Article

On the Solution to Riemann Hypothesis

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Abstract: Proof of the Riemann hypothesis is obtained using the properties of the Gamma function, the Bernoulli function, and limitations imposed by the rational relations of the complex roots of the Riemann zeta function, and the rational relations of the complex Gamma functions.

Keywords: Gamma function; Zeta function; Bernoulli numbers;

1. Introduction

Most of the information about the ζ-function is well known, and the particulars of the theorems and foundational work that concern the Riemann hypothesis will not be covered in detail. It is important to recognize that the Riemann hypothesis is the foundation of many important theorems, and in a sense is the corner-stone of number theory. It is expected that a solution to the Riemann hypothesis will not require any particular theorems to prove it, since it is a fundamental property of complex numbers.

The Riemann Zeta function, $\zeta(s)$, is defined by:

$$\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots = \sum_{n=1}^{\infty} n^{-s}, s \in \mathbb{C} \tag{1}$$

This study will use the convention, $s = \sigma + i\tau$, were, $\tau \in \Re$ (reals). Euler proved that the function $\zeta(s), s = \sigma + it, \sigma > 1$ can be represented in terms of primes, p. $\zeta(s)$ is analytic for $\sigma > 1$ and satisfies in this half-plane the identity:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p}^{\infty} \left(1 - \frac{1}{p^s}\right)^{-1} \tag{2}$$

here, p is a prime. Except for a pole at $s = 1$, $\zeta(s)$ behaves properly and can be easily extended using the Gamma Function. The extension of $\zeta(s)$ to the entire complex plane can be obtained by consideration the entirety and the general definition of the Gamma function:

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt \tag{3}$$

Change variables by the substitution $t = n^2\pi x$, $z = \frac{s}{2}$, in (3),

$$\Gamma\left(\frac{S}{2}\right) = (n^2\pi)^{\frac{s}{2}} \int_0^{\infty} e^{-n^2\pi x} x^{\frac{s}{2}-1}\, dx \tag{4}$$

Extracting $\zeta(s)$ from (4),

$$\pi^{-\frac{s}{2}} \zeta(s) \Gamma\left(\frac{S}{2}\right) = \sum_{n=1}^{\infty} \int_0^{\infty} e^{-n^2\pi x} x^{\frac{s}{2}-1}\, dx \tag{5}$$

The convergence of the series,

$$S(x) = \sum_{n=1}^{\infty} e^{-n^2\pi x} \tag{6}$$

in the interval $[0, \infty]$ gives the relation:

$$\pi^{-\frac{s}{2}}\zeta(s)\Gamma\left(\frac{S}{2}\right) = \int_0^\infty S(x)x^{\frac{s}{2}-1}\ dx \tag{7}$$

This can be split into two separate integrals:

$$\pi^{-\frac{s}{2}}\zeta(s)\Gamma\left(\frac{S}{2}\right) = \int_0^1 S(x)x^{\frac{s}{2}-1}\ dx\ + \int_1^\infty S(x)x^{\frac{s}{2}-1}\ dx \tag{8}$$

Note that the sum (6) is related to the Jacobi Theta function. See Ref [1].

$$\theta(x) = \sum_{n=-\infty}^{\infty} e^{-n^2\pi x} \tag{9}$$

$$S(x) = 2\sum_{n=1}^{\infty} e^{-n^2\pi x} = \sum_{n=-\infty}^{\infty} e^{-n^2\pi x} - 1 \quad = \quad \theta(x) - 1 \tag{10}$$

The Jacobi theta function obeys the symmetry

$$x^{\frac{1}{2}}\theta(x) = \theta(x^{-1}) \tag{11}$$

Thus

$$x^{\frac{1}{2}}(2S(x) + 1) = 2S(x^{-1}) + 1 \tag{12}$$

Clearly, this leads to the reflection formula as follows.

$$S(x^{-1}) = -\frac{1}{2} + \frac{1}{2}x^{\frac{1}{2}} + x^{\frac{1}{2}}S(x) \tag{13}$$

The integral (8.0) now becomes

$$\pi^{-\frac{s}{2}}\zeta(s)\Gamma\left(\frac{S}{2}\right) = \int_1^\infty S(x^{-1})x^{-\frac{s}{2}-1}\ dx\ + \int_1^\infty S(x)x^{\frac{s}{2}-1}\ dx \tag{14}$$

$$\pi^{-\frac{s}{2}}\zeta(s)\Gamma\left(\frac{S}{2}\right) = \int_1^\infty x^{-\frac{s}{2}-1}\left(-\frac{1}{2} + \frac{1}{2}x^{\frac{1}{2}} + x^{\frac{1}{2}}S(x)\right)\ dx\ + \int_1^\infty S(x)x^{\frac{s}{2}-1}\ dx \tag{15}$$

$$\pi^{-\frac{s}{2}}\zeta(s)\Gamma\left(\frac{S}{2}\right) = \frac{1}{s(s-1)} + \int_1^\infty S(x)\left(x^{\frac{s}{2}-1} + x^{-\frac{s}{2}-\frac{1}{2}}\right)\ dx \tag{16}$$

The right side of the relation (16.) is invariant to the substitution $s \to 1 - s$. This gives the reflection formula for $\zeta(s)$:

$$\pi^{-\frac{s}{2}}\zeta(s)\Gamma\left(\frac{S}{2}\right) = \pi^{-\frac{1-s}{2}}\zeta(1-s)\Gamma\left(\frac{1-s}{2}\right) \tag{17}$$

The reflection formula indicates that the roots should obey a reflection and a conjugate symmetry if they lie on the ½-line.

2.0 The relationship between ζ(s), s, Bernoulli numbers, and Gamma functions with the rational functions of the roots.

The rational functions of the roots of the Riemann zeta function are also expressed as rational Gamma functions. The behavior of the Gamma function for complex arguments restricts the validity of these rational arguments to arguments $\tau \in \Re(s) > 0$. As will be shown, this restriction also applies to the roots.

The relationship between $\zeta(s), s$, Bernoulli numbers and Gamma functions with the rational functions of the roots is shown. Consider the rational forms of the ζ-function: Using the Jensen formula [1] (p 1036),

$$\zeta(s) = \left(\frac{2^{s-1}}{2^s - 1}\right)\left(\frac{s}{s-1}\right) + \frac{2}{2^s - 1}\int_0^\infty \left(\frac{\left(\frac{1}{4} + t^2\right)\sin(s\tan^{-1} 2t)}{e^{2\pi t} - 1}\right) dt \tag{18}$$

Substituting $\tan\theta = 2t$,

$$\zeta(s) = \left(\frac{2^{s-1}}{2^s - 1}\right)\left(\frac{s}{s-1}\right) + \frac{2}{2^s - 1}\int_0^{\frac{\pi}{2}} \left(\frac{\left(\frac{1}{2}\sec s\theta\right)\sin(s\theta)\ \left(\frac{1}{2}(\sec\theta)^2\right)}{e^{\pi\tan\theta} - 1}\right) d\theta \tag{19}$$

This reduces to

$$\zeta(s) = \left(\frac{2^{s-1}}{2^s - 1}\right)\left(\frac{s}{s-1}\right) + \frac{2^s}{2^s - 1}\int_0^{\frac{\pi}{2}} \left(\frac{\sin(s\theta)\ \left((\sec\theta)^{2-s}\right)}{e^{\pi\tan\theta} - 1}\right) d\theta \tag{20}$$

Using the ChebyshevT function, [1] (p 993, 8.94

$$\sin(s\cos^{-1}(x)) = \sqrt{1 - \mathrm{ChebyshevT}(s, \cos^{-1}(x))^2} \tag{21}$$

Using $x = \cos\theta$,

$$\zeta(s) = \left(\frac{2^{s-1}}{2^s - 1}\right)\left[\left(\frac{s}{s-1}\right) + 2s\int_0^{\frac{\pi}{2}} \left(\frac{(\cos\theta)^{s-2}\left(1 - \frac{1}{2}\left(\left(\cos\theta + i(1-\cos\theta^2)^{\frac{1}{2}}\right)\right)^s + \frac{1}{2}\left(\left(\cos\theta - i(1-\cos\theta^2)^{\frac{1}{2}}\right)\right)^s\right)}{e^{\frac{\pi\sqrt{1-\cos\theta^2}}{\cos\theta}} - 1}\right) d\theta\right]$$

$$\zeta(s) = \left(\frac{2^{s-1}}{2^s - 1}\right)\left[\left(\frac{s}{s-1}\right) + 2s\int_0^{\frac{\pi}{2}} \left(\frac{(\cos\theta)^{s-2}\left(1 - \frac{1}{2}\left(\left(\cos\theta + i(1-\cos\theta^2)^{\frac{1}{2}}\right)\right)^s + \frac{1}{2}\left(\left(\cos\theta - i(1-\cos\theta^2)^{\frac{1}{2}}\right)\right)^s\right)}{e^{\frac{\pi\sqrt{1-\cos\theta^2}}{\cos\theta}} - 1}\right) d\theta\right]$$

Which can be reduced again by the substitution, x = $\tan\theta$,

$$\zeta(s) = \left(\frac{2^{s-1}}{2^s - 1}\right)\left[\left(\frac{s}{s-1}\right) + 2s\int_0^\infty \left(\frac{\left(\frac{1}{\sqrt{1+x^2}}\right)^{2s}\left(\left(\frac{1}{\sqrt{1+x^2}}\right)^{-2s} - \left(\frac{1}{2}(1+ix)^s + \frac{1}{2}(1-ix)^s\right)^2\right)^{\frac{1}{2}}}{e^{\pi x} - 1}\right) dx\right],$$

Finally, one arrives at the Abel Plena form:

$$\zeta(s) = \left(\frac{2^{s-1}}{2^s - 1}\right)\left[\left(\frac{s}{s-1}\right) + 2\int_0^\infty \left(\frac{i\,(1+ix)^{-s} + (1+ix)^{-s}}{e^{\pi x} - 1}\right) dx\right] \tag{22}$$

The function (22) can now be reduced to a series form, as follows:

$$(1+ix)^{-s} = \sum_{n=0}^\infty \frac{(ix)^n\Gamma(1-s)}{\Gamma(-s-n+1)n!}, (1-ix)^{-s} = \sum_{n=0}^\infty \frac{(-ix)^n\Gamma(1-s)}{\Gamma(-s-n+1)n!} \tag{23}$$

It is convenient at this point to note that the odd terms vanish and one is left with:

$$\zeta(s) = \left(\frac{2^{s-1}}{2^s - 1}\right)\left[\left(\frac{s}{s-1}\right) + \int_0^\infty \left(2i\sum_{n=1}^\infty \frac{(ix)^{2n-1}\Gamma(1-s)}{2\Gamma(-s-n+2)(2n-1)!\,e^{\pi x} - 1}\right) dx\right] \tag{24}$$

$$\zeta(s)=\left(\frac{2^{s-1}}{2^{s}-1}\right)\left[\left(\frac{s}{s-1}\right)+\sum_{n=1}^{\infty}\frac{2\Gamma(1-s)}{\Gamma(-s-n+2)(2n-1)!}\int_{0}^{\infty}\left(\left(\frac{x^{2n-1}}{e^{\pi x}-1}\right)\right)dx\right] \tag{25}$$

Using the relation [1] (p 1038),

$$\zeta(2n)\Gamma(2n)\left[\frac{1}{\pi^{2n}}\right]=\int_{0}^{\infty}\frac{x^{2n-1}}{e^{\pi x}-1}dx \tag{26}$$

$$\zeta(s)=\left(\frac{2^{s-1}}{2^{s}-1}\right)\left[\left(\frac{s}{s-1}\right)+\sum_{n=1}^{\infty}\frac{2\Gamma(1-s)\zeta(2n)\Gamma(2n)}{\pi^{2n}\Gamma(-s-n+2)(2n-1)!}dx\right] \tag{27}$$

$$\Gamma(2n)=(2n-1)!$$

$$\zeta(s)=\left(\frac{2^{s-1}}{2^{s}-1}\right)\left[\left(\frac{s}{s-1}\right)+\sum_{n=1}^{\infty}\frac{2\Gamma(1-s)\zeta(2n)}{\pi^{2n}\Gamma(-s-n+2)}dx\right] \tag{28}$$

Using the relation [1] (p 1038)

$$\zeta(2n)=\frac{(-1)^{n+1}2^{2n-1}\pi^{2n}\mathrm{B}_{2n}}{(2n)!} \tag{29}$$

one arrives at the desired form:

$$\zeta(s)=\left(\frac{2^{s-1}}{2^{s}-1}\right)\left[\left(\frac{s}{s-1}\right)-\sum_{n=1}^{\infty}\frac{2^{2n}\mathrm{B}_{2n}\Gamma(1-s)}{\Gamma(-s-2n+2)(2n)!}\right] \tag{30}$$

Now, a critical observation is that the series in (30) also determines an equal representation of the rational forms of the arguments by the truncated sum:

$$\frac{s}{s-1}=-\sum_{n=0}^{1}\frac{(2^{n})B_{n}\Gamma(1-s)}{\Gamma\big((2-n-s)\big)n!} \tag{31}$$

The function becomes:

$$\zeta(s)=\left(\frac{2^{s-1}}{2^{s}-1}\right)\left[\left(\frac{s}{s-1}\right)-\left(\sum_{n=0}^{\infty}\frac{2^{n}\mathrm{B}_{n}\Gamma(1-s)}{\Gamma(-s-n+2)(n)!}-\sum_{n=0}^{1}\frac{(2^{n})B_{n}\Gamma(1-s)}{\Gamma\big((2-n-s)\big)n!}\right)\right] \tag{32}$$

$$\zeta(s)=\left(\frac{2^{s-1}}{2^{s}-1}\right)\left[\left(\frac{s}{s-1}\right)-\left(\sum_{n=0}^{\infty}\frac{2^{n}\mathrm{B}_{n}\Gamma(1-s)}{\Gamma(-s-n+2)(n)!}+\frac{s}{s-1}\right)\right] \tag{33}$$

$$\zeta(s)=-\left(\frac{2^{s-1}}{2^{s}-1}\right)\left(\sum_{n=0}^{\infty}\frac{(2^{n})B_{n}\Gamma(1-s)}{\Gamma\big((2-n-s)\big)n!}\right) \tag{34}$$

It is clear that when the ζ-function vanishes,

$$\frac{s}{s-1}=\sum_{n=1}^{\infty}\frac{2^{2n}\mathrm{B}_{2n}\Gamma(1-s)}{\Gamma(-s-2n+2)(2n)!}=-\sum_{n=0}^{1}\frac{(2^{n})B_{n}\Gamma(1-s)}{\Gamma\big((2-n-s)\big)n!},$$

and (35)

$$\sum_{n=0}^{\infty}\frac{2^{n}\mathrm{B}_{n}\Gamma(1-s)}{\Gamma(-s-n+2)(n)!}=0$$

This is where the role of the Bernoulli numbers, B_n and the Gamma function come into play with the roots of the Riemann zeta function. Note that the expression (35) is the rational forms of the Γ-function.

3.0 The role of the Gamma Function in the Riemann Hypothesis

Consider the Γ-function:

$$\Gamma(z) = \frac{e^{-\gamma z}}{z}\prod_{k=1}^{\infty}\left(\frac{e^{\frac{z}{k}}}{1+\frac{z}{k}}\right), \qquad \mathbb{R}(z) > 0 \tag{36}$$

where γ, is Euler's constant.

The relation (36) is only valid for $\Re(z) > 0$, i.e., $\Re(z) = \Re(-s-2n+1) > 0$. The Bernoulli numbers and the Γ-function (36) interact only for values of $n \in \{0,1\}$. These are precisely the values that truncate the relation:

$$\frac{s}{s-1} = \sum_{n=1}^{\infty}\frac{2^{2n}B_{2n}\Gamma(1-s)}{\Gamma(-s-2n+2)(2n)!} \to -\sum_{n=0}^{1}\frac{(2^n)B_n\Gamma(1-s)}{\Gamma\big((2-n-s)\big)n!} \tag{37}$$

Put $z = 1-s$, and also, $z = 2-2n-s$ in (36), and since (36) is only valid for $n = 0, 1$, it can be seen that (37) produces the correct result:

$$-\frac{1}{s-1} - 1 + \sum_{n=1}^{\infty}\frac{2^{2n}B_{2n}\Gamma(1-s)}{\Gamma(-s-2n+2)(2n)!} = 0 \tag{38}$$

Which provided,

$$\frac{s}{s-1} = \sum_{n=1}^{\infty}\frac{2^{2n}B_{2n}\Gamma(1-s)}{\Gamma(-s-2n+2)(2n)!} \tag{39}$$

It follows that unless s is a root, the result (39) is not true. Let us expand $\Gamma(1-s)$ using the functional relation (36):

$$\frac{s}{s-1} = \frac{1}{1-s}\left(e^{-\gamma(1-s)}\prod_{k=1}^{\infty}\left(\frac{e^{\frac{1-s}{k}}}{1+\frac{1-s}{k}}\right)\sum_{n=1}^{\infty}\frac{2^{2n}B_{2n}}{\Gamma(-s-2n+2)(2n)!}\right) \tag{40}$$

It can be seen that from the numerator, a root s, can be expressed as

$$s = -e^{-\gamma(1-s)}\prod_{k=1}^{\infty}\left(\frac{e^{\frac{1-s}{k}}}{1+\frac{1-s}{k}}\right)\left[\sum_{n=1}^{\infty}\frac{2^{2n}B_{2n}}{\Gamma(-s-2n+2)(2n)!}\right] \tag{41}$$

This relation must be satisfied by the substitution, $s \to 1-s$ in (41); hence,

$$1-s = -e^{-\gamma(s)}\prod_{k=1}^{\infty}\left(\frac{e^{\frac{s}{k}}}{1+\frac{s}{k}}\right)\left[\sum_{n=1}^{\infty}\frac{2^{2n}B_{2n}}{\Gamma(s-2n+1)(2n)!}\right] \tag{42}$$

Additionally, an equivalent representation of the rational form is obtained:

$$\frac{s}{s-1} = \frac{-e^{-\gamma(1-s)}\prod_{k=1}^{\infty}\left(\frac{e^{\frac{1-s}{k}}}{1+\frac{1-s}{k}}\right)\left[\sum_{n=1}^{\infty}\frac{2^{2n}B_{2n}}{\Gamma(-s-2n+2)(2n)!}\right]}{e^{-\gamma(s)}\prod_{k=1}^{\infty}\left(\frac{e^{\frac{s}{k}}}{1+\frac{s}{k}}\right)\left[\sum_{n=1}^{\infty}\frac{2^{2n}B_{2n}}{\Gamma(s-2n+1)(2n)!}\right]} \tag{43}$$

It is possible to simplify (43) as follows:

$$\frac{s}{s-1} = -e^{-\gamma(1-2s)}\prod_{k=1}^{\infty}\left(\frac{e^{\frac{1-2s}{k}}(k+s)}{(k+1-s)}\right)\left[\frac{\Gamma(s)}{\Gamma(1-s)}\frac{\sum_{n=1}^{\infty}\frac{2^{2n}B_{2n}\Gamma(1-s)}{\Gamma(-s-2n+2)(2n)!}}{\sum_{n=1}^{\infty}\frac{2^{2n}B_{2n}\Gamma(s)}{\Gamma(s-2n+1)(2n)!}}\right] \tag{44}$$

$$\frac{s}{s-1} = -e^{-\gamma(1-2s)}\frac{\Gamma(s)}{\Gamma(1-s)}\prod_{k=1}^{\infty}\left(\frac{e^{\frac{1-2s}{k}}(k+s)}{(k+1-s)}\right)\left[\frac{\frac{s}{s-1}}{\frac{1-s}{s}}\right] \tag{45}$$

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$$\frac{s}{s-1} = -e^{-\gamma(1-2s)}\left[\frac{\Gamma(s)}{\Gamma(1-s)}\right]\prod_{k=1}^{\infty}\left(\frac{e^{\frac{1-2s}{k}}(k+s)}{(1+k-s)}\right)\left[\frac{s}{s-1}\right]^2 \quad (46)$$

$$\frac{s-1}{s} = e^{-\gamma(1-2s)}\left[\frac{\Gamma(s)}{\Gamma(1-s)}\right]\prod_{k=1}^{\infty}\left(\frac{e^{\frac{1-2s}{k}}}{\frac{(1+k-s)}{(k+s)}}\right) \quad (47)$$

$$\frac{s-1}{s} = e^{-\gamma(1-2s)}\left[\frac{\Gamma(s)}{\Gamma(1-s)}\right]\prod_{k=1}^{\infty}\left(\frac{e^{\frac{1-2s}{k}}}{\frac{\left(k+s+(1-2s)\right)}{(k+s)}}\right) \quad (48)$$

and finally,

$$\frac{s-1}{s} = e^{-\gamma(1-2s)}\left[\frac{\Gamma(s)}{\Gamma(1-s)}\right]\prod_{k=1}^{\infty}\left(\frac{e^{\frac{1-2s}{k}}}{1+\frac{1-2s}{(k+s)}}\right) \quad (49)$$

Now, the roots are complex-valued. In this case, one uses the complex representation of the Γ-function:

$$\Gamma(x+iy) = \Gamma(x)\left[\frac{xe^{-iy\gamma}}{x+iy}\right]\prod_{k=1}^{\infty}\left(\frac{e^{\frac{iy}{k}}}{1+\frac{iy}{(x+k)}}\right), \qquad x>0, \quad (50)$$

$$(x+iy) = \frac{\Gamma(x)xe^{-iy\gamma}}{\Gamma(x+iy)}\prod_{k=1}^{\infty}\left(e^{\frac{iy}{k}}\left(\frac{x+k}{x+k+iy}\right)\right) \quad (51)$$

$$(x+iy) = \frac{\Gamma(x+1)e^{-iy\gamma}}{\Gamma(x+iy)}\prod_{k=1}^{\infty}\left(e^{\frac{iy}{k}}\left(\frac{x+k}{x+k+iy}\right)\right) \quad (52)$$

then,

$$(x+iy-1) = -\frac{\Gamma(2-x)e^{iy\gamma}}{\Gamma(1-x-iy)}\prod_{k=1}^{\infty}\left(e^{\frac{-iy}{k}}\frac{(1-x+k)}{1-x-iy+k}\right) \quad (53)$$

One can now express the rational form of the roots as follows:

$$\frac{(x+iy-1)}{\mathrm{x+iy}} = \frac{-\frac{\Gamma(2-x)e^{iy\gamma}}{\Gamma(1-x-iy)}\prod_{k=1}^{\infty}\left(e^{\frac{-iy}{k}}\frac{(1-x+k)}{1-x-iy+k}\right)}{\frac{\Gamma(x+1)e^{-iy\gamma}}{\Gamma(x+iy)}\prod_{k=1}^{\infty}\left(e^{\frac{iy}{k}}\left(\frac{x+k}{x+k+iy}\right)\right)} \quad (54)$$

$$\frac{(x+iy-1)}{x+iy} = \frac{\Gamma(2-x)}{\Gamma(x+1)}\frac{\Gamma(x+iy)}{\Gamma(1-x-iy)}e^{2iy\gamma}\prod_{k=1}^{\infty}\left(e^{-\frac{2iy}{k}}\left(\frac{(1-x+k)}{x+k}\right)\left(\frac{x+k+iy}{1-x+k-iy}\right)\right) \quad (55)$$

From (49),

$$\frac{s-1}{s} = e^{-\gamma(1-2s)}\left[\frac{\Gamma(s)}{\Gamma(1-s)}\right]\prod_{k=1}^{\infty}\left(\frac{e^{\frac{1-2s}{k}}}{1+\frac{1-2s}{(k+s)}}\right) \quad (56)$$

Let $s=\sigma+i\tau$, $\mathbb{R}(s)=\sigma$, $\{\sigma,\tau\}\in\mathbb{R}$. It is possible to obtain the same rational form of the roots as

$$\frac{\sigma+i\tau-1}{\sigma+i\tau} = e^{-\gamma(1-(2\sigma+2i\tau))}\left[\frac{\Gamma(\sigma+i\tau)}{\Gamma(1-\sigma-i\tau)}\right]\prod_{k=1}^{\infty}\left(\frac{e^{\frac{1-(2\sigma+2i\tau)}{k}}}{1+\frac{1-(2\sigma+2i\tau)}{(k+\sigma+i\tau)}}\right) \quad (57)$$

$$\frac{\sigma+i\tau-1}{\sigma+i\tau}=e^{-\gamma(1-(2\sigma+2i\tau))}\left[\frac{\Gamma(\sigma+i\tau)}{\Gamma(1-\sigma-i\tau)}\right]\prod_{k=1}^{\infty}\left(\frac{e^{\frac{1-(2\sigma+2i\tau)}{k}}}{\frac{1-\sigma-i\tau+k}{(k+\sigma+i\tau)}}\right)$$

$$\frac{\sigma+i\tau-1}{\sigma+i\tau}=e^{-\gamma(1-(2\sigma+2i\tau))}\left[\frac{\Gamma(\sigma+i\tau)}{\Gamma(1-\sigma-i\tau)}\right]\prod_{k=1}^{\infty}\left(e^{\frac{1-(2\sigma+2i\tau)}{k}}\left(\frac{k+\sigma+i\tau}{1-\sigma+k-i\tau}\right)\right) \quad (58)$$

Recall (55) and, let $x=\sigma,\ y=\tau$.

$$\frac{\sigma+i\tau}{1-\sigma-i\tau}=\frac{\Gamma(\sigma+1)}{\Gamma(2-\sigma)}\frac{\Gamma(1-\sigma-i\tau)}{\Gamma(\sigma+i\tau)}e^{-2i\tau\gamma}\prod_{k=1}^{\infty}\left(e^{\frac{2iy}{k}}\left(\frac{\sigma+k}{(1-\sigma+k)}\right)\left(\frac{1-\sigma+k-i\tau}{\sigma+k+i\tau}\right)\right) \quad (59)$$

Multiplying (58) by (59), it is possible to reach:

$$1=\left[\frac{\Gamma(\sigma+1)}{\Gamma(2-\sigma)}\right]\left[\frac{\Gamma(1-\sigma-i\tau)}{\Gamma(\sigma+i\tau)}\right]\left[\frac{\Gamma(\sigma+i\tau)}{\Gamma(1-\sigma-iy)}\right]e^{-i\gamma(1-2\sigma)}\prod_{k=1}^{\infty}\left(e^{\frac{1-2\sigma}{k}}\left(\frac{\sigma+k}{(1-\sigma+k)}\right)\right)$$

$$1 \;=e^{-i\gamma(1-2\sigma)+\log\left(\frac{\Gamma(\sigma+1)}{\Gamma(2-\sigma)}\prod_{k=1}^{\infty}\left(e^{\frac{1-2\sigma}{k}}\left(\frac{\sigma+k}{1-\sigma+k}\right)\right)\right)} \quad (60)$$

$$1 \;=e^{-i\gamma(1-2\sigma)+\log\left(\frac{\Gamma(\sigma+1)}{\Gamma(2-\sigma)}\right)+\sum_{k=1}^{\infty}\left(\frac{1-2\sigma}{k}+\log\left(\frac{\sigma+k}{(1-\sigma+k)}\right)\right)} \quad (61)$$

Now, the left-hand side of (62) is unity. However, the right-hand side of (61) can grow due to the fact that the sum

$$\sum_{k=1}^{\infty}\left(\frac{1}{k}\right)=\zeta(1)=\infty$$

Except when $\sigma=\frac{1}{2}$, and one finds that all the terms in (62) reduce to unity. This proves the Riemann hypothesis.

No assumptions have been made about σ.

4.0. Discussion

It is clear that the restrictions for the validity of the Gamma functions (50) and the reflection formula (17), for complex variables leads to the consequences that restrict the real part of rational functions of the roots in the sum (31), and the expressions (58) and (59).

4.1 Conclusion

As expected, *no new theorems* are required since in its fundamental form, the result is as fundamental as the arithmetic that describes numbers. A cornerstone does not need a corner stone! The Riemann hypothesis is proved to be a fundamental property of numbers, the Gamma functions, their relation to the Bernoulli numbers, and to vanishing of the Zeta function at its complex roots, $s=\sigma+i\tau$, and the Riemann Zeta function's pole $s=1$.

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