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Solve the 3x+1 problem

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Abstract

The $3x+1$ problem asks the following: Suppose we start with a positive integer, and if it is odd then multiply it by 3 and add 1, and if it is even, divide it by 2. Then repeat this process as long as you can. Do you eventually reach the integer 1, no matter what you started with? Collatz conjecture (or $3n+1$ problem) has been explored for about 85 years. In this article, we prove the Collatz conjecture by modifying Sharkovsky ordering of positive integers and denote the composition of the collatz function as a algebraic formula about $\frac{3^m}{2^r}$, convert the problem to a algebraic problem, we can solve it completely.

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Key words: $3x+1$ problem, Collatz conjecture, Sharkovsky ordering path

1 Introduction

The $3x+1$ problem is one of the unsolved problems in mathematics. It is also known as the Collatz conjecture, $3x+1$ mapping, Ulam conjecture, Kakutani's problem, Thwaites conjecture, Hasse's algorithm, or Syracuse problem [1]. Paul Erdos (1913-1996) commented on the intractability of problem $3n+1$ [2]: "Mathematics is not ready for those problems yet".

The $2x+1$ problem is that, take any positive integer x, If x is even, divide x by 2. If x is odd, multiply x by 3 and add 1. Repeat this process continuously. The conjecture states that no matter which number you start with, you will always reach 1 eventually.

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2 Terminology and notations

We will use the notations as in [4,7]. we describe a Collatz function as

$$T(n) = \begin{cases} 3n+1, & \text{if } n \text{ is odd number,} \\ \frac{n}{2} & \text{if } n \text{ is even number.} \end{cases} \quad (1)$$

Let N denote the set of positive integers. For $n \in N$, and $k = 0, 1, 2, 3, \cdots$, $T^0(n)$ and $T^{k+1}(n)$ denote n and $T(T^k(n)$, respectively.

The $3x + 1$ problem concerns the behavior of the iterates of the Collatz function, for any integer n, there must exist an integer r, so that

$$T^r(n) = 1.$$

3 Proof of the Collatz Conjecture

3.1 The modified Sarkovskii ordering and integer lattice

We remove the last row number to the first column, get an integer lattice[6] of the modified Sarkovskii ordering as

$$\begin{array}{ccccccccccc}
1, & 3, & 5, & 7, & 9, & 11, & 13, & 15, & 17, & 19, & \cdots \\
2, & 2\cdot 3, & 2\cdot 5, & 2\cdot 7, & 2\cdot 9, & 2\cdot 11, & 2\cdot 13, & 2\cdot 15, & 2\cdot 17, & 2\cdot 19, & \cdots \\
2^2, & 2^2\cdot 3, & 2^2\cdot 5, & 2^2\cdot 7, & 2^2\cdot 9, & 2^2\cdot 11, & 2^2\cdot 13, & 2^2\cdot 15, & 2^2\cdot 17, & 2^2\cdot 19, & \cdots \\
2^3, & 2^3\cdot 3, & 2^3\cdot 5, & 2^3\cdot 7, & 2^3\cdot 9, & 2^3\cdot 11, & 2^3\cdot 13, & 2^3\cdot 15, & 2^3\cdot 17, & 2^3\cdot 19, & \cdots \\
2^4, & 2^4\cdot 3, & 2^4\cdot 5, & 2^4\cdot 7, & 2^4\cdot 9, & 2^4\cdot 11, & 2^4\cdot 13, & 2^4\cdot 15, & 2^4\cdot 17, & 2^4\cdot 19, & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}$$

In the first row, its are odd number from left to right, that are $1, 3, 5, 7, 9, 11, 13, \cdots$, from the second row, each number is multiplying each number in its previous row by 2, and so on.

3.2 The algebraic formula and Collatz graph

If we draw a line segment of arrow between two digits in the lattice of integer in the modified Sarkovskii ordering, those are the original value x, and its

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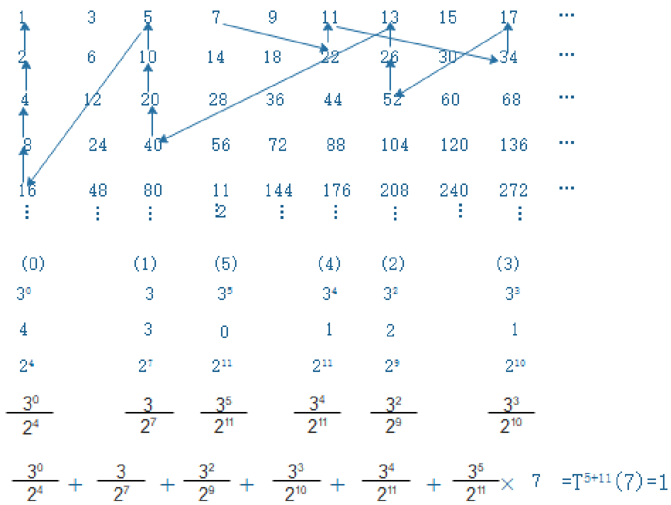

Fig. 1. The Collatz graph of $T^{16}(7) = T(5, 11, 7) = 1$ in the lattice of integers in the modified Sarkovskii ordering and the algebraic formula.

value of Collatz function $T(x)$, and connect $T(x)$ to $T^2(x)$, and so on $T^2(x)$ to $T^3(x), \cdots$, thus we get a graph, which can be called as *Collatz graph*. Using the Collatz function $T(x)$, We obtain an algebraic formula of $\frac{1}{2^4}, \frac{3}{2^7}, \frac{3^2}{2^9}, \cdots, \frac{3^m}{2^r} \cdot x$. Here r is the number of perpendicular segments, m is the oblique segments in the Collatz graph,

$$T^{m+r}(n) = T(m, r, n) = \frac{1}{2^4} + \frac{3}{2^7} + \frac{3^2}{2^9} + \cdots + \frac{3^m}{2^r} \cdot x = 1,$$

For example, $n = 7$, the algebraic formula is

$$T^{16}(7) = T(5, 11, 7) = \frac{1}{2^4} + \frac{3}{2^7} + \frac{3^2}{2^9} + \frac{3^3}{2^{10}} + \frac{3^4}{2^{11}} + \frac{3^5}{2^{11}} \cdot 7 = 1,$$

and the Collatz graph is Fig. 1.

And $n = 36$, the algebraic formula is

$$T^{21}(36) = T(6, 15, 36) = \frac{1}{2^4} + \frac{3}{2^7} + \frac{3^2}{2^9} + \frac{3^3}{2^{10}} + \frac{3^4}{2^{11}} + \frac{3^5}{2^{13}} + \frac{3^6}{2^{15}} \cdot 36 = 1$$

and the Collatz graph is Fig. 2.

4 Numerical example

We propose the following procedure,

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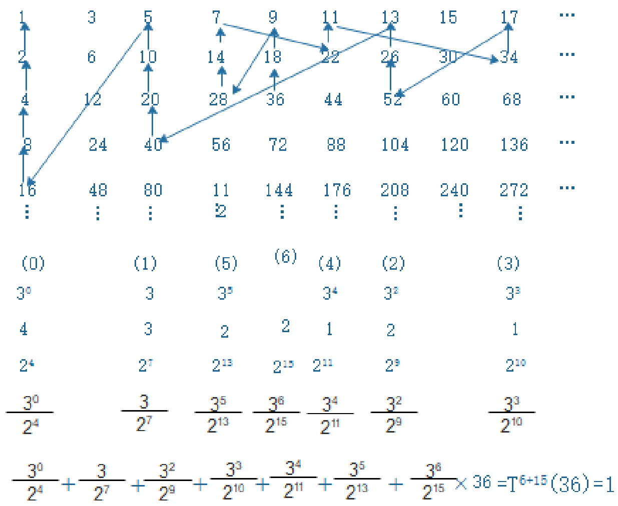

Fig. 2. The Collatz graph of $T^{21}(36) = T(6, 15, 36) = 1$ in the lattice of integers in the modified Sarkovskii ordering and the algebraic formula.

$$T^{20}(18) = T(6, 14, 18) = \frac{1}{2^4} + \frac{3}{2^7} + \frac{3^2}{2^9} + \frac{3^3}{2^{10}} + \frac{3^4}{2^{11}} + \frac{3^5}{2^{13}} + \frac{3^6}{2^{14}} \cdot 18 = 1$$

$$T^{15}(23) = T(4, 11, 23) = \frac{1}{2^4} + \frac{3}{2^9} + \frac{3^2}{2^{10}} + \frac{3^3}{2^{11}} + \frac{3^4}{2^{11}} \cdot 23 = 1$$

$$T^{17}(15) = T(5, 12, 15) = \frac{1}{2^4} + \frac{3}{2^9} + \frac{3^2}{2^{10}} + \frac{3^3}{2^{11}} + \frac{3^4}{2^{12}} + \frac{3^5}{2^{12}} \cdot 15 = 1$$

$$T^{12}(17) = T(3, 9, 17) = \frac{1}{2^4} + \frac{3}{2^7} + \frac{3^2}{2^9} + \frac{3^3}{2^9} \cdot 17 = 1$$

$$T^{16}(397) = T(5, 11, 397) = \frac{1}{2^4} + \frac{3}{2^7} + \frac{3^2}{2^9} + \frac{3^3}{2^{10}} + \frac{3^4}{2^{11}} + \frac{3^5}{2^{17}} + \frac{3^6}{2^{20}} + \frac{3^7}{2^{20}} \cdot 397 = 1$$

$$T^{19}(61) = T(5, 14, 61) = \frac{1}{2^4} + \frac{3}{2^9} + \frac{3^2}{2^{10}} + \frac{3^3}{2^{11}} + \frac{3^4}{2^{14}} + \frac{3^5}{2^{14}} \cdot 61 = 1$$

We observe the three properties,

Note 1 *In the algebraic formula, in numerators, it is* $1, 3, 3^2, 3^3, \cdots, 3^m$*. there is not a lack. But in denominator, there are many lacks,* $2, 2^2, 2^3, \cdots, 2^r$*.*

Note 2 *For positive integers* $i, j, k, l,$ *and* $l_k, l_{k-1}, \cdots, l_1$*, if* $i > j$*, then there is a recurrence relation*

$$T^i(n) = \frac{3^k}{2^l} T^j(n) + \frac{3^{k-1}}{2^{l_k}} + \cdots + \frac{3^2}{2^{l_3}} + \frac{3}{2^{l_2}} + \frac{1}{2^{l_1}}$$

where $k + l = i - j$*, and* $l \geq l_k \geq l_{k-1} \geq \cdots \geq l_1$*.*

Example 1 *There are*

$$T^3(97) = \frac{3}{2^2}\cdot 97 + \frac{1}{2^2} = 73$$

$$T^{18}(97) = \frac{3^7}{2^{11}}\cdot 97 + \frac{3^6}{2^{11}} + \frac{3^5}{2^9} + \frac{3^4}{2^7} + \frac{3^3}{2^6} + \frac{3^2}{2^5} + \frac{3}{2^2} + \frac{1}{2} = 107$$

We can get the recurrence formula about the Collatz function $T(x)$

$$T^{26}(97) = \frac{3^3}{2^5}\cdot T^{18}(97) + \frac{3^2}{2^5} + \frac{3}{2^4} + \frac{1}{2^2} = 91,$$

namely,

$$T^{26}(97) = \frac{3^{10}}{2^{16}}\cdot 97 + \frac{3^9}{2^{16}} + \frac{3^8}{2^{14}} + \frac{3^7}{2^{12}} + \frac{3^6}{2^{11}} + \frac{3^5}{2^{10}} + \frac{3^4}{2^7} + \frac{3^3}{2^6} + \frac{3^2}{2^5} + \frac{3}{2^4} + \frac{1}{2^2} = 91$$

$$T^{31}(97) = \frac{3^2}{2^3}\cdot 91 + \frac{3}{2^3} + \frac{1}{2^2} = 103$$

$$T^{40}(97) = \frac{3^4}{2^5}\cdot 103 + \frac{3^3}{2^5} + \frac{3^2}{2^4} + \frac{3}{2^3} + \frac{1}{2} = 263$$

$$T^{53}(97) = \frac{3^5}{2^8}\cdot 263 + \frac{3^4}{2^8} + \frac{3^3}{2^7} + \frac{3^2}{2^6} + \frac{3}{2^4} + \frac{1}{2} = 251$$

$$T^{53}(97) = \frac{3^9}{2^{13}}\cdot 103 + \frac{3^8}{2^{13}} + \frac{3^7}{2^{12}} + \frac{3^6}{2^{11}} + \frac{3^5}{2^9} + \frac{3^4}{2^8} + \frac{3^3}{2^7} + \frac{3^2}{2^6} + \frac{3}{2^4} + \frac{1}{2} = 251 = T^{53}(97)$$

251	x
754	$T(x) = 3x + 1$
377	$T^2(x) = \frac{3}{2}x + \frac{1}{2}$
1132	$T^3(x) = \frac{3^2}{2}x + \frac{3}{2} + 1$
566	$T^4(x) = \frac{3^2}{2^2}x + \frac{3}{2^2} + \frac{1}{2}$
283	$T^5(x) = \frac{3^2}{2^3}x + \frac{3}{2^3} + \frac{1}{2^2}$
850	$T^6(x) = \frac{3^3}{2^3}x + \frac{3^2}{2^3} + \frac{3}{2^2} + 1$
425	$T^7(x) = \frac{3^3}{2^4}x + \frac{3^2}{2^4} + \frac{3}{2^3} + \frac{1}{2}$
$\vdots$	$\vdots$
958	$T^{64}(97) = \frac{3^5}{2^6}T^{53}(97) + \frac{3^4}{2^6} + \frac{3^3}{2^5} + \frac{3^2}{2^3} + \frac{3}{2^2} + 1$
61	$T^{100}(97) = \frac{3^7}{2^{15}}T^{77}(97) + \frac{3^6}{2^{15}} + \frac{3^5}{2^{14}} + \frac{3^4}{2^{13}} + \frac{3^3}{2^{12}} + \frac{3^2}{2^8} + \frac{3}{2^6} + \frac{1}{2^4}$
1	$T^{119}(97) = T^{100}(97)\cdot\frac{3^5}{2^{14}} + \frac{3^4}{2^{14}} + \frac{3^3}{2^{11}} + \frac{3^2}{2^{10}} + \frac{3}{2^9} + \frac{1}{2^4}$

Note 3 *We observe that*

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$$3^5 \cdot 61 + 3^4 + 3^3 \cdot 2^3 + 3^2 \cdot 2^4 + 3 \cdot 2^5 + 2^{10} = 2^{14}$$

$$3^7 \cdot 397 + 3^6 + 3^5 \cdot 2^3 + 3^4 \cdot 2^9 + 3^3 \cdot 2^{10} + 3^2 + 3 \cdot 2^{13} + 2^{16} = 2^{20}$$

Theorem 2 *We can use the Collatz function $T(x)$, obtain that a series of $\frac{1}{2^4}, \frac{3}{2^7}, \frac{3^2}{2^9}, \cdots, \frac{3^m}{2^r}$, where r is the number of perpendicular segments, m is the oblique segments in the graph. We can get a unique algebra donation about 3^m in numerator and 2^r in denominator, as*

$$T(m, r, n) = 1,$$

So, there is

$$T^{m+r}(n) = T(m, r, n) = \frac{3^m}{2^r} \cdot n + \frac{3^{m-1}}{2^{r_{m-1}}} + \cdots + \frac{3^2}{2^{r_2}} + \frac{3}{2^{r_1}} + \frac{1}{2^4} = 1$$

where $r \geq r_{m-1} \geq r_{m-2} \geq \cdots \geq r_1 > 4$.

and there is a recurrence relation

$$T^i(n) = \frac{3^k}{2^l} T^j(n) + \frac{3^{k-1}}{2^{l_k}} + \cdots + \frac{3^2}{2^{l_3}} + \frac{3}{2^{l_2}} + \frac{1}{2^{l_1}}$$

where $l \geq l_k \geq l_{m-2} \geq \cdots \geq l_1$.

Theorem 3 *For positive integer, n, there must exist positive integer m, r and $r_{m-1}, \cdots, r_1$, such that*

$$2^r = 3^m \cdot n + 3^{m-1} \cdot 2^{m-r_{m-1}} + \cdots + 3^2 \cdot 2^{m-r_2} + 3 \cdot 2^{m-r_1} + 2^{m-4},$$

where $r \geq r_{m-1} \geq r_{m-2} \geq \cdots \geq r_1 \geq 4$. This mean 3 multiply to the initial value n gradually with the digits of the abacus, must equal to 2^r.

Example 4 *For example, for the formula*

$$T(6, 14, 18) = \frac{1}{2^4} + \frac{3}{2^7} + \frac{3^2}{2^9} + \frac{3^3}{2^{10}} + \frac{3^4}{2^{11}} + \frac{3^5}{2^{13}} + \frac{3^6}{2^{14}} \cdot 18 = 1,$$

namely,

$$2^{10} + 3 \cdot 2^7 + 3^2 \cdot 2^5 + 3^3 \cdot 2^4 + 3^4 \cdot 2^3 + 3^5 \cdot 2 + 3^6 \cdot 18 = 2^{14},$$

PROOF. We calculate

$$
\begin{aligned}
3 &= 2+1\\
3^2 &= 2^3+1\\
3^3 &= 2^4+2^3+2+1\\
3^4 &= 2^6+2^4+1\\
3^5 &= 2^7+2^6+2^5+2^4+2+1\\
3^6 &= 2^9+2^7+2^6+2^4+2^3+1\\
18 &= 2^4+2
\end{aligned}
$$

and substitute them in the expression

$$
\begin{aligned}
&3^6\cdot 18+3^5\cdot 2+3^4\cdot 2^3+3^3\cdot 2^4+3^2\cdot 2^5+3\cdot 2^7+2^{10}\\
=&(2^9+2^7+2^6+2^4+2^3+1)\cdot(2^4+2)+(2^7+2^6+2^5+2^4+2+1)\cdot 2\\
&+(2^6+2^4+1)\cdot 2^3+(2^4+2^3+2+1)\cdot 2^4+(2^3+1)\cdot 2^5+(2+1)\cdot 2^7+2^{10},
\end{aligned}
$$

and get the value 2^{14}.

Remark 5 *We can say that $3x+1$ problem is the convert statement of period three implies chaos [4].*

5 Conclusion

In the integer lattice in the modifying the Sarkovskii ordering, denote the composition of the Collatz function as a algebraic formula about the $\frac{3^m}{2^r}$, we give a bridge of algebraic formula with graphs. We completely solve the $3x+1$ problem.

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