associates to each key κ the values $p(\kappa)$ for each event σ_j^i with payload p and activity label a, so to prepare the insertion of sorted records in $\mathsf{AttributeTable}_{\mathcal{L}}^{\kappa}$. Please observe that, by construction, the set of pairs associated with each activity id $\beta(\mathsf{a})$ is already sorted by increasing trace and event id.

We also pre-allocate a `TraceToEventBulkVector` map (represented as a vector of vectors) which will later associate each event trace to an offset on the $\mathsf{ActivityTable}_{\mathcal{L}}$ where such event is stored. KnoBAB will later use this to calculate `Prev` and `Next` in the ActivityTable. After this, KnoBAB knows the number of the traces within the log $|\mathcal{L}|$, the length $|\sigma^j|$ for each trace σ^j, and the number of distinct activity labels $|\Sigma|$ is known, as well as their associated unique id $\beta(\mathsf{a})$ for each $\mathsf{a} \in \Sigma$. We can show that this procedure might be computed in quasi-linear time with respect to the full log size (Lemma S1).

4.2.2. Loading and Indexing

We continue our discussion with LOADINGANDINDEXING. *First*, we can iterate over the activity labels in ascending order of appearance (Line 17). All the tables including the $\mathsf{CountingTable}_{\mathcal{L}}$ have activity ids $\beta(\mathsf{a})$ as their first cell: by further iterating by increasing trace id, we can immediately orderly store the records in $\mathsf{CountingTable}_{\mathcal{L}}$ (Line 20).

Second, we start populating the $\mathsf{ActivityTable}_{\mathcal{L}}$ (Line 23) where each record is associated with an increasing offset of the table (Line 25). We can populate its primary index so to point at the record representing the first event of the first trace with the currently considered activity label. We store such information in the pre-allocated `traceToEventBulkVector` (Line 24), so to later set the currently null pointer ($\uparrow$) `Next` and `Prev` fields.

Third, we start populating each $\mathsf{AttributeTable}_{\mathcal{L}}^{\kappa}$ for each key $\kappa \in K$ associated with at least one value in a payload: as per the previous discussion, each record associates the offset of event $\sigma_j^i = \langle \mathsf{a}, p\rangle$ in the $\mathsf{ActivityTable}_{\mathcal{L}}$ with a value $v = p(\kappa)$ and the activity label id $\beta(\mathsf{a})$ (Line 32). We also populate its secondary index by associating each event offset in the $\mathsf{ActivityTable}_{\mathcal{L}}$ to the current position in the $\mathsf{AttributeTable}_{\mathcal{L}}^{\kappa}$ (Line 33). The last iteration finally populates each $\mathsf{ActivityTable}_{\mathcal{L}}$'s secondary index (Line 50) and sets the `Next` (Line 45) and `Prev` (Line 42) fields through the offset via `TraceToEventBulkVector`. After this, the

Algorithm 1 Populating the Knowledge Base (§4.2)

```
procedure BulkInsertion($\mathcal{L}$)
    $\Sigma, K \leftarrow \emptyset$
    for all $\sigma^i \in \mathcal{L}$ do
        $\Sigma \leftarrow \Sigma \cup \{\mathsf{a}\}$
        for all $\sigma^i_j = \langle \mathsf{a}, p\rangle \in \sigma^i$ do
            $\texttt{CountBulkMap}[\beta(\mathsf{a})][i] = \texttt{CountBulkMap}[\beta(\mathsf{a})][i] + 1$
            $\texttt{ActToEventBulkVector}[\beta(\mathsf{a})].\texttt{put}(\langle i, j\rangle)$
            $\texttt{TraceToEventBulkVector}[i][j] = j$
            for all $\kappa \in \text{dom}(p)$ do
                $K \leftarrow K \cup \{\kappa\}$
                $\texttt{AttBulkMap}_\kappa[\beta(\mathsf{a})][p(\kappa)].\texttt{put}(\langle i, j\rangle)$
            end for
        end for
    end for

procedure LoadingAndIndexing($\mathcal{L}$)
    $\texttt{actTableOffset} \leftarrow 1$
    for all $\beta(\mathsf{a}) \in \{1, \dots, |\Sigma|\}$ do
        $\mathsf{ActivityTable}_\mathcal{L}.\texttt{primary\_index}[\beta(\mathsf{a})] \leftarrow \texttt{actTableOffset}$
        for all $\sigma^i \in \mathcal{L}$ do
            $\mathsf{CountingTable}_\mathcal{L}.\texttt{load}(\langle \beta(\mathsf{a}), i, \texttt{CountBulkMap}[\beta(\mathsf{a})][i]\rangle)$
        end for
        for all $\langle i, j\rangle \in \texttt{ActToEventBulkVector}[\beta(\mathsf{a})]$ do
            $\mathsf{ActivityTable}_\mathcal{L}.\texttt{load}[\langle \beta(\mathsf{a}), i, j, \uparrow, \uparrow\rangle]$
            $\texttt{TraceToEventBulkVector}[i][j] = \texttt{actTableOffset}$
            $\texttt{actTableOffset} \leftarrow \texttt{actTableOffset} + 1$
        end for
    end for
    for all $\kappa \in K$ and $\beta(\mathsf{a}) \in \{1, \dots, |\Sigma|\}$ do
        $\texttt{begin} \leftarrow |\mathsf{AttributeTable}^\kappa_\mathcal{L}|, \quad \texttt{map} \leftarrow \{\}$
        for all $\langle v, lst\rangle \in \texttt{AttBulkMap}_\kappa[\beta(\mathsf{a})]$ and $\langle i, j\rangle \in lst$ do        ▷ $\sigma^i_j = \langle \mathsf{a}, p\rangle$ with $v = p(\kappa)$
            $\texttt{offset} \leftarrow \texttt{TraceToEventBulkVector}[i][j]$
            $\mathsf{AttributeTable}^\kappa_\mathcal{L}.\texttt{load}(\langle \beta(\mathsf{a}), v, \texttt{offset}\rangle)$
            $\mathsf{AttributeTable}^\kappa_\mathcal{L}.\texttt{secondary\_index}[\texttt{offset}] \leftarrow |\mathsf{AttributeTable}^\kappa_\mathcal{L}|$
        end for
        $\mathsf{AttributeTable}^\kappa_\mathcal{L}.\texttt{primary\_index}[\beta(\mathsf{a})] \leftarrow \langle \texttt{begin}, |\mathsf{AttributeTable}^\kappa_\mathcal{L}|\rangle$
    end for
    for all $\sigma^i \in \mathcal{L}$ and $\sigma^i_j \in \sigma^i$ do
        $\texttt{curr} \leftarrow \texttt{TraceToEventBulkVector}[i][j]$
        if $j = 1$ then
            $\mathsf{ActivityTable}_\mathcal{L}.\texttt{secondary\_index}[i] \leftarrow \langle \texttt{curr}, \texttt{TraceToEventBulkVector}[i][|\sigma^i|]\rangle$
        else
            $\mathsf{ActivityTable}_\mathcal{L}[\texttt{curr}](\texttt{Prev}) \leftarrow \texttt{TraceToEventBulkVector}[i][j-1]$
        end if
        if $j < |\sigma^i|$ then
            $\mathsf{ActivityTable}_\mathcal{L}[\texttt{curr}](\texttt{Next}) \leftarrow \texttt{TraceToEventBulkVector}[i][j1]$
        end if
    end for

function ReconstructLog($\mathcal{L}$)
    $\mathcal{L}' \leftarrow \emptyset$
    for all $\langle i, \langle \texttt{begin}, \texttt{end}\rangle\rangle \in \mathsf{ActivityTable}_\mathcal{L}.\texttt{secondary\_index}$ do
        $\varsigma^i \leftarrow [];\quad j \leftarrow 1$
        repeat
            $r \leftarrow \mathsf{ActivityTable}_\mathcal{L}[\texttt{begin}]$
            $\mathsf{a} \leftarrow \beta^{-1}(r(\texttt{Activity}))$
            $p \leftarrow \{\}$
            for all $\kappa \in K$ s.t. $\exists o.\ \langle \texttt{begin}, o\rangle \in \mathsf{AttributeTable}_\kappa.\texttt{secondary\_index}$ do
                $p(\kappa) \leftarrow \mathsf{AttributeTable}_\kappa[o](\texttt{Value})$        ▷ $\mathsf{AttributeTable}_\kappa[o](\texttt{Offset}) = \texttt{begin}$
            end for
            $\varsigma^i_j \leftarrow \langle \mathsf{a}, p\rangle;\quad \sigma^i.\texttt{put}(\varsigma^i_j)$
            $\texttt{begin} \leftarrow r(\texttt{Next});\quad j \leftarrow j + 1$
        until $\texttt{begin} \neq \uparrow$
        $\mathcal{L}'.\texttt{put}(\varsigma^i)$
    end for
    return $\mathcal{L}'$
```

Algorithm 2 Atomization Pipeline (§5.1.1)

```
1: procedure AtomizationPipeline($\mathcal{M}$, strategy)
2:     CollectIntervals($\mathcal{M}$)                                                  ▷ See Algorithm S1
3:     $\mathcal{D}_\varphi$-encoding( )                                                ▷ See Algorithm S1
4:     for all clause$_l$(A, $p$, B, $p'$) where $\Theta$ $\in \mathcal{M}$ do
5:         if p=True and (strategy=AtomizeOnlyOnDataPredicates or $ak$(A) = {A}) then   ▷ Defining $S_A$ for clause$_l$
6:             clause$_l$.left ← {A}
7:         else
8:             clause$_l$.left ← $ak$(A) ∩ Atom$_{\mu,ad}$(A, $p$)                      ▷ Equation S1
9:         end if
10:        if p′=True and (strategy=AtomizeOnlyOnDataPredicates or $ak$(B) = {B}) then  ▷ Defining $S_T$ for clause$_l$
11:            clause$_l$.right ← {B}
12:        else
13:            clause$_l$.right ← $ak$(B) ∩ Atom$_{\mu,ad}$(B, $p'$)                    ▷ Equation S1
14:        end if
15:    end for
```

relational database is fully loaded in primary memory. The overall time complexity grows linearly to the whole log representation (Lemma S2).

5. Query Processing and Optimization

This section shows how a declarative model $\mathcal{M}$ is compiled to a query plan consisting of **xt** LTL$_f$ *operators (§5.1) so it can be run (§5.2) on top of the primary memory data described in the previous section.*

5.1. Query Compiler

The conversion of a declarative model $\mathcal{M}$ into its corresponding **xt** LTL$_f$ *query plan is structured into three main phases. First, the atomization pipeline calls the preliminary $\mathcal{D}_\varphi$-encoding from [25] for rewriting the data predicates appearing in each declarative clause as a disjunction of mutually exclusive atoms (§5.1.1). Second, we (ii) rewrite each Declare constraint as a* **xt** LTL$_f$ *formula from which we obtain a preliminary query plan represented as a* Direct Acyclic Graph *(DAG) (§5.1.2). Third, we compute the scheduling order for the operators' execution over the DAG, thus preparing the execution to a potential parallel evaluation of the query (§5.1.3).*

5.1.1. Atomization Pipeline

The atomization pipeline (Algorithm 2) represents each activation and target condition as a set of disjunct atoms or activity labels. KnoBAB can always be configured in two ways: to either fully represent each possible activation (or target) condition with activity label a as a disjunction of atoms (or activity labels) if there exists at least one declarative clause where a is also associated with a non-trivial payload condition (strategy=`AtomizeEverything`), or to restrict atomization to data conditions appearing in a clause (strategy=`AtomizeOnlyOnDataPredicates`). Both can be setted through the AtomizationPipeline procedure in Algorithm 2. The $\mathcal{D}_\varphi$-encoding step guarantees that each activation or target condition will be associated with at least one atom or activity label. While the former approach will maximise the access to the AttributeTable$_\mathcal{L}$, the latter will maximise the access to the ActivityTable$_\mathcal{L}$. Correlation conditions do not undergo this rewriting step. We discuss the effects of each different strategy on the query runtime via empirical benchmarks in §7.4. We can show that this step has a polynomial complexity with respect to the model, key set, and element intervals' maximum size (Lemma S3).

Example 10. *With reference to Figure 2a, we might observe that, as no activation or target is ever associated with payload conditions, the atomisation pipeline will never express each activation or target condition as a disjunction of atoms, as no elementary interval is collected. Therefore, those will be only associated with activity labels.*

Example 11. *With reference to Example 6, Figure 3 shows the atomized version of the declarative model, where each activation and target condition is associated, in this case, with just one atom.*

Table 5. Declare templates illustrated as their associated xtLTL$_f$ semantics. S_A (and S_T) denote the disjunction of collected atoms and activity labels (represented as sets) associated to the activation (and target) condition. The Atomization Pipeline will return these sets. For declarative clauses that can be directly represented as xtLTL$_f$ operators, we might have two different possible operators dependingly on the atomisation result.

Exemplifying clause (c_l)	xtLTL$_f$ Semantics	
	$S_A = \{\mathsf{A}\}, S_T = \{\mathsf{B}\}$ $\mathsf{A}, \mathsf{B} \in \Sigma$	Otherwise (e.g., atomisation)
$\mathsf{Init}(S_A)$	$\mathsf{Init}^{\mathcal{L}}_{A}(\mathsf{A})$	$\mathsf{Init}(S_A)$
Exists(S_A, n)	$\mathsf{Exists}^{\mathcal{L}}_{A}(\mathsf{A}, n)$	$\mathsf{Exists}_n(S_A)$
Absence$(S_A, n+1)$	$\mathsf{Absence}^{\mathcal{L}}_{A}(\mathsf{A}, n)$	$\mathsf{Absence}_{n+1}(S_A)$
Precedence(S_A, S')	$\mathsf{Or}_{\mathbf{True}}(\mathsf{Until}(\complement S', S_A), \textbf{Absence}(S', 1))$	
$\mathsf{ChainPrecedence}(S_A, S_T)$ **where** Θ	$\mathsf{Globally}(\mathsf{Or}^{\tau}_{\mathbf{True}}(\mathsf{Or}^{\tau}_{\mathbf{True}}(\mathsf{Last}^{\mathcal{L},\tau}, \mathsf{Next}^{\tau}(\complement S_T)), \mathsf{And}^{\tau}_{\Theta}(\mathsf{Next}^{\tau}(S_A), S_T)))$	
$\mathsf{Choice}(S_A, S_{A'})$	$\mathsf{Or}_{\mathbf{True}}(S_A, S_{A'})$	
Response(S_A, S_T) where Θ	$\mathsf{Globally}(\mathsf{Or}^{\tau}_{\mathbf{True}}(\complement S_A, \mathsf{AndFuture}^{\tau}_{\Theta}(S_A, S_T)))$	
$\mathsf{ChainResponse}(S_A, S_T)$ **where** Θ	$\mathsf{Globally}(\mathsf{Or}^{\tau}_{\mathbf{True}}(\complement S_A, \mathsf{And}^{\tau}_{\Theta}(S_A, \mathsf{Next}^{\tau}(S_T))))$	
RespExistence(S_A, S_T) where Θ	$\mathsf{Or}_{\mathbf{True}}(\textbf{Absence}(S_A, 1), \mathsf{And}_{\Theta}(S_A, S_T))$	
$\mathsf{ExlChoice}(S_A, S_{A'})$	$\mathsf{And}_{\mathbf{True}}(\mathsf{Or}_{\mathbf{True}}(\textbf{Exists}(S_A, 1), \textbf{Exists}(S_{A'}, 1)), \mathsf{Or}_{\mathbf{True}}(\textbf{Absence}(S_A, 1), \textbf{Absence}(S_{A'}, 1)))$	
$\mathsf{CoExistence}(S_A, S_T)$ **where** Θ	$\mathsf{And}_{\mathbf{True}}(\textbf{RespExistence}(S_A, S_T)$ **where** Θ, $\textbf{RespExistence}(S_{A'}, S_{T'})$ **where** $\Theta^{-1})$ s.t. $S_{A'} = S_T$ and $S_{T'} = S_A$	
$\mathsf{Succession}(S_A, S_T)$ **where** Θ	$\mathsf{And}_{\mathbf{True}}(\textbf{Precedence}(S_A, S'), \textbf{Response}(S_A, S_T)$ **where** $\Theta)$ s.t. $S' = S_T$	
$\mathsf{ChainSuccession}(S_A, S_T)$ **where** Θ	$\mathsf{Globally}(\mathsf{And}^{\tau}_{\mathbf{True}}(\mathsf{Or}^{\tau}_{\mathbf{True}}(\mathsf{Or}^{\tau}_{\mathbf{True}}(\mathsf{Last}^{\mathcal{L},\tau}, \mathsf{Next}^{\tau}(\complement S_{T'})), \mathsf{And}^{\tau}_{\Theta^{-1}}(\mathsf{Next}^{\tau}(S_{A'}), S_{T'})),$ $\mathsf{Or}^{\tau}_{\mathbf{True}}(\complement S_A, \mathsf{And}^{\tau}_{\Theta}(S_A, \mathsf{Next}^{\tau}(S_T))))))$ s.t. $S_{A'} = S_T$ and $S_{T'} = S_A$	
$\mathsf{AltResponse}(S_A, S_T)$ **where** Θ	$\mathsf{Globally}(\mathsf{Or}^{\tau}_{\mathbf{True}}(\complement S_A, \mathsf{And}^{\tau}_{\Theta}(S_A, \mathsf{Next}^{\tau}(\mathsf{Until}^{\tau}_{\mathbf{True}}(\complement S_A, S_T)))))$	
$\mathsf{AltPrecedence}(S_A, S_T)$ **where** Θ	$\mathsf{And}_{\mathbf{True}}(\textbf{Precedence}(S_A, S_T), \mathsf{Globally}(\mathsf{Or}^{\tau}_{\mathbf{True}}(\complement S_A, \mathsf{And}^{\tau}_{\Theta}(S_A, \mathsf{Next}^{\tau}(\mathsf{Or}^{\tau}_{\mathbf{True}}(\mathsf{Until}^{\tau}(\complement S_A, S_T), \mathsf{Globally}^{\tau}(\complement S_A)))))))$	
$\mathsf{NotCoExistence}(S_A, S_T)$ **where** Θ	$\mathsf{Not}(\mathsf{And}_{\Theta}(S_A, S_T))$	
$\mathsf{NotSuccession}(S_A, S')$	$\mathsf{Globally}(\mathsf{Or}_{\mathbf{True}}(\complement S_A, \mathsf{AndGlobally}^{\tau}_{\mathbf{True}}(S_A, \complement S_T)))$	

Example 12. *Continuing with Example 7, where we discussed the outcome of the $\mathcal{D}_{\varphi}$-encoding phase for a model $\mathcal{M}'$ in Equation 2, we obtain the following atomization:*

$$\begin{aligned}\{&\mathsf{Choice}_1(\mathit{left} = \{p_{12}, p_{17}\}, \mathit{right} = \{p_4, p_9\}),\\ &\mathsf{Absence}_2(\mathit{left} = \{p_1, \dots, p_5, p_{16}, \dots, p_{20}\}, \mathit{n} = 1),\\ &\mathsf{Absence}_3(\mathit{left} = \{p_1, p_3, p_5, p_6, p_8, p_{10}, p_{11}, p_{13}, p_{15}, p_{16}, p_{18}, p_{20}\}, \mathit{n} = 1)\}\end{aligned}$$

5.1.2. Query Optimizer

The query optimizer[9] *consists of three steps: (i) loading the* **xt** LTL$_f$ *formulæ associated to each declarative clause at warm-up, (ii) exploiting the outcome of the Atomization Pipeline to instantiate the* **xt** LTL$_f$ *formulæ, (iii) and coalescing the single* **xt** LTL$_f$ *into one compact abstract syntax DAG. Our query plan will not be represented as a tree as we merge as many nodes computing the same result as possible, thus computing the same sub-expression at most once.*

First, we load the translation map `xtTemplates` (Table 5) at warm-up through an external script[10] providing the temporal semantics associated with the clauses of interest via partially-instantiated xtLTL$_f$ expressions. Such representation also supports negated activation or target conditions, thus avoiding the need to compute a Not operator stripping the information of either activation or target conditions. These are marked in the previous table via set complementation, $\complement$.

Second, we exploit the aforementioned map to convert each declarative clause into its xtLTL$_f$ semantics ψ. If the clause is met for the first time, we proceed with its instantiation by recursively visiting ψ until the leaves are reached: at this level, we potentially replace the activation and target placeholders with the associated set of atoms. Disjunctions of atoms and activity labels associated with leaf nodes as returned by the previous pipeline are minimised by ensuring that each shared $\mathsf{Or}_{\mathbf{True}}$ computation across all of the atoms and activity labels is computed at most once. If an atom is met, we decompose it into its defining compound conditions (Line 14), thus guaranteeing that each compound condition

9 Further optimization and implementation details are deferred to Supplement V.1 and its Algorithm S6.

10 At the time of the writing, the scripts provide the xtLTL$_f$ semantics for Declare templates. Future investigations will express other temporal declarative languages such as [39] in xtLTL$_f$, as well as other LTL$_f$ extensions including "past" operators [16,18].

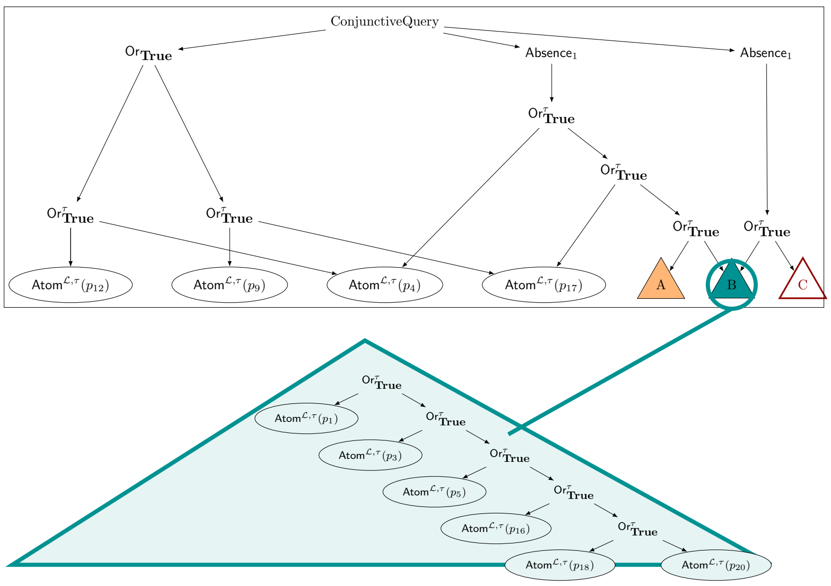


Figure 5. In depth representation of the query plan associated to the model described in Example 15.

is evaluated via $\texttt{Compound}^{\mathcal{L},\tau}_{A/T}$ at most once across all of the atoms occurring in the $\texttt{xtLTL}_f$ formula when running the query (§5.2.1).

Third, we complete the process by coalescing shared disjunct sub-expressions via a map (*queryCache*) guaranteeing that all of the equivalent sub-expression are all replaced by just one instance of these. Last, we associate each sub-expression referring to each clause to the final query operator representing the expression's root (*queryRoot*), either presenting an aggregation or a conjunctive query.

Example 13. *The model in Figure 2a, when compiled and associated to a conjunctive query, might produce the following xLTLf expression:*

$$\mathsf{And}_{\textbf{True}}\Big(\mathsf{Globally}\Big(\mathsf{Or}^{\tau}(\mathsf{Not}^{\tau}(\mathsf{Activity}^{\mathcal{L},\tau}(\mathsf{rec})), \mathsf{AndFuture}^{\tau}_{\textbf{True}}(\mathsf{Activity}^{\mathcal{L},\tau}_{A}(\mathsf{rec})), \mathsf{Activity}^{\mathcal{L},\tau}_{T}(\mathsf{weap}))\Big)\Big),$$
$$\mathsf{And}_{\textbf{True}}\Big(\mathsf{Absence}(\mathsf{iiot\_sh},1),$$
$$\mathsf{Or}_{\textbf{True}}(\mathsf{Activity}^{\mathcal{L}}(\mathsf{comm}), \mathsf{Activity}^{\mathcal{L}}(\mathsf{act}))\Big)\Big)$$

We might observe that this expression cannot be further minimised, as there are neither shared atoms nor sub-expression in common. This can neither be achieved by rewriting $\mathsf{Not}^{\tau}(\mathsf{Activity}^{\mathcal{L},\tau}(\mathsf{rec}))$ *as* $\mathsf{Or}^{\tau}_{\textbf{True}\ \mathsf{a}\in\Sigma,\ \mathsf{a}\neq\mathsf{rec}}\ \mathsf{Activity}^{\mathcal{L},\tau}(\mathsf{a})$*, as the* comm *and* act *atoms associated to the choice clause are untimed, while the former rewriting only included timed* Activity *operators. As these two different flavours of operators do not necessarily return the same result, these nodes are not merged.*

Example 14. *With reference to Example 11 and Table 1, as the* Response *clause was associated to the same activation and target condition to* Succession*, the former is indeed a subquery of the latter. For this reason, those queries are fused together, thus guaranteeing that the result for* Response *is computed at most once. As the query root requires the computation of Max-SAT, this one is always going to be linked to the sub-expression being the representation of an original declarative clause. Green arrows in Figure 3 remark operators' output shared among operators.*

Example 15. *This last example shows the effect of the reduction of the number of shared timed union operators at the leaf level. By recalling the atomised model discussed in Example 12, we need to represent each set of atoms as a timed disjunction of* Atom *operators. While doing so, we observe*

Algorithm 3 Query Scheduler (§5.1.3)

```
function QUERYSCHEDULER($\mathcal{G}$)
    $layer \leftarrow \{\}$
    $V \leftarrow$ REVERT(TOPOLOGICALSORT($\mathcal{G}$))
    for all $\psi \in V$ do
        for all $\psi' \in \psi$.args do
            $\psi'$.distance $\leftarrow \max(\psi'$.distance, $\psi$.distance $+ 1)$
        end for
    end for
    for all $\psi \in V$ do
        $layer[\psi$.distance$]$.put($\psi$)
    end for
    return $layer$
```

that Choice and the first Absence condition share atoms p_4 and p_{17}, while the two Absence clauses share all the atoms in $\{p_1, p_3, p_5, p_{16}, p_{18}, p_{20}\}$. Not ensuring that the timed unions associated with these last elements are computed only once will result in both multiple data access to our relational tables, as well as a considerable increase in run time as union operations are run twice. The detection and minimisation of such kind of shared sub-queries cannot be merely computed through a simple caching mechanism, thus requiring a more sophisticated algorithm for determining the maximal common subset shared among all of the possible sets of atoms (and potentially activity labels). We discuss the solution to this problem in Supplement V.1. Figure 5 showcases the result of the application of such an algorithm while generating unique xtLTL$_f$ *expressions. Such an algorithm also guarantees the non-repetition of single-leaf operators appearing in different clauses.*

5.1.3. Enabling Intraquery Parallelism

The query scheduler (Algorithm 3) takes as an input the query compiled in the previous phase and returns the scheduling order for achieving *intraquery parallelism* [37]. The previously generated expression might not be considered as an abstract syntax tree, rather than an abstract syntax DIRECT ACYCLIC GRAPH (DAG) rooted in the entry-point operator *queryRoot*, as we guarantee that sub-expressions appearing multiple times are replaced by unique instances of these. Therefore, we can freely represent the query plan as a DAG $\mathcal{G}$ in our pseudocode notation, where each root operator in ψ is a single node while edges connect parent operators to the siblings' (ψ.args) root operator. Graph edges induce the execution order, where any ancestor node needs to be run after all of its immediate siblings. A reversed topological sort (Line 3) induces the order in which the operations should be run. To know which of these operators can be run contemporarily (i.e., scheduled together [40]) as they share no interdependencies, we compute for each node its maximum distance from *queryRoot* (Line 6). This generates a *layer*ing [41] guaranteeing that all of the nodes at the same levels share no mutual dependencies (Line 10). This enables the level-wise parallelisation of the tasks' execution (also referred to as Intraquery Parallelism [37]), thus showing how such a problem can be reduced into an embarrassingly parallel problem by parallelising the computation of each operator in the same given layer. This procedure runs in linear time with respect to the number of operators appearing in the xtLTL$_f$ query plan (Lemma S4). We benchmark query plan parallelization with different task scheduling policies in §7.3.

Example 16. *The DAG in Figure 3 depicts a query plan, where operators' dependencies are suggested as arrows starting from the ancestors. The graph is also already represented as a layered graph, as all of the nodes having the same maximum distance from the query root are aligned horizontally. We might observe that none of the nodes within each layer shares dependencies.*

5.2. Execution Engine

The execution engine (Algorithm 4) runs the previously compiled query (§5.1) on top of the relational model populated from the XES log (§4.2). The computation will start from the DAG

Algorithm 4 Execution Engine (§5.2)

```
function ExecutionEngine($layer, \mathcal{L}, \mathcal{A}$)
    for all $\psi \in atomQ$ (parallel) do $\psi$.result $\leftarrow \mathcal{A}(\psi)$
    Run$\mathcal{D}_\varphi$-EncodingAtoms($\mathcal{L}$)                          ▷ Algorithm S7
    for all $\langle distance, \Psi\rangle \in layer$ do
        for all $\psi \in \Psi$ (parallel) do
            if $\psi.atom = \{p_i\}$ and $p_i \in \bigcup_{a\in\Sigma} ak(a)$ then
                $\psi$.result $\leftarrow \mathcal{A}(\text{Atom}^{\mathcal{L},\tau}_{\cdot})(\psi)$          ▷ Algorithm S7
            else if $\psi$.atom $= \{a\} \wedge a \in \Sigma$ then
                continue                                            ▷ Already run in Line [TODO]
            else
                $\psi$.result $\leftarrow (\mathcal{A}(\psi))(\{\psi'.\text{result} | \psi' \in \psi.\text{args}\})$
            end if
        end for
    end for
    $queryRoot \leftarrow layer[0]$
    return $queryRoot$.result
```

query leaves directly accessing the relational database (§5.2.1) for then propagating the results until the root of the DAG is reached (§5.2.2). At this point, we can perform the final conjunctive or aggregation queries (§5.2.3).

*At each stage, we exploit a functor $\mathcal{A}$ associating to each **xt** LTL$_f$ operator an algorithm which will take the result from the ψ's operands as an input while returning the expected output by formal definition in an intermediate result ρ. This abstraction enables the separation between **xt** LTL$_f$ syntax and multiple possible algorithmic implementations. Some algorithmic implementations for such operators are discussed in §6.*

For this step, we will not discuss the computational complexity of evaluating the query plan as this is heavily dominated by the computation of every single operator, the model of choice, and the log size. For this reason, we only conducted empirical analysis by benchmarking the run time of the whole execution engine, where models either only contain Activity$^{\mathcal{L},\tau}_{A/T}$ (§7.2) operators or mainly Atom$^{\mathcal{L},\tau}_{A/T}$ ones (§7.5).

5.2.1. Basic Operators' execution

Among all of the possible DAG node leaves, we first (Line 2) execute the leaves either *(i)* directly associated with an activity label, or *(ii)* First and Last. For the former *(i)*, each activity label a is run through its correspondent Activity$^{\mathcal{L},\tau}_{A/T}$(a) operator, whether either A or T or none are going to be set dependingly to the fact that such atom refers to an activation (ψ.`isActivation`) or target (ψ.`isTarget`) condition, or whether the associated result should be ignored as a whole **(**ψ.`isNeither`**)**. For the latter *(ii)*, we directly access the data tables and retrieve the data from those. As the tables are already sorted by trace and event id, no further post-processing besides the insertion of activation or target label in the nested component L of the intermediate representation is required.

Next, we evaluate the intermediate result associated with each atom generated by the $\mathcal{D}_\varphi$-encoding (Line 3). Intuitively[11], this requires three subsequent phases. *First*, we get the compound conditions grouped by key and activity label as collected at query compile time, and we exploit those to pipeline multiple range queries over each AttributeTable$^k_{\mathcal{L}}$. The associated results are cached. *Second*, we compute the results for each atom by intersecting the previously cached results before actually computing the actual Atom$^{\mathcal{L},\tau}_{A/T}$. This also guarantees that shared intersections are run at most once across all of the previously cached results. *Third*, we exploit the former result to compute the Atom$^{\mathcal{L},\tau}_{A/T}$ operator at the leaf level on our DAG, while associating either an activation or a target mark in L depending on the prior definition of our leaf-level operator.

[11] Please refer to Supplement V.2 for a more in-depth discussion with pseudocode.

5.2.2. Results Propagation

After running the basic operators and their derived counterparts (e.g., $\mathsf{Atom}^{\mathcal{L},\tau}_{A/T}$), the only $\mathtt{xtLTL_f}$ operators that KnoBAB runs are the ones not accessing the relational tables. KnoBAB implements three different $\mathcal{A}$-s which are only sharing the implementation for the aforementioned operators: one set is either strictly abiding by the formal definition and completely ignoring the fact that the intermediate results are provided as an ordered set of tuples or providing slower algorithms overall, one will leverage appropriate data representation, thus outperforming the former operations, while the other will implement hybrid algorithms for selecting the best performant implementation depending on the data conditions through hybrid algorithms. An in-depth discussion of how different operators might have different algorithmic implementations is postponed to a specific section (§6).

While computing these, we associate a temporary primary-memory cache[12] to each intermediate representation being computed (ψ.`result`).

5.2.3. Conjunctive and Aggregation Queries

The first version of KnoBAB supports the CONJUNCTIVE QUERY of the model as well as three aggregation queries: MAX-SAT, CONFIDENCE, and SUPPORT. While the former requires a further untimed $\mathsf{And}_{\mathbf{True}}$ among all the intermediate results associated with the computation to each clause, the aggregation requires just an iteration over the provided results. The conjunctive query is formulated as follows:

$$\textsc{ConjunctiveQuery}(\rho_1,\dots,\rho_n) = \mathsf{And}_{\mathbf{True}}(\rho_1,\dots\mathsf{And}_{\mathbf{True}}(\rho_{n-1},\rho_n))$$

The Max-SAT will calculate the ratio of the intermediate results ρ_l associated with each clause c_l, over the total number of model clauses $|\mathcal{M}|$. $\mathsf{ActLeaves}(\rho_l)$ is the untimed union of the intermediate results yielded by activation conditions for the declare clause $c_l \in \mathcal{M}$. For c_l, the CONFIDENCE represents the ratio between the number of traces returned by ρ_l and the total number of traces that contain activation conditions. When the same numerator is on the other hand divided by the total log traces, we have SUPPORT. Following the computation of each ρ_l per clause c_l, the aggregation functions can be expressed as follows:

$$\text{Max-SAT}(\rho_1,\dots,\rho_n) = \left(\frac{|\{\, l \mid \exists j, L.\, \langle i,j,L\rangle \in \rho_l \,\}|}{|\mathcal{M}|}\right)_{\sigma^i\in\mathcal{L}}$$

$$\textsc{Confidence}(\rho_1,\dots,\rho_n) = \left(\frac{|\{\, i \mid \exists j, L.\, \langle i,j,L\rangle \in \rho_l \,\}|}{|\mathsf{ActLeaves}(\rho_l)|}\right)_{c_l\in\mathcal{M}}$$

$$\textsc{Support}(\rho_1,\dots,\rho_n) = \left(\frac{|\{\, i \mid \exists j, L.\, \langle i,j,L\rangle \in \rho_l \,\}|}{|\mathcal{L}|}\right)_{c_l\in\mathcal{M}}$$

The execution of such queries is performed in a non-parallel way, as each aggregation query will appear at the top of the query plan, and this will be associated with the latest execution run of the scheduler (Line 15). We then return and prompt the result associated with the root node of our query plan (Line 16).

Example 17. *Figure 2a establishes our high-level intrusion scenario, where our model defines a successful attack on a system. For this example, we provide the calculations for Max-SAT, Conjunctive Query, Support and Confidence. By exploiting the previous formulæ, we can compute the metrics as Table 6. These metrics may provide some insight of correlations between events. For example, clause Ⓑ had Support(Confidence) values as 1.0, while clause Ⓒ had* 1/3 *(1.0). This therefore indicates that the activation of the latter occurred much less than that of the former, however every time the activation occurred, the clause was always fulfilled. Conclusions such as these can*

[12] We can completely free each intermediate cache if we are not computing a CONFIDENCE query and if the furthest ancestor has already accessed it, or if the cache is unassociated with any activation required by CONFIDENCE.

Algorithm 5 xtLTL$_f$ pseudocode implementation for Future and Globally

```
1: function FUTURE($\rho$)                                              ▷ $O(|\mathcal{L}|\epsilon^2)$
2:     for all $\langle i,j,L\rangle \in \rho$ do yield $\langle i,j,\bigcup\{\, L' \mid \langle i,j',L'\rangle \in \rho$ and $j' \geq j\,\}\rangle$
3:     end for
4: function GLOBALLY($\rho$)
5:     for all $\langle i,j,L\rangle \in \rho$ do
6:         $E \leftarrow \{\, j' \mid \langle i,j',L'\rangle \in \rho$ and $j' \geq j\,\}$
7:         if $|E| = \ell_t - j$ then yield $\langle i,j,\bigcup\{\, L' \mid \langle i,j',L'\rangle \in \rho$ and $j' \in E\,\}\rangle$ end if
8:     end for
```

help to identify any weaknesses/strengths within the model and the system itself (here, the metrics obtained from Ⓒ may suggest that comm/act contain a correlation that needs investigating).

6. Algorithmic Implementations

In this section, we show how the relational model and the proposed intermediate result representation enable the definition of different operators boosting the query performance compared to an equivalent **xt** LTL$_f$ *expression obtained through the straightforward translation procedure entailed by the lemmas in Appendix A.1* (LTL$_f$*-rewriting). Each subsection is going to discuss different possible algorithms for implementing some operators, as well as discussing its associated pseudocode and computational complexity.*

6.1. Timed and Untimed Or/And

Algorithm 6 shows the implementation of the timed version of the $\mathsf{And}^{\tau}_{\Theta}$ (Line 27) and $\mathsf{Or}^{\tau}_{\Theta}$ (Line 28) operators, for then generalising this concept for the implementation of the untimed And_{Θ}. We omit the discussion related to the implementation of the untimed Or_{Θ} operator for the sake of conciseness.

As we see from their formal definition, any binary xtLTL$_f$ operator supports Θ conditions. And (and Or) resembles a sorted set intersection (or union, Line 11), where we use both trace (i) and event (j) id information from the intermediate result triplet as preliminary equality condition for the match. We also use a Θ binary predicate to be tested over the activated and targeted events in the third component (L). The event shared among the operands is returned if either Θ is always true (Line 7) or, from this point in time, if there exists one activated future activated event (in a L coming from the left operand) as well as a targeted one (in a L coming from the right operand) satisfying the correlation (Line 4). The match is then represented as a marked correlation condition $M(h,k)$, which is then collected in the L associated with the returned event (Line 5).

For the untimed And_{Θ} operator, we require to return one single trace i as $\langle i,1,L\rangle$ if either Θ is true and each operator has an event from σ^i, or if there exists at least one event per operand from the same trace performing the match. This can be implemented in two different ways: we can either group the records by trace id (Line 31 and 32) and then scan the intermediate result's records (Line 38) associated to the same trace id (Line 36, SLOWUNTIMEDAND) or straightforwardly scan those by trace id without exploiting the preliminary aggregation (FASTUNTIMEDAND). This latter implementation is possible as the

Table 6. Conjunctive and Aggregation queries for Figure 2a.

(a) Metric calculations per trace.

Trace	MAX-SAT	in Conjunctive Query
σ_1	$\frac{\lvert\{\, c_1,c_2,c_3\,\}\rvert}{\lvert M\rvert} = 1.0$	**true**
σ_2	$\frac{\lvert\{\, c_2\,\}\rvert}{\lvert M\rvert} = 1/3$	**false**
σ_3	$\frac{\lvert\{\, c_1,c_2\,\}\rvert}{\lvert M\rvert} = 2/3$	**false**

(b) Metric calculations per clause.

Clause	Support	Confidence
Ⓐ	$\frac{\lvert\{\, \sigma_1,\sigma_3\,\}\rvert}{\lvert\mathcal{L}\rvert} = 2/3$	$\frac{\lvert\{\, \sigma_1,\sigma_3\,\}\rvert}{\lvert\{\, \sigma_1,\sigma_2,\sigma_3\,\}\rvert} = 2/3$
Ⓑ	$\frac{\lvert\{\, \sigma_1,\sigma_2,\sigma_3\,\}\rvert}{\lvert\mathcal{L}\rvert} = 1.0$	$\frac{\lvert\{\, \sigma_1,\sigma_2,\sigma_3\,\}\rvert}{\lvert\{\, \sigma_1,\sigma_2,\sigma_3\,\}\rvert} = 1.0$
Ⓒ	$\frac{\lvert\{\, \sigma_1\,\}\rvert}{\lvert\mathcal{L}\rvert} = 1/3$	$\frac{\lvert\{\, \sigma_1\,\}\rvert}{\lvert\{\, \sigma_1\,\}\rvert} = 1.0$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 13 January 2023 doi:10.20944/preprints202301.0254.v1

Algorithm 6 $\texttt{xt}\mathrm{LTL}_f$ pseudocode implementation for And_Θ and Or_Θ operators

```
function $\mathcal{T}_\Theta^{E,i}(L, L')$
    $L'' \leftarrow \emptyset$; $hasMatch \leftarrow \Theta = $ True        ▷ (Explicitly) computing $\mathcal{T}_\Theta^{E,i}$
    if $\Theta \neq$ True and $L \neq \emptyset$ and $L' \neq \emptyset$ then
        for all $A(m) \in L$ and $T(n) \in L'$ s.i. $\Theta(m, n)$ do
            $L'' \leftarrow L'' \cup \{ M(m, n) \}$; $hasMatch \leftarrow$ true
        end for
    else
        $L'' \leftarrow L'' \cup L' \cup L$
    end if
    if $hasMatch$ then return $L''$ else return False

function TimedIntersection$_\Theta(\rho, \rho', isUnion)$
    $it \leftarrow$Iterator$(\rho)$, $it' \leftarrow$Iterator$(\rho')$
    while $it \neq \uparrow$ and $it' \neq \uparrow$ do
        $\langle i, j, L\rangle \leftarrow \texttt{current}(it)$, $\langle i', j', L'\rangle \leftarrow \texttt{current}(it')$
        if $i = i'$ and $j = j'$ then
            $tmp \leftarrow \mathcal{T}_\Theta^{E,i}(L, L')$
            if $tmp \neq$ False then yield $\langle i, j, tmp\rangle$
            $\texttt{next}(it)$; $\texttt{next}(it')$;
        else if $i < i'$ or ($i = i'$ and $j < j'$) then
            if $isUnion$ then yield $\langle i, j, L\rangle$ end if
            $\texttt{next}(it)$
        else
            if $isUnion$ then yield $\langle i', j', L'\rangle$ end if
            $\texttt{next}(it')$
        end if
    end while

function And$_\Theta^\tau(\rho, \rho')$ TimedIntersection$_\Theta(\rho, \rho', $false$)$

function Or$_\Theta^\tau(\rho, \rho')$ TimedIntersection$_\Theta(\rho, \rho', $true$)$

function SlowUntimedAnd$_\Theta(\rho, \rho')$
    $leftOperand \leftarrow \{\}$; $rightOperand \leftarrow \{\}$
    for all $\langle i, j, L\rangle \in \rho$ do $rightOperand[i].\texttt{put}(\langle i, j, L\rangle)$
    for all $\langle i, j, L\rangle \in \rho'$ do $rightOperand[i].\texttt{put}(\langle i, j, L\rangle)$
    $it \leftarrow$Iterator$(leftOperand)$, $it' \leftarrow$Iterator$(rightOperand)$
    while $it \neq \uparrow$ and $it' \neq \uparrow$ do
        $\langle i, R\rangle \leftarrow \texttt{current}(it)$; $\langle i', R'\rangle \leftarrow \texttt{current}(it')$
        if $i = i'$ then
            $L'' \leftarrow \emptyset$; $hasMatch \leftarrow \Theta = $ True
            for all $\langle i, j, L\rangle \in R$ and $\langle i, j', L'\rangle \in R'$ do
                $tmp \leftarrow \mathcal{T}_\Theta^{E,i}(L, L')$
                if $tmp \neq$ False then
                    $hasMatch \leftarrow$ true; $L'' \leftarrow L'' \cup tmp$
                end if
            end for
            if $hasMatch$ then yield $\langle i, 1, L''\rangle$;
        else if $i < i'$ then $\texttt{next}(it)$
        else $\texttt{next}(it')$
        end if
    end while

function FastUntimedAnd$_\Theta(\rho, \rho')$
    $it \leftarrow$Iterator$(\rho)$, $it' \leftarrow$Iterator$(\rho')$
    while $it \neq \uparrow$ and $it' \neq \uparrow$ do
        $\langle i, \iota, \lambda\rangle \leftarrow \texttt{current}(it)$; $\langle i', \iota', \lambda'\rangle \leftarrow \texttt{current}(it')$
        if $i = i'$ then
            $L'' \leftarrow \emptyset$; $canOptimize \leftarrow$ false
            $it_* \leftarrow it$
            while $it_* \neq \uparrow$ do
                $\langle i, j, L\rangle \leftarrow \texttt{current}(it_*)$; $it'_* \leftarrow it'$
                if not $canOptimize$ then
                    while $it'_* \neq \uparrow$ do
                        $\langle i', j', L'\rangle \leftarrow \texttt{current}(it'_*)$
                        $tmp \leftarrow \mathcal{T}_\Theta^{E,i}(L, L')$
                        if $tmp \neq$ False then
                            $hasMatch \leftarrow$ true; $L'' \leftarrow L'' \cup tmp$
                        end if
                        $\texttt{next}(it'_*)$
                    end while
                    if $\Theta = $ True then $canOptimize \leftarrow$ true
                else$L'' \leftarrow L'' \cup L$
                end if
                $\texttt{next}(it_*)$
            end while
            if $hasMatch$ then yield $\langle i, 1, L''\rangle$;
            $it \leftarrow it_*$; $it' \leftarrow it'_*$;
        else if $i < i'$ then $\texttt{next}(it)$
        else $\texttt{next}(it')$
        end if
    end while
```

intermediate results records are already sorted, thus allowing the results' aggregation while scanning the intermediate results without the need for any preliminary aggregation. We show that the faster version is always faster than computing it with its slower counterpart in Corollary S1.

Similar considerations can be also applied for the untimed Or operation, for which we implemented equivalent SLOWUNTIMEDOR and FASTUNTIMEDOR, as we only need to pay an additional linear scan for the unmatched traces.

6.2. Choice and Untimed Or

We prelude our analysis of derived operators by firstly discussing the difference in computational complexity between providing the straightforward translation from LTL_f to $\texttt{xt}\mathrm{LTL}_f$ and to exploiting equivalent expression rewriting in $\texttt{xt}\mathrm{LTL}_f$. We remind the reader that the definition of Choice (see Table 1) states that either one condition or another should occur anytime in the trace.

This requirement can be interpreted in two distinct ways: by either returning all the traces satisfying the first condition or the second separately and then merging those, or rather collecting all of the events satisfying either the former or the latter condition while jointly scanning both operands, for then returning the traces where any one of those is met. After observing[13] that the SLOWUNTIMEDOR is actually slower than FASTUNTIMEDOR and that the latter actually implements the Choice declarative clause (Corollary A1), the time complexity of computing the LTL_f rewriting of Choice in its LTL_f-rewriting is almost equivalent to the time complexity of FASTUNTIMEDOR, as we can have an asymptotic constant speed-up in the best case scenario (Corollary S3). As the untimed Or_Θ behaves by computing a Future operator (Algorithm 5) on each of its operands, the computation of an additional Future operator for each of its operands becomes an omittable overhand.

6.3. Untimed Until(s)

We show how different data access policies for scanning the intermediate results affect the overall computational complexity as well as their associated run time. Algorithm 7 provides two possible variants for the untimed until: all optimizations happen when the activation condition coming from the second operand do not occur at the beginning of a trace (Lines 34 and 61). In the first variant, we calculate, for all of the events in the first operand starting from the beginning of the trace (Line 29, and Line 51 for the second variant), the position of the last activated event preceding the current target condition with a logarithmic scan with respect to the length of the first operand (Line 34). On the other hand, the second variant directly discards the traces not starting with a target condition (Line 59) and, otherwise, it moves the scan of the first operand – from that initial position – by an offset equal to the distance from the event preceding activation (Line 61): if that position does not correspond to an activation condition preceding the current activation condition, then we completely discard the trace (Line 65). The matching conditions between activations and target are implemented similarly (Line 37-40 and 67-69). Lemma S7 shows that the second variant is better asymptotically only for bigger datasets.

6.4. Derived Operators

Our previous observation for the untimed Or_Θ led us to the definition of additional derived operators with the hope of easing the overall computational complexity. We walked in the same footsteps of relational algebra, where it was customary to merge multiple operators into one single new operator if the latter might be implemented through a more performant algorithm than computing an equivalent expression being the straightforward translation of LTL_f formulae into LTL_f (LTL_f *rewriting*).

For example, we can implement TIMEDANDFUTURE by extending the fast implementation of the timed AND operator, and considering all of the trace events from the

[13] Please also refer to the experiments in §7.1 for the empirical evidence of such theoretical claims.

Algorithm 7 Two implementations for the untimed $\texttt{xt}\mathrm{LTL_f}$ Until_Θ.

```
function $A^i_\Theta(\langle it', bEnd\rangle, \langle it, aEnd\rangle)$
    $\langle i', j', L'\rangle \leftarrow \texttt{current}(it')$; $L'' \leftarrow \emptyset$;
    if $\Theta \neq$ True and $L' \neq \emptyset$ then
        for all $A(k), M(k,k') \in L'$ do
            $aBeg \leftarrow it$
            while $aBeg \neq aEnd$ do
                $\langle i, j, L\rangle \leftarrow \texttt{current}(aBeg)$
                if $L = \emptyset$ then $L'' \leftarrow L'' \cup L$
                else
                    $anyMatch \leftarrow$ false
                    for all $T(h) \in L$ s.t. $\Theta(\sigma^i_k, \sigma^i_h)$ do $anyMatch \leftarrow$ true; $L'' \leftarrow L'' \cup \{M(k,h)\}$
                    end for
                    if not $anyMatch$ then return False
                end if
            end while
        end for
    else
        while $aBeg \neq aEnd$ do
            $\langle i, j, L\rangle \leftarrow \texttt{current}(aBeg{++})$; $L'' \leftarrow L'' \cup L$
        end while
        $L' \leftarrow L'' \cup L'$
    end if
    return $L''$

function UntimedUntil$^1_\Theta(\rho, \rho')$
    $it \leftarrow$ Iterator$(\rho)$, $it' \leftarrow$ Iterator$(\rho')$
    while $it' \neq \uparrow$ do
        $\langle i', j', L'\rangle \leftarrow \texttt{current}(it')$; $bend \leftarrow$ UpperBound$(\rho', it', \uparrow, \langle i', |\sigma^{i'}|+1, \emptyset\rangle)$
        $it \leftarrow$ LowerBound$(\rho, it, \uparrow, \langle i', 1, \emptyset\rangle)$
        $atLeastOneResult \leftarrow$ false; $L'' \leftarrow \emptyset$
        while $it' < bend$ do
            if $j' = 1$ then
                $atLeastOneResult \leftarrow$ true; $L'' \leftarrow L'' \cup L$; $it'{++}$
            else
                $aEnd \leftarrow$ UpperBound$(\rho, it, \uparrow, \langle i', j'-1, \top_\Omega\rangle)$
                if $it = aEnd$ or Distance$(aEnd-1, it)+1 \neq j'-1$ then break
                else                                                    ▷ $i = i'$. Computing partial $\mathcal{T}^{A,i}_\Theta$
                    $tmp \leftarrow A^i_\Theta(\langle it', bend\rangle, \langle it, aEnd\rangle)$
                    $atLeastOneResult \leftarrow atLeastOneResult$ or $tmp \neq$ False
                    if $tmp \neq$ False then $L'' \leftarrow L'' \cup tmp$;
                    $it'{++}$
                end if
            end if
        end while
        if $atLeastOneResult$ then yield $\langle i, 1, L''\rangle$
        $it' \leftarrow bend$
    end while

function UntimedUntil$^2_\Theta(\rho, \rho')$
    $it \leftarrow$ Iterator$(\rho)$, $it' \leftarrow$ Iterator$(\rho')$
    while $it' \neq \uparrow$ do
        $\langle i', j', L'\rangle \leftarrow \texttt{current}(it')$; $bend \leftarrow$ UpperBound$(\rho', it', \uparrow, \langle i', |\sigma^{i'}|+1, \emptyset\rangle)$
        $it \leftarrow$ LowerBound$(\rho, it, \uparrow, \langle i', 1, \emptyset\rangle)$
        $atLeastOneResult \leftarrow$ false; $L'' \leftarrow \emptyset$
        while $it' < bend$ do
            if $j' = 1$ then
                $atLeastOneResult \leftarrow$ true; $L'' \leftarrow L'' \cup L$; $it'{++}$
            else if $it = \uparrow$ then break
            else
                $\langle i, j, L\rangle \leftarrow \texttt{current}(it)$;
                if $j > 1$ then break
                else
                    $aEnd \leftarrow$ MoveForward$(it, j'-1)$;                ▷ $(it)+j'-1$
                    if $aEnd = \uparrow$ then break
                    else
                        $\langle i_e, j_e, L_e\rangle \leftarrow \texttt{current}(aEnd)$
                        if $i_e > i'$ or $j_e \neq j'-1$ then break
                        else                                            ▷ $i = i' = i_e$. Computing partial $\mathcal{T}^{A,i}_\Theta$
                            $tmp \leftarrow A^i_\Theta(\langle it', bend\rangle, \langle it, aEnd\rangle)$
                            $atLeastOneResult \leftarrow atLeastOneResult$ or $tmp \neq$ False
                            if $tmp \neq$ False then $L'' \leftarrow L'' \cup tmp$;
                            $it'{++}$
                        end if
                    end if
                end if
            end if
        end while
        if $atLeastOneResult$ then yield $\langle i, 1, L''\rangle$
        $it' \leftarrow bend$
    end while
```

Table 7. Range of datasets used for benchmarking.

Competitor	Dataset	Traces $\lvert\mathcal{L}\rvert$	Events	Distinct Activities $\lvert\Sigma\rvert$
SQL Miner	BPIC 2011 (original)	1143	150 291	624
	BPIC 2011 (10)	10	2613	158
	BPIC 2011 (100)	100	12 195	276
	BPIC 2011 (1000)	1000	133 935	607
Declare Analyzer	BPIC 2012 (original)	13087	262 200	24

second operand succeeding the events from the first operand within the same trace. Similar considerations can be carried out with TIMEDANDGLOBALLY, where in the former we need to count whether all of the events from the current time until the end of the trace are present in the rightmost operand, while in the latter we also need to skip the matched event from the rightmost operand and start scanning from the following ones.

For simplicity's sake, we postpone the discussion of these operands' pseudocode as well as the discussion of their computational complexity in Supplement IV.2, where we show that these two operators might come with two different algorithms, for which there always exists one of them having a lower running time with respect to the equivalent xtLTL$_f$ expression containing no derived operators. We can show formally that while the first implementation (*variant*) works better for smaller datasets, the second works better for reasonably long traces when the number of the traces is upper bounded by an exponential number of events (Corollary S4).

7. Results and Discussions

Our benchmarks exploited a Razer Blade Pro on Ubuntu 20.04: Intel Core i7-10875H CPU @ 2.30GHz - 5.10 GHz, 16GB DDR4 2933MHz RAM, 450GB free disk space. All of our datasets used for benchmarking (synthetic data generation (§7.1), BPIC_2011 (§7.2, §7.3), BPIC_2012 (§7.4, §7.5) and our proposed cancer example (§1.1) are publicly available[14].

7.1. Comparing different operators' algorithms

We advocate that the choice of representing the intermediate representation as an ordered record set allows the exploitation of efficient algorithms through which we might avoid costly counting and aggregation operations [42]. From these comparisons, the operators fully assuming that the data is sorted greatly outperform naïve operators. Walking in the footsteps of relational algebra, we show that the computational complexity of so-called derived operators outperforms the computation of an equivalent expression evaluated through either naïve or fast algorithms. The experiments are discussed in order of presentation of the algorithms in the previous section.

To create a suitable testing environment, we synthetically generate data-less logs, where the trace and log lengths are increased 10-fold at a time from $10^1 - 10^4$, with the resulting sets $|\mathcal{L}| \in \{ 10, 100, 1000, 10000 \}$ $\epsilon \in \{ 10, 100, 1000, 10000 \}$, with the most extreme log consisting of 10^8 events. In some cases we exceeded 16GB of primary memory on the testing machine; in the following results (Figure 6-9), **M+** denotes an out of memory exception. We chose to generate our data in place of using existing real-world logs [14], as the controlled scenario allows for identifying the location and extent of any possible speed-ups. This data was up-sampled, guaranteeing that a given log configuration was always a subset of the larger. The data generation randomly assigned events from the universal alphabet ($\Sigma = \{ A, B, C, D, E \}$), up to the maximum length for the set in consideration, and we stored the resulting logs as tab-separated files.

Our operators consider correlations between timed events A and B, where the computed speed-up is *per* operator. Given this, we denote $\rho_1 = \mathsf{Activity}_A^{\mathcal{L},\tau}(A), \rho_2 = \mathsf{Activity}_T^{\mathcal{L},\tau}(B)$,

[14] https://dx.doi.org/10.17605/OSF.IO/2CXR7

Table 8. Proposed operator semantics vs traditional.

Operator	LTL$_f$ rewriting	Optimized
Choice	$\mathsf{Or}_\Theta(\mathsf{Future}(\rho_1), \mathsf{Future}(\rho_2))$	$\mathsf{Or}_\Theta(\rho_1, \rho_2)$
TIMEDANDFUTURE	$\mathsf{And}_\Theta(\rho_1, \mathsf{Future}^\tau(\rho_2))$	$\mathsf{AndFuture}^\tau_\Theta(\rho_1, \rho_2)$
TIMEDANDGLOBALLY	$\mathsf{And}_\Theta(\rho_1, \mathsf{Globally}^\tau(\rho_2))$	$\mathsf{AndGlobally}^\tau_\Theta(\rho_1, \rho_2)$

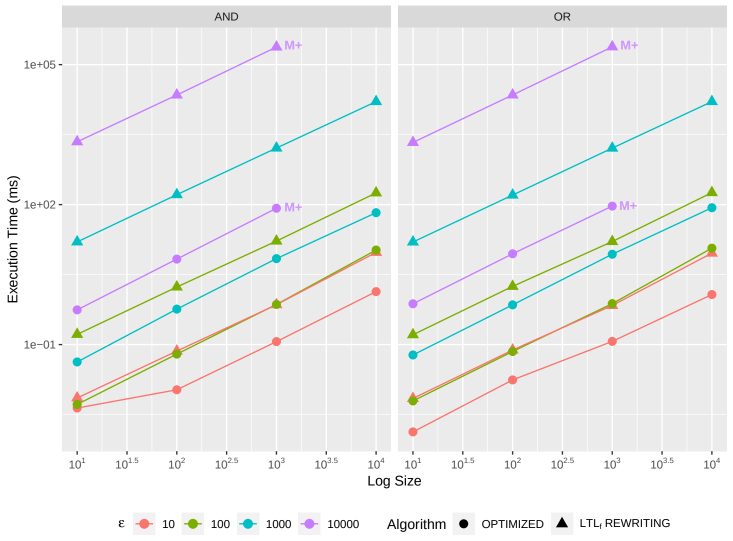


Figure 6. Results for the fast set operations §6.1 against the traditional logical implementation.

prior to benchmarking, and we ignore the time required for accessing the data on the knowledge base, as the focus of the present benchmarks is solely on the operators. Details of how the custom clauses/derived operators are run are demonstrated in Table 8, while singular operators are run sequentially.

Untimed Or/And. The first group of experiments aim to challenge different possible algorithms for the same `xtLTL`$_f$ operators, $\mathsf{And}_{\mathbf{True}}$ and $\mathsf{Or}_{\mathbf{True}}$ as discussed in §6.1. The outcome of such experiments is given in Figure 6: our experiments reveal that, in every case, the FAST- operators are *always* more performant than their logical counterparts. Our benchmark confirm the cost of overhead encumbered by the SLOW- implementation, which conforms **linearly** to increased log size, almost **polynomially** with trace length. This aggregation is upper bounded with a **quadratic** with respect to trace length ϵ (Line 31 and 32); in the most extreme case ($\epsilon = 10^4$), the cost is over one order of magnitude versus the algorithm without aggregation. From now on, we always exploit our FAST- operators in place of the SLOW- equivalent for representing non-derived `xtLTL`$_f$ operators, which usually suffer the cost caused by the preliminary aggregation as per previous experiments.

Choice and Untimed Or. The next set of experiments is to evaluate the customary declarative clause implementation, where we hypothesise reformulating the semantics associated with Choice to provide performance gains from the absence of preliminary aggregations via the UntimedFuture operator. In fact, the proposed optimization derives from the omittance of the Future operators for ρ_1, ρ_2, which formally comply with the logical definition. For the untimed Future §3.2.2 operator, bounded scans can be exploited, as the data is sorted with respect to trace id, and all the events that satisfy ρ for the current trace id are included in the result. Therefore, we expect an overhead that grows linearly with log size. Figure 7 shows that, in the best case ($\epsilon = 10$), we gain 0.5 orders of magnitude in performance. The findings affirm that log size has a greater influence on computational overhead than trace length. For $\epsilon \geq 10^3$, the overhead resulting from the Future operators

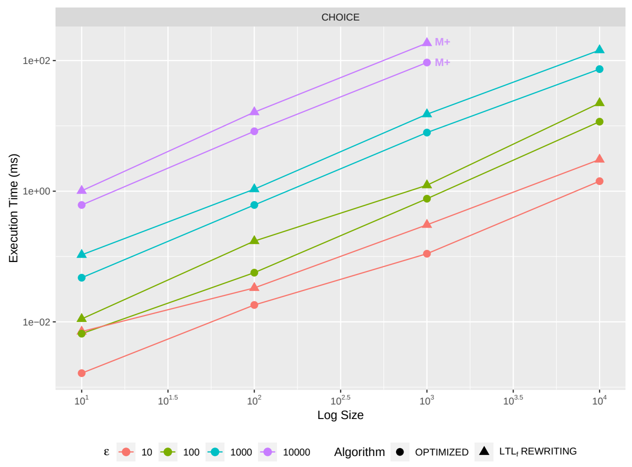


Figure 7. Results for the custom declarative clause implementations §6.2 against the traditional logical implementation.

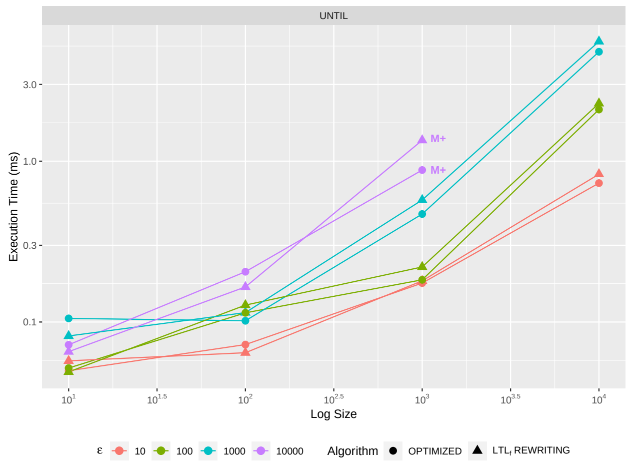


Figure 8. Results for the **Until** operator (§ 6.3). We provide the execution times for the logical and our fast implementations, as

steadily increases while both the trace length ϵ and the log size $|\mathcal{L}|$ grows, albeit this is negligible in the logarithmic scale.

Untimed Until(s). Benchmarks from Figure 8 show that the first variant is almost always more performant than the second one for considerably short traces, while the latter becomes more efficient when ϵ increases. With significant increases to log size, the latter becomes more performant; when $|\mathcal{L}| = 10^4$, all cases show improved running times, regardless of ϵ. The plots also show that the operator's running time is polynomial with respect to the number of traces in the log, as a consequence of the increased scans within every single trace.

Derived Operators. The final set of experiments is to test whether the newly proposed derived operators achieve more optimized results than those from their LTL$_f$ rewriting counterpart (Table 8). For example, TimedAndGlobally can be optimized with the customary algorithms replacing one single operator with the execution of multiple pipelined operators. Computations from LTL$_f$ rewriting demonstrate worse performance than the derived counterparts across all operators; in the most extreme case TimedAndGlobally there is over $10^{1.5}$ speed-up for $\epsilon = 10^4$. We were able to conclude that different impersonations to the internal data storage of the optimized algorithm may provide better results depending on the log size. As for UntimedUntil, we provide two implementations

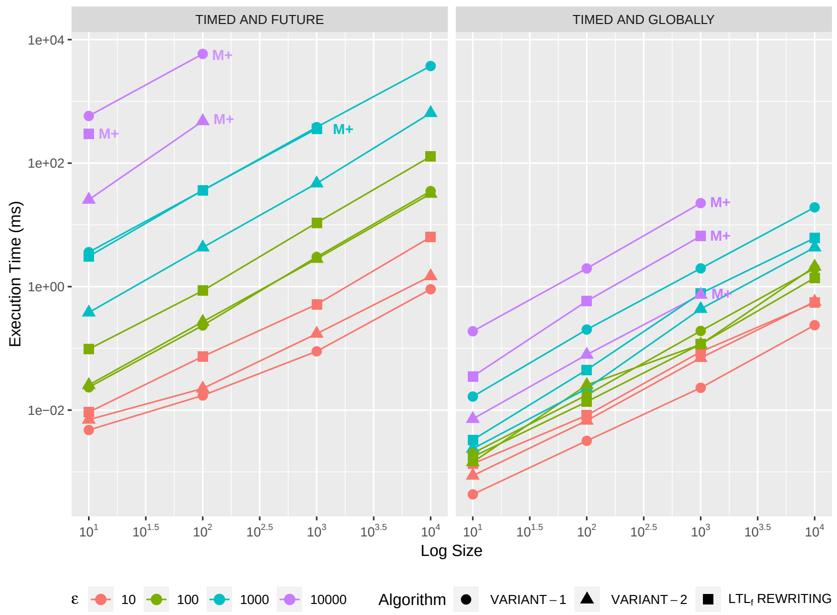


Figure 9. Results for the derived operators TIMEDANDFUTURE and TIMEDANDGLOBALLY §6.4. We include both variants of the fast implementations to analyse the environments where each thrive.

Algorithm 8 Hybrid Algorithms

```
1:  function HybridUntimedUntil$^{\eta}_{\Theta}(\rho, \rho')$
2:      if $|\mathcal{L}| \geq \eta$ then return UntimedUntil$^{2}_{\Theta}(\rho, \rho')$        ▷ Algorithm 7
3:      else return UntimedUntil$^{1}_{\Theta}(\rho, \rho')$                                  ▷ Algorithm 7
4:      end if

5:  function HybridAndFuture$^{\gamma}_{\Theta}(\rho, \rho')$
6:      if $\epsilon > \gamma$ then return AndFuture$^{2}_{\Theta}(\rho, \rho')$              ▷ Algorithm S2
7:      else return AndFuture$^{1}_{\Theta}(\rho, \rho')$                                      ▷ Algorithm S3
8:      end if

9:  function HybridAndGlobally$^{\gamma}_{\Theta}(\rho, \rho')$
10:     if $\epsilon \geq \gamma$ then return AndGlobally$^{2}_{\Theta}(\rho, \rho')$         ▷ Algorithm S2
11:     else return AndGlobally$^{1}_{\Theta}(\rho, \rho')$                                   ▷ Algorithm S3
12:     end if
```

for TIMEDANDGLOBALLY and TIMEDANDFUTURE, VARIANT-1 (Algorihtm S2) and VARIANT-2(Algorihtm S3); with the latter exploiting bounded reversed scans on the data.

TIMEDANDGLOBALLY: by merging the AND join operation with Globally, we only consider elements within the same trace **after** the first operand. The logical implementation performs these operations separately, and so cannot reap the benefits of a merged join [43]. Figure 9 shows that, in most cases, there is a linear performance gain with log size. VARIANT-2 aims to exploit potential gains from a reversed scan of a trace while VARIANT-1 provides a forwards scan for every activation. By performing a reverse scan, the latter is able to prune further events from any activations happening in the past, as the condition did not hold for the current time. For smaller trace lengths ($\epsilon \leq 10^1$), the VARIANT-1 demonstrate better performance than VARIANT-2. With increased trace length, the latter operators outperform the former, sometimes by over an order of magnitude ($\epsilon = 10^4$). In some cases the VARIANT-1 performs slower than their LTL$_f$-rewriting counterparts ($\epsilon \geq 10^3$).

TIMEDANDFUTURE: the principal optimization gains from this operator follow the same reasoning as TIMEDANDGLOBALLY, however the implementations of the variants follow a unique approach. By exploiting the allocation of intermediate data structures in reverse, VARIANT-2 also provides improved performance for larger $|\mathcal{L}|$. As with TIMEDANDGLOBALLY, VARIANT-1 outperforms the former for smaller trace lengths.

We conclude that VARIANT-1 (VARIANT-2) of **TimedAndFuture** and **TimedAndGlobally** outperform each other for small (large) trace lengths. In addition, the first variant of Until proves to be more performant than our second variant for smaller log lengths. We design a mechanism for always running the fastest algorithm under the previously-observed circumstances. We then need to calculate the average trace length and the log size at data loading time (this only needs to happen once per log). Then, at query time, the most optimal operator is chosen based on these values. We define a HYBRID TRACE QUERY THRESHOLD γ of $10^2/2$ (Line 5 and 9) and a HYBRID LOG QUERY THRESHOLD η of $10^3/2$ (Line 1); values exceeding these thresholds will execute the operators more tailored towards large trace (log) sizes. The pseudocode provided as Algorithm 8 demonstrates how two different variants can be engulfed in one single parametric algorithm.

7.2. Relational Temporal Mining

We now move from synthetic data, required to tune hybrid algorithms and thoroughly test our operators, towards real data benchmarks with no data payload conditions. We contextualise our experiments for data-intensive model mining operations that can be also run on a relational model. While doing so, we compare our runtimes both with hybrid operators with the one from the previous paper [4], as well as run times from the relational model with traditional SQL queries.

SQLMiner, provided by Schonig et al. [5], utilises database architectures for declarative process mining. We chose to test our hypothesis of engineering a custom database architecture against state-of-the-art traditional relational databases (**PostgreSQL 14.2**). For this set of experiments, we exploited the BPIC 2011 (Dutch academic hospital log) dataset [14], as used in [5]. This log contained data payload information, though the queries executed as [5] comprised of data-less events. The original dataset was sampled into sub-logs containing 10, 100 and 1000 traces, and the sampling approach adopted the same behaviour as the synthetic dataset from the previous set-up, where each sub-log is guaranteed to be a subset of the greater ones. Increased sizes of datasets exhibited exponential increases in primary memory requirements and so justifies our sampling approach. Schönig [44] provides the templated implementations for mining eight declarative clauses. As these are only templates, the models were instantiated from the resulting combinations of the five most occurring events. Therefore we generated eight models, each consisting of 25 clauses. SQLMiner simulated this by creating a secondary Actions table, with each row containing the instantiated declare template. SQLMiner provides the Support values associated to each clause. We extend this to also provide trace information, where each clause also contains the traces satisfying it. We also want to test our hypothesis that our proposed hybrid operator pipeline (§7.1) can outperform the pipeline set up from our previous work [4] that does not exploit the potential gains that can be made from picking the best algorithm according to the data conditions, and only uses our defined VARIANT-1 operators. The outcome of these experiments are shown in Figure 10, where each plot represents the execution times for a given elected template, with the more complex queries located on the first row.

SQLMiner results. In the worst case, our running time is comparable with SQLMiner (Response). Even for this case, SQLMiner returns only the Support information, while KnoBAB also returns (for the same execution time) trace information. In SQL, providing the least possible query alterations to provide the trace information causes $10^{1.5}$ run time increase, thus demonstrating that we are more performant on the same conditions. Conversely, in the best case, we outperform SQLMiner by over five orders of magnitude. By exploiting efficient database design, our custom query plan can minimise data access and our computation avoided explicit computations of aggregations. In addition, guaranteeing that the intermediate results are always sorted allows for **linear** scanning cost for counting operations. Responded Existence is a clear candidate for demonstrating the gains from custom database design: with access to our proposed CountingTable$_{\mathcal{L}}$, our solution requires only a table look-up, while SQLMiner requires an aggregation requiring an **entire** scan of the Log table. Combining this with the extended xtLTL$_f$ operators allows for much more

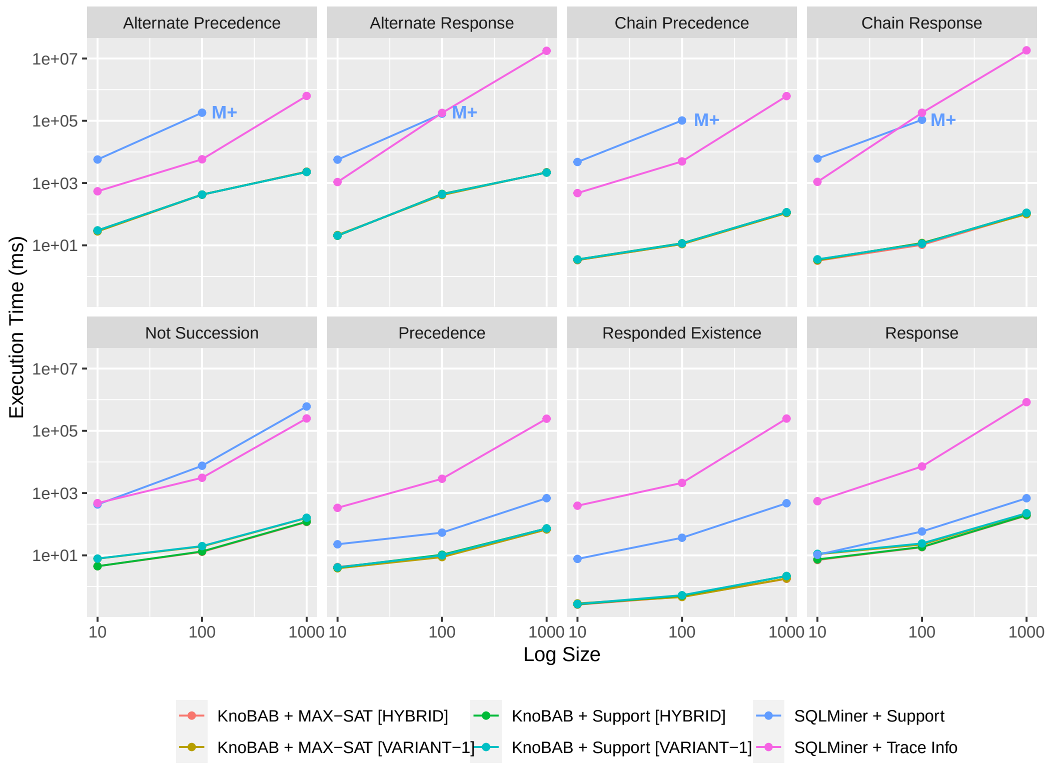


Figure 10. Results for relational temporal mining §7.2.

optimized query times; this is shown in the results, where KnoBAB is consistently at least two orders of magnitude more performant with queries returning trace information. As $|\mathcal{L}|$ increases beyond 10^2, the more complex queries were unable to finish to completion for SQLMiner, exceeding the 16GB primary memory of the benchmarking machine.

Pipeline results. The execution times for KnoBAB + Support and KnoBAB + Max-SAT are comparable, while there is much greater variation for SQLMiner + Support and SQLMiner + Trace Info. As support requires only an aggregation over intermediate results (§5.2.3), we guarantee that we suffer at most a cost proportional to the model size, so we expect a constant overhead based on model size. The large fluctuation in results for SQLMiner is a culprit of the query rewriting provided by the PostgreSQL query engine; in some cases returning trace information yielded better results. In these experiments we combined the alternate ensemble methods with our proposed HYBRID operators. The results demonstrate that, for most operators, there is a marginal improvement in time complexity. For NotSuccession and Response, the improvement is more apparent, with the former, for $|\mathcal{L}| = 10$ providing 20% improvement against VARIANT-1. The reader is encouraged to refer back to Figure 9 to explain this. The faster operators thrive with $|\mathcal{L}| > 10^3$, while for traces within the region of 10^2 the gain is much less apparent. The BPIC_2011 dataset has a corresponding average trace length of ∼220: exploiting the VARIANT-2 operators within this region will therefore yield lesser benefit than much larger $|\mathcal{L}|$.

7.3. Query Plan Parallelisation

By keeping the immediately preceding experimental setting while considering the whole log as well as extending the model size, we now benchmark our solution in a multithreaded environment, where we perform intra-query parallelism by running each operator laying in the same layer in parallel as per previous discussions.

The correctness of our proposed parallelisation approach is guaranteed by the fact that each thread in a given layer can operate independently with no interdependencies requiring costly mutual exclusions. In place of directly using the `pthread` C++ library on multiple tasks, we utilised a thread pool proposed by [45], to minimise the thread creation overhead, while feeding the pool with the tasks denoted by **for . . . (parallel) do** statements in our pseudocode Algorithm 4. We extended the library to support both static and dynamic scheduling approaches proposed by the OpenMP specifications [46]; those are:

- **BLOCKED STATIC**: aims to balance the chunk sizes per thread by distributing any leftover iterations.
- **BLOCK-CYCLIC STATIC**. Does not utilise balancing as the former. Instead, work blocks are cyclically allocated over the threads.
- **GUIDED DYNAMIC**: aims to distribute large chunks when there is lots of work still to be done; tasks are split into smaller chunks as the work load diminishes.
- **MONOTONIC DYNAMIC**: uses a single centralized counter that is incremented when a thread performs an iteration of work. The schedule issues iterations to threads in an increasing manner.

In addition to those, we also implemented two different scheduling policies splitting the tasks to be run in parallel while estimating the running time that each operator will take dependingly on the size of its associated operands (if any).

- **TASK SIZE PREDICTION BLOCK STATIC** provides an estimation of work required per chunk. Then, these chunks are sorted in ascending work load, with the last providing the greatest amount of computation. Threads are then assigned chunks through a distribution algorithm, distributing the first and last chunk of the sorted work to the first thread, the second and penultimate to the second etc.. The algorithm aims to distribute equal amounts of work to each thread, though assumes that the workload is strictly increasing while workload sizes are evenly distributed.
- **TASK SIZE PREDICTION UNBALANCED DYNAMIC**: unlike the former, we assume that the incoming work is not balanced. Instead, a chunk is taken, its work size estimated and assigned to a thread. Then, the next thread will recursively receive chunks until the summed work load is approximate to that of the former. The next thread is then pulled from the pool and the process repeated until all chunks are assigned.

For this set of experiments, we exploited the full BPIC 2011 (Dutch hospital log) dataset. We want to determine how varying the total number of threads affects execution time, and therefore use only the original dataset with no sampling. This also demonstrates the performance against the real world scenario. Similarly to the previous mining approach in §7.2, we generated models from the most occurring events labels. Here, we extended the model size to consider the top 15 events for the same eight declare templates, thus resulting in 225 clauses. Extending the model size as such allows a better scalability analysis on the large; in fact, a smaller model size would not be able to reap the benefits of the dissected query plan, as it becomes more likely that there will not be enough work to allocate; as more threads might be left idle in the pool, no speed-up can be achieved.

The results of our experiments are shown in Figure 11. Across all instances, the parallelization pipeline (line with data-points) proves more performant than any single threaded executions (horizontal vertical bar). There also appears to be a great variation in speed-up for different scheduling policies; MONOTONIC DYNAMIC, TASK SIZE PREDICTION UNBALANCED DYNAMIC and GUIDED DYNAMIC consistently perform worse than all others. In addition to this, the former schedules grant almost no gain with trace number, indicating that dynamic scheduling is not only less performant than static in our use case scenario, but also bears no potential gains by through thread scalability. This is especially true in the case of Alternate Precedence, where all static policies have improved performance by at least an order of magnitude. Schedules also show different degrees of speed-ups. For the dynamic and BLOCK-CYCLIC STATIC schedules, increasing the number of threads has little effect on performance. In fact, adding threads proves to be **detrimental** in some cases (BLOCK-CYCLIC STATIC & Chain Precedence). Conversely, the other static schedules

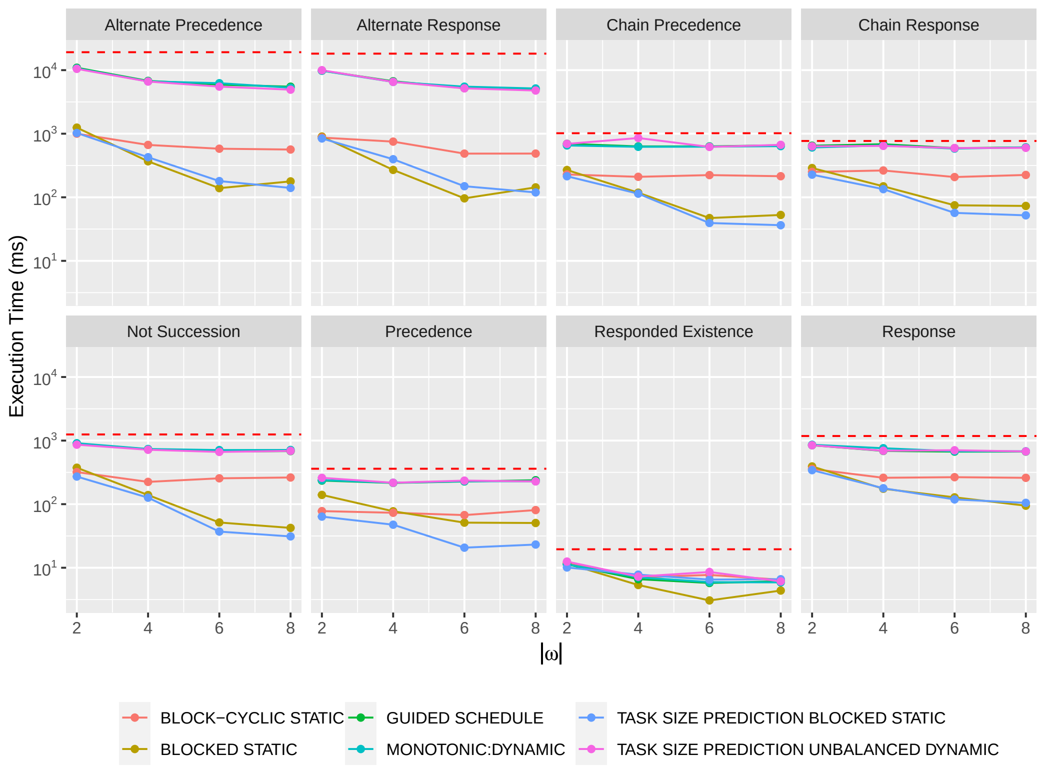


Figure 11. Results for parallelization §7.3. ω indicates the set of threads in the thread pool, and the red dashed horizontal lines indicate running times for single threaded instances.

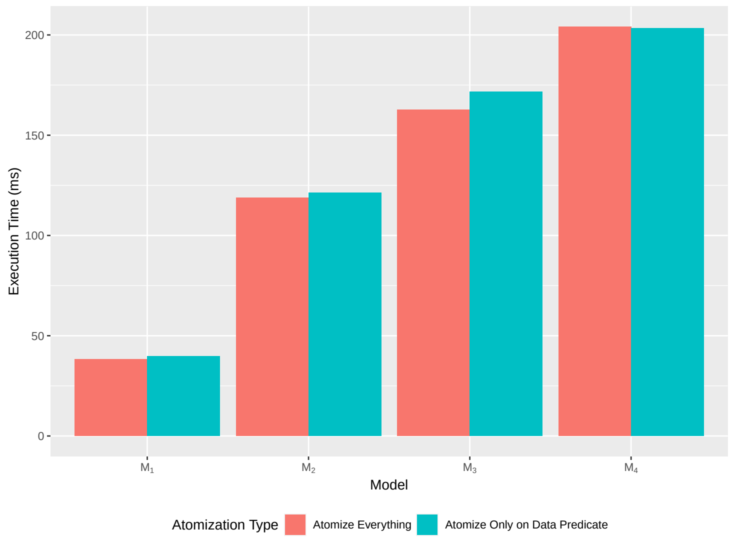


Figure 12. Running times over different models (Table S1a) for different atomisation strategies.

(BLOCKED STATIC and TASK SIZE PREDICTION BLOCK STATIC) achieve a super-linear speed-up [47–49], as the thread count increases. The greatest gains in performance were found for Alternate Precedence and Alternate Response with thread sizes of eight, there is over two orders of magnitude improvement against a single threaded instance, and almost the same speed up compared with the static schedules. As our problem is heavily bounded on data access and on the size of it, reducing the task allocation size will create an overall increase of cache misses, while these are minimised by associating each thread a greater amount of tasks.

7.4. $\mathcal{D}_\varphi$-encoding atomization strategies

We now want to test how distinct query atomisation strategies affect the query run time. For this, we exploit a different dataset while we hardcoded some models suitable for remarking such differences.

While the `AtomizeEverything` strategy guarantees that all activation and targets undergo the atomizaiton step if a clause is found that contains a data payload predicate, the `AtomizeOnlyOnDataPredicate` atomizes only those conditions containing a data payload and consider the others as activity labels. As a consequence, the former is expected to have more weighted access to $\mathsf{AttributeTable}_\mathcal{L}$, while the latter to $\mathsf{ActivityTable}_\mathcal{L}$. We analyse the execution times over the same models $M_1 - M_5$, where each model differs from the other in the number of clauses as well as in data conditions.

For these experiments, we exploited the full BPIC 2012 (Dutch loan company) dataset. This contained event/trace payload information and comprised of activities occurring for a loan transaction. The models exploited are visualised in Supplement Table S1a . We define four models, increasing by five clauses, where each is a sub-model of the latter. This clauses consisted of both data and data-less payload conditions, so to adhere to our benchmarking hypothesis.

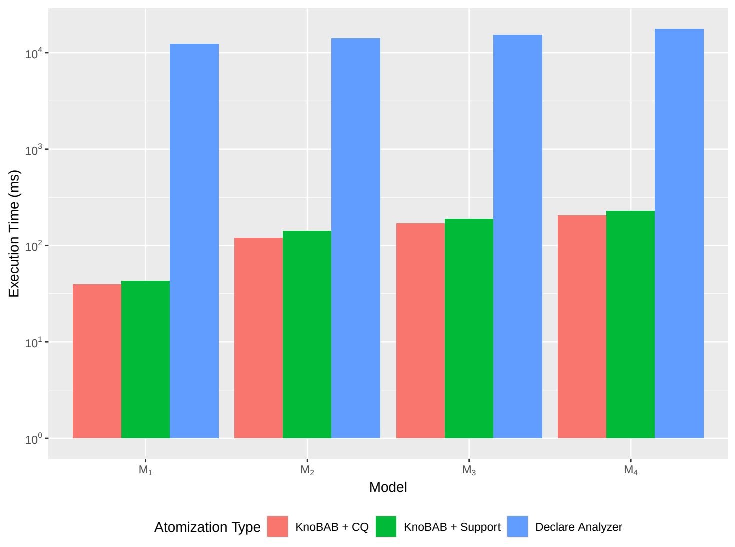
Results are shown in Figure 12 for both configurations, where there is a positive correlation between model size and execution time, with a constant increase with each additional set of clauses. For the smaller model size, `AtomizeEverything` outperforms `AtomizeOnlyOnDataPredicate`, though the former exhibits greater increases in running time as more clauses are added. This therefore suggests that accessing the $\mathsf{ActivityTable}_\mathcal{L}$ becomes more expensive than the $\mathsf{AttributeTable}_\mathcal{L}$ as the number of activation/target conditions increases. To explain this, the reader is encouraged to refer back to Supplement Table S1a, which defines the clauses that are added to each model, and therefore the new activities and atoms that may require decomposition. With increased model sizes, `AtomizeOnlyOnDataPredicate` suffers from duplicated memory access; as some events (e.g. A_SUBMITTED) are accessed in both tables: while returning the events satisfying an atom requires the access to the $\mathsf{AttributeTable}^k_\mathcal{L}$ for any given attribute k of interest, returning all of the events having a given activity label requires accessing the $\mathsf{ActivityTable}_\mathcal{L}$. The data access for the atomized queries may duplicate access to the $\mathsf{ActivityTable}_\mathcal{L}$, which becomes more costly as our model size increases. Conversely, `AtomizeEverything` will atomize A_SUBMITTED from q_1, as clauses q_2 and q_3 contain payload conditions. Therefore, these queries only ever access the $\mathsf{AttributeTable}_\mathcal{L}$, and the duplication of data access is removed. For the smaller model size M_1, this gain is less apparent as the duplicated data access becomes negligible.

7.5. Data-Aware Conformance Checking

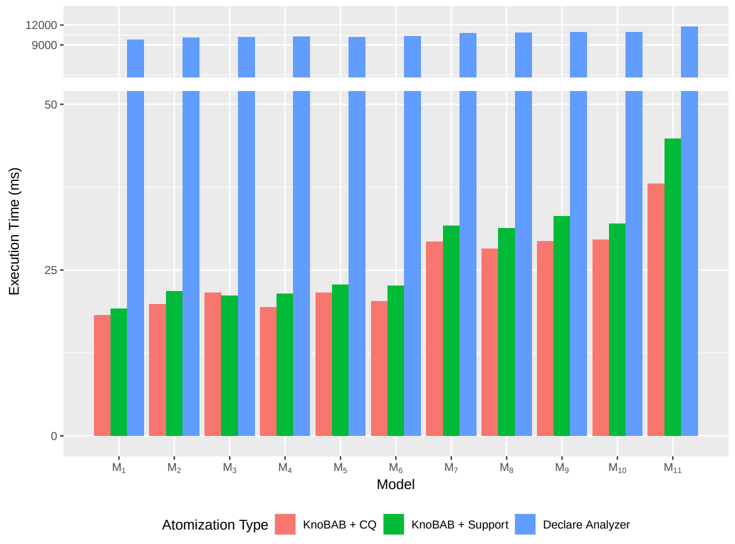
We now consider another state of the art solution, Declare Analyzer [6] for conformance checking with payload information. This solution is tested against two different sets of models of increasing sizes, each of these providing either the worst or the best case scenario for KnoBAB. These experiments exploit the same dataset as in the former experimental set-up, and also used in [6].

We represented the log for Declare Analyzer via MapDB[15], thus reflecting a relational model representation. The authors do not consider trace payloads, and therefore propose injecting trace payload as an extension of each event payload. On the other hand, KnoBAB injects the trace payload as a *unique* event at the beginning of the trace (§2), thus reducing the overhead of testing an activation/target condition *per event* while minimising data loading time. We wanted to investigate our solution's performance among the best/worst cases regarding the clauses of choice. Therefore, we provide two scenarios. The first scenario (SCENARIO 1), also described in our seminal paper [4], provides our *worst* case scenario models (Table S1a) where each additional set of clauses consist of entirely novel activity labels and clauses and, within each sub-model, each clause is distinguished by

15 https://mapdb.org/



(a) Scenario 1 (Table S1a).



(b) Scenario 2 (Table S2).

Figure 13. Running times for data-aware conformance checking.

data payload conditions. Consequently, the query plan cannot exploit gains made from data access minimisation as every condition is considered a unique disjunction of atoms. Conversely, the second (SCENARIO 2) novel scenario describes our best case. We encourage the reader to refer to this, where activation and target conditions appear several times in different clauses (Table S2). So, there are many more instances where data access can be minimised; for example, the model $q_1 \wedge q_2 \wedge q_3 \wedge q_4 \wedge q_5$ considers the activity label A_SUBMITTED across five instances. Following strategies as [23], this can be reduced to one access. SCENARIO 1 (SCENARIO 2) results are shown from Figure 13a (Figure 13b). For either scenario, we average 2-3 orders of magnitude more performant than Declare Analyzer; even in the worst case (M_4) we are over an order of magnitude more performant. For both scenarios we compute the following metrics: Conjunctive Query (CQ) and Support, to analyse any variations between the ensemble methods. KnoBAB + CQ outperforms KnoBAB + Support in all cases, where the cost increase is linear with model size.

SCENARIO 1. For Declare Analyzer, increases in model size results in a constant slope of $3.47 \cdot 10^2$ms per model size, while our solution demonstrates an initial slope of $2 \cdot 10^1$ms per model size, followed by a constant slope of $6 \cdot 10^0$ms per model size. To explain this abrupt behaviour, the reader is encourage to refer to Supplement Table S1a and the query plan from Figure 3. KnoBAB thrives when data access is minimised; if this is cannot be achieved (due to the addition of novel activation/target conditions), potential gains cannot be exploited. **Every** clause from M_2 contains new activation/target labels/payload conditions than M_1. As a result, the number of atoms and leaves in the query plan is doubled. However, M_3 contains the activity label O_CANCELLED. This atom has already been considered in the previous model, and so data access is optimised. Therefore, the time increase from M_2 to M_3 is much less than that of the former. Subsequently, as M_3 is a sub-model of M_4, the same gains are seen here (M_4 contains entirely novel conditions). Overall, the results show that we are not bounded by model size unlike Declare Analyzer, which must perform an entire log scan per clause, while we can ignore irrelevant traces via bounding/indexing across our tabular representation available to the relational model. Still, our running times reflect the formal definition stated in §5.2.3, where queries still need to scan each model clause and therefore their expected running time is proportional to the model size.

SCENARIO 2. We now want to test whether clauses providing similar queries lead to lower running times. Here, the model sizes are smaller than the previous example, so as to demonstrate the potential optimization from even small examples. The former contains only a single clause, while the latter consists of seven clauses. The slope between these models is $3.3 \cdot 10^0$ms per model size, an order of magnitude less than the worst case scenario. To clarify the results, the reader is encouraged to compare the models q_1 vs $q_1 \wedge q_2 \wedge q_3 \wedge q_4 \wedge q_5$. All atoms in the former are included in the latter, so we can have much greater data access minimisation, which these results confirm. Of course, a hand-made model is unlikely to contain such overlapping elements, but these results demonstrate the potential gains to be made, even for less bespoke scenarios such as data mining, where a huge amount of overlap might still occur while testing multiple clauses combinations.

8. Conclusions

By summarizing the contributions of our paper, we showed how to express temporal logic through ad hoc temporal algebra (xtLTL$_\mathrm{f}$) based on the relational model. The latter, defined both in its logical and physical model, has been suitably extended for log and operators' result representation. We showed how it is possible to load data on this model using suitable algorithms and how it is possible to represent a sequence of operations with a parallelizable query plan providing super-linear speed-up. As a new contribution to our previous work, we have also shown different implementations for the xtLTL$_\mathrm{f}$ operators thus showing how there is always a faster non-trivial implementation exploiting both the properties of the intermediate result representation as well as query rewriting. Our proposed solution, KnoBAB, leverages all of the aforementioned feature thus providing higher performance than current conformance checking and mining solutions, be it data or data-less.

This work encourages future KnoBAB developments and implementations, including more efficient data model mining algorithms and the use of views to reduce further the cost of allocating intermediate results. Furthermore, secondary memory representation of the log according to the percepts of Near Data Processing is in its infancy. Future developments will explore the possibility of using KnoBAB to learn temporal models from data and the ability to fully support trace repair operations so to make deviant traces compliant to the given model.

9. Patents

Supplementary Materials: The supplementary material is attached to the present paper and starts from Page 54.

Author Contributions: *Conceptualization*[16], Samuel Appleby, Giacomo Bergami; *methodology*, Giacomo Bergami; *software*, Samuel Appleby, Giacomo Bergami; *validation*, Samuel Appleby, Giacomo Bergami; *formal analysis*, Giacomo Bergami; *investigation*, Samuel Appleby; *resources*, Giacomo Bergami, Graham Morgan; *data curation*, Samuel Appleby; *writing—original draft preparation*, Samuel Appleby, Giacomo Bergami; *writing—review and editing*, Giacomo Bergami, Graham Morgan; *visualization*, Samuel Appleby; *supervision*, Giacomo Bergami, Graham Morgan; *project administration*, Giacomo Bergami; *funding acquisition*, Giacomo Bergami. All authors have read and agreed to the published version of the manuscript.

Funding: Samuel Appleby's work is supported by Newcastle University.

Institutional Review Board Statement: Not applicable

Informed Consent Statement: Not applicable

Data Availability Statement: The dataset associated with the presented experiments is available online: https://dx.doi.org/10.17605/OSF.IO/2CXR7

Conflicts of Interest: The authors declare no conflict of interest.

Sample Availability: The most up-to-date version of KnoBAB is available on GitHub: https://github.com/datagram-db/knobab.

Abbreviations

The following abbreviations are used in this manuscript:

DAG	Direct Acyclic Graph
KnoBAB	KNOwledge Base for Alignments and Business process modelling
$\mathrm{LTL_f}$	Linear Temporal Logic over Finite traces
RDBMS	Relational Database Management System
XES	eXtensible Event Stream
$\mathtt{xt}\mathrm{LTL_f}$	eXTended Linear Temporal Logic over Finite traces

Appendix A

We now show some equivalence and correctness lemmas.

Appendix A.1

First, we want to show that $\mathtt{xt}\mathrm{LTL_f}$ is at least as expressive as $\mathrm{LTL_f}$. To support this claim, we need to prove the two following lemmas where, as $\mathrm{LTL_f}$ does not support explicit activation and target conditions with Θ correlation conditions over the payload data, we are always going to assume $\Theta = \mathbf{True}$ and that the atomic operators are never associated to an activation/target label, thus always returning an empty third component of the intermediate result. As we might observe, the following lemma entails that, differently from standard $\mathrm{LTL_f}$ semantics applied to each event trace at a time, $\mathtt{xt}\mathrm{LTL_f}$ semantics returns all of the events for which the given temporal condition holds. This becomes very relevant for minimising the data access while scanning our relational representation of the log, as well as allowing better intermediate result reuse for any incoming sub-expression. The following lemma also entails a correspondence between timed $\mathtt{xt}\mathrm{LTL_f}$ operators and $\mathrm{LTL_f}$ formulae.

Lemma A1. *For each* LTL_f *formula* φ*, it exists a timed* $\mathtt{xt}\,\mathrm{LTL}_f$ *expression* ψ^τ *evaluated over an intended relational model representing a log* $\mathcal{L}$ *of finite and non-empty traces for which the latter returns* $\langle i, j, L\rangle$ *iff.* $\sigma^i_j \vDash \varphi$*. More formally:*

$$\forall \sigma^i_j \in \sigma^i, \sigma^i \in \mathcal{L}.\ \forall\varphi \in \mathrm{LTL}_f.\ \exists\psi^\tau : \mathtt{timed}\ \mathtt{xt}\mathrm{LTL}_f.(\langle i, j, L\rangle \in \psi^\tau \Leftrightarrow \sigma^i_j \vDash \varphi)$$

[16] CRediT taxonomy

Proof. The constructive proof proceeds by structural induction over ψ^τ. We need first to consider a rewriting lemma stating that $\langle \beta(\mathsf{a}), i, j, \pi, \chi\rangle \in \mathsf{ActivityTable}_{\mathcal{L}}$ iff. it exists a p such that $\sigma^i_j = \langle a, p\rangle$. Now, we can start the proof by induction.

$\varphi = \mathsf{a}$**:** By applying the aforementioned rewriting lemma (from now on simply referred as *by construction of ActivityTable*), we can immediately close the goal by choosing $\psi^\tau = \mathsf{Activity}^\tau(\mathsf{a})$ as the model will only return data associated to the log of choice:

$$\langle i, j, L\rangle \in \mathsf{Activity}^\tau(\mathsf{a}) \Leftrightarrow \exists p. \sigma^i_j = \langle \mathsf{a}, p\rangle \Leftrightarrow \sigma^i_j \vDash \mathsf{a}$$

$\varphi = \mathsf{a} \wedge q$**:** If the compound condition is also atomic for which q can be expressed as an interval query $low \leq \kappa \leq up$ for some payload key κ, we can follow a similar proof from the former case and choose the atom $\psi^\tau = \mathsf{Compound}^\tau(\mathsf{a}, \kappa, low, up)$, thus closing the goal as follows:

$$\begin{aligned}\langle i, j, L\rangle \in \mathsf{Compound}^\tau(\mathsf{a}, \kappa, [low, up]) &\Leftrightarrow \exists p. \sigma^i_j = \langle a, p\rangle \wedge low \leq p(\kappa) \leq up \\ &\Leftrightarrow \sigma^i_j \vDash \mathsf{a} \wedge (low \leq \kappa \wedge \kappa \leq up)\end{aligned}$$

$\varphi = \bigcirc\varphi'$**:** by inductive hypothesis, we know the ρ $\texttt{xtLTL}_\mathrm{f}$ expression returning ρ, which contains $\langle i, j+1, L\rangle$ when $\sigma^i_{j+1} \vDash \varphi'$. For this, we choose as $\psi^\tau = \mathsf{Next}^\tau(\rho)$, which also guarantees that j never exceeds the trace's length ($j \leq |\sigma^i|$). We can therefore expand the definition of our proposed operator by obtaining:

$$\begin{aligned}\langle i, j, L\rangle \in \mathsf{Next}^\tau(\rho) &\Leftrightarrow \langle i, j+1, L\rangle \in \rho \wedge 1 < j+1 \leq |\sigma^i| \\ &\overset{IH}{\Leftrightarrow} \sigma^i_{j+1} \vDash \varphi \wedge 0 < j < |\sigma^i| \\ &\Leftrightarrow \varphi^i_j \vDash \bigcirc\varphi\end{aligned}$$

$\varphi = \Box\varphi'$**:** The application of the induction is similar to the former and, similarly to the former case, we also proceed by expanding the definition of the relational operator. We can hereby choose $\psi^\tau = \mathsf{Globally}^\tau(\rho)$ where the induction is applied over ρ and φ'. We can close the goal as follows:

$$\begin{aligned}\langle i, j, L\rangle \in \mathsf{Globally}^\tau(\rho) &\Leftrightarrow \langle i, j, L_j\rangle \in \rho \wedge |\sigma^i| - j + 1 = |\{\langle i, k, L_k\rangle \in \rho | j \leq k \leq |\sigma^i|\}| \\ &\Leftrightarrow \forall j \leq k \leq |\sigma^i|.\, \langle i, k, L_k\rangle \in \rho \\ &\overset{IH}{\Leftrightarrow} \forall j \leq k \leq |\sigma^i|. \sigma^i_k \vDash \varphi' \\ &\Leftrightarrow \sigma^i_j \vDash \Box\varphi'\end{aligned}$$

$\varphi = \Diamond\varphi'$**:** Similarly to globally, we obtain $\psi^\tau = \mathsf{Future}^\tau(\rho)$ for a ρ corresponding to φ' by inductive hypothesis.

$\varphi = \neg\varphi'$**:** Similarly to the previous unary operators, we choose as $\texttt{xtLTL}_\mathrm{f}$ operator $\psi^\tau = \mathsf{Not}^\tau(\rho)$ where the inductive hypothesis links ρ to φ'. We can therefore close the goal as follows:

$$\begin{aligned}\langle i, j, L\rangle \in \mathsf{Not}^\tau(\rho) &\Leftrightarrow \sigma^i_j \in \sigma^i \wedge \sigma^i \in \mathcal{L} \wedge \langle i, j, L\rangle \notin \rho \\ &\overset{IH}{\Leftrightarrow} \sigma^i_j \in \sigma^i \wedge \sigma^i \in \mathcal{L} \wedge \sigma^i_j \vDash \neg\varphi \\ &\Leftrightarrow \sigma^i_j \vDash \neg\varphi\end{aligned}$$

This is doable as stating $\langle i,j,L\rangle \in \psi^\tau \Leftrightarrow \sigma^i_j \vDash \varphi$ is equivalent to $\langle i,j,L\rangle \notin \psi^\tau \Leftrightarrow \sigma^i_j \nvDash \varphi$ where the latter can be rewritten as $\sigma^i_j \vDash \neg\varphi$.

$\varphi = \varphi' \wedge \varphi''$: As we have that two inductive hypotheses associating ρ' and ρ'' respectively to φ' and φ'', we choose the xtLTL$_f$ formula $\psi^\tau = \mathsf{And}^\tau_{\mathbf{True}}(\rho',\rho'')$ to be associated to $\varphi' \wedge \varphi''$. For this xtLTL$_f$ operator, we can state that a result $\langle i,j,\ell\rangle$ is returned by such an operator if and only if $\langle i,j,\varnothing\rangle \in \rho'$ and $\langle i,j,\varnothing\rangle \in \rho''$ per definition of operators never returning explicit activation or target condition. We close the goal as follows:

$$\begin{aligned}\langle i,j,L\rangle \in \mathsf{And}^\tau_{\mathbf{True}}(\rho,\rho') &\Leftrightarrow \langle i,j,\varnothing\rangle \in \rho \wedge \langle i,j,\varnothing\rangle \in \rho' \\ &\overset{IH}{\Leftrightarrow} \sigma^i_j \vDash \varphi \wedge \sigma^i_j \vDash \varphi' \\ &\Leftrightarrow \sigma^i_j \vDash \varphi \wedge \varphi'\end{aligned}$$

$\varphi = \varphi' \vee \varphi''$: We can firstly observe that $(A \wedge B) \vee (A \wedge \neg B) \vee (\neg A \wedge B)$ in the classical semantics is equivalent to $A \vee B$ for any possible proposition A and B (*OrRwLem*). After observing that the current operator is defined by extension of the previously proved one, we can exploit the previous one as a rewriting lemma. As we have that two inductive hypothesis associating ρ' and ρ'' respectively to φ' and φ'', we choose the xtLTL$_f$ formula $\psi^\tau = \mathsf{Or}^\tau_{\mathbf{True}}(\rho',\rho'')$ to be associated to $\varphi' \vee \varphi''$. We close the goal:

$$\begin{aligned}\langle i,j,L\rangle \in \mathsf{Or}^\tau_{\mathbf{True}}(\rho,\rho') &\overset{IH}{\Leftrightarrow} \langle i,j,L\rangle \in \mathsf{And}^\tau(\rho,\rho') \vee (\sigma^i_j \vDash \varphi \wedge \sigma^i_j \nvDash \varphi') \vee (\sigma^i_j \vDash \varphi' \wedge \sigma^i_j \nvDash \varphi) \\ &\Leftrightarrow \sigma^i_j \vDash \varphi \wedge \varphi' \vee (\sigma^i_j \vDash \varphi \wedge \sigma^i_j \nvDash \varphi') \vee (\sigma^i_j \vDash \varphi' \wedge \sigma^i_j \nvDash \varphi) \\ &\overset{OrRwLem}{\Leftrightarrow} \sigma^i_j \vDash \varphi \vee \varphi'\end{aligned}$$

$\varphi = \varphi' \,\mathcal{U}\, \varphi''$: as both the results from the third element of the intermediate results are always empty by construction and preliminary assumption, and we have inductive hypothesis associating ρ and ρ' respectively to φ and φ', we can immediately close the goal after choosing the xtLTL$_f$ formula $\psi^\tau = \mathsf{Until}^\tau_{\mathbf{True}}(\rho',\rho'')$ to be associated to $\varphi = \varphi' \,\mathcal{U}\, \varphi''$.

□

The next lemma is required for closing the generic lemma stated at the beginning of this sub-section, as LTL$_f$ starts assessing the formulae from the beginning of each trace. We need to show that the former lemma applies to xtLTL$_f$ operators in a stricter version which is the following one:

Lemma A2. *For each* LTL$_f$ *formula φ satisfied from the beginning of the trace, it exists an* xtLTL$_f$ *expression ψ returning a $\langle i,1,L\rangle$, thus remarking that the condition holds from the beginning of the trace. More formally:*

$$\forall\sigma^i \in \mathcal{L}.\forall\varphi\colon \mathrm{LTL}_f.\exists\psi \in \mathit{xt}\mathrm{LTL}_f.(\sigma^i \vDash \varphi \Rightarrow \exists L.\,\langle i,1,L\rangle \in \psi)$$

Proof. Similarly to the previous lemma, as LTL$_f$ cannot express activation and target conditions to be tested in Θ correlation conditions, we always choose $\Theta = \mathbf{True}$ and we decide to use base xtLTL$_f$ operators where none of these conditions is returned. Differently from the previous lemma, we now have to go by inductive structure over the LTL$_f$ formulærather than on the xtLTL$_f$ ones. We can therefore consider the following inductive cases:

$\varphi = \mathsf{a}$: By definition of the Init operator, it is sufficient to consider $\psi = \mathsf{Init}(\mathsf{a})$.

$\varphi = \mathsf{a} \wedge p$: Under the assumption that the compound condition corresponds to an atomic query with $p := low \leq \kappa \leq up$, we can formulate the former as follows: $\psi = \mathsf{Init}(\mathsf{Compound}^{\mathcal{L},\tau}(\mathsf{a}, \kappa, low, up))$.

$\varphi = \bigcirc\varphi'$: By rewriting this definition, this implies to prove that $\varphi^i_2 \vDash \varphi'$. As the Next^τ operator is a timed one and we cannot assess φ' from the beginning of the trace, we cannot exploit the inductive hypothesis for φ', but we need to apply the previously proven lemma for the conditions happening at any point in the trace. From the application of the previous lemma, we have that $\varphi^i_2 \vDash \varphi \Leftrightarrow \langle i, 2, L\rangle \in \rho$ for some $\texttt{xt}\mathrm{LTL}_\mathrm{f}$ expression returning ρ. From this follows that $\langle i, 1, L\rangle \in \mathsf{Next}^\tau(\rho)$. By its definition, Next^τ returns all events preceding the ones stated in ρ while, for $\sigma^i \vDash \bigcirc\varphi$, we are only interested in restricting all of the possible results of Next^τ to the ones also corresponding to the beginning of the trace. For this reason, we need to consider ψ as $\mathsf{And}^\tau(\mathsf{First}^\tau, \mathsf{Next}^\tau(\rho))$.

$\varphi = \Box\varphi'$: Similarly to the previous operator, φ' is timed and should be checked for all events σ^i_j of interest within the trace σ^i. Even in this case, we need to apply the previous lemma for φ', thus guaranteeing that it exists an $\texttt{xt}\mathrm{LTL}_\mathrm{f}$ expression ρ containing $\langle i, j, L\rangle$ whenever $\sigma^i_j \vDash \varphi'$. As globally requires that all of the events satisfy φ', we have that $\mathsf{Globally}(\rho)$ responds by the intended semantics, and therefore we choose this as our ψ.

$\varphi = \Diamond\varphi'$: Similarly to the previous operator, we choose $\mathsf{Future}(\rho)$ when ρ is linked to the evaluation of φ' for any possible trace event by the previous lemma.

$\varphi = \neg\varphi'$: In this other scenario, we can directly apply the previous lemma, as the evaluation of φ' will always start from the beginning of the trace. After recalling that $\nexists x.P(x) \Leftrightarrow \forall x.\neg P(x)$, we rewrite the definition of φ while applying the inductive hypothesis for the present lemma over some ρ semantically linked to φ' as follows:

$$\sigma^i \vDash \neg\varphi \Leftrightarrow \sigma^i_1 \nvDash \varphi' \overset{IH}{\Leftrightarrow} \forall L.\, \langle i, 1, L\rangle \notin \rho$$

Per inductive hypothesis, ρ contains all of the records $\langle i, 1, L\rangle$ for which $\sigma^i_1 \vDash \varphi'$; as the untimed negation will return a record $\langle \iota, 1, \varnothing\rangle$ if and only if there is no event associated to the trace ι in the provided operand, we can choose $\psi = \mathsf{Not}(\rho)$ and close the goal as follows:

$$\langle i, 1, \varnothing\rangle \in \mathsf{Not}(\rho) \Leftrightarrow \forall j, L.\, \langle i, j, L\rangle \notin \rho \Leftrightarrow \forall L.\, \langle i, 1, L\rangle \notin \rho$$

$\varphi = \varphi' \,\mathcal{U}\, \varphi''$: Similarly to the former operators, both φ' and φ'' required a timed evaluation of the events along the trace of interest, for which we need to exploit the former lemma, thus obtaining timed $\texttt{xt}\mathrm{LTL}_\mathrm{f}$ expressions ρ' and ρ''. We can immediately close the lemma by choosing $\psi = \mathsf{Until}_{\mathbf{True}}(\rho', \rho'')$.

$\varphi = \varphi' \wedge \varphi''$: Similarly to the negation operator, we can directly apply the inductive hypothesis on φ' and φ'', as also these sub-operators will be assessed from the beginning of a trace; these will be associated respectively to the $\texttt{xt}\mathrm{LTL}_\mathrm{f}$ expressions ρ' and ρ'' having $\langle i, 1, \varnothing\rangle \in \rho'$ and $\langle i, 1, \varnothing\rangle \in \rho''$ as we exploit neither activation nor target conditions. As per construction ρ' and ρ'' will contain no record $\langle i, j+2, L\rangle$ for some natural number $j \geq 0$, we chose $\psi = \mathsf{And}_{\mathbf{True}}(\rho', \rho'')$.

$\varphi = \varphi' \vee \varphi''$: By exploiting similar consideration from the former operator, we chose $\psi = \mathsf{Or}_{\mathbf{True}}(\rho', \rho'')$ for some ρ' and ρ'' respectively associated by inductive hypothesis to φ' and φ''.

$\Box$

As a corollary of the two given lemmas, we have that $\texttt{xt}\mathrm{LTL}_\mathrm{f}$ is at least as expressive as LTL_f, as any LTL_f formula can be always computed through an equivalent $\texttt{xt}\mathrm{LTL}_\mathrm{f}$ formula. This validates the decision from our previous work [4] where we expressed the semantics of each template in Declare through a correspondent $\texttt{xt}\mathrm{LTL}_f$ expression. These were also checked through automated testing[17]. At this stage, we want also to ascertain that the untimed and timed operators work as expected, that is, that we can mimic the outcome of the timed operators over the timed ones if, for each event $\langle i, j, L\rangle$, we evaluate the corresponding untimed operator over the suffix $\sigma^i_j, \dots, \sigma^i_{|\sigma^i|}$. This can be proven as follows:

Lemma A3. *For each timed $\mathbf{\mathit{xt}}\,\mathrm{LTL}_f$ operator ψ^τ containing a result $\langle i, j, L\rangle$ over a relational representation of $\mathcal{L}$, generate a log of suffixes $\mathcal{L}' = \{\sigma^{i\oplus j}\}$ where $\sigma^{i\oplus j} := \sigma^i_j, \dots, \sigma^i_{|\sigma^i|}$ of σ^i and each event is defined as $\sigma^{i\oplus j}_k := \sigma^i_{j+k-1}$ for each $1 \leq k \leq |\sigma^i| - j + 1$. For this, it always exists an $\mathbf{\mathit{xt}}\,\mathrm{LTL}_f$ expression ψ evaluated over the relational representation of $\mathcal{L}'$ such that $\langle i \oplus j, 1, L\rangle \in \psi$.*

Proof. We prove the lemma by induction over ψ^τ by considering all of the timed operators having an untimed counterpart. Please observe that we discard the negation Not from our considerations, as we have previously remarked that the timed and untimed versions of this serve different purposes. We also provide an implementation[17] of such proofs via automated testing.

$\psi^\tau = \textbf{Activity}^{\mathcal{L},\tau}_{A/T}(\mathsf{a})$: This can be trivially closed by choosing $\mathsf{Init}^{\mathcal{L}'}_{A/T}(\mathsf{a})$.

$\psi^\tau = \textbf{Compound}^{\mathcal{L},\tau}_{A/T}(\mathsf{a}, k, \boldsymbol{low}, \boldsymbol{up})$: This can be trivially closed by choosing $\mathsf{Init}(\mathsf{Compound}^{\mathcal{L}'}_{A/T}(\mathsf{a}, k, [low, up]))$.

$\psi^\tau = \textbf{Globally}^\tau(\rho)$: After observing that $|\sigma^{i\oplus j}| = |\sigma^i| - j + 1$, we obtain the following condition by operator's expansion, where ρ' is evaluated over $\mathcal{L}'$ as per inductive hypothesis:

$$\begin{aligned}
\langle i,j,L\rangle \in \mathsf{Globally}^\tau(\rho) &\Leftrightarrow L := \cup_{\substack{j\leq k\leq |\sigma^i|,\\ \langle i,k,L_k\rangle\in\rho}} L_k \wedge\ |\sigma^i| - j + 1 = \left|\{\langle i,k,L_k\rangle \in \rho | j \leq k \leq |\sigma^i|\}\right| \\
&\Leftrightarrow L := \cup_{\substack{1\leq k\leq |\sigma^{i\oplus j}|,\\ \langle i\oplus j,k,L_k\rangle\in\rho}} L_k \wedge\ |\sigma^{i\oplus j}| = \left|\{\langle i\oplus j,k,L_k\rangle \in \rho | 1 \leq k \leq |\sigma^{i\oplus j}|\}\right| \\
&\Leftrightarrow L := \cup_{\langle i\oplus j,k,L_k\rangle\in\rho} L_k \wedge\ |\sigma^{i\oplus j}| = \left|\{\langle i\oplus j,k,L_k\rangle \in \rho\}\right| \\
&\Leftrightarrow \langle i\oplus j, 1, L\rangle \in \mathsf{Globally}(\rho')
\end{aligned}$$

$\psi^\tau = \textbf{Future}^\tau(\rho)$: By following similar consideration as per the former operator, we have:

$$\begin{aligned}
\langle i,j,L\rangle \in \mathsf{Future}^\tau(\rho) &\Leftrightarrow L := \cup_{\substack{j\leq k\leq |\sigma^i|\\ \langle i,k,L_k\rangle\in\rho}} L_k \wedge \exists h \geq j, L.\ \langle i,h,L_h\rangle \in \rho \\
&\Leftrightarrow L := \cup_{\substack{1\leq k\leq |\sigma^{i\oplus j}|\\ \langle i\oplus j,k,L_k\rangle\in\rho}} L_k \wedge \exists h \geq 1, L.\ \langle i,h,L_h\rangle \in \rho \\
&\Leftrightarrow L := \cup_{\langle i\oplus j,k,L_k\rangle\in\rho} L_k \wedge \exists h, L.\ \langle i,h,L_h\rangle \in \rho \\
&\Leftrightarrow \langle i\oplus j, 1, L\rangle \in \mathsf{Future}(\rho')
\end{aligned}$$

$\psi^\tau = \textbf{And}^\tau_\Theta(\rho_1, \rho_2)$: By rewriting the definition of the timed And operator, we obtain the following:

$$\begin{aligned}
\langle i,j,L\rangle \in \mathsf{And}^\tau_\Theta(\rho_1,\rho_2) \Leftrightarrow \exists L_1, L_2.\ \langle i,j,L_1\rangle \in \rho_1 \wedge \langle i,j,L_2\rangle \in \rho_2 \wedge \\
L := \mathcal{T}^{E,i}_\Theta([j \mapsto L_1],[j \mapsto L_2]) \wedge L \neq \textbf{False}
\end{aligned}$$

[17] https://github.com/datagram-db/knobab/blob/main/tests/ltlf_operators_test.cpp

If And contains for both of its operands an event σ_j^i, it follows that there should be at least one match $\sigma_1^{i\oplus j}$ over the corresponding untimed operator $\mathsf{And}_\Theta(\rho', \rho'')$ evaluated over $\mathcal{L}'$. For the latter operator, we can therefore ensure that it exists a j and a j' being $j = j' = 1$ and L as well as L' for which the following condition holds:

$$\begin{aligned}\langle i,j,L\rangle \in \mathsf{And}^\tau_\Theta(\rho_1,\rho_2) \Rightarrow\ & \exists L_1, L_2.\, \langle i\oplus j,1,L_1\rangle \in \rho' \wedge \langle i\oplus j,1,L_2\rangle \in \rho'' \wedge \\ & L := \mathcal{T}^{E,i}_\Theta([1\mapsto \{L_j | \langle i,j,L_j\rangle \in \rho'\}],[1\mapsto \{L_j | \langle i,j,L_j\rangle \in \rho''\}]) \wedge \\ & L \neq \textbf{False} \\ \Leftrightarrow\ & \langle i\oplus j,1,L\rangle \in \mathsf{And}_\Theta(\rho',\rho'')\end{aligned}$$

$\psi^\tau = \mathbf{Or}^\tau_\Theta(\rho_1,\rho_2)$**:** As this operator is derived from the definition of And^τ_Θ, we can directly close the goal by the previous inductive step if the result represents a match between the elements of the first and second operand. If there were no events that might have been matched, the data comes either from the first or from the second operand. As the two cases are symmetric, we just provide proof for the former case. In this situation, we have a $\langle i,j,L\rangle \in \mathsf{Or}^\tau_\Theta(\rho_1,\rho_2)$ corresponding to a $\langle i,j,L\rangle \in \rho_1$ for which there is no L' such that $\langle i,j,L'\rangle \in \rho_2$. If there still exists a j' and L' such that $\langle i,j',L'\rangle \in \rho_2$ for which there might be a match between L and L', then this case falls under the untimed And_Θ over $\mathcal{L}'$, and we still have some τ for which the latter returns $\langle i\oplus j,1,\tau\rangle$; if match is never possible or no of such j' exists, then the untimed Or_Θ operator will return a $\langle i\oplus j,1,\cup\{L|\exists k.\,\langle i\oplus j,k,L\rangle \in \rho_2\}\rangle$ by definition.

$\psi^\tau = \mathbf{Until}^\tau(\rho_1,\rho_2)$**:** This is a mere rewriting exercise, as the untimed version of Until is a mere instantiation of the latter where only the case $k = 1$ is considered.

□

Appendix A.2

At this stage, we provide some rewriting lemmas motivating the introduction of derived operators. First, we want to show that the untimed $\mathsf{And}_\Theta(\rho,\rho')$ operator can be also exploited to compute $\mathsf{And}_\Theta(\mathsf{Future}(\rho),\mathsf{Future}(\rho'))$, thus motivating the peculiar definition of such operator with an existential interpretation over all of the possible matches in the future. We can formally prove this as follows:

Lemma A4.

$$\forall \rho,\rho'. And_\Theta(Future(\rho), Future(\rho')) = And_\Theta(\rho,\rho')$$

Proof. By expanding the definition of the operators, we obtain:

$$\begin{aligned}\langle i,1,L''\rangle \in \mathsf{And}_\Theta(\mathsf{Future}(\rho),\mathsf{Future}(\rho')) \Leftrightarrow\ & \exists L,L'.\,\langle i,1,L\rangle \in \mathsf{Future}(\rho) \wedge \langle i,1,L'\rangle \in \mathsf{Future}(\rho') \\ & L'' := \mathcal{T}^{E,i}_\Theta([1\mapsto \cup\{L_j|\langle i,j,L_j\rangle \in \rho_1\}],[1\mapsto \cup\{L_j|\langle i,j,L_j\rangle \in \rho_2\}]), \\ & L'' \neq \textbf{False} \\ \Leftrightarrow\ & \exists j,j',L,L'.\,\langle i,j,L\rangle \in \rho \wedge \langle i,j',L'\rangle \in \rho' \\ & L'' := \mathcal{T}^{E,i}_\Theta([1\mapsto \cup\{L_j|\langle i,j,L_j\rangle \in \rho_1\}],[1\mapsto \cup\{L_j|\langle i,j,L_j\rangle \in \rho_2\}]), \\ & L'' \neq \textbf{False} \\ \Leftrightarrow\ & \langle i,1,L''\rangle \in \mathsf{And}_\Theta(\rho,\rho')\end{aligned}$$

□

Please remember that the untimed And operator is also compliant with the $\mathrm{LTL_f}$ semantics as per our previous lemmas. We can therefore exploit the versatile definition of such operation to reduce the computational overhead provided by the additional and unrequired aggregation provided by Future. Given the previous lemma, we have as a Corollary that the semantics associated with the Choice Declare clause, i.e. $\mathsf{Or}_\Theta(\mathsf{Future}(\rho),\mathsf{Future}(\rho'))$, can be equivalently be computed by $\mathsf{Or}_\Theta(\rho,\rho')$. The following proof motivates the choice of exploiting E^i_Θ as a correlation matching semantics for both And_Θ and Or_Θ.

Corollary A1.

$$\forall\rho,\rho'.Or_\Theta(Future(\rho),Future(\rho')) = Or_\Theta(\rho,\rho')$$

Proof. By expanding the definition of the untimed Or_Θ, we obtain:

$$\begin{aligned}\mathsf{Or}_\Theta(\mathsf{Future}(\rho),\mathsf{Future}(\rho')) = \mathsf{And}_\Theta(\mathsf{Future}(\rho),\mathsf{Future}(\rho')) &\cup \{\, \langle i,1,\cup\{L|\exists j.\,\langle i,j,L\rangle \in \mathsf{Future}(\rho)\}\rangle \mid \nexists j,L'.\,\langle i,j,L'\rangle \in \mathsf{Future}(\rho')\,\}\\ &\cup \{\, \langle i,1,\cup\{L|\exists j.\,\langle i,j,L\rangle \in \mathsf{Future}(\rho')\}\rangle \mid \nexists j,L'.\,\langle i,j,L'\rangle \in \mathsf{Future}(\rho)\,\}\end{aligned}$$

For the previous Lemma, this becomes:

$$\begin{aligned}\mathsf{Or}_\Theta(\mathsf{Future}(\rho),\mathsf{Future}(\rho')) = \mathsf{And}_\Theta(\rho,\rho') &\cup \{\, \langle i,1,\cup\{L|\exists j.\,\langle i,j,L\rangle \in \mathsf{Future}(\rho)\}\rangle \mid \nexists j,L'.\,\langle i,j,L'\rangle \in \mathsf{Future}(\rho')\,\}\\ &\cup \{\, \langle i,1,\cup\{L|\exists j.\,\langle i,j,L\rangle \in \mathsf{Future}(\rho')\}\rangle \mid \nexists j,L'.\,\langle i,j,L'\rangle \in \mathsf{Future}(\rho)\,\}\end{aligned}$$

At this stage, we only need to test the contribution of the second component of the union, as the third one is symmetrical (ρ and ρ' are just inverted). As the elements of the second component of the union come from a Future operators, we can rewrite such as follows:

$$\{\, \langle i,1,\cup\{L|\,\langle i,1,L\rangle \in \mathsf{Future}(\rho)\}\rangle \mid \nexists L'.\,\langle i,1,L'\rangle \in \mathsf{Future}(\rho')\,\}$$

We can also observe that $\langle i,1,L\rangle \in \mathsf{Future}(\rho)$ for a given L if there exists a j and L'' for which $\langle i,j,L''\rangle \in \rho$. Similar considerations come from the negated counterpart ($\langle i,1,L\rangle \notin \mathsf{Future}(\rho)$). For this expansion, we can therefore close our goal. □

The remaining lemmas, show the correctness of the logical formulation of the derived operators, thus motivating their adoption when possible. These lemmas were also tested in our implementation[18]. The supplementary materials (§IV) show that it is possible to implement such derived operators so that they are faster than their corresponding $\mathrm{LTL_f}$ rewriting counterpart.

Lemma A5.

$$\forall\rho,\rho'.And^\tau_\Theta(\rho_1,Future^\tau(\rho_2)) = AndFuture^\tau_\Theta(\rho_1,\rho_2)$$

Proof.

$$\begin{aligned}\langle i,j,L\rangle \in \mathsf{And}^\tau_\Theta(\rho_1,\mathsf{Future}^\tau(\rho_2)) &\Leftrightarrow \exists L_1,L_2.\,\langle i,j,L_1\rangle \in \rho_1 \wedge \langle i,j,L_2\rangle \in \mathsf{Future}^\tau(\rho_2)\wedge\\ &\qquad L := \mathcal{T}^{E,i}_\Theta([j\mapsto L_1],[j\mapsto L_2]) \wedge L \neq \mathbf{False}\\ &\Leftrightarrow \exists L_1,L_2.\,\langle i,j,L_1\rangle \in \rho_1 \wedge \exists h\geq j,L.\,\langle i,h,L\rangle \in \rho_2\wedge\\ &\qquad L := \mathcal{T}^{E,i}_\Theta([j\mapsto L_1],[j\mapsto \cup_{\substack{j\leq k\leq|\sigma^i|\\ \langle i,k,L_k\rangle\in\rho}} L_k]) \wedge L \neq \mathbf{False}\\ &\Leftrightarrow \langle i,j,L\rangle \in \mathsf{AndFuture}^\tau_\Theta(\rho_1,\rho_2)\end{aligned}$$

□

Lemma A6.

$$\forall\rho,\rho'.And^\tau_\Theta(\rho_1,Globally^\tau(\rho_2)) = AndGlobally^\tau_\Theta(\rho_1,\rho_2)$$

[18] See the end of https://github.com/datagram-db/knobab/blob/main/tests/until_test.cpp

Proof.

$$\begin{aligned}
\langle i,j,L\rangle \in \mathsf{And}^{\tau}_{\Theta}(\rho_1,\mathsf{Globally}^{\tau}(\rho_2)) \Leftrightarrow\ & \exists L_1,L_2.\,\langle i,j,L_1\rangle \in \rho_1 \wedge \langle i,j,L_2\rangle \in \mathsf{Globally}^{\tau}(\rho_2)\wedge\\
& L := \mathcal{T}^{E,i}_{\Theta}([j\mapsto L_1],[j\mapsto L_2]) \wedge L \neq \textbf{False}\\
\Leftrightarrow\ & \exists L_1.\,\langle i,j,L_1\rangle \in \rho_1 \wedge \langle i,j,L_j\rangle \in \rho_2\wedge\\
& |\sigma^i| - j + 1 = \left|\{\langle i,k,L_k\rangle \in \rho | j \leq k \leq |\sigma^i|\}\right.\wedge\\
& L := \mathcal{T}^{E,i}_{\Theta}([j\mapsto L_1],[j\mapsto \cup_{\substack{j\leq k\leq|\sigma^i|\\ \langle i,k,L_k\rangle\in\rho}} L_k]) \wedge L \neq \textbf{False}\\
\Leftrightarrow\ & \exists L_1.\,\langle i,j,L_1\rangle \in \rho_1 \wedge \forall j \leq k \leq |\sigma^i|.\exists L'.\,\langle i,k,L_k\rangle \in \rho_2\wedge\\
& L := \mathcal{T}^{E,i}_{\Theta}([j\mapsto L_1],[j\mapsto \cup_{\substack{j\leq k\leq|\sigma^i|\\ \langle i,k,L_k\rangle\in\rho}} L_k]) \wedge L \neq \textbf{False}\\
\Leftrightarrow\ & \langle i,j,L\rangle \in \mathsf{AndGlobally}^{\tau}_{\Theta}(\rho_1,\rho_2)
\end{aligned}$$

□

References

1. Agrawal, R.; Imieliński, T.; Swami, A. Mining Association Rules between Sets of Items in Large Databases. *SIGMOD Rec.* **1993**, *22*, 207–216.
2. Bergami, G.; Maggi, F.M.; Montali, M.; Peñaloza, R. Probabilistic Trace Alignment. In Proceedings of the 2021 3rd International Conference on Process Mining (ICPM), 2021, pp. 9–16. https://doi.org/10.1109/ICPM53251.2021.9576856.
3. Schön, O.; van Huijgevoort, B.; Haesaert, S.; Soudjani, S. Correct-by-Design Control of Parametric Stochastic Systems **2022**.
4. Appleby, S.; Bergami, G.; Morgan, G. Running Temporal Logical Queries on the Relational Model. In Proceedings of the International Database Engineered Applications Symposium (IDEAS'22). ACM, 2022, pp. 222–231.
5. Schönig, S.; Rogge-Solti, A.; Cabanillas, C.; Jablonski, S.; Mendling, J. Efficient and Customisable Declarative Process Mining with SQL. In Proceedings of the CAiSE. Springer, 2016.
6. Burattin, A.; Maggi, F.M.; Sperduti, A. Conformance checking based on multi-perspective declarative process models. *Expert Syst. Appl.* **2016**, *65*, 194–211.
7. Pesic, M.; Schonenberg, H.; van der Aalst, W.M.P. DECLARE: Full Support for Loosely-Structured Processes. In Proceedings of the EDOC. IEEE Computer Society, 2007, pp. 287–300.
8. Musser, D.R. Introspective Sorting and Selection Algorithms. *Softw. Pract. Exp.* **1997**, *27*, 983–993.
9. Naldurg, P.; Sen, K.; Thati, P. A Temporal Logic Based Framework for Intrusion Detection. In Proceedings of the Formal Techniques for Networked and Distributed Systems - FORTE 2004, 24th IFIP WG 6.1 International Conference, Madrid Spain, September 27-30, 2004, Proceedings; de Frutos-Escrig, D.; Núñez, M., Eds. Springer, 2004, Vol. 3235, *Lecture Notes in Computer Science*, pp. 359–376.
10. Ray, Indrani. Security Vulnerabilities in Smart Contracts as Specifications in Linear Temporal Logic. Master's thesis, 2021.
11. Buschjäger, S.; Hess, S.; Morik, K. Shrub Ensembles for Online Classification. In Proceedings of the AAAI. AAAI Press, 2022, pp. 6123–6131.
12. Huo, X.; Hao, K.; Chen, L.; song Tang, X.; Wang, T.; Cai, X. A dynamic soft sensor of industrial fuzzy time series with propositional linear temporal logic. *Expert Systems with Applications* **2022**, *201*, 117176.
13. Bergami, G.; Francescomarino, C.D.; Ghidini, C.; Maggi, F.M.; Puura, J. Exploring Business Process Deviance with Sequential and Declarative Patterns. *CoRR* **2021**, *abs/2111.12454*.
14. Zhou, H.; Milani Fard, A.; Makanju, A. The State of Ethereum Smart Contracts Security: Vulnerabilities, Countermeasures, and Tool Support. *Journal of Cybersecurity and Privacy* **2022**, *2*, 358–378.
15. Szabo, N. Smart contracts: building blocks for digital markets. *EXTROPY: The Journal of Transhumanist Thought* **1996**, *18*.
16. Fionda, V.; Greco, G.; Mastratisi, M.A. Reasoning About Smart Contracts Encoded in LTL. In Proceedings of the AIxIA; Springer International Publishing: Cham, 2021; pp. 123–136.
17. Bank, H.S.; D'souza, S.; Rasam, A. Temporal Logic (TL)-Based Autonomy for Smart Manufacturing Systems. *Procedia Manufacturing* **2018**, *26*, 1221–1229. 46th SME North American Manufacturing Research Conference, NAMRC 46, Texas, USA.
18. Mao, X.; Li, X.; Huang, Y.; Shi, J.; Zhang, Y. Programmable Logic Controllers Past Linear Temporal Logic for Monitoring Applications in Industrial Control Systems. *IEEE Transactions on Industrial Informatics* **2022**, *18*, 4393–4405.
19. Boniol, P.; Linardi, M.; Roncallo, F.; Palpanas, T.; Meftah, M.; Remy, E. Unsupervised and scalable subsequence anomaly detection in large data series. *The VLDB Journal* **2021**.
20. Xu, H.; Pang, J.; Yang, X.; Yu, J.; Li, X.; Zhao, D. Modeling clinical activities based on multi-perspective declarative process mining with openEHR's characteristic. *BMC Medical Informatics Decis. Mak.* **2020**, *20-S*, 303.

21. Rovani, M.; Maggi, F.M.; de Leoni, M.; van der Aalst, W.M.P. Declarative process mining in healthcare. *Expert Systems with Applications* **2015**, *42*, 9236–9251.
22. Bertini, F.; Bergami, G.; Montesi, D.; Veronese, G.; Marchesini, G.; Pandolfi, P. Predicting Frailty Condition in Elderly Using Multidimensional Socioclinical Databases. *Proceedings of the IEEE* **2018**, *106*, 723–737.
23. Bellatreche, L.; Kechar, M.; Bahloul, S.N. Bringing Common Subexpression Problem from the Dark to Light: Towards Large-Scale Workload Optimizations. In Proceedings of the IDEAS. ACM, 2021.
24. De Giacomo, G.; Maggi, F.M.; Marrella, A.; Patrizi, F. On the Disruptive Effectiveness of Automated Planning for LTLf-Based Trace Alignment. In Proceedings of the AAAI'17. AAAI press, 2017.
25. Bergami, G.; Maggi, F.M.; Marrella, A.; Montali, M. Aligning Data-Aware Declarative Process Models and Event Logs. In Proceedings of the Business Process Management, 2021, pp. 235–251.
26. Bergami, G. A Logical Model for joining Property Graphs. *CoRR* **2021**, *abs/2106.14766*.
27. Zhu, S.; Pu, G.; Vardi, M.Y. First-Order vs. Second-Order Encodings for LTLf-to-Automata Translation. *CoRR* **2019**, *abs/1901.06108*.
28. Ceri, S.; Gottlob, G. Translating SQL Into Relational Algebra: Optimization, Semantics, and Equivalence of SQL Queries. *IEEE Trans. Software Eng.* **1985**, *11*, 324–345.
29. Calders, T.; Lakshmanan, L.V.S.; Ng, R.T.; Paredaens, J. Expressive power of an algebra for data mining. *ACM Trans. Database Syst.* **2006**, *31*, 1169–1214.
30. Li, J.; Pu, G.; Zhang, Y.; Vardi, M.Y.; Rozier, K.Y. SAT-based explicit LTLf satisfiability checking. *Artificial Intelligence* **2020**, *289*, 103369.
31. Petermann, A.; Junghanns, M.; Müller, R.; Rahm, E. FoodBroker - Generating Synthetic Datasets for Graph-Based Business Analytics. In Proceedings of the WBDB'14, 2014.
32. Bergami, G. On Declare MAX-SAT and a finite Herbrand Base for data-aware logs. *CoRR* **2021**, *abs/2106.07781*.
33. Pichler, P.; Weber, B.; Zugal, S.; Pinggera, J.; Mendling, J.; Reijers, H.A. Imperative versus Declarative Process Modeling Languages: An Empirical Investigation. In Proceedings of the BPM Workshops, 2011, pp. 383–394.
34. Wang, J.; Ntarmos, N.; Triantafillou, P. GraphCache: A Caching System for Graph Queries. In Proceedings of the EDBT. OpenProceedings.org, 2017, pp. 13–24.
35. Keller, A.M.; Basu, J. A Predicate-based Caching Scheme for Client-Server Database Architectures. *VLDB J.* **1996**, *5*, 35–47.
36. Davey, B.A.; Priestley, H.A. *Introduction to Lattices and Order, Second Edition*; Cambridge University Press, 2002.
37. Elmasri, R.; Navathe, S.B. *Fundamentals of Database Systems*, 7th ed.; Pearson, 2015.
38. Idreos, S.; Groffen, F.; Nes, N.; Manegold, S.; Mullender, K.S.; Kersten, M.L. MonetDB: Two Decades of Research in Column-oriented Database Architectures. *IEEE Data Eng. Bull.* **2012**, *35*, 40–45.
39. Polyvyanyy, A.; ter Hofstede, A.H.M.; Rosa, M.L.; Ouyang, C.; Pika, A. Process Query Language: Design, Implementation, and Evaluation; 2019; Vol. abs/1909.09543.
40. Jr., E.G.C.; Graham, R.L. Optimal Scheduling for Two-Processor Systems. *Acta Informatica* **1972**, *1*, 200–213.
41. Sugiyama, K.; Tagawa, S.; Toda, M. Methods for Visual Understanding of Hierarchical System Structures. *IEEE Transactions on Systems, Man, and Cybernetics* **1981**, *11*, 109–125.
42. Bergami, G. On Efficiently Equi-Joining Graphs. In Proceedings of the IDEAS. ACM, 2021.
43. Dittrich, J. *Patterns in Data Management: A Flipped Textbook*; CreateSpace Independent Publishing Platform, 2016.
44. Schönig, S. SQL Queries for Declarative Process Mining on Event Logs of Relational Databases. *CoRR* **2015**, *abs/1512.00196*.
45. Shoshany, B. A C++17 Thread Pool for High-Performance Scientific Computing. *arXiv e-prints* **2021**, [arXiv:cs.DC/2105.00613]. https://doi.org/10.5281/zenodo.4742687.
46. Klemm, M.; Cownie, J., 8 Scheduling parallel loops. In *High Performance Parallel Runtimes*; De Gruyter Oldenbourg: Berlin, Boston, 2021; pp. 228–258.
47. Ristov, S.; Prodan, R.; Gusev, M.; Skala, K. Superlinear speedup in HPC systems: Why and when? In Proceedings of the 2016 Federated Conference on Computer Science and Information Systems (FedCSIS), 2016, pp. 889–898.
48. Yan, B.; Regueiro, R.A. Superlinear speedup phenomenon in parallel 3D Discrete Element Method (DEM) simulations of complex-shaped particles. *Parallel Computing* **2018**, *75*, 61–87.
49. Nagashima, U.; Hyugaji, S.; Sekiguchi, S.; Sato, M.; Hosoya, H. An experience with super-linear speedup achieved by parallel computing on a workstation cluster: Parallel calculation of density of states of large scale cyclic polyacenes. *Parallel Computing* **1995**, *21*, 1491–1504. https://doi.org/https://doi.org/10.1016/0167-8191(95)00026-K.
50. de Berg, M.; Cheong, O.; van Kreveld, M.J.; Overmars, M.H. *Computational geometry: algorithms and applications, 3rd Edition*; Springer, 2008.
51. Augusto, A.; Awad, A.; Dumas, M. Efficient Checking of Temporal Compliance Rules Over Business Process Event Logs. *CoRR* **2021**, *abs/2112.04623*.
52. Maier, D. The Complexity of Some Problems on Subsequences and Supersequences. *J. ACM* **1978**, *25*, 322–336. https://doi.org/10.1145/322063.322075.
53. Davey, B.A.; Priestley, H.A. *Introduction to Lattices and Order*, 2 ed.; Cambridge University Press, 2002.

Short Biography of Authors

Giacomo Bergami received the PhD degree in Computer Science and Engineering from the University of Bologna, Bologna, Italy, in 2018. He is currently a Lecturer (Assistant Professor) in Game Technology at Newcastle University. He is known for designing algorithms for big data solutions outperforming relational, document-oriented, and graph databases. His research interests span from IoT simulation towards data integration and machine learning with logical guarantees. He hosts more than 100 GitHub projects through his several personal accounts. He is part of the Program Committee of IDEAS and SimBIG and is the editor of the SOCA journal.

Samuel Appleby received the MSc in Computer Game Engineering from Newcastle University, Newcastle, U.K., in 2021, where he is a second year Ph.D. Student. His research focuses on Artificial Intelligence with a special interest in Video-Game Technology. During his Master's Degree, he was hired by Lucid Games Ltd, where he worked as a Junior Network Programmer until applying for his Ph.D. for the year commencing 2022.

Graham Morgan received the Ph.D. degree in computing science from Newcastle University, Newcastle, U.K., in 1999. He has worked and led industry-focussed research for more than 20 years in the areas of distributed systems, cloud computing, real-time simulations (video game technologies), and digital healthcare. He is the Director of Networked and Ubiquitous Systems Engineering (NUSE) Group, jointly with Professor Rajiv Ranjan, in the School of Computing. He is well known for establishing and leading the Game Technology Laboratory, which hosts the industry-led Game Engineering M.Sc. He has served on more than 50 international conference committees, edited and reviewed for top international journals, acted as a project reviewer for a number of governments, presented invited talks across a number of continents, and been invited to attend many events at several learned societies.

Supplementary Material

Part

Table of Contents

Supplement I LTL$_f$ Semantics

The First Order Logic semantics associated with the LTL$_f$ syntax provided in Equation 1 is defined as follows:

- an event satisfies the **activity label** a iff. its activity labels is a: $\sigma^i_j \vDash \mathsf{a} \Leftrightarrow \sigma^i_j = \langle \mathsf{a}, p\rangle$
- an event satisfies the **negated** formula iff. the same event does not satisfy the non-negated formula: $\sigma^i_j \vDash \neg\varphi \Leftrightarrow \sigma^i_j \nvDash \varphi$
- an event satisfies the **disjunction** of LTL$_f$ sub-formulæ iff. the event satisfies one of the two sub-formulæ: $\sigma^i_j \vDash \varphi \vee \varphi' \Leftrightarrow \sigma^i_j \vDash \varphi \vee \sigma^i_j \vDash \varphi'$
- an event satisfies the **conjunction** of LTL$_f$ formulæ iff. the event satisfies all of the sub-formulæ: $\sigma^i_j \vDash \varphi \wedge \varphi' \Leftrightarrow \sigma^i_j \vDash \varphi \wedge \sigma^i_j \vDash \varphi'$
- an event satisfies a formula at the **next** step iff. the formula is satisfied in the incoming event if present: $\sigma^i_j \vDash \bigcirc\varphi \Leftrightarrow i < |\sigma_j| \wedge \sigma^i_{j+1} \vDash \varphi$
- an event **globally** satisfies a formula iff. the formula is satisfied in all the following events, including the current one: $\sigma^i_j \vDash \Box\varphi \Leftrightarrow \forall j \leq x \leq |\sigma^i|.\sigma^i_x \vDash \varphi$
- an event **eventually** satisfies a formula iff. the formula is satisfied in either the present or in any future event: $\sigma^i_j \vDash \Diamond\varphi \Leftrightarrow \exists j \leq x \leq |\sigma^i|.\sigma^i_x \vDash \varphi$
- an event satisfies φ **until** φ' holds iff. φ holds at least until φ' becomes true, which must hold at the current or a future position: $\sigma^i_j \vDash \varphi\ \mathcal{U}\ \varphi' \Leftrightarrow \exists j \leq y \leq |\sigma^i|.\sigma^i_y \vDash \varphi' \wedge (\forall x \leq z < y.\sigma^i_z \vDash \varphi)$

Supplement II $\mathcal{D}_\varphi$-encoding pipeline

This section describes in greater details the algorithmic details of $\mathcal{D}_\varphi$-encoding [25].

Given an LTL$_f$ expression φ containing compound conditions, we denote $\mathcal{D}_\varphi$ as the set of distinct compound conditions in φ. We refer to the items in $\mathcal{D}_\varphi$ as *atoms* iff. for each pair of distinct compound conditions in it, those never jointly satisfy any possible payload

p[19]. [25] shows a procedure how any formula φ can be rewritten into an equivalent one φ' by ensuring $\mathcal{D}_{\varphi'}$ contains atoms. This can be achieved by constructing $\mathcal{D}_{\varphi'}$ first from φ (Algorithm S1), and then converting each compound conditions in φ as disjunctions of atoms in $\mathcal{D}_{\varphi'}$, thus obtaining φ'.

We collect all the conjunctions referring to the same payload key into a map $\mu(\mathsf{a},\kappa)$ (Line 4). After doing so, we can construct a Segment Tree [50] from the intervals in $\mu(\mathsf{a},\kappa)$, thus identifying the *elementary intervals* partitioning the collected intervals (Line 14). These elementary intervals also partition the payload data space associated with events for each activity label a. This can be achieved by combining each elementary interval in each dimension κ for a (Line 16) and then associating it with a new atom representing such a partition (Line 18) that is then guaranteed to be an atom by construction. This entails that each interval $low_\kappa \leq \kappa \leq up_\kappa$ will be characterised by the disjunction of all of the atoms p_i comprising such interval (Line 21). Given this, we can then associate to each activation condition A that is associated with an activation payload condition p the disjunction of atoms that are collected by the following formula:

$$\mathsf{Atom}_{\mu,ad}(\mathsf{A},p) := \bigcup_{conj\in p}\ \bigcap_{(low\leq\kappa\leq up)\in conj}\ \bigcup_{I\in\mu(\mathsf{A},\kappa).\mathtt{findElementaryIntervals}(low,up)} ad(\mathsf{A},I) \quad \text{(S1)}$$

Time Complexity. If we assume that the dimension of $\mu(\mathsf{a},\kappa)$ for each $\mathsf{a}\in\Sigma$ and $\kappa\in K$ is at most m, our implementation[20] builds such trees in $\sum_{1\leq i\leq m}\log(i)+m \in O(m\cdot\log(m))$ time, as we first insert the intervals into the data structure and then we guarantee to minimise the tree representation, requiring a linear visit cost to the whole tree data structure. The time complexity of $\mathcal{D}_\varphi$-ENCODING() is $m|K|(1+\log m+|\Sigma|)\in O(m|K||\Sigma|)$.

19 More formally, $\forall p.\forall \mathsf{a}\in\Sigma.\forall \mathsf{a}\wedge q, \mathsf{a}\wedge q' \in \mathcal{D}_\varphi.(q\neq q') \Rightarrow (q(p) \Rightarrow \neg q'(p))$

20 https://github.com/datagram-db/knobab/blob/main/include/yaucl/structures/query_interval_set/structures/segment_partition_tree.h

Algorithm S1 $\mathcal{D}_\varphi$-encoding pipeline [25].

```
1: global $\mu \leftarrow \{\}$; $ad \leftarrow \{\}$; $ak \leftarrow \{\}$

2: procedure CollectIntervals(a, DNF)        ▷ DNF:= $\bigvee_{1\leq i\leq n}\bigwedge_{1\leq k\leq m(i)} low_{i,k}\leq k_{i,k}\leq up_{i,k}$
3:     for all $conj \in$ DNF and $low \leq k \leq up \in conj$ do
4:         $\mu(\mathsf{a},k)$.put($[low,up]$)
5:     end for

6: procedure CollectIntervals($\mathcal{M}$)        ▷ $\mathcal{M} := \bigwedge_{1\leq i\leq|\mathcal{M}|} \mathsf{clause}_i(\mathsf{A},p,\mathsf{B},p')$
7:     for all $\mathsf{clause}_i(\mathsf{A},p,\mathsf{B},p') \in \mathcal{M}$ do
8:         if $p \neq$ True then CollectIntervals(A, $p$)
9:         if $p' \neq$ True then CollectIntervals(B, $p'$)
10:    end for

11: procedure $\mathcal{D}_\varphi$-Encoding( )
12:    for all a $\in \Sigma$ do
13:        for all $k \in K$ do
14:            $\mu(\mathsf{a},k) \leftarrow$SegmentTree($\mu(\mathsf{a},k)$)
15:        end for
16:        for all $partition \in \times_{k\in K} \mu(\mathsf{a},k)$.elementaryIntervals() do        ▷ $partition := (low_k \leq k \leq up_k)_{k\in K}$
17:            $p_i \leftarrow$new atom()
18:            $p_i := \mathsf{a} \wedge partition$
19:            $ak(\mathsf{a})$.put($p_i$)
20:            for all $low_k \leq k \leq up_k \in partition$ do
21:                $ad(\mathsf{a},\ low_k \leq k \leq up_k)$.put($p_i$)
22:            end for
23:        end for
24:        if $ak(\mathsf{a}) = \emptyset$ then
25:            $ak(\mathsf{a}) \leftarrow \{\mathsf{a}\}$
26:        end if
27:    end for
```

Supplement III Matching Requirements.

Depending on the binary $\texttt{xtLTL}_\text{f}$ operator of choice, we might set a successful match if at least two events in a trace, one per operator, match anytime in the future (E^i_Θ), or if correlation conditions hold across activated and targeted events within the temporal window of interest (e.g., *Until*; A^i_Θ). Given some maps M_1 and M_2 associating events of interest from the same trace σ^i to a list L providing the activation and target conditions to be tested, we define such two distinct matching requirements as follows:

$$E^i_\Theta(M_1, M_2) = \Big\{ M[h,k] \;\Big|\; \exists j \in \text{dom}(M_1) \cap \text{dom}(M_2), L_1 \in M_1(j), L_2 \in M_2(j). \\ (A(h) \in L_1 \vee \exists h'.M[h,h'] \in L_1), \\ (T(k) \in L_2 \vee \exists k'.M[k,k'] \in L_2),\ \Theta(\sigma^i_h, \sigma^i_k) \Big\} \tag{S2}$$

$$A^i_\Theta(M_1, M_2) = \begin{cases} \varnothing & \exists j \in \text{dom}(M_1) \cap \text{dom}(M_2). E^i_\Theta([j \mapsto M_1(j)], [j \mapsto M_2(j)]) = \varnothing \\ E^i_\Theta(M_1, M_2) & \text{oth.} \end{cases} \tag{S3}$$

where i indicates the trace id (e.g., of σ^i) from which the events in L should be considered while reconstructing their associated payload while determining the binary predicate Θ.

After expressing the different matching semantics as such, we skip testing correlations when (e.g.) only one of the two operands will contain activation (or target) events. We distinguish the case where the activation and target conditions of interest are missing because some basic operators did not return those from the scenario where the correlation condition was tested but failed. While in the former we return either an empty set or a set of just activations or targets, in the latter we want to return a placeholder **False** to remember that the correlation condition was falsified. We can then define a *testing functor* as follows:

$$\mathcal{T}^{F,i}_\Theta(M_1, M_2) = \begin{cases} \text{cod}(M_1) \cup \text{cod}(M_2) & \text{dom}(M_1) = \varnothing \vee \text{dom}(M_2) = \varnothing \\ F^i_\Theta(M_1, M_2) & F^i_\Theta(M_1, M_2) \neq \varnothing \\ \textbf{False} & \text{oth.} \end{cases} \tag{S4}$$

where F subsumes one of the two matching semantics of choice, either E^i_Θ or A^i_Θ.

Supplement IV Time Complexity

Supplement IV.1 Pipeline Components

We now discuss the first two procedures from Algorithm 1.

Lemma S1. BULKINSERTION *can be computed in quasi-linear time with respect to the full log size (thus including the trace size, as well as the size of their associated payload).*

Proof. As most of the data structures are vectors (Lines 7 and 8) and the only two maps are a hashmap (Line 6) and a hashmap of ordered maps (Line 11), this phase has a quasi-linear time computational complexity with respect to the size of the log, thus showing that no additional overhead for pre-calculating the semantics associated to the declarative clauses is required in [51]. In fact, bulk inserting the data requires scanning all the traces and their associated events, as well as its possible associated payload, while the insertion in the $\texttt{AttBulkMap}_k$ performs a sorted insertion for each value $p(k)$ (Line 11); this pays a log-linear cost over the number of distinct values per key k. □

Lemma S2. LOADINGANDINDEXING *has linear time complexity with respect to the log size (trace and data payload inclusive).*

Proof. We show that the first outer for loop has a time complexity in $O(|\mathcal{L}|\epsilon)$. The initialization of the primary index of $\textsf{ActivityTable}_{\mathcal{L}}$ requires a linear scan of every unique activity label of the log, with index assignments being performed in constant time (Line 18). The loading of the aforementioned table (minus assignments of `Prev(Next)`) requires a scan of every element of the log, thus providing a linear complexity with respect to the log size (inclusive of trace lengths, Line 23). The loading of the $\textsf{CountingTable}_{\mathcal{L}}$ has complexity $O(|\Sigma||\mathcal{L}|)$ (Line 20), which is always dominated by loading of the $\textsf{ActivityTable}_{\mathcal{L}}$, as it is guranteed that $|\Sigma| \leq |\mathcal{L}|\epsilon$. In fact, in the worst-case scenario, every single event has a unique activity label, and therefore $|\Sigma| = |\mathcal{L}|\epsilon = |\texttt{ActToEventBulkVector}|$. In the best-case scenario, there is only one activity label ($|\Sigma| = 1$) providing a linear complexity with log size.

In the best case scenario, no trace comes with a data payload, and therefore the entire outermost second loop is skipped as the set of keys is completely empty, and therefore no $\textsf{AttributeTable}^{\kappa}_{\mathcal{L}}$ will be initialised. Otherwise, loading an $\textsf{AttributeTable}^{\kappa}_{\mathcal{L}}$ associated to a key $\kappa \in K$ requires a linear scan over each activity label a from Σ. While doing so, we cals access the `AttBulkMap` for every unique activity label and payload key, from which we also scan all the events from *lst* associated to a same value v. The initialisation of each $\textsf{AttributeTable}^{k}_{\mathcal{L}}$ primary index is linear with respect to the number of unique activity labels (Line 35); the loading and secondary indexing of $\textsf{AttributeTable}^{\kappa}_{\mathcal{L}}$ is bounded by the number of events associating a value v to each key κ of interest. In the worst-case scenario, every event contains the same key labels. Overall, this provides a complexity dominated by the number of key-value associations in each event and therefore by $|K|$, as each event can contain an unlimited number of payload keys. This boils down to a worst-case time complexity of $O(|\mathcal{L}|\epsilon|K|)$.

We show that the last outer for loop also has a time complexity in $O(|\mathcal{L}|\epsilon)$. This includes the assignment of the `Prev` (`Next`) pointers and the initialisation of the $\textsf{ActivityTable}_{\mathcal{L}}$'s secondary index , requiring only a linear scan of the log (trace inclusive). No other data structures are iterated, therefore we always perform computations in $O(|\mathcal{L}|\epsilon)$ time.

Overall, in the best case scenario no event is associated to a payload, and therefore the time complexity is dominated by the first two loops, which provide a time complexity linear to the log size, trace size included: $O(|\mathcal{L}|\epsilon)$. Otherwise, we have to scan each value associated to each event which, in the worst case scenario, requires a full scan of $|K|$ for each event. In this other scenario, the time complexity is heavily dominated by the data payload and therefore is $O(|\mathcal{L}|\epsilon|K|)$. □

Last, we discuss the time complexity for the query processing algorithms.

Lemma S3. ATOMIZATIONPIPELINE *(Algorithm 2) has polynomial time complexity with respect to the model, key-set, and elementary intervals' maximum size.*

Proof. In the best-case scenario, all the clauses have no payload conditions, for which the activation condition will only contain the activation (or target) activity label. For this, we only pay the linear cost of scanning the model, $O(|\mathcal{M}|)$. In the worst-case scenario, each clause has a non-trivial payload condition for activation and target (when available): this implies always computing an intersection over unordered sets, which has the computational cost of scanning the smaller set and checking which elements are contained in the larger, where each operation takes $O(1)$ time. We can assume that each p is described in disjunctive normal form and that its size is negligible if compared to the size of both the collected elementary intervals and the model. Given that usually $ak(\mathsf{B})$ contains all of the atoms referring to the data predicates to B and therefore $\textsf{Atom}_{\mu,ad}(\mathsf{B},\kappa) \subseteq ak(\mathsf{B})$ for any $\kappa \in K$, we have a computational complexity of $O(m|K||\mathcal{M}|)$ under the assumption that each clause's data payload condition contains predicates for any payload key in $O(|K|)$ and that each interval has a size comparable to the maximum size of $m = \mu(\mathsf{a},\kappa)$ for $\mathsf{a} \in \Sigma$ and $\kappa \in K$. □

Lemma S4. QUERYSCHEDULER *(Algorithm 3) has a linear time complexity with respect to the number of operators appearing in the compiled query plan.*

Proof. If we exclude the root node, the branching factor of our graph is at most 2, as each operator (represented as a node in the graph) might contain at most two sub-expressions; therefore, we can estimate that the number of edges of $\mathcal{G}$ is linear with respect to the graph nodes. As the scheduler is represented through a vector of vectors and the distance computation requires visiting each edge associated with each non-leaf node, while filling the scheduler layer-wise just requires visiting each operator, the time complexity of this step is linear with respect to the operators appearing in the formula. □

Supplement IV.2 Operators' algorithmic implementation

We now discuss the computational complexity of the $\texttt{xt}\mathrm{LTL_f}$ operators. Let us assume that ℓ is the size of the collected events in L that need to be matched. Let us also remember that the intermediate results ρ and ρ' are represented as vectors of triplets sorted by trace and event id. Let us also assume all traces have a maximum length of ϵ. In the worst case scenario, we have $|\rho| = |\rho'| = |\mathcal{L}|\epsilon$.

Lemma S5. *$\mathsf{Future}(\rho)$ has linear time complexity with respect to the operand size while $\mathsf{Future}^{\tau}(\rho)$ has quadratic time complexity with respect to the maximum trace length size.*

Proof. The computational complexity associated with the untimed Future operator boils down to performing an aggregation for each trace represented in the operand as well as its represented event, where its collected activation/target conditions are all associated with the first event of the trace. Therefore, this has a linear time complexity with respect to the size of each operand, i.e. $|\rho| \in O(|\mathcal{L}|\epsilon)$.

On the other hand, its timed counterpart (Future^{τ}) needs both to list all of the events in the operand and, for each of them, to associate to it also the activation and target conditions happening in the future, thus requiring a quadratic cost over the size of each trace. This boils down to $O(|\mathcal{L}|\epsilon^2)$. □

Lemma S6. TIMEDINTERSECTION *has computational complexity in $O(\ell^2|\mathcal{L}|\epsilon)$.*

Proof. The time complexity of the TIMEDINTERSECTION is in the worst-case scenario in $O(\ell^2|\mathcal{L}|\epsilon)$ if we assume that each matched event needs to be tested and that ρ and ρ' have non-empty Ls. If either Θ is always true or the Ls of the two operands are empty, the time complexity boils down to $O(|\mathcal{L}|\epsilon)$. □

Corollary S1. *For $\Theta = \textbf{\textit{True}}$,* FASTUNTIMEDAND$(\rho, \rho')$ *is faster than* SLOWUNTIMEDAND(ρ, ρ'). *For $\ell \approx 1$, the theoretical speed-up is proportional to the increase of both trace length and number of traces.*

Proof. The time complexity of the SLOWUNTIMEDAND is in the worst case scenario in $O(\ell^2|\mathcal{L}|\epsilon^2)$: as the former is a minor adaptation of TIMEDINTERSECTION where, nevertheless, we have to pay a quadratic cost for scanning the whole traces as in timed future (Future^{τ}). This has also an additional cost of $O(|\mathcal{L}|\log|\mathcal{L}|\epsilon)$ gained while grouping the operands' results by trace id.

On the other hand, its FASTUNTIMEDAND avoids a quadratic cost if $\Theta = \textbf{True}$ by only collecting the activation and target conditions through a linear scan of both operands, thus boiling down its computational complexity to $O(2|\mathcal{L}|\epsilon\ell)$.

At this point, we can compute the speed-up of the allegedly slower version over the faster one by checking when the following condition holds:

$$\frac{\ell^2|\mathcal{L}|\epsilon^2 + 2|\mathcal{L}|\log|\mathcal{L}|\epsilon}{2|\mathcal{L}|\epsilon\ell} = \frac{\ell\epsilon}{2} + \frac{\log|\mathcal{L}|}{\ell} > 1$$

We can close the goal after observing that the aforementioned condition always holds, and that is heavily dominated by the average trace length. □

Corollary S2. *Computing* FASTUNTIMEDAND(ρ, ρ') *is more efficient than its equivalent* FASTUNTIMEDAND$(\mathsf{Future}(\rho), \mathsf{Future}(\rho'))$ *by a theoretical constant speed up.*

Proof. By expanding and composing the definitions of the former computational complexities, we get that the speed-up is almost constant for small ℓ values:

$$\frac{2|\mathcal{L}|\epsilon + 2\mathcal{L}\epsilon\ell}{2\mathcal{L}\epsilon\ell} = \frac{1}{\ell} + 1 > 1$$

□

Corollary S3. *Computing* FASTUNTIMEDOR(ρ, ρ') *is more efficient than* Choice *as per its original Declare semantics, i.e.* FASTUNTIMEDOR$(\mathsf{Future}(\rho), \mathsf{Future}(\rho'))$ *by a theoretical constant speed up.*

Proof. As when no traces match we have no additional matching costs as they are just collected in the final results with a linear scan, the worst-case scenario boils down to performing trace matches as per the previous corollary. □

Lemma S7. *For $\Theta = \textbf{\textit{True}}$,* UNTIMEDUNTIL$^2(\rho, \rho')$ *is faster than* UNTIMEDUNTIL$^1(\rho, \rho')$ *for greater log sizes.*

Proof. In both implementations of the untimed Until operator, for each log trace, we perform a logarithmic scan for reaching the end of each trace within the operand. Trace by trace, the computation time decreases with the number of the operand's traces being visited. This can be expressed as $\sum_{i=0}^{|\mathcal{L}|-1} \log(|\mathcal{L} - i|\epsilon) = \sum_{i=1}^{|\mathcal{L}|} \log(i\epsilon)$. Given $\sum_{i=0}^{n} \log i = \log(i!)$, we can rewrite the computational cost of such operation as follows:

$$\sum_{i=1}^{|\mathcal{L}|} \log(i\epsilon) = \sum_{i=1}^{|\mathcal{L}|} (\log i + \log \epsilon) = \log(|\mathcal{L}|!) + |\mathcal{L}| \log \epsilon \leq |\mathcal{L}|(\log |\mathcal{L}|\epsilon)$$

In its worst-case scenario, UNTIMEDUNTIL^1 always run the same scan over each trace's targeted events for each activated event in the second operand, while its Fast- counterpart performs constant access for each of the ϵ trace events. So, while UNTIMEDUNTIL^2 has an overall time complexity of $\epsilon|\mathcal{L}|$, the former has the upper bound of:

$$= \sum_{i=1}^{|\mathcal{L}|} (\log i + \sum_{j=1}^{\epsilon} \log j) \leq |\mathcal{L}| \log |\mathcal{L}| + |\mathcal{L}|\epsilon \log \epsilon = |\mathcal{L}| \log(|\mathcal{L}|\epsilon^{\epsilon})$$

Furthermore, for both implementations we pay at most a quadratic cost for scanning each targeted event that needs to hold until the activation is met, thus including the cost of inserting the matched conditions in the intermediate result set. This implies a cost of $O(\epsilon^2|\mathcal{L}|)$ despite $\Theta = \textbf{True}$. As for the speed-up we can neglect the sub-operations having the same computational complexity[21], we can rewrite the asymptotic speed-up as:

$$\frac{|\mathcal{L}| \log(|\mathcal{L}|\epsilon^{\epsilon})}{\epsilon|\mathcal{L}|} = \frac{\log |\mathcal{L}|}{\epsilon} + \log \epsilon > 1$$

which is always true only for adequately large traces, $\varepsilon \gg 0$. We can see that if ϵ is fixed, the speed-up grows proportionally with the log size. As the equation is always

[21] $\frac{A+B}{A+C} > 1 \leftrightarrow B > C \leftrightarrow \frac{B}{C} > 1$ when $A + B > 0$, $A + C > 0$, and $A > 0$ are always true.

Algorithm S2 xtLTL$_f$ pseudocode implementation for derived operators, First Variant

```
function TIMEDANDFUTURE$^1_\Theta(\rho, \rho')$
    $it$ ←Iterator$(\rho)$, $it'$ ←Iterator$(\rho')$
    while $it \neq\uparrow$ and $it' \neq\uparrow$ do
        $\langle t, e, L\rangle \leftarrow$ current$(it)$, $\langle t', e'', \lambda\rangle \leftarrow$ current$(it')$
        if $t = t'$ and $e \geq e''$ then
            $L'' \leftarrow \emptyset$; $hasMatch \leftarrow \Theta =$ True
            $it'_* \leftarrow it'$
            while $it'_* \neq\uparrow$ do
                $\langle t', e', L'\rangle \leftarrow$ current$(it'_*)$;
                if $t' \neq t$ then break;
                $tmp \leftarrow \mathcal{T}^{E,i}_\Theta(L, L')$                ▷ Algorithm 6
                if $tmp \neq$ False then
                    $hasMatch \leftarrow$ true; $L'' \leftarrow L'' \cup tmp$
                end if
                next$(it'_*)$
            end while
            if $hasMatch$ then yield $\langle t, e, L''\rangle$;
            next$(it)$; next$(it')$;
        else if $t < t'$ or $(t = t'$ and $e < e')$ then
            next$(it)$
        else
            next$(it')$
        end if
    end while

function TIMEDANDGLOBALLY$^1_\Theta(\rho, \rho')$
    $it$ ←Iterator$(\rho)$, $it'$ ←Iterator$(\rho')$
    while $it \neq\uparrow$ and $it' \neq\uparrow$ do
        $\langle t, e, L\rangle \leftarrow$ current$(it)$, $\langle t', e'', \lambda\rangle \leftarrow$ current$(it')$
        if $t = t'$ and $e = e''$ then
            $L'' \leftarrow \emptyset$; $hasMatch \leftarrow \Theta =$ True; $count \leftarrow 0$
            $it'_* \leftarrow it'$
            while $it'_* \neq\uparrow$ do
                $\langle t', e', L'\rangle \leftarrow$ current$(it'_*)$
                if $t' \neq t$ then break;
                $tmp \leftarrow \mathcal{T}^{E,i}_\Theta(L, L')$                ▷ Algorithm 6
                if $tmp \neq$ False then
                    $hasMatch \leftarrow$ true; $L'' \leftarrow L'' \cup tmp$; $count \leftarrow count+1$
                end if
                next$(it'_*)$
            end while
            if $hasMatch$ and $count = |\sigma^t| - e + 1$ then yield $\langle t, e, L''\rangle$;
            next$(it)$; next$(it')$;
        else if $t < t'$ or $(t = t'$ and $e < e')$ then
            next$(it)$
        else
            next$(it')$
        end if
    end while
```

true for non-empty datasets, the fast implementation is always faster than the slower counterpart. □

Corollary S4. *For $\Theta = $ **True**, the second variant of the (timed)* ANDGLOBALLY *(Algorithm S3) is theoretically more performant than the first one (Algorithm S2) when the number of traces is exponentially upper bounded by the maximum trace length size.*

Proof. In the second variant of the operator, for each trace, we always perform a logarithmic scan for both operators for determining the end of the trace. As seen in the previous lemma, this corresponds to $O(|\mathcal{L}|\log(|\mathcal{L}|\epsilon^\epsilon))$. While scanning the trace events backwards we have that, in the worst-case scenario, each event on the left operand corresponds an event on the right one. This operation, as well as the gradual backward creation of the result, boils down to an additional time complexity of $2\epsilon|\mathcal{L}|$.

In the first algorithm for the operator (referred to as "variant"), we scan all of the events per single trace, this reducing to $2|\mathcal{L}|\epsilon$, but, for each event in the left operand, we have to scan all of the events in the right one until the end of the trace, thus adding up to a quadratic computational complexity with respect to the trace size: $\epsilon|\mathcal{L}|$.

Algorithm S3 xtLTL$_f$ pseudocode implementation for derived operators, Second Variant

```
function TimedAndFuture$^2_\Theta(\rho, \rho')$
    if $\rho' = \emptyset$ then return $\emptyset$                                        ▷ Yielding no records
    $it \leftarrow$Iterator$(\rho), it' \leftarrow$Iterator$(\rho')$
    while $it \neq \uparrow$ do
        $toBeReversed \leftarrow \emptyset$; $\langle i_c, j_c, L\rangle \leftarrow \texttt{current}(it)$; $\langle i', j', L'\rangle \leftarrow \texttt{current}(it')$
        if $i_c > i'$ or $(i = i'$ and $j_c > j')$ then
            $it' \leftarrow$LowerBound$(\rho, it', \uparrow, \langle i_c, j_c, \top_\Omega\rangle)$; if $it' = \uparrow$ break
        else if $i < i'$ then
            $it \leftarrow$LowerBound$(\rho, it, \uparrow, \langle i', 1, \top_\Omega\rangle)$;
        else
            if $it' = \uparrow$ break; $tmp \leftarrow \emptyset$
            $bend \leftarrow$ UpperBound$(\rho', it', \uparrow, \langle i', |\sigma^{i'}| + 1, \top_\Omega\rangle)$; $aend \leftarrow$ UpperBound$(\rho, it, \uparrow, \langle i_c, |\sigma^{i_c}| + 1, \top_\Omega\rangle)$
            if $\texttt{current}(it' - 1).i \neq i$ then
                $it \leftarrow aend$;
                while$(it \neq \uparrow)$ and $\texttt{current}(it).i = i$ do $it$++
                continue
            end if
            $it \leftarrow$--$bend$; $it' \leftarrow$--$aend$; $\langle i, j, L\rangle \leftarrow \texttt{current}(it)$; $\langle i', j', L'\rangle \leftarrow \texttt{current}(it')$
            for $J \leftarrow j'$ downto 1 by 1 do
                if $bend \geq it'$ and $J = j'$ then
                    $tmp \leftarrow tmp \cup L'$; $it'$--; $\langle i', j', L'\rangle \leftarrow \texttt{current}(it')$
                end if
                while $i = i'$ and $j > j'$ do
                    $aend$--; $\langle i, j, L\rangle \leftarrow \texttt{current}(it)$.
                end while
                if $i \neq i'$ then break
                if $j < j_c$ or $j < J$ then continue
                $tmp \leftarrow \mathcal{T}^{E,i}_\Theta(L, L')$                                     ▷ Algorithm 6
                if $tmp \neq$ False then $toBeReversed$.add$(\langle i, j, tmp\rangle)$
            end for
            forall $\langle i, j, L''\rangle \in$ Reversed$(toBeReversed)$ do yield $\langle i, j, L''\rangle$
            $it \leftarrow aend$; if $it' = \uparrow$ then break;
        end if
    end while

function TimedAndGlobally$^2_\Theta(\rho, \rho')$
    if $\rho' = \emptyset$ then return $\emptyset$                                        ▷ Yielding no records
    $it \leftarrow$Iterator$(\rho), it' \leftarrow$Iterator$(\rho')$
    while $it \neq \uparrow$ do
        $toBeReversed \leftarrow \emptyset$; $\langle i, j, L\rangle \leftarrow \texttt{current}(it)$; $\langle i', j', L'\rangle \leftarrow \texttt{current}(it')$
        if $i > i'$ or $(i = i'$ and $j > j')$ then
            $it$++; if $it' = \uparrow$ break
        else
            if $it' = \uparrow$ break; $tmp \leftarrow \emptyset$
            $bend \leftarrow$ UpperBound$(\rho', it', \uparrow, \langle i', |\sigma^{i'}| + 1, \top_\Omega\rangle)$; $aend \leftarrow$ UpperBound$(\rho, it, \uparrow, \langle i, |\sigma^{i}| + 1, \top_\Omega\rangle)$
            $it \leftarrow$--$bend$; $it' \leftarrow$--$aend$; $\langle i, j, L\rangle \leftarrow \texttt{current}(it)$; $\langle i', j', L'\rangle \leftarrow \texttt{current}(it')$
            $J \leftarrow j'$
            for $J \leftarrow j'$ downto 1 by 1 do
                if $bend \geq it'$ and Distance$(it', bend) = |\sigma^{i'}| - J + 1$ then
                    $tmp \leftarrow tmp \cup L'$; $it'$--; $\langle i', j', L'\rangle \leftarrow \texttt{current}(it')$
                else break
                end if
                while $i = i'$ and $j > j'$ do
                    $it$--; $\langle i, j, L\rangle \leftarrow \texttt{current}(it)$.
                end while
                if $i \neq i'$ then break
                if $j < J$ then continue
                $tmp \leftarrow \mathcal{T}^{E,i}_\Theta(L, L')$                                     ▷ Algorithm 6
                if $tmp \neq$ False then $toBeReversed$.add$(\langle i, j, tmp\rangle)$
            end for
            forall $\langle i, j, L''\rangle \in$ Reversed$(toBeReversed)$ do yield $\langle i, j, L''\rangle$
            $it \leftarrow aend$; if $it' = \uparrow$ then break
        end if
    end while
```

The asymptotic and theoretical speed-up can be then computed as follows:

$$\frac{\epsilon^2|\mathcal{L}|}{2|\mathcal{L}|\log(|\mathcal{L}|\epsilon^\epsilon)} = \frac{\epsilon^2}{2(\log|\mathcal{L}| + \epsilon\log\epsilon)} > 1 \Leftrightarrow \frac{\epsilon^2}{2} - \epsilon\log\epsilon > \log|\mathcal{L}|$$
$$\Leftrightarrow |\mathcal{L}| < 2^{\epsilon^2/2 - \epsilon\log\epsilon} \leq \sqrt{2^{\epsilon^2}}$$

We can therefore conclude that the first variant will be faster when traces are shorter, while the other one will be always faster for "reasonably" long traces. We can also observe that, when the log size is fixed, the speed-up grows most quadratically with respect to the maximum trace length. □

Supplement V Maximal Common Subset Problem

In order to minimise union and intersection operations over a sequence of atoms, we had to solve multiple times the MAXIMAL COMMON SUBSET PROBLEM[22]. This problem, different from the maximal common subsequence for sequences (e.g., strings) [52], can be stated as follows:

Problem S1. *Given a set of sets $\mathcal{S} = \{ S_1, \dots, S_n \}$, we want to compute the maximal common subsets $\Delta = \{ \delta_1, \dots, \delta_k \}$ so that each set $S_\iota \in \mathcal{S}$ can be defined as the union of some pairwise disjoint sets $\delta_{\iota_1}, \dots, \delta_{\iota_m}$ in Δ. This decomposition has to guarantee that there exists no other decomposition containing non-overlapping supersets of the sets given in Δ providing such decomposition.*

Please observe that, if all of the sets in $\mathcal{S}$ are pairwise disjoint, then $\mathcal{S} = \Delta$.

Example S1. *Given $\mathcal{S} = \{ \{1,2,3\}, \{2,3\}, \{4,5\}, \{1,2,3,4,5\}, \{2,3,4,5\} \}$, we obtain the maximal common subsets $\Delta = \{ \{2,3\}, \{1\}, \{4,5\} \}$. Given this, we might characterize the sets in $\mathcal{S}$ from the constituents in Δ, and therefore* decomposition *returned by the* MAXIMALCOMMONSUBSET *function allows the characterization of $\mathcal{S}$ as $\{ \delta_2 \cup \delta_1,\ \delta_1,\ \delta_3,\ \delta_2 \cup \delta_1 \cup \delta_3,\ \delta_1 \cup \delta_3 \}$ via the returned decomposition $\mathcal{I}$.*

Algorithm S4 sketches a solution to the maximal common subsets problem: by remembering the sets S_i in which each set item is contained (Line 4) and by subsequently remembering the items that are shared among the sets in $\mathcal{S}$ (Line 7), the latter map identifies the desired maximal common subsets. Therefore, each set S_ι can be defined through the union of the maximal common subsets $\delta_k \in \Delta$ (Line 10).

The primary aim of this algorithm in KnoBAB is to minimize the computations of Or or And operators where the computation of further intermediate subsets is appropriate. As we might indeed see from the former example, the reconstruction of the original sets in $\mathcal{S}$ from their constituents in Δ implies evaluating the same unions multiple times, e.g. $\delta_2 \cup \delta_1$ and $\delta_1 \cup \delta_3$. This overhead can be limited as follows: first, we might sort the decomposed itemsets by their inclusion relationship (Line 17): this is possible as the set of all possible subsets is a partially ordered set (poset) where the partial order is the subset-equal relationship [53]. Last, we iterate over all the possible decomposed itemsets I which do not appear as a singleton, as their replacement is trivial (Line 18): if the decomposition I' for a set S_j contains the decomposition I for a set S_i, we can elect I as a new refined subset δ_k (Line 25) which is now a component for I' (Line 27).

Example S2. *Let us continue the former example: the refinement step computes a new refinement $\delta_4 \in \Delta'$ which is defined as $\delta_1 \cup \delta_2$; the characterization of $\mathcal{S}$ has now become $\{\delta_4, \delta_1, \delta_3, \delta_4 \cup \delta_3, \delta_1 \cup \delta_3\}$.*

Example S3. *Let us now suppose to have an array of sets containing* `xt` LTL$_f$ *formulæ$\mathcal{S} = \{ \{\psi_1, \psi_2, \psi_3\}, \{\psi_2, \psi_3\}, \{\psi_4, \psi_5\}, \{\psi_1, \psi_2, \psi_3, \psi_4, \psi_5\}, \{\psi_2, \psi_3, \psi_4, \psi_5\} \}$, where we want to express each set in $\mathcal{S}$ as a timed disjunction $Or^{\mathcal{I}}_{True}$ formulæ. While doing so, we want also to minimise as possible the computation of unions that are shared among different sets. Figure S1 shows the desired result, where Δ and Δ' contain the intermediate untimed unions or just one single* `xt` LTL$_f$ *formula, while S will contain the nodes that will perform the final union operations.*

[22] This problem was coded as the `partition_sets` function in https://github.com/datagram-db/knobab/blob/main/include/yaucl/structures/set_operations.h.

Algorithm S4 Maximal Common Subsets

```
function MAXIMALCOMMONSUBSET($\mathcal{S}$)
    setsToSharedValues$\leftarrow$ {}; itemInSets $\leftarrow$ {}; $\mathcal{I} \leftarrow \{\}$; $k \leftarrow 1$; $\Delta \leftarrow \varnothing$
    for all $S_i \in \mathcal{S}$ and item $\in S_i$ do
        itemInSets[item].put($i$)
    end for
    for all $\langle$item, $\{i, j, \dots\}\rangle$ $\in$itemInSets do
        setsToSharedValues[$\{i, j, \dots\}$].put(item)                ▷ item$\in S_i \cap S_j \cap \dots$
    end for
    for all $\langle\{i, j, \dots\}, \{\text{item}_\alpha, \text{item}_\beta, \dots\}\rangle$ $\in$setsToSharedValues and $\iota \in \{i, j, \dots\}$ do
        $\mathcal{I}[\iota]$.put($k$)                                      ▷ $\delta_k \subseteq S_\iota, S_\iota \in \mathcal{S}$
        $\Delta$.put($\{\text{item}_\alpha, \text{item}_\beta, \dots\}$)     ▷ $\delta_k = \{\text{item}_\alpha, \text{item}_\beta, \dots\}$
        $k \leftarrow k + 1$
    end for
    return $\langle\Delta, \mathcal{I}\rangle$

function MCSREFINEMENT($\Delta, \mathcal{I}$)
    $\Delta' \leftarrow \varnothing$; $k \leftarrow |\Delta| + 1$; elemMap $\leftarrow$ {}
    SORT($\mathcal{I}$, ($\langle s, I\rangle$, $\langle s', I'\rangle$) $\rightarrow I \subseteq I'$ )
    for all $\langle s, I\rangle_i \in \mathcal{I}$ from $i = 1$ to $|\mathcal{I}|$ s.t. $|I| \neq 1$ do
        hasElem $\leftarrow$ False;
        for all $\langle s', I'\rangle_j \in \mathcal{I}$ from $j = |\mathcal{I}| - 1$ to $i + 1$ s.t. $|I'| \neq 1$ do
            if $|I| = |I'|$ then break
            else if $I \subseteq I'$ then
                if not hasElem then
                    hasElem $\leftarrow$ True
                    $\Delta'$.put($I$)
                end if
                $I' \leftarrow I' \backslash I \cup \{k\}$
            end if
        end for
        if hasElem then
            elemMap[$i$] $\leftarrow k$
            $k \leftarrow k + 1$
        end if
    end for
    return $\langle\Delta, \Delta', \mathcal{I}\rangle$
    for all $\langle i, k\rangle \in$ elemMap do
        $\mathcal{I}[i] \leftarrow \{k\}$
    end for
```

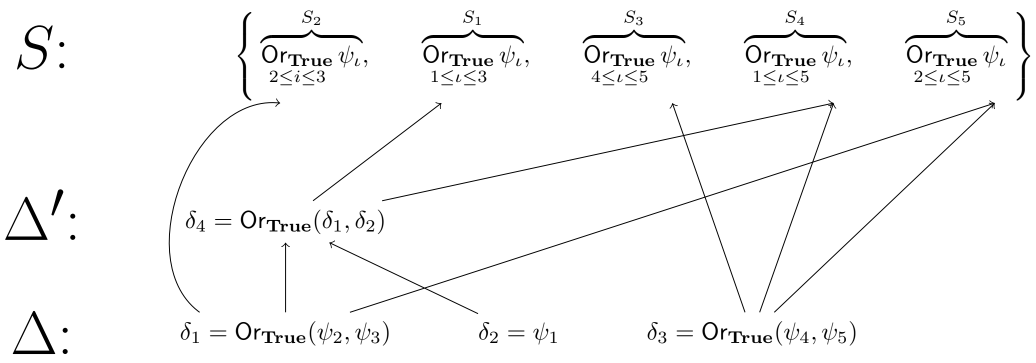

S:
S_2 $\mathsf{Or}_{\mathbf{True}}\,\psi_\iota$, $2\leq i\leq 3$
S_1 $\mathsf{Or}_{\mathbf{True}}\,\psi_\iota$, $1\leq\iota\leq 3$
S_3 $\mathsf{Or}_{\mathbf{True}}\,\psi_\iota$, $4\leq\iota\leq 5$
S_4 $\mathsf{Or}_{\mathbf{True}}\,\psi_\iota$, $1\leq\iota\leq 5$
S_5 $\mathsf{Or}_{\mathbf{True}}\,\psi_\iota$ $2\leq\iota\leq 5$
Δ′:
$\delta_4 = \mathsf{Or}_{\mathbf{True}}(\delta_1, \delta_2)$
Δ:
$\delta_1 = \mathsf{Or}_{\mathbf{True}}(\psi_2, \psi_3)$
$\delta_2 = \psi_1$
$\delta_3 = \mathsf{Or}_{\mathbf{True}}(\psi_4, \psi_5)$

Figure S1. Decomposing untimed unions of xtLTL$_f$ formulæ into maximal shared sub-expressions via an abstract syntax DAG, thus attempting at computing each untimed union shared among different expressions at most once.

FINITARYSETOPERATIONS($\mathcal{S}$, Or$_{\textit{True}}$) *returns the desired* **xt** LTL$_f$ *DAG in Figure S1, rooted in each distinct formula associated to a set in* $\mathcal{S}$.

Time Complexity. For MAXIMALCOMMONSUBSET, if we assume that each subset is at most of size m, then the time complexity of the first loop is in $O(|\mathcal{S}|m)$; the second loop iterates over the whole possible items contained in each set in $\mathcal{S}$, and therefore has a time complexity of $O(|\bigcup\mathcal{S}|)$; the last iteration performs a number of iterations which is comparable to the size of Δ which, in the worst-case scenario, is composed of singletons of size $k = 1$ comprising elements from $\bigcup\mathcal{S}$, and therefore has a time complexity in $O(|\bigcup\mathcal{S}|)$. We also assume that all the insertion operations can be performed in $O(1)$ time, as all the

Algorithm S5 Finitary set operations for commutative and associative set operations $\oplus$

```
function FINITARYSETOPERATIONS($\mathcal{S}, \bigoplus$)                    ▷ $\oplus \in \{\bigcup, \bigcap\}$
    $\Delta, \mathcal{I} \leftarrow$ MAXIMALCOMMONSUBSETS($\mathcal{S}$)
    $\Delta, \Delta', \mathcal{I} \leftarrow$ MCSREFINEMENT($\Delta, \mathcal{I}$)
    for all $s' \in \Delta'$ (parallel) do
        $\delta_{s'} \leftarrow \bigoplus_{\delta_\iota \in \Delta} \delta_\iota$
    end for
    for all $\langle i, D \rangle \in \mathcal{I}$ do     ▷ Decomposition $D$ of $S_i$ through intermediate sets $\delta_s$ with $s \in D$
        yield $\langle i, \bigoplus_{s \in D} \delta_s \rangle$                  ▷ Result associated to $S_i \in S$
    end for

function FINITARYSETOPERATIONS($\mathcal{S}, Results, \bigoplus$)           ▷ $\oplus \in \{\bigcup, \bigcap\}$
    $\Delta, \mathcal{I} \leftarrow$ MAXIMALCOMMONSUBSETS($\mathcal{S}$)
    $\Delta, \Delta', \mathcal{I} \leftarrow$ MCSREFINEMENT($\Delta, \mathcal{I}$)
    for all $s \in \Delta$ (parallel) do
        $\delta_s \leftarrow \bigoplus_{\iota \in s} Results[\iota]$
    end for
    for all $s' \in \Delta'$ (parallel) do
        $\delta_{s'} \leftarrow \bigoplus_{\iota \in s'} \delta_\iota$
    end for
    for all $\langle i, D \rangle \in \mathcal{I}$ do     ▷ Decomposition $D$ of $S_i$ through intermediate sets $\delta_s \in \Delta \cup \Delta'$
        yield $\langle i, \bigoplus_{s \in D} \delta_s \rangle$
    end for
```

data structures that might be exploited are either hash maps or vectors. Overall, this makes a worst case scenario time complexity of $O(|\mathcal{S}|m + |\bigcup \mathcal{S}|)$.

In MCSREFINEMENT, as checking the subset relationship among itemsets has at most a cost proportional to the size k of each itemset, this makes such sorting $O(|\bigcup \mathcal{S}| \log(|\bigcup \mathcal{S}|))$ when $k = 1$ and $O(|\bigcup S| \log(c))$ when $k = {|\bigcup \mathcal{S}|}/{c}$ for c constant; so, even in this scenario, we have the worst case scenario computational complexity for $k = 1$. Last, in the worst case scenario, the double for loop has a computational complexity comparable to $O(|\mathcal{S}|^2)$, as we perform an iteration over all the possible characterizations of the sets in $\mathcal{S}$ while trying to aggregate the remaining shared union operations further.

The overall time complexity of both phases becomes $O(|\mathcal{S}|(m+|\mathcal{S}|) + |\bigcup \mathcal{S}| \log(|\bigcup \mathcal{S}|))$.

Supplement V.1 Use Case: Query Optimizer

This subsection describes in greater detail the Query Optimizer pipeline (§5.1.2) sketched in Algorithm S6.

Line 11 refers to the *first* phase and shows the point in the code where we associate each negated leaf with the complementary set of atoms appearing after the decomposition process. With respect to the *second* phase, Line 65 shows the rewriting of the Declare clause into an intermediate xtLTL$_f$ by recursively visiting it in each of its operands until the leaves are reached (Line 5). If during this visit we meet a binary operator marked as being the "tester" for the correlation condition, we associate to it the Θ coming from the declarative clause (Line 4); otherwise, the operator keeps the default **True**. Concerning the leaves, for unary clauses, we consider the sole activation condition, while for binary clauses, we might also consider target conditions. If the leaf node is associated with an S_A (or S_T) containing more than one activity label or atom, we need to keep track of all of those while representing such a leaf as a disjunction of those (Lines 18-25). Next, we optimize each disjunction of atoms and activity labels so to minimize the number of shared union computations (Line 48); such optimization is performed after fully visiting the xtLTL$_f$ expression, thus ensuring that each appearing disjunction is actually collected (Line 69).

Line 14 shows where we collect atoms representing compound conditions while guaranteeing that its associated $\texttt{Compound}^{\mathcal{L},\tau}_{A/T}$ operator is computed only once, as well as decomposing it in its constituent compound conditions.

Third, method PUTINCACHE extends the *queryCache* map by guaranteeing that each distinct disjunction of atoms is also represented at most once within the query plan.

Example S4. *The upper box from Figure S2 (compare the box with Figure 5) show a query plan where common union operations are shared across sub-trees by representing each sub-tree at most*

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 13 January 2023 doi:10.20944/preprints202301.0254.v1

Algorithm S6 Query Optimizer

```
global $declare2\mathtt{xtLTL}_f \leftarrow \{\}$; $queryCache \leftarrow \{\}$; $collectUnions \leftarrow \{\}$; $\mathcal{Q} \leftarrow \{\}$; $atomQ \leftarrow \emptyset$
global keyToLabelToSortedIntervals $\leftarrow \{\}$; $\mathcal{S}_\Sigma \leftarrow \{\}$; $Results \leftarrow \{\}$

function INSTANTIATE($\psi, \Theta, S_A, S_T$)
    if $\psi$.hasTheta then $\psi$.theta $\leftarrow \Theta$
    if $\psi$.arg$= \emptyset$ then                                    ▷ $\psi$ is a leaf
        if $\psi$.isActivation or $\psi$.isNeither then
            $\psi$.atom $\leftarrow S_A$
        else if $\psi$.isTarget then
            $\psi$.atom $\leftarrow S_T$
        end if
        if $\psi$.negated then $\psi$.atom $\leftarrow \complement\, \psi$.atom     ▷ Complementing the atoms from the universe set upon negation
        for all $atom \in \psi$.atom do
            if $atom \in \bigcup_{\mathsf{a}\in\Sigma} ak(\mathsf{a})$ then        ▷ The atom is generated from $\mathcal{D}_\varphi$-encoding
                RETRIEVEINTERVALS($atom$)
            else $atomQ$.put($atom$)
            end if
        end for
        if $|\psi$.atom$| > 1$ then
            $disj \leftarrow \emptyset$
            for all $atom \in \psi$.atom do
                $\psi' \leftarrow$ new $\mathtt{xtLTL}_f$ ()
                $\psi'$.atom $= \{atom\}$
                $disj$.put($atom$)
            end for
            $collectUnions[disj]$.put($\psi$)
        else
        end if
    else
        for all arg $\in \psi$ do
            arg $\leftarrow$INSTANTIATE(arg$, \Theta, S_A, S_T$)
        end for
    end if

procedure COLLECTUNIONS( )                          ▷ DAG over the leaves undergoing union operations.
    for all $\langle atomSet, \psi' \rangle \in$ FINITARYSETOPERATIONS($collectUnions$, $\mathsf{Or}_{\mathbf{True}}$) do
        for all $\psi \in collectUnions[atomSet]$ do
            $queryCache[\psi] \leftarrow \psi'$
        end for
    end for

procedure PUTINCACHE($\psi$)
    if $\exists\psi'.\, \langle \psi, \psi' \rangle \in queryCache$ then
        return $\psi'$
    else
        for all arg $\in \psi$.args do
            arg $\leftarrow$PUTINCACHE(arg)
        end for
        $\psi' \leftarrow$ new $\mathtt{xtLTL}_f$ ()
        $\psi' \leftarrow \psi$
        $queryCache[\psi] \leftarrow \psi'$
        return $\psi'$
    end if

procedure RETRIEVEINTERVALS($p_i$)                                    ▷ $p_i := \mathsf{a} \wedge partition$
    for all $low_\kappa \leq \kappa \leq up_\kappa \in partition$ do          ▷ $p_i = \bigwedge_{\kappa\in K} low_\kappa \leq \kappa \leq up_\kappa$
        if $\exists h.\, \langle low_\kappa \leq \kappa \leq up_\kappa, h \rangle \in$ keyToLabelToSortedIntervals$[\kappa][\mathsf{a}]$ then
            $\mathcal{S}_\Sigma[p_i]$.put($h$)
        else
            $Results$.put($\emptyset$)
            $\mathcal{S}_\Sigma[p_i]$.put($|Results|$)
            keyToLabelToSortedIntervals$[\kappa][\mathsf{a}]$.put($\langle low_\kappa \leq \kappa \leq up_\kappa, |Results| \rangle$)
        end if
    end for

function QUERYOPTIMIZER($\mathcal{M}$,$queryRoot$)
    for all clause$_l$($\mathsf{A}, p, \mathsf{B}, q$) where $\Theta \in \mathcal{M}$ do
        if $\exists\psi$: $\mathtt{xtLTL}_f$. $\langle$clause$_l$($\mathsf{A}, p, \mathsf{B}, q$) where $\Theta, \psi\rangle \in declare2\mathtt{xtLTL}_f$ then $\mathcal{Q}$.push($\psi$)
        else
            $\psi \leftarrow$ INSTANTIATE(xtTemplates[clause$_l$], $\Theta$, clause$_l$.left, clause$_l$.right)
            $\mathcal{Q}$.push($\psi$)
        end if
    end for
    COLLECTUNIONS( )
    $queryRoot$.args $\leftarrow$ { PUTINCACHE($\psi$) | $\psi \in \mathcal{Q}$ }
    return $queryRoot$
```

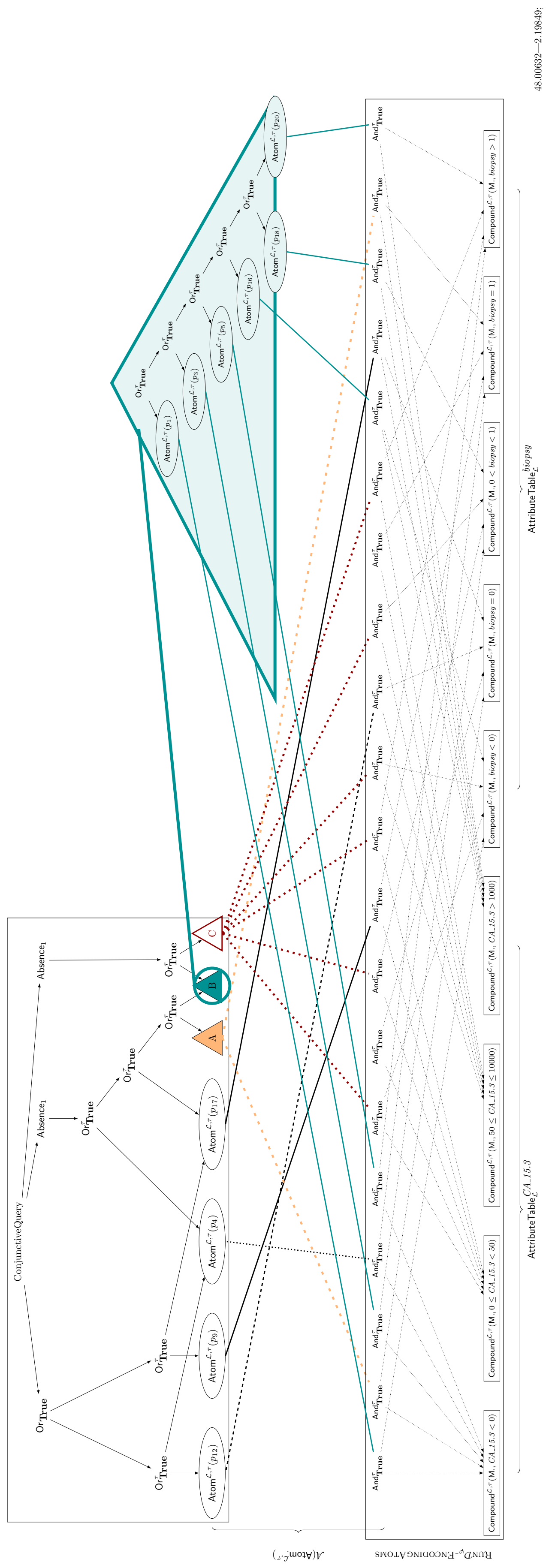


Figure S2. Detailed view of the query plan represented in Figure 5 for Example 15.

Algorithm S7 Query Plan Initialisation and Execution for $\mathcal{D}_\varphi$-encoding p_i atoms.

```
1: global keyToLabelToSortedIntervals ← {}; $\mathcal{S}_\Sigma$ ← {}; $Results$ ← {}; $Cache$ ← {}

2: procedure ExecuteRangeQueries($\mathcal{L}$)
3:     for all $\langle\kappa, map\rangle \in$ keyToLabelToSortedIntervals do
4:         if table AttributeTable$^\kappa_\mathcal{L}$ not exists then continue
5:         for all $\langle a, L\rangle \in map$ (parallel) do
6:             if $\exists\langle beg, end\rangle . \langle beg, end\rangle \in$ AttributeTable$^\kappa_\mathcal{L}$.primary_index[$\beta$(a)] then
7:                 Sort($L$)
8:                 for all $\langle low_\kappa \le \kappa \le up_\kappa, idx\rangle \in L$ do      ▷ Computing Compound$^{\mathcal{L},\tau}_?$(a, $\kappa$, [$low_\kappa, up_\kappa$])
9:                     $beg$ ← LowerBound(AttributeTable$^\kappa_\mathcal{L}$, $beg$, $low_\kappa$, $end$)
10:                    $mid$ ← UpperBound(AttributeTable$^\kappa_\mathcal{L}$, $beg$, $up_\kappa$, $end$)
11:                    while $beg$<$mid$ and $beg$<$end$ do
12:                        $i, j$ ← ActivityTable$_\mathcal{L}$[AttributeTable$^\kappa_\mathcal{L}$[$beg$](Offset)](Trace, Event)
13:                        $Results$[$idx$].put($\langle i, j\rangle$)
14:                        $beg$++;
15:                    end while
16:                    $beg$ ← $mid$; Sort($Results$[$idx$])
17:                    if $beg \ge end$ then break
18:                end for
19:            end if
20:        end for
21:    end for

22: procedure Run$\mathcal{D}_\varphi$-EncodingAtoms($\mathcal{L}$)
23:     ExecuteRangeQueries($\mathcal{L}$)
24:     for all $\langle p_i, Result\rangle \in$FinitarySetOperations($\mathcal{S}_\Sigma$, $Results$, $\bigcap$) do      ▷ Algorithm S5
25:         $Cache$[$p_i$] ← $Result$
26:     end for
27:     $\mathcal{S}_\Sigma$ ← {}; $Results$ ← {}; keyToLabelToSortedIntervals ← {}

28: function $\mathcal{A}$(Atom$^{\mathcal{L},\tau}_?$)($\psi$)      ▷ require $\psi$.atom = {$p_\iota$} with $p_\iota$ from the $\mathcal{D}_\varphi$-encoding pipeline
29:     if $\psi$.isActivation then      ▷ Atom$^{\mathcal{L},\tau}_A$($p_\iota$)
30:         return { $\langle i, j, \{A(j)\}\rangle \mid \langle i, j\rangle \in Cache[p_\iota]$ }
31:     else if $\psi$.isTarget then      ▷ Atom$^{\mathcal{L},\tau}_T$($p_\iota$)
32:         return { $\langle i, j, \{T(j)\}\rangle \mid \langle i, j\rangle \in Cache[p_\iota]$ }
33:     else      ▷ Atom$^{\mathcal{L},\tau}$($p_\iota$)
34:         return { $\langle i, j, \emptyset\rangle \mid \langle i, j\rangle \in Cache[p_\iota]$ }
35:     end if
```

once (compare to Figure S1). These are actually represented in the query plan as opposed to the evaluation associated with the atoms, which is discussed in the following subsection. The compound operators are therefore placed at the bottom of Figure S2, and then connect with their corresponding atom could be tracked by navigating the edges backwards.

Supplement V.2 Use Case: Running Atom$^{\mathcal{L},\tau}_{A/T}$ operators

This subsection describes how efficiently computing the results associated to the Atom$^{\mathcal{L},\tau}_{A/T}$ operator also requires the computation of the MaximalCommonSubset problem. This is achieved not only by performing all the Compound$^{\mathcal{L},\tau}_{A/T}$ queries to be run on the same AttributeTable$^k_\mathcal{L}$ in one single scan of the former but by also running the final result associated to the Atom$^{\mathcal{L},\tau}_{A/T}$ by computing the shared intersections among different atoms' definition at most once. This detailed pseudocode is shown in Algorithm S7.

First, we evaluate each compound condition over the AttributeTable$^k_\mathcal{L}$, as each atom is a conjunction of interval queries expressed as compound conditions. As the query compiler grouped such intervals by AttributeTable$^k_\mathcal{L}$ and associated activity labels by calling the RetrieveIntervals method in the keyToLabelToSortedIntervals map, we now need to iterate over such a map and access the AttributeTable$^\kappa_\mathcal{L}$ for each key κ appearing in it (Line 4). Each AttributeTable$^k_\mathcal{L}$ is then accessed by activity label a through its primary index (Line 6): if the primary index points to a specific region in the table delimited by a beginning and end table offset, we can access it and query it via the sorted compound condition. For each of those (representing elementary intervals), we get the offset beg (and end) to the first element within the block equal or greater to the lower bound low_κ (or strictly greater to the upper bound up_κ). Such bounds are returned in Lines 9 and 10. For each record existing in such a range, we reconstruct the trace and event id containing the desired value (Line 12) and then store it into a preliminary result[23] (Line 13). As both the elementary intervals and the table are sorted in ascending order by value and given that each elementary interval is non-overlapping by construction, we can query the next interval in L from the remaining portion of the table, thus steadily reducing the table visiting cost by further reducing the data over which the scan is performed. As these results are not sorted by increasing trace and event id rather than by increasing value in V, we have to sort them accordingly (Line 16) to be compatible with the intermediate results ρ requirements. Please observe that despite each of these intervals could appear in the definition of multiple atoms, this construction guarantees that each of these queries is performed at most once per occurring interval.

Second, we can compute the events associated with each p_i atom returned from the $\mathcal{D}_\varphi$-encoding pipeline by intersecting the result associated with each of the intervals. The result of such intersections is then stored in an intermediate cache (Line 25). We can perform these intersections efficiently by guaranteeing the computation of each sub-intersection shared across different atoms at most once through the calculation of maximal common subsets occurring in intersections' sub-expressions.

Third, while evaluating each atom from our scheduled operator in ψ (Line 7 from Algorithm 4), we can retrieve the result from the cache and, dependingly on whether this expression was associated to an activation (Line 29 from Algorithm S7), a target condition (Line 31), or neither of these (Line 34), we can now effectively represent the result as an intermediate result with the appropriate label markers in L.

Example S5. *The lowermost boxes from Figure S2 represent the compound conditions extracted from the atoms appearing on the leaf nodes of the query plan, as well as their associated result stored in "Results" after running the* ExecuteRangeQueries*. The result of the intersection between all of the possible compound conditions as per atom definitions is then computed in the* ***for*** *loop in* Run$\mathcal{D}_\varphi$-EncodingAtoms*. At this point, the execution of the leaf nodes on the query plan boils down to accessing the upper nodes in the box above and converting such results according to the activation or target condition associated with them. This mechanism guarantees that the results and the tables are accessed at most once, independently from the activation/target mark associated with the atom nodes.*

Supplement VI Data And Models

We wanted to test our claim that exploiting data access minimisation techniques to can quicken conformance checking as well as model mining time. In addition, we want to prove that even in the worst-case scenario, we still achieve orders of magnitude of gains versus state-of the art solutions. Therefore, we provide two models, Table S2 and Table S1a (also available at [14]). The former was adapted from Burattin et al. [6], where we exploited the constraints they provided, allowing for a direct comparison. These tables are displayed under the assumption from [6], where the trace payloads are injected into every event. In our actual implementation, we inject a `__trace_payload`event at the beginning every trace. Therefore, M_{11} from Table S1a has a KnoBAB representation as Table S1b, where the resulting model size is five (seven) for Burattin (KnoBAB).

By performing conjunctions among these, we were able to provide 11 models, the largest consisting of five (seven, due to trace payload injection) clauses. The queries constructing these larger models are similar, for example, even the model $q_1 \wedge q_2$ hold the clause Response(`A_SUBMITTED`,**true**,`A_ACCEPTED`,**true**) twice. These clauses need not be duplicated in the query plan, and therefore these models should better demonstrate the gains from data access minimisation. Conversely, Table S1a denotes an entirely novel model, that avoids similar queries. With each sub-model, entirely new events are considered, and even within those payload conditions vary. As such, our pipeline can gain much less from

[23] Please observe: **not** the intermediate representation ρ!

Table S1. Worst-case scenario (SCENARIO 1) model representation for the BPIC_2012 dataset.

(a) Clause constituents of each model, where each sub-model is guaranteed to be a subset of the larger.

Model	*Clauses*
$M_1 =$	$q_1 :=$ Response(A_SUBMITTED, **true**, A_ACCEPTED, **true**)
	$q_2 :=$ Response(A_SUBMITTED, AMOUNT_REQ $\geq 10^3$, A_ACCEPTED, **true**)
	$q_3 :=$ Response(A_SUBMITTED, AMOUNT_REQ $< 10^3$, A_ACCEPTED, **true**)
	$q_4 := q_1$ where A_SUBMITTED.org:resource $=$ A_ACCEPTED.org:resource
	$q_5 := q_1$ where A_SUBMITTED.org:resource $\neq$ A_ACCEPTED.org:resource
$M_2 = M_1 +$	$q_6 :=$ Response(W_Completeren aanvraag, **true**, W_Valideren aanvraag, **true**)
	$q_7 :=$ Response(W_Completeren aanvraag, **true**, O_CANCELLED, **true**)
	$q_8 := q_6$ where W_Valideren aanvraag.org:resource $\neq$ W_Valideren aanvraag.org:resource
	$q_9 :=$ Response(W_Valideren aanvraag, AMOUNT_REQ $= 5 \cdot 10^3$, O_CANCELLED, **true**)
	$q_{10} := q_9$ where W_Valideren aanvraag.org:resource $=$ O_CANCELLED.org:resource
$M_3 = M_2 +$	$q_{11} :=$ Response(O_SELECTED, **true**, O_CANCELLED, **true**)
	$q_{12} := q_{11}$ where O_SELECTED.org:resource $=$ O_CANCELLED.org:resource
	$q_{13} :=$ Response(O_SELECTED, AMOUNT_REQ $< 8 \cdot 10^3$, O_CANCELLED, **true**)
	$q_{14} := q_{13}$ where O_SELECTED.org:resource $=$ O_CANCELLED.org:resource
	$q_{15} :=$ Response(O_SELECTED, AMOUNT_REQ $> 10^3$, O_CANCELLED, **true**) where O_SELECTED.org:resource $\neq$ O_CANCELLED.org:resource
$M_4 = M_3 +$	$q_{16} :=$ Response(A_PARTLYSUBMITTED, **true**, A_DECLINED, **true**)
	$q_{17} := q_{16}$ where A_PARTLYSUBMITTED.org:resource $=$ A_DECLINED.org:resource
	$q_{18} :=$ Response(A_PARTLYSUBMITTED, AMOUNT_REQ $> 2 \cdot 10^4$, A_DECLINED, **true**)
	$q_{19} :=$ Response(A_PARTLYSUBMITTED, AMOUNT_REQ $> 2 \cdot 10^4$, A_CANCELLED, **true**)
	$q_{20} := q_{18}$ where A_PARTLYSUBMITTED.org:resource $=$ A_DECLINED.org:resource

(b) Distinguishing consisting clauses of from Table S2, where [6] must inject the trace payload into each event.

Clause	*Declare Analyzer Representation*	*KnoBAB representation*
q_1	Response(A_SUBMITTED, **true**, A_ACCEPTED, **true**)	Response(A_SUBMITTED, **true**, A_ACCEPTED, **true**)
q_2	Response(A_SUBMITTED, AMOUNT_REQ $\geq 10^3$, A_ACCEPTED, **true**)	Response(A_SUBMITTED, **true**, A_ACCEPTED, **true**) Exists(__trace_payload, AMOUNT_REQ $\geq 10^3, \geq 1$)
q_3	Response(A_SUBMITTED, AMOUNT_REQ $< 10^3$, A_ACCEPTED, **true**)	Response(A_SUBMITTED, **true**, A_ACCEPTED, **true**) Exists(__trace_payload, AMOUNT_REQ $< 10^3, \geq 1$)
q_4	Response(A_SUBMITTED, **true**, A_ACCEPTED, **true**) where A_SUBMITTED.org:resource $=$ A_ACCEPTED.org:resource	Response(A_SUBMITTED, **true**, A_ACCEPTED, **true**) where A_SUBMITTED.org:resource $=$ A_ACCEPTED.org:resource
q_5	Response(A_SUBMITTED, **true**, A_ACCEPTED, **true**) where A_SUBMITTED.org:resource $\neq$ A_ACCEPTED.org:resource	Response(A_SUBMITTED, **true**, A_ACCEPTED, **true**) where A_SUBMITTED.org:resource $\neq$ A_ACCEPTED.org:resource

Table S2. Declare Models for a data access best-case scenario for the BPIC_2012 dataset. These queries are performed under the assumption the trace payload has been injected into every event of the trace.

Model Name	*Conjunctive Query*
M_1	$q_1 :=$ Response(A_SUBMITTED, **true**, A_ACCEPTED, **true**)
M_2	$q_2 :=$ Response(A_SUBMITTED, AMOUNT_REQ $\geq 10^3$, A_ACCEPTED, **true**)
M_3	$q_3 :=$ Response(A_SUBMITTED, AMOUNT_REQ $< 10^3$, A_ACCEPTED, **true**)
M_4	$q_4 := q_1$ where A_SUBMITTED.org:resource $=$ A_ACCEPTED.org:resource
M_5	$q_5 := q_1$ where A_SUBMITTED.org:resource $\neq$ A_ACCEPTED.org:resource
M_6	$q_1 \wedge q_2$
M_7	$q_1 \wedge q_2 \wedge q_4$
M_8	$q_1 \wedge q_3 \wedge q_4$
M_9	$q_1 \wedge q_2 \wedge q_5$
M_{10}	$q_1 \wedge q_3 \wedge q_5$
M_{11}	$q_1 \wedge q_2 \wedge q_3 \wedge q_4 \wedge q_5$

DAG optimization techniques. This model, therefore, provides a more appropriate case for analysing these proposed enhancements.