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Article

Delicate Comparison of the Central and Non-central Lyapunov Ratios with Applications to the Berry–Esseen Inequality for Compound Poisson Distributions

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Abstract: For each $t \in (-1, 1)$, exact values of the least upper bounds

$$H(t) = \sup_{\mathbb{E}X=t,\,\mathbb{E}X^2=1} \frac{\mathbb{E}|X|^3}{\mathbb{E}|X-t|^3}, \qquad \sup_{\mathbb{E}X=t,\,\mathbb{E}X^2=1} \frac{L_1(X)}{L_1(X-t)}$$

are obtained, where $L_1(X) = \mathbb{E}|X|^3/(\mathbb{E}X^2)^{3/2}$ is the non-cental Lyapunov ratio. It is demonstrated that these values are attained only at two-point distributions. As a corollary, S. Shorgin's conjecture is proved that states that the exact value is

$$\sup \frac{L_1(X)}{L_1(X-\mathbb{E}X)} = \frac{\sqrt{17+7\sqrt{7}}}{4} = 1.4899\ldots,$$

where the supremum is taken over all non-degenerate distributions of the random variable X with the finite third moment. Also, in terms of the central Lyapunov ratio $L_1(X-\mathbb{E}X)$, an analog of the Berry–Esseen inequality is proved for Poisson random sums of independent identically distributed random variables with the constant

$$0.3031 \cdot H\left(\frac{\mathbb{E}X}{\sqrt{\mathbb{E}X^2}}\right)\left(1-\frac{(\mathbb{E}X)^2}{\mathbb{E}X^2}\right)^{3/2} \leqslant 0.3031 \cdot \frac{\sqrt{17+7\sqrt{7}}}{4} < 0.4517.$$

where $\mathscr{L}(X)$ is the common distribution of the summands.

Keywords: Lyapunov fraction; extreme problem, moment inequality; central limit theorem; Berry–Esseen inequality; compound Poisson distribution; normal approximation.

1. Introduction

Let $X, X_1, X_2, \ldots$ be independent identically distributed random variables (i.i.d.r.v.s), N_λ be a r.v. having the Poisson distribution with parameter $\lambda > 0$ and independent of the sequence $\{X_n\}_{n\geqslant 1}$ for each $\lambda > 0$. The r.v.

$$S_\lambda = X_1 + X_2 + \ldots + X_{N_\lambda},$$

is called a *Poisson random sum* and its distribution is called *compound Poisson*. Here for definiteness it is assumed that $\sum_{k=1}^{0}(\,\cdot\,) = 0$. Poisson random sums S_λ are popular mathematic models in many fields. In particular, in the classical collective risk model [1], the r.v. S_λ has the meaning of the total insurance claim amount per a time unit, if the intensity of the claim arrivals is λ. Many examples of applied problems that assume using Poisson random

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sums as reasonable models can be found, e. g., in the books [2–4]. As a rule, these problems can be successfully solved only if the distribution function of the r.v. S_λ is either known, or is estimated accurately enough.

Assume that $\mathbb{E}X^2 \in (0,\infty)$. Denote

$$\widetilde{S}_\lambda = \frac{S_\lambda - \mathbb{E}S_\lambda}{\sqrt{\mathbb{D}S_\lambda}} = \frac{S_\lambda - \lambda\mathbb{E}X}{\sqrt{\lambda\mathbb{E}X^2}},$$

$$F_\lambda(x) := \mathbb{P}(\widetilde{S}_\lambda < x), \quad \Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}\,dt, \quad x \in \mathbb{R}.$$

As is well known, under the assumption made above, the compound Poisson distributions are asymptotically normal:

$$\Delta_\lambda(X) := \sup_x |F_\lambda(x) - \Phi(x)| \to 0, \quad \lambda \to \infty.$$

Therefore, irrespective of the common distribution of the summands X_j, the distribution of the Poisson random sum S_λ can be approximated by the normal law with the corresponding location and scale parameters under 'convenient' (computable) estimates $\Delta_\lambda \leqslant \overline{\Delta}_\lambda$ for the uniform distance:

$$\Phi(x) - \overline{\Delta}_\lambda \leqslant F_\lambda(x) \leqslant \Phi(x) + \overline{\Delta}_\lambda, \quad x \in \mathbb{R}.$$

If no other conditions are imposed, then Δ_λ may tend to zero arbitrarily slowly, as was demonstrated in [5, Theorems 5, 8]. Some possible upper bounds for Δ_λ in this situation were presented in [6]. However, under some additional moment-type conditions, the rate of convergence of Δ_λ to zero can be rather universally estimated by a 'convenient' power-type function. For example, if $\mathbb{E}|X|^{2+\delta} < \infty$ for some $\delta \in (0,1]$, then $\Delta_\lambda = O(\lambda^{-\delta/2})$, as $\lambda \to \infty$. A particular form of $O(\ldots)$ is determined by the available moment characteristics of the distribution of X.

Main attention was traditionally paid to the case $\delta = 1$, since for $\delta > 1$, in general, the convergence rate remains the same as for $\delta = 1$. Moreover, by analogy with convergence rate bounds for sums of a non-random number of independent r.v.'s, central moments were initially used in the moment-type bounds for Δ_λ, since these bounds themselves were obtained by a more or less ingenious application of the formula of total probability in order to extend to random sums the bounds initially constructed for non-random sums. These bounds had a rather cumbersome form, e. g. see [7,8]).

However, in the construction of the estimates of the accuracy of the normal approximation to compound Poisson distributions it turned out convenient and reasonable to use non-central moments. In these terms the bounds take a pretty simple form [9,10]

$$\Delta_\lambda(X) \leqslant \frac{C_1}{\sqrt{\lambda}} \cdot L_1(X), \quad \lambda > 0, \tag{1}$$

where

$$L_1(X) = \frac{\mathbb{E}|X|^3}{(\mathbb{E}X^2)^{3/2}} \tag{2}$$

is the so-called *non-central Lyapunov ratio* or *non-central Lyapunov fraction*. Estimate (1) is an analog of the Berry–Esseen inequality for Poisson random sums (or for compound Poisson distributions).

First upper bounds for the constant C_1 [9–11] were greater than the then best upper bounds for the absolute constant C in the Berry–Esseen inequality

$$\sup_{x\in\mathbb{R}} \left| \mathbb{P}\left(\frac{X_1 + \ldots + X_n - n\mathbb{E}X}{\sqrt{n\mathbb{D}X}} < x \right) - \Phi(x) \right| \leqslant \frac{C}{\sqrt{n}} \cdot L_0(X), \quad n \in \mathbb{N},$$

where

$$L_0(X) = \frac{\mathbb{E}|X-\mathbb{E}X|^3}{(\mathbb{D}X)^{3/2}} = L_1(X-\mathbb{E}X) \tag{3}$$

is the *central Lyapunov ratio* or *central Lyapunov fraction*. Michel [12] was the first to prove that $C_1 \leqslant C$ (four years later this result was independently re-proved in [13]). Finally, the authors of [14] succeeded in proving that actually $C_1 < C$. Namely, in that paper the upper bound $C_1 \leqslant 0.345$ was obtained which was strictly less than the lower bound $C_E := (\sqrt{10}+3)/(6\sqrt{2\pi}) = 0.4097\ldots$ [15] of the absolute constant C. Later this bound was lowered to $C_1 \leqslant 0.3041$ [16,17] and $C_1 \leqslant 0.3031$ [18, Theorem 4]. The first lower bound $C_1 \geqslant 0.2344$ was obtained in the paper [19]. In [5, Theorem 5] and [20, Chapter 3, p. 50] this estimate was improved to

$$C_1 \geqslant \sup_{\gamma>0, m\in\mathbb{N}_0} \sqrt{\gamma}\left(e^{-\gamma}\sum_{k=0}^{m}\frac{\gamma^k}{k!} - \Phi\left(\frac{m-\gamma}{\sqrt{\gamma}}\right)\right) \geqslant 0.266012\ldots = \frac{2}{3\sqrt{2\pi}} + 0.0000505\ldots$$

(In [5], an intermediate estimate was obtained in terms of the least upper bound in γ and m, whereas in [20] exact values $\gamma = 6.4206$, $m = 6$, were found that provide the lower bound for this supremum. Moreover, if we let $\gamma = m \to \infty$, then the limit value is $2/(3\sqrt{2\pi}))$ only. The lower bound for the constant C_1 is presented here with the separation of the term $2/(3\sqrt{2\pi})$, since this number plays the same asymptotic role in inequality (1) as the Esseen lower bound C_E in the classical Berry–Esseen inequality. For more details concerning the asymptotically exact constants see [5,21]). A detailed survey of the moment-type bounds for the accuracy of the normal approximation to the compound Poisson distribution including both the case $0 \leqslant \delta < 1$ and asymptotic settings, e. g., see [5]. Also see [17, Section 3] and [22, Section 2.4].

It should be noted that the estimate (1) in terms of the non-central Lyapunov ratio $L_1(X)$ implies a similar estimate in terms of the central Lyapunov ratio

$$\Delta_\lambda(X) \leqslant \frac{C_0}{\sqrt{\lambda}} \cdot L_0(X), \quad \lambda > 0, \tag{4}$$

where C_0 is an absolute constant, but not vice versa. Namely, let

$$J(X) = J(\mathscr{L}(X)) := \frac{L_1(X)}{L_0(X)} = \frac{\mathbb{E}|X|^3}{\mathbb{E}|X-\mathbb{E}X|^3}\left(\frac{\mathbb{D}X}{\mathbb{E}X^2}\right)^{3/2},$$

and let $\mathscr{P}$ be the class of all non-degenerate distributions on $\mathbb{R}$ with finite third moments. In 1996 S. Shorgin [23] proved that for any $\mathscr{L}(X) \in \mathscr{P}$

$$J(X) \leqslant 2\sqrt{2} < 2.8285 \quad \text{and} \quad \inf_{\mathscr{L}(X)\in\mathscr{P}} J(X) = 0,$$

whence, with the account of the upper bound $C_1 \leqslant 0.3031$ [18], it follows that $C_0 \leqslant 2\sqrt{2}C_1 < 0.8573$, and also that inequality (4) does not imply (1), that is, bound (1) in terms of the non-central Lyapunov ratio is not only more natural than (4), but is also more accurate.

In 2001 S. Shorgin [24] suggested that

$$\sup_{\mathscr{L}(X)\in\mathscr{P}} J(X) = \sup_{\mathscr{L}(X)\in\mathscr{P}} \frac{L_1(X)}{L_0(X)} = \frac{\sqrt{17+7\sqrt{7}}}{4} = 1.48997\ldots =: C_{\mathrm{SH}} \tag{5}$$

and described the hypothetical extreme two-point distribution of the r.v. X.

In 2011 in the paper [25] it was demonstrated that it suffices to look for $\sup_{\mathscr{L}(X)\in\mathscr{P}} J(X)$ in the class of distributions concentrated in at most three points, and the estimate $\sup_{\mathscr{L}(X)\in\mathscr{P}} J(X) \leqslant$

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1.49 was numerically computed, that implies $C_0 < 1.49C_1 \leqslant 1.49 \cdot 0.3031 < 0.4517$, also see [22, Section 2.4].

In the present paper, a complete proof of the hypothesis (5) is given, but the main result consists in the solution of this problem in a somewhat more accurate setting. Namely, it is suggested here to fix the value of the normalized mathematical expectation $\mathbb{E}X/\sqrt{\mathbb{E}X^2} = t \in (-1,1)$ and, instead of the unconditional optimization problem (5), to solve the problem of conditional optimization

$$\sup_{\substack{\mathscr{L}(X)\in\mathscr{P}:\\ \mathbb{E}X=t\sqrt{\mathbb{E}X^2}}} J(X) = \sup_{\substack{\mathscr{L}(X)\in\mathscr{P}:\\ \mathbb{E}X=t,\,\mathbb{E}X^2=1}} \frac{L_1(X)}{L_1(X-t)} = \left(1-t^2\right)^{3/2} \sup_{\substack{\mathscr{L}(X)\in\mathscr{P}:\\ \mathbb{E}X=t,\,\mathbb{E}X^2=1}} \frac{\mathbb{E}|X|^3}{\mathbb{E}|X-t|^3}, \tag{6}$$

that makes it possible to take possible smallness of the centering parameter $\mathbb{E}X/\sqrt{\mathbb{E}X^2}$ into account and majorize the ratio $L_1(X)/L_0(X)$ by a quantity that is close to unity in this case, that is almost one and a half times more accurate, than is allowed by (5) The values $t = \pm 1$ are not considered here, because the only solution satisfying the conditions $\mathbb{E}X = t = \pm 1$ and $\mathbb{E}X^2 = 1$, is the degenerate distribution concentrated in the point t. The solution of the conditional optimization problem (6) reduces to the calculation of the least upper bound

$$H(t) := \sup\left\{\frac{\mathbb{E}|X|^3}{\mathbb{E}|X-t|^3} : \mathscr{L}(X) \in \mathscr{P},\ \mathbb{E}X = t,\ \mathbb{E}X^2 = 1\right\}, \quad -1 < t < 1. \tag{7}$$

In the present paper, $H(t)$ is calculated for each value of the centering parameter $t \in (-1,1)$ (Theorem 1 and Table 1), and the hypothesis (5) is proved by writing $\sup J(X)$ in the form

$$\sup_{\mathscr{L}(X)\in\mathscr{P}} J(X) = \sup_{t\in(-1,1)} H(t)\left(1-t^2\right)^{3/2}$$

and calculating this supremum in $t \in (-1,1)$ (Theorem 2 and Table 1). In particular, from (7) it follows that for any $\mathscr{L}(X) \in \mathscr{P}$ we have

$$J(X) \leqslant H\left(\frac{\mathbb{E}X}{\sqrt{\mathbb{E}X^2}}\right)\left(1 - \frac{(\mathbb{E}X)^2}{\mathbb{E}X^2}\right)^{3/2},$$

and hence, inequality (4) actually holds for any distribution $\mathscr{L}(X) \in \mathscr{P}$ with known value of the normalized first moment $\mathbb{E}X/\sqrt{\mathbb{E}X^2} = t \in (-1,1)$ with a more accurate value of the constant

$$C_0 = C_0(t) := C_1 \cdot H(t)\left(1-t^2\right)^{3/2} \leqslant 0.3031 \cdot \frac{\sqrt{17+7\sqrt{7}}}{4} < 0.4517.$$

The values $C_0(t)$ rounded above up to the fourth digit are presented for some $t \in [0,1)$ in the fourth column of Table 1. Also, in Theorem 3 the form of the constant C_0 is presented for the case where only an upper bound $|\mathbb{E}X|/\sqrt{\mathbb{E}X^2} \leqslant t$ is known for the normalized expectation for each $t \in (-1,1)$.

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t	$H(t)$	$H(t)(1-t^2)^{3/2}$	$C_0(t)$	$p(t)$
0	1	1	0.3031	$\frac{3-\sqrt{3}}{6}$
0.001	1.00111	1.00111	0.3035	0.2116
0.01	1.0108	1.0107	0.3064	0.21405
0.05	1.057	1.053	0.3192	0.22494
0.1	1.1225	1.1057	0.3352	0.23856
0.2	1.285	1.20871	0.3664	0.26593
0.3	1.5034	1.3051	0.3956	0.29365
0.4	1.805	1.3896	0.4212	0.32205
0.5	2.2392	1.4544	0.4409	0.35168
0.6	2.9067	1.4882	0.4511	0.38345
$\sqrt{\frac{5-\sqrt{7}}{6}}$	$\frac{1+2\sqrt{7}}{2}$	$\frac{\sqrt{17+7\sqrt{7}}}{4}$	0.4517	$\frac{5-\sqrt{7}}{6}$
0.7	4.04901	1.4747	0.447	0.41691
0.8	6.4739	1.3984	0.4239	0.44833
0.9	15.041	1.2457	0.3776	0.47783
$1-$	$+\infty$	1	0.3031	0.5

Table 1. The values of the functions $H(t)$, $H(t)(1-t^2)^{3/2}$, $C_0(t) = 0.3031 \cdot H(t)(1-t^2)^{3/2}$ and the mass $p(t)$ of one of the atoms of the extreme distribution rounded above, for some $t \in [0, 1)$.

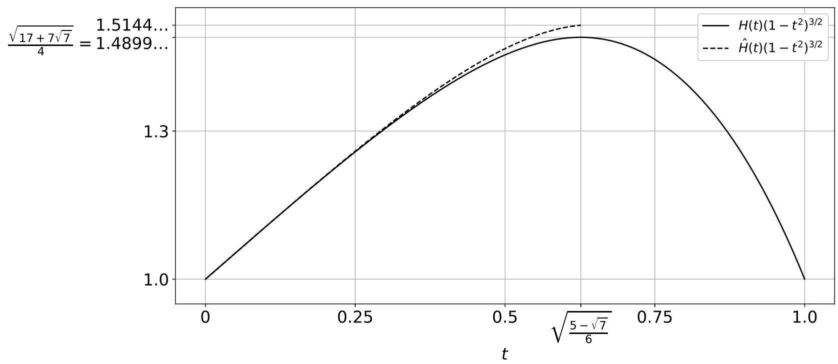

Figure 1. The plots of the functions $H(t)(1-t^2)^{3/2}$ and $\hat{H}(t)(1-t^2)^{3/2}$.

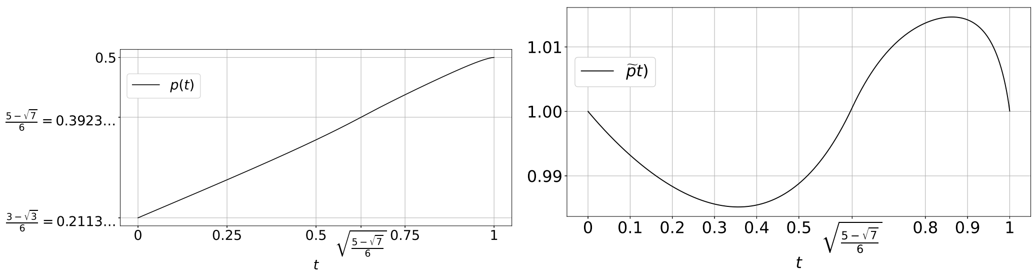

Figure 2. The functions $p(t)$ and $\widetilde{p}(t)$ defined in (13) and (15) correspondingly.

As regards the methods, computation of the least upper bound in (7) is implemented in the two steps: reduction to the distributions concentrated in at most two points (see Section 3) and analysis of two-point distributions (see Section 4), the last step being, in fact, most difficult one from the technical point of view. It also should be noted here that the standard technique based on the works [26–28] allows reduction only to three-point distributions, since there are totally three linear conditions on $\mathscr{L}(X)$ in (6) and (7): two moment conditions $\mathbb{E}X = t$, $\mathbb{E}X^2 = 1$ plus one probability normalization condition

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$\mathbb{E}X^0 \equiv \mathbb{E}1 = 1$. In fact, the same moments should be fixed in (5) to make the objective function

$$L_1(X) - C_{\rm SH} \cdot L_0(X) = \frac{\mathbb{E}|X|^3}{(\mathbb{E}X^2)^{3/2}} - C_{\rm SH} \cdot \frac{\mathbb{E}|X - \mathbb{E}X|^3}{(\mathbb{D}X)^{3/2}},$$

linear with respect to $\mathscr{L}(X)$, and hence, no further reduction in (5) can be allowed just by the standard techniques. Therefore, we used an alternative approach based on the construction of a special lower bound to $|x-t|^3$ with two tangency points as a linear combination of the functions $1, x, x^2, |x|^3$ generating the required moment conditions $\mathbb{E}1 = 1, \mathbb{E}X = t, \mathbb{E}X^2 = 1$, $\mathbb{E}|X|^3 < \infty$ (lemma 1 in Section 3), and than integrating the obtained inequality with respect to x (lemma 2 in Section 3). This trick allows immediately to reduce calculation of the least upper bound in (7) to the analysis of two-point distributions which is implemented in lemma 4 of Section 4.

Section 2 contains accurate formulations of the main results and Section 5 contains their proofs.

To conclude this introductory overview, note that, as well, an 'opposite' problem of comparing central and non-central absolute moments

$$\frac{\mathbb{E}|X - \mathbb{E}X|^p}{\mathbb{E}|X|^p} \longrightarrow \sup$$

was considered in the papers [29] with $p = 3$ and [30] with arbitrary $p > 1$ and for a wider class of functions of X and $X - \mathbb{E}X$ including $|\cdot|^p$, and also in [31] with $p = 3$ under an additional restriction $\mathbb{E}X/\sqrt{\mathbb{E}X^2} = t$ for each $t \in (-1, 1)$.

2. Formulations of main results

Theorem 1. *For each* $t \in (-1, 1)$

$$H(t) := \sup_{\substack{\mathscr{L}(X)\in\mathscr{P}:\\ \mathbb{E}X = t\sqrt{\mathbb{E}X^2}}} \frac{\mathbb{E}|X|^3}{\mathbb{E}|X-t|^3} = \begin{cases} 1, & t = 0, \\ 1 + \dfrac{3t^2}{1-t^2} \cdot \dfrac{1 - z^2(t)}{1 - 3z^2(t)}, & 0 < |t| < t_0, \\ \dfrac{2}{z(t)(3 - z^2(t))}, & t_0 \leqslant |t| < 1, \end{cases} \tag{8}$$

where $t_0 = \sqrt{\frac{5-\sqrt{7}}{6}} = 0.6263\ldots$,

$$z(t) = \begin{cases} u(t), & 0 < t < t_0, \\ v(t), & t_0 \leqslant t < 1, \end{cases} \tag{9}$$

$u(t), 0 < t < \sqrt{3}/2 = 0.8660\ldots$, *is the unique root of the equation*

$$\frac{4u\sqrt{1-u^2}}{3u^2 - 1} = \frac{4t^2 - 3}{3t\sqrt{1-t^2}} \tag{10}$$

on the interval $0 < u < \frac{\sqrt{3}}{3}$, *and* $v(t), t \in (0, 1)$, *is the unique root of the equation*

$$\frac{2(1-v^2)^{3/2}}{v(3-v^2)} = \frac{t(3-2t^2)}{(1-t^2)^{3/2}} \tag{11}$$

on the interval $0 < v < 1$. *Moreover, the function* $H(t)$ *continuously and monotonically increases on* $[0, 1)$ *with*

$$\lim_{t \to 1-} H(t) = +\infty.$$

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For $0 < t < 1$ the supremum in (7) is attained, at that, only on the two-point distribution of the form

$$\mathbb{P}(X = x) = p = 1 - \mathbb{P}(X = y) =: 1 - q, \tag{12}$$

where

$$x = x(t) = t + \sqrt{\tfrac{q}{p}(1-t^2)}, \quad y = y(t) = t - \sqrt{\tfrac{p}{q}(1-t^2)},$$
$$p = p(t) = \frac{1 - z(t)}{2}, \quad t \in (0,1). \tag{13}$$

If $0 \leqslant t \leqslant t_0$, then the function H admits the upper bound

$$H(t) \leqslant \hat{H}(t) := \frac{5t + 6\sqrt{2}}{2(1-t^2)(3\sqrt{2} - 2t)}, \tag{14}$$

moreover $\hat{H}$ is continuous, monotonically increases on $[0, t_0]$ and

$$\lim_{t\to 0+} \frac{\hat{H}(t) - 1}{H(t) - 1} = 1.$$

The values of the functions $H(t)$ and $p(t)$ for some $t \in [0,1)$ rounded above up to the fourth digit are presented in the second and fifth columns of Table 1. Since the function $p(t)$ is close to linear (see the left graph on Fig. 2), for more clarity, the right graph on Fig. 2 also represents the normalized function

$$\widetilde{p}(t) := \frac{p(t)}{p(0+) + t(p(1-) - p(0+))} = \frac{p(t)}{\frac{3-\sqrt{3}}{6} + t\left(\frac{1}{2} - \frac{3-\sqrt{3}}{6}\right)}, \quad 0 < t < 1. \tag{15}$$

Theorem 2. *For each $t \in (-1,1)$*

$$\sup_{\substack{\mathscr{L}(X)\in\mathscr{P}:\\ \mathbb{E}X = t\sqrt{\mathbb{E}X^2}}} J(X) = \sup_{\substack{\mathscr{L}(X)\in\mathscr{P}:\\ \mathbb{E}X = t\sqrt{\mathbb{E}X^2}}} \frac{\mathbb{E}|X|^3}{\mathbb{E}|X-t|^3}(1-t^2)^{3/2} = H(t)(1-t^2)^{3/2}.$$

Moreover, the function $H(t)(1-t^2)^{3/2}$ is even and continuous on the interval $(-1,1)$, increases on the interval $0 < t < t_0 := \sqrt{\frac{5-\sqrt{7}}{6}} = 0.6263\ldots$, decreases on the interval $t_0 < t < 1$, and

$$\sup_{\mathscr{L}(X)\in\mathscr{P}} J(X) = \sup_{-1<t<1} \sup_{\substack{\mathscr{L}(X)\in\mathscr{P}:\\ \mathbb{E}X = t\sqrt{\mathbb{E}X^2}}} J(X) = H(t_0)(1-t_0^2)^{3/2} =$$

$$= \frac{\sqrt{1-t_0^2}}{1 - 2t_0^2 + 2t_0^4} = \frac{\sqrt{17 + 7\sqrt{7}}}{4} = 1.489971\ldots.$$

Furthermore, the supremum is attained, at that, only on the two-point distribution of the form

$$\mathbb{P}(X = \pm t_0^{-1}) = t_0^2, \quad \mathbb{P}(X = 0) = 1 - t_0^2.$$

The limit values of $H(t)(1-t^2)^{3/2}$ at the endpoints of the interval $(-1,1)$ are

$$\lim_{t\to\pm 1} H(t)(1-t^2)^{3/2} = 1.$$

If $0 \leqslant t \leqslant t_0$, then the function $H(t)(1-t^2)^{3/2}$ admits the upper bound

$$H(t)(1-t^2)^{3/2} < \hat{H}(t)(1-t^2)^{3/2} = \frac{\sqrt{1-t^2}(5t + 6\sqrt{2})}{2(3\sqrt{2} - 2t)}, \tag{16}$$

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with the function $\hat{H}$ defined in Theorem 1 (see (14)), moreover, $\hat{H}(t)(1-t^2)^{3/2}$ is also continuous, monotonically increases on $[0,t_0]$ and

$$\hat{H}(t_0)(1-t_0^2)^{3/2} = H(t_0)(1-t_0^2)^{3/2} + 0.0244\ldots = 1.5144\ldots,$$

$$\lim_{t\to 0+}\frac{\hat{H}(t)(1-t^2)^{3/2}-1}{H(t)(1-t^2)^{3/2}-1} = 1.$$

The values of the function $H(t)(1-t^2)^{3/2}$ for some $t\in[0,1)$, rounded above up to the fourth digit, are presented in the third column of Table 1. The plots of the functions $H(t)(1-t^2)^{3/2}$ and $\hat{H}(t)(1-t^2)^{3/2}$ are shown on Fig. 1.

Theorem 2 and inequality (1) directly imply the following estimate of the accuracy of the normal approximation to the distribution of a Poisson random sum in terms of central moments of the summands X_j.

Theorem 3. *For each $t\in(-1,1)$ and for any common distribution of summands $\mathscr{L}(X)\in\mathscr{P}$ with $\mathbb{E}X = t\sqrt{\mathbb{E}X^2}$ we have*

$$\Delta_\lambda(X) \leqslant \frac{C_0(t)}{\sqrt{\lambda}}\cdot L_0(X),\quad \lambda>0, \tag{17}$$

where

$$C_0(t) = C_1\cdot H(t)(1-t^2)^{3/2} \leqslant 0.3031\cdot\frac{\sqrt{17+7\sqrt{7}}}{4} < 0.4517,\quad t\in(-1,1).$$

If $|\mathbb{E}X| \leqslant t\sqrt{\mathbb{E}X^2}$, then inequality (17) holds for each $t\in[0,1)$ with $C_0(t)$ replaced by $C_0(t\wedge t_0)$, where $t_0 = \sqrt{\frac{5-\sqrt{7}}{6}} = 0.6263\ldots$ was defined in Theorem 1. Moreover, $C_0(t)$ admits the estimate

$$C_0(t) \leqslant 0.3031\cdot\frac{\sqrt{1-t^2}(5t+6\sqrt{2})}{2(3\sqrt{2}-2t)},\quad 0\leqslant t\leqslant t_0,$$

the right-hand side of which monotonically increases on $(0,t_0)$.

The values of $C_0(t)$, rounded above up to the fourth digit, are presented for some $t\in[0,1)$ in the fourth column of Table 1.

Before turning to the proofs of these theorems note that we obviously have $H(0)=1$,

$$H(t) = \sup_{\substack{\mathscr{L}(X)\in\mathscr{P}:\\ \mathbb{E}X=t,\\ \mathbb{E}X^2=1}}\frac{\mathbb{E}|X|^3}{\mathbb{E}|X-t|^3} = \sup_{\substack{\mathscr{L}(X)\in\mathscr{P}:\\ \mathbb{E}X=t,\\ \mathbb{E}X^2=1}}\frac{\mathbb{E}|(-X)|^3}{\mathbb{E}|(-X)-(-t)|^3} = \sup_{\substack{Y\in\mathscr{P}:\\ \mathbb{E}Y=-t,\\ \mathbb{E}Y^2=1}}\frac{\mathbb{E}|Y|^3}{\mathbb{E}|Y-(-t)|^3} = H(-t),$$

and hence, it suffices to consider only $t\in(0,1)$.

3. Reduction to the case of two-point distributions

Lemma 1. *Let $t\in\mathbb{R}\setminus\{0\}$. Then for all $u,v\in\mathbb{R}$ such that*

$$\begin{cases} u+v>0,\\ u<1<v,\end{cases}$$

there holds the inequality

$$|x-t|^3 \geqslant a+bx+cx^2+d|x|^3,\quad x\in\mathbb{R}, \tag{18}$$

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where

$$a = a_t(u,v) = |t|^3 a_1(u,v), \tag{19}$$
$$b = b_t(u,v) = t|t| b_1(u,v), \tag{20}$$
$$c = c_t(u,v) = |t| c_1(u,v), \tag{21}$$
$$d = d(u,v), \tag{22}$$

$$a_1(u,v) = \begin{cases} -\dfrac{(2uv-u-v)\left(2u^2v^2-2u^2v-u^2-2uv^2+4uv-v^2\right)}{(u-v)^3}, & u \geqslant 0, \\ \dfrac{6u^4v^2-u^4-12u^3v^2+4u^3v+6u^2v^4-12u^2v^3+6u^2v^2+4uv^3-v^4}{(u-v)(u+v)(u^2-4uv+v^2)}, & u < 0, \end{cases}$$

$$b_1(u,v) = \begin{cases} 3\left(2u^3v^2-4u^3v+u^3+2u^2v^3-4u^2v^2+\right. \\ \qquad \left.+5u^2v-4uv^3+5uv^2-4uv+v^3\right)/(u-v)^3, & u \geqslant 0, \\ -\dfrac{3\left(4u^3v-u^3-4u^2v^2-3u^2v+4uv^3-3uv^2+4uv-v^3\right)}{(u-v)(u^2-4uv+v^2)}, & u < 0, \end{cases}$$

$$c_1(u,v) = \begin{cases} \dfrac{3\left(u^3-4u^2v^2+5u^2v-4u^2+5uv^2-4uv+2u+v^3-4v^2+2v\right)}{(u-v)^3}, & u \geqslant 0, \\ \dfrac{3\left(u^4+4u^3v-4u^3-6u^2v^2+2u^2+4uv^3+v^4-4v^3+2v^2\right)}{(u-v)(u+v)(u^2-4uv+v^2)}, & u < 0, \end{cases}$$

$$d(u,v) = \begin{cases} -\dfrac{(u+v-2)\left(u^2-4uv+2u+v^2+2v-2\right)}{(u-v)^3}, & u \geqslant 0, \\ \dfrac{(u+v-2)\left(u^2-4uv+2u+v^2+2v-2\right)}{(u+v)(u^2-4uv+v^2)}, & u < 0, \end{cases}$$

with equality attained exactly in the two points ut and vt.

Remark 1. *In [31, Lemma 1] it was demonstrated that for any $t \in \mathbb{R} \setminus \{0\}$ and real u, v such that*

$$\begin{cases} u+v<0, \\ v>1, \end{cases}$$

the inequality

$$|x-t|^3 \leqslant a_t(u,v) + b_t(u,v)x + c_t(u,v)x^2 + d_t(u,v)|x|^3, \quad x \in \mathbb{R},$$

holds with the same functions a_t, b_t, c_t, d_t as in Lemma 1 for the case $u < 0$ with equality attained in exactly the two points ut and vt.

Let
$$f(x) = |x-1|^3 \quad \text{and} \quad g(x) = a + bx + cx^2 + d|x|^3, \quad x \in \mathbb{R},$$
be the left-hand and right-hand sides of (18) with $t = 1$ correspondingly. Figs. 3–5 illustrate that several variants of the location of tangency points of the functions f and g with respect to the stationary points of g are possible. On the left side of these figures there are the plots of $f(x)$ (solid line) and $g(x)$ (dotted line), whereas on the right side, for clarity, there is the plot of the difference $f(x) - g(x)$.

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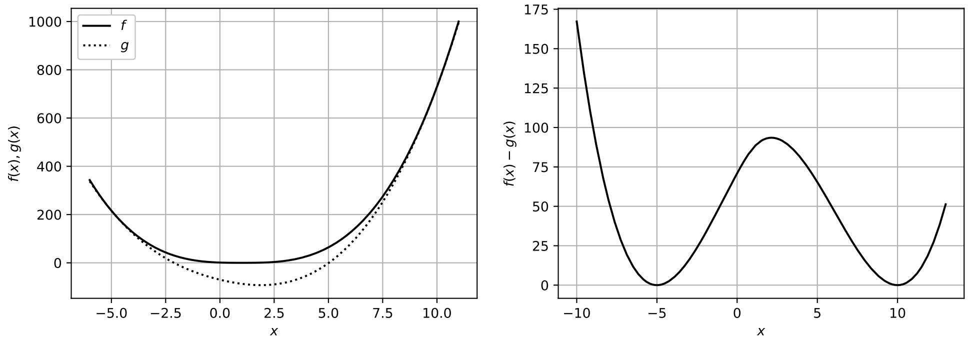

Figure 3. The graphs of the functions $f(x) = |x-1|^3$ and $g(x) = a + bx + cx^2 + d|x|^3$ from Lemma 1 (left), and the graph of the difference $f - g$ (right) for $u = -5$, $v = 10$ $(d > 0)$. The unique minimum point of g lies between the tangency points u and v.

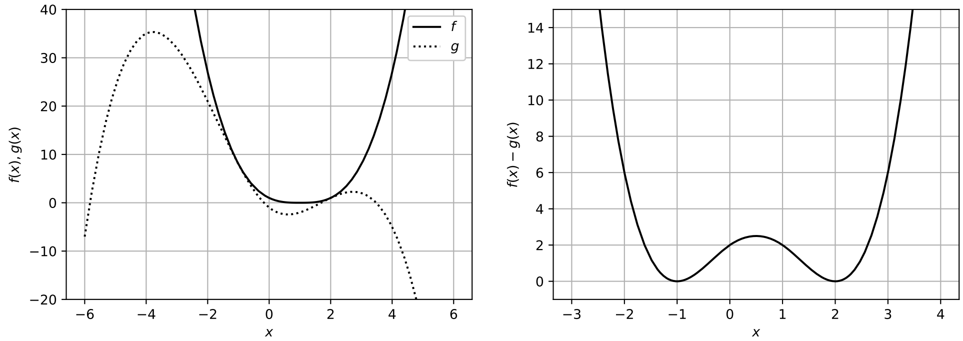

Figure 4. The graphs of the functions $f(x) = |x-1|^3$ and $g(x) = a + bx + cx^2 + d|x|^3$ from Lemma 1 (left), and the graph of the difference $f - g$ (right) for $u = -1$, $v = 2$ $(d < 0)$. The unique minimum point of g lies between the tangency points u and v. The maximum points lie to the left from the point u and to the right from the point v.

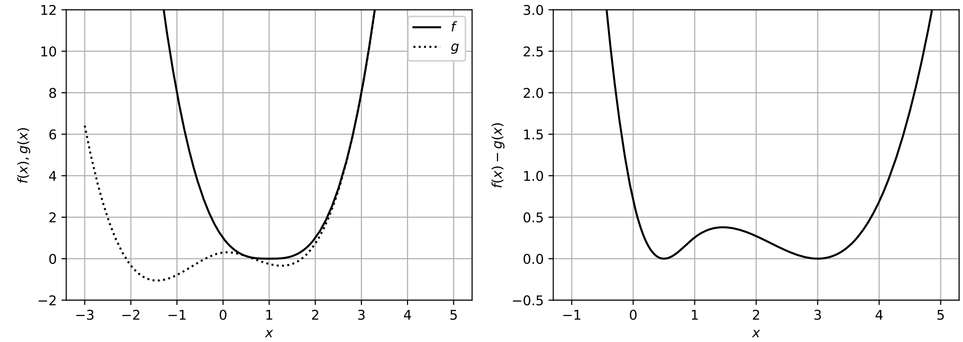

Figure 5. The graphs of the functions $f(x) = |x-1|^3$ and $g(x) = a + bx + cx^2 + d|x|^3$ from Lemma 1 (left), and the graph of the difference $f - g$ (right) for $u = 0.5$, $v = 3$ $(d > 0)$. The unique minimum point of g lies between the tangency points u and v. Two more stationary points, minimum and maximum points, lie to the left from the point u.

Proof. By virtue of relations (19), (20), (21), (22), the problem is reduced to the case $t = 1$ by the scale transformation. Let

$$f(x) = |x-1|^3, \quad g(x) = a + bx + cx^2 + d|x|^3, \quad h(x) = f(x) - g(x), \quad x \in \mathbb{R}.$$

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The coefficients a, b, c, d given in the formulation of the lemma, are constructed so that the points u and v are the tangency points of the functions $g(x)$ and $f(x)$, that is, these coefficients are the solutions of the system of four linear equations

$$\begin{cases} g(u) &= f(u), \\ g'(u) &= f'(u), \\ g(v) &= f(v), \\ g'(v) &= f'(v), \end{cases} \iff \begin{cases} a+bu+cu^2+d|u|^3 &= (1-u)^3, \\ b+2cu+3du|u| &= -3(1-u)^2, \\ a+bv+cv^2+dv^3 &= (v-1)^3, \\ b+2cv+3dv^2 &= 3(v-1)^2. \end{cases}$$

Prove that $h(x) \geqslant 0$ for any $x \in \mathbb{R}$. 86

1. Let $0 \leqslant u < 1$. Then

$$a_1(u,v) = -\frac{(2uv-u-v)\big(2u^2v^2-2u^2v-u^2-2uv^2+4uv-v^2\big)}{(u-v)^3},$$
$$b_1(u,v) = \frac{3\big(2u^3v^2-4u^3v+u^3+2u^2v^3-4u^2v^2+5u^2v-4uv^3+5uv^2-4uv+v^3\big)}{(u-v)^3},$$
$$c_1(u,v) = \frac{3\big(u^3-4u^2v^2+5u^2v-4u^2+5uv^2-4uv+2u+v^3-4v^2+2v\big)}{(u-v)^3},$$
$$d(u,v) = -\frac{(u+v-2)\big(u^2-4uv+2u+v^2+2v-2\big)}{(u-v)^3}.$$

1 a) Let $x \geqslant 1$. We have

$$h(x) = \frac{2(u-1)^2(x-v)^2(2uv+ux-3u-3vx+v+2x)}{(u-v)^3}.$$

Since

$$2(u-1)^2(x-v)^2 \geqslant 0, \quad (u-v)^3 < 0,$$

it suffices to show that

$$s_1(x) := 2uv+ux-3u-3vx+v+2x \leqslant 0.$$

We have

$$s_1(1) = 2(u-1)(v-1) < 0,$$
$$s_1'(x) = u-3v+2 < 0, \quad \text{since } v > 1 \geqslant \frac{u+2}{3},$$

therefore $s_1(x) < 0$ and $h(x) \geqslant 0$, moreover, $h(x) = 0$ if and only if $x = v$. 87

1 b) Let $0 \leqslant x < 1$. Then

$$h(x) = \frac{2(x-u)^2(v-1)^2(2uv-3ux+u+vx-3v+2x)}{(u-v)^3}.$$

Since

$$2(x-u)^2(v-1)^2 \geqslant 0, \quad (u-v)^3 < 0,$$

it suffices to show that

$$s_2(x) := (v-3u+2)x+2uv+u-3v \leqslant 0.$$

We have

$$s_2(0) = 2uv+u-3v < 0, \quad \text{since } v > 1 > \frac{u}{3-2u},$$
$$s_2(1) = 2(u-1)(v-1) < 0,$$

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$$\min\{s_2(0), s_2(1)\} \leqslant s_2(x) \leqslant \max\{s_2(0), s_2(1)\},$$

therefore $s_2(x) < 0$ and $h(x) \geqslant 0$, moreover, $h(x) = 0$ if and only if $x = u$. 88

1 c) Let $x < 0$. Then

$$h'(x) = -3(1-x)^2 - b - 2cx + 3dx^2,$$
$$h''(x) = 6(d-1)x + 2(3-c),$$
$$h(0) \geqslant 0,$$

moreover, $h(0) = 0$ if and only if $u = 0$ (as it was proved above),

$$h'(0) = -b - 3 = \frac{6u(v-1)^2(u^2+uv-2v)}{(v-u)^3},$$

$$d - 1 = \frac{2(u-1)^2(u-3v+2)}{(v-u)^3},$$

$$3 - c = \frac{6(v-1)^2(2u^2-u-v)}{(u-v)^3}.$$

With the account of relations

$$\frac{6u(v-1)^2}{(v-u)^3} \geqslant 0,$$

$$u^2 + uv - 2v < 0, \quad \text{since } v > 1 > \frac{u^2}{2-u},$$

we have $h'(0) \leqslant 0$, moreover, $h'(0) = 0$ if and only if $u = 0$. Note that

$$\frac{2(u-1)^2}{(v-u)^3} > 0,$$

$$u - 3v + 2 < 0, \quad \text{since } v > 1 > \frac{u+2}{3},$$

$$\frac{6(v-1)^2}{(u-v)^3} < 0,$$

$$2u^2 - u - v < 0, \quad \text{since } v > 1 > 2u^2 - u,$$

therefore $d - 1 < 0$, $3 - c > 0$ and $h''(x) > 0$. Hence, $h'(x)$ increases, whence with the 89
account of $h'(0) \leqslant 0$ we obtain that $h'(x) < 0$ for $x < 0$, that is, $h(x)$ monotonically 90
decreases for $x < 0$. Since $h(0) \geqslant 0$, we have $h(x) > 0$ for $x < 0$. 91

2. Now let $u < 0$. We have

$$a_1(u,v) = \frac{6u^4v^2 - u^4 - 12u^3v^2 + 4u^3v + 6u^2v^4 - 12u^2v^3 + 6u^2v^2 + 4uv^3 - v^4}{(u-v)(u+v)(u^2-4uv+v^2)},$$
$$b_1(u,v) = -\frac{3(4u^3v - u^3 - 4u^2v^2 - 3u^2v + 4uv^3 - 3uv^2 + 4uv - v^3)}{(u-v)(u^2-4uv+v^2)},$$
$$c_1(u,v) = \frac{3(u^4 + 4u^3v - 4u^3 - 6u^2v^2 + 2u^2 + 4uv^3 + v^4 - 4v^3 + 2v^2)}{(u-v)(u+v)(u^2-4uv+v^2)},$$
$$d(u,v) = \frac{(u+v-2)(u^2-4uv+2u+v^2+2v-2)}{(u+v)(u^2-4uv+v^2)}.$$

2 a) Let $x \geqslant 1$. Then

$$h(x) = \frac{2(x-v)^2 s_3(x)}{(v-u)(u+v)(u^2-4uv+v^2)},$$

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where

$$s_3(x) = 3u^4 - 6u^3 + 3u^2v^2 + 6u^2vx - 6u^2v - 3u^2x + 3u^2 - 6uv^2x + 4uv + 2ux + 3v^2x - v^2 - 2vx.$$

Note that

$$\frac{2(x-v)^2}{(v-u)(u+v)(u^2-4uv+v^2)} \geqslant 0,$$

moreover, the left-hand side of this inequality turns into zero if and only if $x = v$. Therefore, it suffices to show that $s_3(x) > 0$. But this follows from the relations

$$s_3(1) = 3u^4 - 6u^3 + 3u^2v^2 - 2u(v-1)(3v+1) + 2v(v-1) > 0,$$

$$s_3'(x) = 3u^2(2v-1) - 2u(3v^2-1) + v(3v-2) > 0.$$

2 b) Let $x \leqslant 0$. Then

$$h(x) = \frac{2(x-u)^2 s_4(x)}{(v-u)(u+v)(u^2-4uv+v^2)},$$

where

$$s_4(x) = 3u^2v^2 - 6u^2vx + 3u^2x - u^2 + 6uv^2x - 6uv^2 + 4uv - 2ux + 3v^4 - 6v^3 - 3v^2x + 3v^2 + 2vx.$$

Note that

$$\frac{2(x-u)^2}{(v-u)(u+v)(u^2-4uv+v^2)} \geqslant 0,$$

moreover, the left-hand side of this inequality turns into zero if and only if $x = u$. Therefore, it suffices to show that $s_4(x) > 0$. But this follows from the relations

$$s_4(0) = u^2(3v^2-1) - 2uv(3v-2) + 3v^2(v-1)^2 > 0,$$

$$s_4'(x) = (v-u)(3u(2v-1) - (3v-2)) < 0.$$

2 c) Let $0 < x < 1$. For all $u < 0$, $v > 1$, $u + v > 0$ we have

$$h(0) = \frac{2u^2\Big(u^2(3v^2-1) - 2uv(3v-2) + 3v^2(v-1)^2\Big)}{(v-u)(u+v)(u^2-4uv+v^2)} > 0,$$

$$h(1) = \frac{2(v-1)^2\Big(3u^4 - 6u^3 + 3u^2v^2 - 2u\big(3v^2-2v-1\big) + 2v(v-1)\Big)}{(v-u)(u+v)(u^2-4uv+v^2)} > 0,$$

$$h'(0) = -\frac{6u\Big(u^2(2v-1) - uv(2v-1) + 2v(v-1)^2\Big)}{(v-u)(u^2-4uv+v^2)} > 0.$$

Moreover,

$$h'''(x) = \frac{12(1-u-v)(u^2+u(1-4v)+v^2+v-2)}{(u+v)(u^2-4uv+v^2)} \geqslant 0 \quad \Longleftrightarrow \quad u+v \leqslant 1.$$

Consider the case $u + v \geqslant 1$. Since $h'''(x) \leqslant 0$, the function h' is concave in this case. Since $h'(0) > 0$, the function h' has at most one root x_0 on the interval $0 \leqslant x \leqslant 1$, moreover, $h'(x) \geqslant 0$ for $0 \leqslant x \leqslant x_0$ and $h'(x) \leqslant 0$ for $x_0 \leqslant x \leqslant 1$. Therefore, either $h(x)$ does not decrease on $0 \leqslant x \leqslant 1$ (if h' is nonnegative), or it does not decrease on $0 \leqslant x \leqslant x_0$ and does not increase on $x_0 \leqslant x \leqslant 1$. Hence it follows that

$$\min_{0 \leqslant x \leqslant 1} h(x) = \min\{h(0), h(1)\}.$$

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Since $h(0) > 0$ and $h(1) > 0$, we have $h(x) > 0$.

Now consider the case $0 < u + v < 1$. In this case h' is convex. Note that

$$h'(1) = \frac{6(v-1)(2-u-v)\Big(2u^3 - 2u^2v + u\big(2v^2 - v - 1\big) - v(v-1)\Big)}{(v-u)(u+v)(u^2 - 4uv + v^2)} < 0.$$

Since $h'(0) > 0,\ h'(1) < 0$ and h' is convex, the function h' has exactly one root x_1 on the interval $0 \leqslant x \leqslant 1$, moreover, $h'(x) \geqslant 0$ for $0 \leqslant x \leqslant x_1$ and $h'(x) \leqslant 0$ for $x_1 \leqslant x \leqslant 1$. So, the function h does not decrease on the interval $0 \leqslant x \leqslant x_1$ and does not increase on $x_1 \leqslant x \leqslant 1$. Therefore

$$\min_{0 \leqslant x \leqslant 1} h(x) = \min\{h(0), h(1)\}.$$

With the account of $h(0) > 0$ and $h(1) > 0$ we have $h(x) > 0$ for all $0 \leqslant x \leqslant 1$. □

Lemma 1 obviously implies the following statement.

Lemma 2. *For any $\mathscr{L}(X) \in \mathscr{P}$, $t \in \mathbb{R} \setminus \{0\}$ and any $u, v \in \mathbb{R}$ such that*

$$\begin{cases} u + v > 0, \\ u < 1 < v, \end{cases}$$

there holds the inequality

$$\mathbb{E}|X - t|^3 \geqslant a_t(u,v) + b_t(u,v)\mathbb{E}X + c_t(u,v)\mathbb{E}X^2 + d(u,v)\mathbb{E}|X|^3,$$

with equality attained if and only if the distribution of the r.v. X is concentrated in two points ut and vt.

By $\mathscr{P}_2$ we denote the class of all non-degenerate two-point distributions. Obviously, $\mathscr{P}_2 \subset \mathscr{P}$.

Lemma 3. *For any $t \in (0,1)$*

$$H(t) := \sup_{\substack{\mathscr{L}(X) \in \mathscr{P}:\\ \mathbb{E}X = t,\\ \mathbb{E}X^2 = 1}} \frac{\mathbb{E}|X|^3}{\mathbb{E}|X - t|^3} = \sup_{\substack{\mathscr{L}(X) \in \mathscr{P}_2:\\ \mathbb{E}X = t,\\ \mathbb{E}X^2 = 1}} \frac{\mathbb{E}|X|^3}{\mathbb{E}|X - t|^3},$$

moreover, the supremum on the right-hand side can be attained only on the two-point distributions.

Proof. It suffices to prove that for any $\varrho \geqslant 1$ and any r.v. X with

$$\mathbb{E}X = t, \quad \mathbb{E}X^2 = 1, \quad \mathbb{E}|X|^3 = \varrho$$

there exists a two-point r.v. Y with

$$\mathbb{E}Y = t, \quad \mathbb{E}Y^2 = 1, \quad \mathbb{E}|Y|^3 = \varrho,$$

satisfying the inequality

$$\mathbb{E}|X - t|^3 \geqslant \mathbb{E}|Y - t|^3,$$

since these conditions imply

$$H(t) = \sup_{\varrho \geqslant 1} \sup_{\substack{\mathscr{L}(X) \in \mathscr{P}:\ \mathbb{E}X = t,\\ \mathbb{E}X^2 = 1,\, \mathbb{E}|X|^3 = \varrho}} \frac{\varrho}{\mathbb{E}|X - t|^3} \leqslant \sup_{\varrho \geqslant 1} \sup_{\substack{Y \in \mathscr{P}_2:\ \mathbb{E}Y = t,\\ \mathbb{E}Y^2 = 1,\, \mathbb{E}|Y|^3 = \varrho}} \frac{\varrho}{\mathbb{E}|Y - t|^3}, \quad 0 < t < 1,$$

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where actually only the equality is possible, since $\mathscr{P}_2 \subset \mathscr{P}$.

1) Let $\varrho > 1$. Consider the two-point r.v. Y_p taking values $x > y$ with probabilities p and $q = 1 - p$ correspondingly. Assume that $\mathbb{E}Y_p = t, \mathbb{E}Y_p^2 = 1$. Then we necessarily have

$$x = x(p) = t + \sqrt{(1-t^2)q/p}, \quad y = y(p) = t - \sqrt{(1-t^2)p/q}.$$

Show that $x + y > 0$ if and only if $p < \frac{1+t}{2}$. We have

$$x + y > 0 \iff \frac{2t\sqrt{pq} + \sqrt{1-t^2}(q-p)}{\sqrt{pq}} > 0 \iff 2t\sqrt{p(1-p)} > \sqrt{1-t^2}(2p-1).$$

The last inequality obviously holds for $0 < p \leqslant \frac{1}{2}$, since the left-hand side is positive, whereas the right-hand side is non-positive. If $\frac{1}{2} < p < 1$, then both sides of the inequality are positive, therefore, they can be squared:

$$4t^2p(1-p) > (1-t^2)(4p^2 - 4p + 1) \iff t^2 > (2p-1)^2 \iff p < \frac{1+t}{2}.$$

Unifying the intervals under consideration, we obtain the desired statement. Note that on the interval $0 < p < \frac{1+t}{2}$ the function

$$\widetilde{\varrho}(p) \equiv \mathbb{E}|Y_p|^3 = p\Big(t + \sqrt{\tfrac{q}{p}(1-t^2)}\Big)^3 + q\Big|t - \sqrt{\tfrac{p}{q}(1-t^2)}\Big|^3$$

takes all values from the interval $(1, +\infty)$, because for any $0 < t < 1$ we have

$$\widetilde{\varrho}\Big(\frac{1+t}{2}\Big) = 1, \quad \lim_{p\to 0+} \widetilde{\varrho}(p) = +\infty$$

and $\widetilde{\varrho}(p)$ is continuous, whence for any $\varrho > 1$ there exists $p_0 \in \big(0, \frac{1+t}{2}\big)$ such that $\mathbb{E}|Y_{p_0}|^3 = \varrho$. Further, note that

$$\begin{cases} y(p_0) < t < x(p_0), \\ x(p_0) + y(p_0) > 0, \end{cases}$$

and hence the couple

$$u = \frac{y(p_0)}{t}, \quad v = \frac{x(p_0)}{t}$$

satisfies the conditions of Lemma 2, according to which with the account of the definition of the r.v. Y_{p_0} we have

$$\mathbb{E}|X - t|^3 \geqslant a_t(u,v) + b_t(u,v)t + c_t(u,v) + d(u,v)\varrho = \mathbb{E}|Y_{p_0} - t|^3,$$

with the inequality turning into equality if and only if the distribution of the r.v. X is concentrated in exactly two points ut and vt. But with the account of the moment conditions this is possible if and only if $X \stackrel{d}{=} Y_{p_0}$. Therefore, the desired statement holds for the r.v. $Y \stackrel{d}{=} Y_{p_0}$.

2) Now let $\varrho = 1$. By virtue of the Jensen inequality, for the convex function $f(x) = x^{3/2}$, $x \geqslant 0$, we have

$$1 = \mathbb{E}|X|^3 = \mathbb{E}f(X^2) \geqslant f(\mathbb{E}X^2) = f(1) = 1,$$

moreover, since f is strictly convex, the equality is possible if and only if

$$\mathbb{P}(X^2 = \mathbb{E}X^2) = 1, \quad \text{i.e. } \mathbb{P}(|X| = 1) = 1,$$

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whence with the account of the condition $\mathbb{E}X = t$ it follows that X has the two-point distribution of the form

$$\mathbb{P}(X=1) = \frac{1+t}{2}, \quad \mathbb{P}(X=-1) = \frac{1-t}{2}.$$

So, the desired statement holds for $Y \stackrel{d}{=} X$. $\square$

4. The analysis of two-point distributions

Recall that by $\mathscr{P}_2$ we denoted the class of all nondegenerate two-point distributions.

Lemma 4. **(a)** *For any* $t \in (0,1)$

$$\sup_{\substack{\mathscr{L}(X)\in\mathscr{P}_2:\\ \mathbb{E}X=t,\\ \mathbb{E}X^2=1}} \frac{\mathbb{E}|X|^3}{\mathbb{E}|X-t|^3} = \max_{-1<z<1} M(z,t) = M(z(t),t), \tag{23}$$

where the function $z(t)$, $t \in (0,1)$, *was defined in Theorem 1 (see (9))*

$$M(z,t) = \begin{cases} M_1(z,t), & -1<z<1-2t^2, \\ M_2(z,t), & 1-2t^2 \leqslant z < 1, \end{cases}$$

$$M_1(u,t) = 1 + \frac{3t^2}{1-t^2} \cdot \frac{1-u^2-a(t)u\sqrt{1-u^2}}{1+u^2}, \quad u \in (-1,1),$$

$$M_2(v,t) = \frac{b(t)\sqrt{1-v^2}+2v}{v^2+1}, \quad v \in (-1,1),$$

$$a(t) = \frac{4t^2-3}{3t\sqrt{1-t^2}}, \quad b(t) = \frac{t(3-2t^2)}{(1-t^2)^{3/2}}, \quad t \in (0,1),$$

moreover, the supremum in (23) is attained only on the two-point distribution defined in Theorem 1 (see (12)).

(b) *The functions* M, M_1, M_2 *are differentiable in the domain* $(z,t) \in (-1,1) \times (0,1)$ *and have continuous derivatives there.*

(c) *There hold the equalities*

$$\lim_{z\to-1+} M_1(z,t) = \lim_{z\to 1-} M_1(z,t) = 1, \quad t \in (0,1),$$

$$\lim_{z\to-1+} M_2(z,t) = -1, \quad \lim_{z\to 1-} M_2(z,t) = 1, \quad t \in (0,1),$$

$$\lim_{t\to 0} M_1(z,t) = 1, \quad \lim_{t\to 0} M_2(z,t) = \frac{2z}{z^2+1}, \quad z \in (-1,1),$$

$$\lim_{t\to 1} M_2(z,t) = +\infty, \quad z \in (-1,1).$$

(d) *The function* $z(t)$ *is continuously differentiable and monotonically decreases on the interval* $t \in (0,1)$ *taking the values*

$$z(0+) = \frac{\sqrt{3}}{3}, \quad z(t_0) = \frac{\sqrt{7}-2}{3}, \quad z(1-) = 0.$$

Moreover, the inequalities

$$z(t) \leqslant 1-2t^2, \quad t \in (0,t_0],$$

$$z(t) \geqslant 1-2t^2, \quad t \in [t_0,1),$$

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hold so that the equality in each of them is attained only in the point $t = t_0 := \sqrt{\frac{5-\sqrt{7}}{6}} = 0.6263\ldots,$ *defined in Theorem 1.*

Proof. **(a)** Consider the two-point distribution

$$\mathbb{P}(X = x) = p = 1 - \mathbb{P}(X = y) = 1 - q, \tag{24}$$

with some $x > t > y,\ \ p \in (0,1)$. From the conditions

$$\mathbb{E}X = t, \quad \mathbb{E}X^2 = 1$$

we find that

$$x = t + \sqrt{\tfrac{q}{p}(1-t^2)}, \quad y = t - \sqrt{\tfrac{p}{q}(1-t^2)}. \tag{25}$$

Denote

$$\widetilde{H}(p,t) = \frac{\mathbb{E}|X|^3}{\mathbb{E}|X-t|^3} = \frac{p|x|^3 + q|y|^3}{p(x-t)^3 + q(t-y)^3}, \quad p \in (0,1),\ \ t \in (0,1). \tag{26}$$

Then

$$\widetilde{H}(t) := \sup_{\substack{\mathscr{L}(X)\in\mathscr{P}_2:\\ \mathbb{E}X=t,\\ \mathbb{E}X^2=1}} \frac{\mathbb{E}|X|^3}{\mathbb{E}|X-t|^3} = \sup_{0<p<1} \widetilde{H}(p,t). \tag{27}$$

Show that the last supremum can be written in the form (23) with $z(t)$ defined in (9).

For $0 < p \leqslant t^2$ we have $y \geqslant 0$ and

$$\widetilde{H}(p,t) = \frac{px^3 + qy^3}{p(x-t)^3 + q(t-y)^3} = \frac{t(3-2t^2) + \frac{q-p}{\sqrt{pq}}(1-t^2)^{3/2}}{\frac{p^2+q^2}{\sqrt{pq}}(1-t^2)^{3/2}} = \frac{b(t)\sqrt{pq} + (q-p)}{p^2+q^2}.$$

For $t^2 < p < 1$ we have $y < 0$ and

$$\widetilde{H}(p,t) = 1 + \frac{t(4t^2-3)(p-q) + 6t^2\sqrt{1-t^2}\sqrt{pq}}{\frac{p^2+q^2}{\sqrt{pq}}(1-t^2)^{3/2}} = 1 + \frac{3t^2}{1-t^2}\cdot\frac{a(t)\sqrt{pq}(p-q) + 2pq}{p^2+q^2}.$$

Introduce the new variable

$$z = q - p = 1 - 2p, \tag{28}$$

then

$$pq = \frac{1-z^2}{4}, \quad p^2 + q^2 = \frac{1+z^2}{2},$$

and the maximization of the function $\widetilde{H}(p,t)$ with respect to $p \in (0,1)$ is equivalent to that of the function $\widetilde{H}\left(\frac{1-z}{2}, t\right)$ with respect to $z \in (-1,1)$. Note that

$$\widetilde{H}\left(\frac{1-z}{2}, t\right) = M(z,t), \tag{29}$$

therefore, there holds the equality

$$\widetilde{H}(t) = \sup_{-1<z<1} M(z,t), \quad t \in (0,1).$$

Show that $z(t)$ is the unique global maximum point of the function $M(\cdot\,,t)$ for each $t \in (0,1)$, whence with the account of relations (24), (25), (28) the item (a) of the lemma follows.

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For M_1 we have

$$\frac{\partial M_1(u,t)}{\partial u} = \frac{3t^2}{1-t^2} \cdot \frac{a(t)(3u^2-1) - 4u\sqrt{1-u^2}}{\sqrt{1-u^2} \cdot (1+u^2)^2},$$

and hence, the stationary points of $M_1(\,\cdot\,,t)$ can be determined from the equation

$$g(u) := \frac{4u\sqrt{1-u^2}}{3u^2-1} = a(t),$$

which coincides with (10). 119

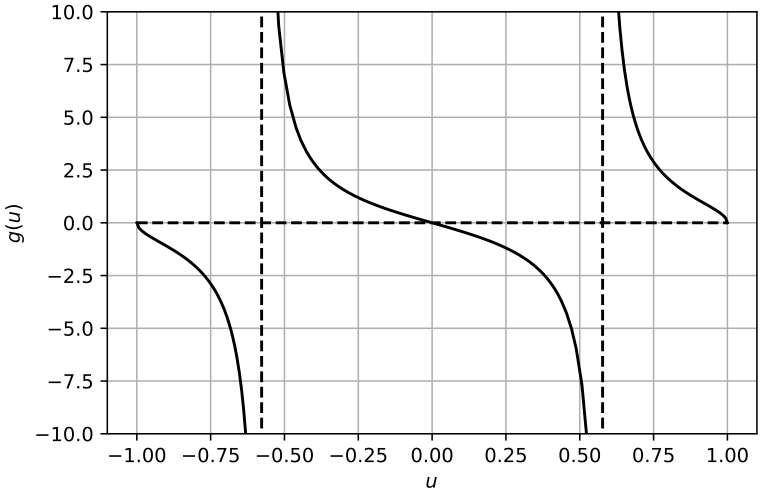

Figure 6. The plot of the function $g(u) = \frac{4u\sqrt{1-u^2}}{3u^2-1}$.

Note that the function $g(u)$ is even, continuously differentiable and monotonically decreasing on the intervals $(-1, -\sqrt{3}/3)$, $(-\sqrt{3}/3, \sqrt{3}/3)$, $(\sqrt{3}/3, 1)$ and has discontinuity points of the second kind in the points $u = \pm\sqrt{3}/3$ (see the plot of $g(u)$ on Fig. 6). Therefore, there exist the inverse functions

$$g_1^{-1} : (-\infty, 0) \to (-1, -\sqrt{3}/3),$$
$$g_2^{-1} : \mathbb{R} \to (-\sqrt{3}/3, \sqrt{3}/3),$$
$$g_3^{-1} : (0, +\infty) \to (\sqrt{3}/3, 1),$$

each of which is differentiable and monotonically decreases on its domain. 120

If $a(t) = 0$ (that is, $t = \sqrt{3}/2$), then it is easy to make sure that $u = 0$ is the unique 121
maximum point and unique stationary point of the function $M_1(\,\cdot\,,t)$ on $(-1,1)$. 122

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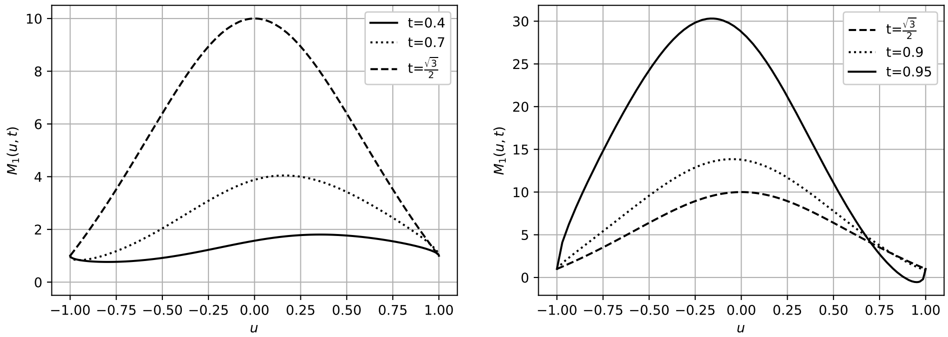

Figure 7. The plots of the functions $M_1(\,\cdot\,,t)$ for some t.

Now let $a(t) \neq 0$. By $u_1(t) < u_2(t)$ denote the roots of the equation $g(u) = a(t)$ on the interval $u \in (-1,1)$. If $a(t) > 0$ (that is, $t > \sqrt{3}/2$), then $u_1(t) = g_2^{-1}(a(t))$, $u_2(t) = g_3^{-1}(a(t))$ are respectively the points of local maximum and minimum of the function $M_1(\,\cdot\,,t)$ (see the plots of the function $M_1(\,\cdot\,,t)$ for some t on Fig. 7), moreover, $a(t)u_1(t) < 0$ and $M_1(u_1(t),t) > 1$. Since $a(t)$ is continuously differentiable and monotonically increases, then $u_1(\cdot)$ and $u_2(\cdot)$ are continuously differentiable and monotonically decrease. Moreover,

$$u_1(\sqrt{3}/2+) = 0, \quad u_1(1-) = -\sqrt{3}/3,$$

$$u_2(\sqrt{3}/2+) = 1, \quad u_2(1-) = \sqrt{3}/3.$$

And if $a(t) < 0$ (that is, $t < \sqrt{3}/2$), then $u_1(t) = g_1^{-1}(a(t))$, $u_2(t) = g_2^{-1}(a(t))$ are the points of local minimum and maximum respectively. Moreover $a(t)u_2(t) < 0$ and $M_1(u_2(t),t) > 1$. Since $a(t)$ is continuously differentiable and monotonically increases, then $u_1(t)$ and $u_2(t)$ are continuously differentiable and monotonically decrease on the interval $t \in (0, \sqrt{3}/2)$, and

$$u_1(0+) = -\sqrt{3}/3, \quad u_1(\sqrt{3}/2-) = -1,$$

$$u_2(0+) = \sqrt{3}/3, \quad u_2(\sqrt{3}/2-) = 0.$$

Since $M_1(\pm 1,t) = 1$, then the local maximum point of the function $M_1(\,\cdot\,,t)$ is the point of its global maximum on the whole interval $u \in (-1,1)$.

So, for an arbitrary $s \in (-1,1)$ we have

$$\sup_{-1<u<s} M_1(u,t) = \begin{cases} M_1(0 \wedge s, t), & a(t) = 0, \\ M_1(u_1(t) \wedge s, t), & a(t) > 0, \\ M_1(u_2(t) \wedge s, t) \vee 1, & a(t) < 0 \end{cases}$$

(here the symbols $\vee$ and $\wedge$ correspondingly denote maximum and minimum). For $s = 1 - 2t^2$ we obtain

$$\sup_{-1<u<1-2t^2} M_1(u,t) = \begin{cases} M_1\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right) = \frac{32}{5}, & t = \sqrt{3}/2, \\ M_1(u_1(t) \wedge (1-2t^2), t), & \sqrt{3}/2 < t < 1, \\ M_1(u_2(t) \wedge (1-2t^2), t) \vee 1, & 0 < t < \sqrt{3}/2. \end{cases}$$

Compare $1 - 2t^2$ with $u_2(t)$ for $0 < t < \frac{\sqrt{3}}{2}$ and with $u_1(t)$ for $\frac{\sqrt{3}}{2} < t < 1$. If $\frac{\sqrt{3}}{2} < t < 1$, then, as it has already been noted, $-\sqrt{3}/3 < u_1(t) < 0$, and hence, $u_1(t) > -\sqrt{3}/3 \geqslant 1 - 2t^2$ obviously for $\sqrt{\frac{1}{2} + \frac{\sqrt{3}}{6}} \leqslant t < 1$. And if $t \in \left(\frac{\sqrt{3}}{2}, \sqrt{\frac{1}{2} + \frac{\sqrt{3}}{6}}\right) = (0.866\ldots, 0.888\ldots)$,

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then $1-2t^2 \in \left(-\frac{\sqrt{3}}{3}, -\frac{1}{2}\right) \subset \left(-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$, that is, the point $1-2t^2$ belongs to the same interval $(-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3})$ of the monotonic decrease of the function $g(u)$ as $u_1(t)$, and hence, on the interval of the values of t under consideration we have

$$1-2t^2 \leqslant u_1(t) \quad \Leftrightarrow \quad g(1-2t^2) \geqslant g(u_1(t)) \equiv a(t) \quad \Leftrightarrow$$
$$\Leftrightarrow \quad \frac{4t(2t^2-1)\sqrt{1-t^2}}{-6t^4+6t^2-1} \geqslant \frac{4t^2-3}{3t\sqrt{1-t^2}} \quad \Leftrightarrow$$
$$\Leftrightarrow \quad 12t^2(2t^2-1)(1-t^2) \geqslant (4t^2-3)(-6t^4+6t^2-1) \quad \Leftrightarrow \quad 6t^4-10t^2+3 \leqslant 0 \quad \Leftrightarrow$$
$$\Leftrightarrow \quad t \in \left[\sqrt{\tfrac{5-\sqrt{7}}{6}}, \sqrt{\tfrac{5+\sqrt{7}}{6}}\right] \cap \left(\tfrac{\sqrt{3}}{2}, \sqrt{\tfrac{1}{2}+\tfrac{\sqrt{3}}{6}}\right) = \left(\tfrac{\sqrt{3}}{2}, \sqrt{\tfrac{1}{2}+\tfrac{\sqrt{3}}{6}}\right).$$

(On the third step here we also took into account that $-6t^4+6t^2-1>0$ for $t \in \left(\sqrt{\frac{1}{2}-\frac{\sqrt{3}}{6}}, \sqrt{\frac{1}{2}+\frac{\sqrt{3}}{6}}\right) \supset \left(\frac{\sqrt{3}}{2}, \sqrt{\frac{1}{2}+\frac{\sqrt{3}}{6}}\right)$). So, unifying the obtained interval with the domain $t \geqslant \sqrt{\frac{1}{2}+\frac{\sqrt{3}}{6}}$, we finally arrive at

$$u_1(t) > 1-2t^2 \text{ for all } t \in \left(\tfrac{\sqrt{3}}{2}, 1\right).$$

It remains to compare $1-2t^2$ with $u_2(t)$ for $0<t<\frac{\sqrt{3}}{2}$. For these t, as it has already been noted, $0<u_2(t)<\frac{\sqrt{3}}{3}$, and hence, $u_2(t)>0 \geqslant 1-2t^2$ a fortiori for $0.707\ldots = \frac{\sqrt{2}}{2} \leqslant t < \frac{\sqrt{3}}{2}$ and $u_2(t) < \sqrt{3}/3 \leqslant 1-2t^2$ for $0<t \leqslant \sqrt{\frac{1}{2}-\frac{\sqrt{3}}{6}} = 0.459\ldots$ And if $t \in \left(\sqrt{\frac{1}{2}-\frac{\sqrt{3}}{6}}, \frac{\sqrt{2}}{2}\right)$, then $1-2t^2 \in \left(0, \frac{\sqrt{3}}{3}\right) \subset (-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3})$, that is, the point $1-2t^2$ belongs to the same interval $(-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3})$ of the monotonic decrease of the function $g(u)$ as $u_2(t)$, and hence,

$$1-2t^2 \leqslant u_2(t) \quad \Leftrightarrow \quad g(1-2t^2) \geqslant g(u_2(t)) \equiv a(t).$$

Further calculations completely coincide with what has been done for the comparison of $u_1(t)$ and $1-2t^2$, including the remark concerning the positiveness of the polynomial $-6t^4+6t^2-1$ on the interval $t \in \left(\sqrt{\frac{1}{2}-\frac{\sqrt{3}}{6}}, \frac{\sqrt{2}}{2}\right)$. Therefore, for these t we have

$$u_2(t) \geqslant 1-2t^2 \quad \Leftrightarrow \quad t \in \left[\sqrt{\tfrac{5-\sqrt{7}}{6}}, \sqrt{\tfrac{5+\sqrt{7}}{6}}\right] \bigcap \left(\sqrt{\tfrac{1}{2}-\tfrac{\sqrt{3}}{6}}, \tfrac{\sqrt{2}}{2}\right) = \left[\sqrt{\tfrac{5-\sqrt{7}}{6}}, \tfrac{\sqrt{2}}{2}\right),$$
$$u_2(t) < 1-2t^2 \quad \Leftrightarrow \quad t \in \left(\sqrt{\tfrac{1}{2}-\tfrac{\sqrt{3}}{6}}, \sqrt{\tfrac{5-\sqrt{7}}{6}}\right).$$

Unifying the obtained domains of the values of t, we finally arrive at

$$u_2(t) \geqslant 1-2t^2 \text{ on the interval } t \in \left(0, \tfrac{\sqrt{3}}{2}\right) \quad \Leftrightarrow \quad \sqrt{\tfrac{5-\sqrt{7}}{6}} =: t_0 \leqslant t < \tfrac{\sqrt{3}}{2},$$

with equality attained only in the point $t = t_0$.

Taking into account that $u_2(t)$ is the global maximum point of the function $M_1(\,\cdot\,,t)$ for $0<t<\sqrt{3}/2$, and also that

$$M_1(1-2t^2,t)\Big|_{t=\sqrt{3}/2} = M_1\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right),$$

we conclude that

$$\max_{-1<u<1-2t^2} M_1(u,t) = \begin{cases} M_1(u_2(t),t), & 0 < t < t_0, \\ M_1(1-2t^2,t) \vee 1, & t_0 \leqslant t < \frac{\sqrt{3}}{2}, \\ M_1(1-2t^2,t), & \frac{\sqrt{3}}{2} \leqslant t < 1. \end{cases} \tag{30}$$

Now consider the behaviour of the function $M_2(\,\cdot\,,t)$. Since the functions $\sqrt{1-v^2}/(v^2+1)$ and $v/(v^2+1)$ increase for $v \in (-1,0]$, then $M_2(v,t)$ increases in $v \in (-1,0]$ for each $t \in (0,1)$.

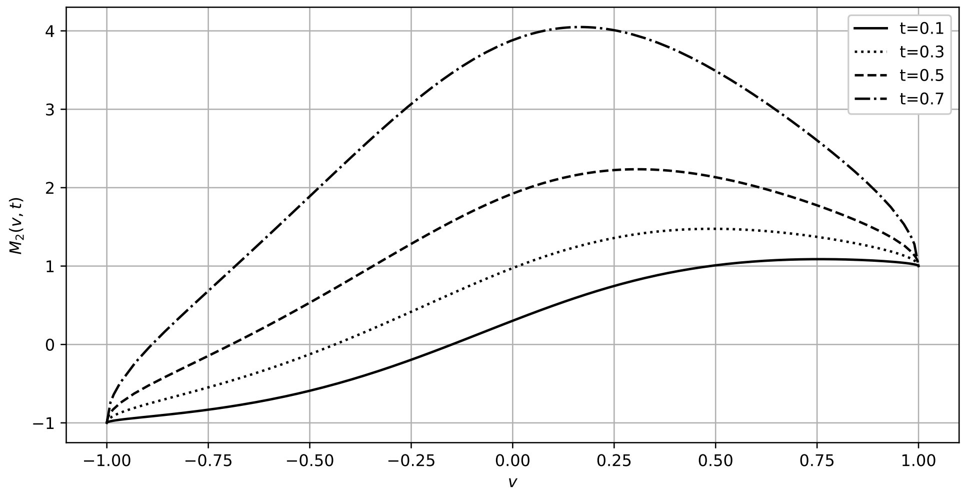

Figure 8. The plots of the function $M_2(\,\cdot\,,t)$ for some t.

The numerator of the derivative

$$\frac{\partial M_2(v,t)}{\partial v} = \frac{b(t)v(v^2-3)+2(1-v^2)^{3/2}}{\sqrt{1-v^2}\cdot(v^2+1)^2}$$

monotonically decreases on the interval $v \in (0,1)$, moreover, its values in the endpoints of the interval are respectively equal to $2 > 0$ and $-2b(t) < 0$. The equation $\frac{\partial M_2}{\partial v} = 0$ takes the form

$$f(v) := \frac{2(1-v^2)^{3/2}}{v(3-v^2)} = b(t) \tag{31}$$

and coincides with (11). The function $f(v)$ is continuously differentiable and monotonically decreasing on the interval $v \in (0,1)$, moreover, $f(+0) = +\infty$, $f(1-) = 0$. Therefore, there exists the inverse function

$$f^{-1} : (0,+\infty) \to (0,1),$$

which is also continuously differentiable and monotonically decreasing. The function $b(t)$ is continuously differentiable and monotonically increases on the interval $t \in (0,1)$. Therefore, equation (31) has the unique root

$$v(t) = f^{-1}(b(t)),$$

which is the global maximum point of the function $M_2(\,\cdot\,,t)$ on the whole interval $(-1,1)$ (see the plots of the function $M_2(\,\cdot\,,t)$ for some t on Fig. 8). Moreover, $v(t)$ is continuously differentiable and monotonically decreases for $t \in (0,1)$ as a superposition of two continuously differentiable functions, one of which ($b(t)$) monotonically increases, whereas the

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other $(f^{-1}(b))$ monotonically decreases. By direct calculations we make sure that $b(0) = 0$, 133
$b(1-) = +\infty$, and hence, $v(0+) = 1$, $v(1-) = 0$. 134

So, for an arbitrary $s \in (-1, 1)$ we have

$$\sup_{s \leqslant v < 1} M_2(v, t) = M_2(v(t) \vee s, t).$$

In particular, for $s = 1 - 2t^2$ we obtain

$$\sup_{1-2t^2 \leqslant v < 1} M_2(v, t) = M_2(v(t) \vee (1 - 2t^2), t).$$

Compare $v(t)$ and $1 - 2t^2$. Since $v(t) \in (0, 1)$ for all $t \in (0, 1)$ by definition, then a fortiori $v(t) > 0 \geqslant 1 - 2t^2$ for $\frac{\sqrt{2}}{2} \leqslant t < 1$. In the domain $0 < t < \frac{\sqrt{2}}{2}$ we have

$$1 - 2t^2 \leqslant v(t) \quad \Leftrightarrow \quad f(1 - 2t^2) \geqslant f(v(t)) \equiv b(t) \quad \Leftrightarrow$$
$$\Leftrightarrow \quad \frac{8t^3(1 - t^2)^{3/2}}{(2t^2 - 1)(2t^4 - 2t^2 - 1)} \geqslant \frac{t(3 - 2t^2)}{(1 - t^2)^{3/2}} \quad \Leftrightarrow$$
$$\Leftrightarrow \quad 8t^2(1 - t^2)^3 \geqslant (3 - 2t^2)(2t^2 - 1)(2t^4 - 2t^2 - 1) \quad \Leftrightarrow \quad 6t^4 - 10t^2 + 3 \leqslant 0$$
$$\Leftrightarrow \quad t \in \left[\sqrt{\tfrac{5-\sqrt{7}}{6}}, \sqrt{\tfrac{5+\sqrt{7}}{6}}\right] \cap \left(0, \tfrac{\sqrt{2}}{2}\right) = \left[\sqrt{\tfrac{5-\sqrt{7}}{6}}, \tfrac{\sqrt{2}}{2}\right) = [0.626\ldots, 0.707\ldots).$$

(On the third step here we also took into account the fact that $2t^2 - 1 < 0$, $2t^4 - 2t^2 - 1 < 0$ in the domain of the values of t under consideration). Thus, unifying the obtained interval with the domain $t \geqslant \sqrt{2}/2$, we arrive at

$$v(t) \geqslant 1 - 2t^2 \text{ on the interval } t \in (0, 1) \quad \Leftrightarrow \quad t_0 \leqslant t < 1,$$

with equality attained only in the point $t = t_0$. 135

So, for $s = 1 - 2t^2$ we finally obtain

$$\max_{1-2t^2 \leqslant v < 1} M_2(v, t) = \begin{cases} M_2(1 - 2t^2, t), & 0 < t < t_0, \\ M_2(v(t), t), & t_0 \leqslant t < 1. \end{cases} \tag{32}$$

As a by-product we showed that

$$u_2(t_0) = v(t_0) = 1 - 2t_0^2 = \frac{\sqrt{7} - 2}{3} = 0.21525\ldots. \tag{33}$$

Also immediately note that

$$M_1(1 - 2t^2, t) = \frac{1}{(1 - t^2)(2t^4 - 2t^2 + 1)} = M_2(1 - 2t^2, t), \quad t \in (0, 1), \tag{34}$$

moreover, the function

$$\frac{1}{(1 - t^2)(2t^4 - 2t^2 + 1)}$$

monotonically increases on the interval $t_0 \leqslant t \leqslant \frac{\sqrt{3}}{2}$. 136

Finally, from (27), (29), (30), (32), (34) it follows that

$$\widetilde{H}(t) = \sup_{-1 < z < 1} \widetilde{H}\left(\frac{1 - z}{2}, t\right) = \max\left\{ \max_{-1 < u < 1 - 2t^2} M_1(u, t), \max_{1 - 2t^2 \leqslant v < 1} M_2(v, t) \right\} =$$

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$$= \begin{cases} M_1(u_2(t),t) \vee M_2(1-2t^2,t), & 0 < t < t_0, \\ M_1(1-2t^2,t) \vee M_2(v(t),t) \vee 1, & t_0 \leqslant t < \frac{\sqrt{3}}{2}, \\ M_1(1-2t^2,t) \vee M_2(v(t),t), & \frac{\sqrt{3}}{2} \leqslant t < 1. \end{cases} =$$

$$= \begin{cases} M_1(u_2(t),t) \vee M_1(1-2t^2,t), & 0 < t < t_0, \\ M_2(1-2t^2,t) \vee M_2(v(t),t) \vee 1, & t_0 \leqslant t < \frac{\sqrt{3}}{2}, \\ M_2(1-2t^2,t) \vee M_2(v(t),t), & \frac{\sqrt{3}}{2} \leqslant t < 1. \end{cases}$$

Taking into account that

$$M_1(1-2t_0^2,t_0) = M_2(1-2t_0^2,t_0) = \frac{54}{8\sqrt{7}-4} = 3.14575\ldots > 1,$$

we obtain

$$\widetilde{H}(t) = \begin{cases} M_1(u_2(t),t) \vee M_1(1-2t^2,t), & 0 < t < t_0, \\ M_2(1-2t^2,t) \vee M_2(v(t),t), & t_0 \leqslant t < 1. \end{cases}$$

Remembering that $v(t)$ is the unique point of global maximum of $M_2(v,t)$ on the interval $v \in (-1,1)$, and $u_2(t)$ is the unique point of global maximum of $M_1(u,t)$ on the interval $u \in (-1,1)$ for $t \in (0,t_0] \subset \left(0, \frac{\sqrt{3}}{2}\right)$ (when $a(t) < 0$), we conclude that

$$\widetilde{H}(t) = \begin{cases} M_1(u_2(t),t), & 0 < t < t_0, \\ M_2(v(t),t), & t_0 \leqslant t < 1, \end{cases} = M(z(t),t).$$

Thus, the function $u_2(t)$ defined for $t \in (0, \sqrt{3}/2)$ (that corresponds to the case $a(t) < 0$) and monotonically decreasing in its domain, acts as the function $u(t)$ mentioned in the formulation of the lemma being proved and Theorem 1, whereas the function $u(t) = u_2(t)$ (for $t \in (0,t_0)$) and the function $v(t)$ (for $t \in [t_0,1)$) act as the point $z(t)$ of the global maximum of the function $M(\,\cdot\,,t)$ which completely agrees with (9).

(b) The functions M_1 and M_2 are obviously differentiable in the domain $(z,t) \in (-1,1) \times (0,1)$ and have continuous partial derivatives. From (25) and (26) it is obvious that the function $\widetilde{H}(p,t)$ is differentiable in the domain $(p,t) \in (0,1) \times (0,1)$ and has continuous partial derivatives. By virtue of (29) we obtain that M is differentiable in the domain $(z,t) \in (-1,1) \times (0,1)$ and has continuous partial derivatives there.

(c) This statement can be verified directly.

(d) Show that $z(t)$ is continuously differentiable and decreases on the interval $t \in (0,1)$. Since $u_2(t)$ is continuously differentiable and monotonically decreases on the interval $t \in (0, \frac{\sqrt{3}}{2}) \supset (0,t_0]$ and the function $v(t)$ is continuously differentiable and monotonically decreases on the interval $t \in (0,1) \supset [t_0,1]$, then, taking into account (33) we obtain that the function $z(t)$ continuously and monotonically decreases on the interval $t \in (0,1)$. Furthermore, the function $z(t)$ is continuously differentiable on each of the intervals $(0,t_0)$ and $(t_0,1)$. Show that $u_2'(t_0) = v'(t_0)$, whence it will follow that the function z is continuously differentiable in the point t_0, and hence, on the whole interval $t \in (0,1)$. In the neighbourhood of t_0 we have

$$u_2(t_0) = g^{-1}(a(t_0)), \quad v(t_0) = f^{-1}(b(t_0)),$$

therefore,

$$u_2'(t_0) = \frac{a'(t_0)}{g'(u_2(t_0))}, \quad v'(t_0) = \frac{b'(t_0)}{f'(v(t_0))},$$

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whence by virtue of (33) we obtain

$$u_2'(t_0) = \frac{a'(t_0)}{g'(1-2t_0^2)}, \quad v'(t_0) = \frac{b'(t_0)}{f'(1-2t_0^2)}.$$

By direct calculations we make sure that

$$u_2'(t_0) = v'(t_0) = -2\sqrt{\tfrac{3-\sqrt{7}}{3}} = -0.687263\ldots.$$

Thus, the function $z(t)$ is differentiable on the interval $t \in (0,1)$.

Now to complete the proof of item **(d)** it remains to remember that

$$z(0+) = u_2(0+) = \frac{\sqrt{3}}{3}, \quad z(1-) = v(1-) = 0, \quad z(t_0) = u_2(t_0) = v(t_0) = \frac{\sqrt{7}-2}{3},$$

and that (see the proof of item **(a)**) each of the equations

$$u_2(t) = 1-2t^2, \quad t \in (0, \sqrt{3}/2),$$

$$v(t) = 1-2t^2, \quad t \in (0,1),$$

has the unique root $t = t_0$. □

5. Proofs of main results

Proof of Theorem 1. It is obvious that $H(0) = 1$ and H is an even function. Since $J(X)$ is invariant with respect to a scale transform of X, the single non-linear condition in (8) can be replaced by the two linear conditions: $\mathbb{E}X = t$, $\mathbb{E}X^2 = 1$. Further, from Lemmas 3 and 4, **(a)**, **(d)** it follows that for $t \in (0,1)$ we have

$$H(t) = M(z(t),t) = \begin{cases} M_1(z(t),t), & -1 < z(t) < 1-2t^2, \\ M_2(z(t),t), & 1-2t^2 \leqslant z(t) < 1, \end{cases} =$$

$$= \begin{cases} 1 + \dfrac{3t^2}{1-t^2} \cdot \dfrac{1-z^2(t) - a(t)z(t)\sqrt{1-z^2(t)}}{1+z^2(t)}, & 0 < t < t_0, \\ \dfrac{b(t)\sqrt{1-z^2(t)} + 2z(t)}{z^2(t)+1}, & t_0 \leqslant t < 1. \end{cases}$$

By the definition of the function $z(t)$,

$$a(t) = \frac{4z(t)\sqrt{1-z^2(t)}}{3z^2(t)-1}, \quad 0 < t < t_0,$$

$$b(t) = \frac{2(1-z^2(t))^{3/2}}{z(t)(3-z^2(t))}, \quad t_0 \leqslant t < 1,$$

hence,

$$H(t) = \begin{cases} 1 + \dfrac{3t^2}{1-t^2} \cdot \dfrac{1-z^2(t)}{1-3z^2(t)}, & 0 < t < t_0, \\ \dfrac{2}{z(t)(3-z^2(t))}, & t_0 \leqslant t < 1, \end{cases}$$

that coincides with (8). The form and uniqueness of the extreme distribution were proved in Lemma 4 **(a)**.

It remains to prove that the function H continuously and monotonically increases on the interval $t \in [0,1)$, and that $H(1-) = +\infty$. By virtue of Lemma 4 **(a)** for $t \in (0,1)$ we have $H(t) = M(z(t),t)$, moreover, M is continuous in the domain $(z,t) \in (-1,1) \times (0,1)$,

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whereas z is continuous on the interval $t \in (0,1)$, hence $H(t)$ is continuous on the interval $t \in (0,1)$. Since

$$H(0+) = M(z(0+), 0+) = 1 = H(0),$$

then H is also continuous in zero.

Finally, prove that the function H monotonically increases. From the definition of the function $z(t)$ it follows that

$$\frac{1}{1-3z^2(t)} = \frac{3-4t^2}{12tz(t)\sqrt{(1-t^2)(1-z^2(t))}}, \quad t \in (0, t_0),$$

and hence, we can write

$$H(t) = \begin{cases} 1 + \dfrac{t(3-4t^2)\sqrt{1-z^2(t)}}{4z(t)(1-t^2)^{3/2}}, & 0 < t < t_0, \\ \dfrac{2}{z(t)(3-z^2(t))}, & t_0 \leqslant t < 1. \end{cases}$$

Note that the function $t(3-4t^2)(1-t^2)^{-3/2}$ is positive and monotonically increases on the interval $0 < t < t_0$, whereas the function $z^{-1}\sqrt{1-z^2}$ is positive and monotonically decreases on the interval $0 < z < 1$. Since the function $z(t)$ monotonically decreases on the interval $0 < t < t_0$ as well, we conclude that H monotonically increases on the interval $(0, t_0)$ as a product of two positive monotonically increasing functions (up to an additive constant). Further, since the function $2/\big(z(3-z^2)\big)$ monotonically decreases on the interval $0 < z < 1$ and the function $z(t)$ monotonically decreases on the interval $t_0 \leqslant t < 1$, the function $H(t)$ monotonically increases on the interval $t_0 \leqslant t < 1$ as a superposition of two decreasing functions. Finally, the existence of infinite limit of $H(t)$ as $t \to 1-$ follows from that $z(t) \to 0+$ as $t \to 1-$.

Now prove the upper estimate for the function $H(t)$ as $t \in [0, t_0]$. By virtue of continuity of H and $\hat{H}$ it suffices to prove the desired inequality only on the interval $(0, t_0)$. By the definition of $z(t)$ for $0 < t < t_0$ as a unique root of the equation

$$g(z) := \frac{4z\sqrt{1-z^2}}{3z^2-1} = \frac{4t^2-3}{3t\sqrt{1-t^2}} =: a(t),$$

on the interval $0 < z < \sqrt{3}/3$ we have

$$\lim_{t\to 0+} z(t) = \frac{\sqrt{3}}{3}, \quad \lim_{t\to 0+} z'(t) = \lim_{t\to 0+} \frac{a'(t)}{g'(z(t))} = -\frac{2\sqrt{6}}{9},$$

hence, by the Lagrange theorem we obtain

$$z(t) = \tilde{z}(t) + o(t), \quad \tilde{z}(t) := \frac{\sqrt{3}}{3} - \frac{2\sqrt{6}}{9}t.$$

Show that $z(t) < \tilde{z}(t)$ for $0 < t < t_0$. By virtue of the monotonic decrease of $g(u)$ for $0 < u < \sqrt{3}/3$ we have

$$z(t) < \tilde{z}(t) \quad \Longleftrightarrow \quad g(z(t)) > g(\tilde{z}(t)) \quad \Longleftrightarrow$$

$$\Longleftrightarrow \quad -\frac{3-4t^2}{3t\sqrt{1-t^2}} > -\frac{\left(3\sqrt{2}-4t\right)\sqrt{-4t^2+6\sqrt{2}t+9}}{3t\left(3\sqrt{2}-2t\right)} \quad \Longleftrightarrow$$

$$\Longleftrightarrow \quad (3-4t^2)(3\sqrt{2}-2t) < (3\sqrt{2}-4t)\sqrt{(-4t^2+6\sqrt{2}+9)(1-t^2)} \quad \Longleftrightarrow$$

$$\Longleftrightarrow \quad 96\sqrt{2}t^5 + t^4\big(-328-96\sqrt{2}\big) + t^3\big(24\sqrt{2}+288\big) + t^2\big(306-12\sqrt{2}\big)+$$

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$$+t\big(-288-108\sqrt{2}\big)+108\sqrt{2}=: s(t)>0.$$

Show that $s(t)>0$ for $0<t<t_0$. We have

$$s'(t)=480\sqrt{2}t^4+t^3\big(-1312-384\sqrt{2}\big)+t^2\big(72\sqrt{2}+864\big)+t\big(612-24\sqrt{2}\big)-288-108\sqrt{2},$$

$$s''(t)=1920\sqrt{2}t^3+t^2\big(-3936-1152\sqrt{2}\big)+t\big(144\sqrt{2}+1728\big)-24\sqrt{2}+612,$$

$$s^{(3)}(t)=5760\sqrt{2}t^2+t\big(-7872-2304\sqrt{2}\big)+144\sqrt{2}+1728,$$

$$s^{(4)}(t)=11520\sqrt{2}t-7872-2304\sqrt{2}<0,\quad t\in(0,t_0),$$

therefore, $s^{(3)}(t)$ decreases for $t\in(0,t_0)$. Since

$$s^{(3)}(0+)=144\sqrt{2}+1728>0,\quad s^{(3)}(t_0-)=-1844.1499\ldots<0,$$

then $s''(t)$ has a unique stationary point on the interval $t\in(0,t_0)$, namely, the local maximum point. Taking into account that

$$s''(0+)=612-24\sqrt{2}>0,\quad s''(t_0-)=271.7769\ldots>0,$$

we conclude that $s''(t)>0$ for $t\in(0,t_0)$, and hence, $s'(t)$ increases for $t\in(0,t_0]$. Since $s'(t_0)=-51.1066<0$, then $s'(t)<0$ for all $t\in(0,t_0]$ and hence, $s(t)$ decreases for $t\in(0,t_0]$. Finally, $s(t_0)=10.8876\ldots$, therefore, $s(t)>0$ for $t\in(0,t_0]$. So, the inequality $z(t)<\widetilde{z}(t)$ is proved for $t\in(0,t_0)$.

Note that

$$H(t)=M(z(t),t)=1+\frac{3t^2}{1-t^2}\cdot\frac{1-z^2(t)}{1-3z^2(t)}$$

for $0<t<t_0$, moreover, the function $M(\,\cdot\,,t)$ increases on the interval $0<z<\sqrt{3}/3$, therefore, taking into account that $0<z(t)<\widetilde{z}(t)<\sqrt{3}/3$ and

$$\widetilde{z}^2(t)=\frac{1}{3}-\frac{4\sqrt{2}}{9}t+\frac{24}{81}t^2,$$

we obtain

$$H(t)\leqslant M(\widetilde{z}(t),t)=1+\frac{3t^2}{1-t^2}\cdot\frac{\frac{2}{3}+\frac{4\sqrt{2}}{9}t-\frac{24}{81}t^2}{\frac{4\sqrt{2}}{3}t-\frac{24}{27}t^2}=\frac{5t+6\sqrt{2}}{2(1-t^2)(3\sqrt{2}-2t)}=\hat{H}(t).$$

The function $\hat{H}(t)$ obviously increases on $(0,t_0)$. Now it remains to note that, as $t\to 0$,

$$z^2(t)=\Big(\frac{\sqrt{3}}{3}-\frac{2\sqrt{6}}{9}t+o(t)\Big)^2=\frac{1}{3}-\frac{4\sqrt{2}}{9}t+o(t)=\widetilde{z}^2(t),$$

$$\begin{aligned}H(t)-1=\frac{3t^2}{1-t^2}\cdot\frac{1-z^2(t)}{1-3z^2(t)}&=\frac{3t^2}{1-t^2}\cdot\frac{\frac{2}{3}+\frac{4\sqrt{2}}{9}t+o(t)}{\frac{4\sqrt{2}}{3}t+o(t)}=\\&=\frac{3t}{1-t^2}\cdot\frac{2+o(1)}{4\sqrt{2}+o(1)}=\frac{3t^2}{1-t^2}\cdot\frac{1-\widetilde{z}^2(t)}{1-3\widetilde{z}^2(t)}=\hat{H}(t)-1,\end{aligned}\tag{35}$$

and hence,

$$\lim_{t\to 0+}\frac{\hat{H}(t)-1}{H(t)-1}=\lim_{t\to 0+}\frac{2+o(1)}{4\sqrt{2}+o(1)}\cdot\frac{4\sqrt{2}+o(1)}{2+o(1)}=1.$$

Theorem 1 is completely proved. □

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Proof of Theorem 2. Lemma 4 **(a)** implies that $H(t) = M(z(t), t)$, where $z(t)$, $t \in (0,1)$, is the unique global maximum point of the function $M(\cdot, t)$, $t \in (0,1)$, moreover, the function $M(z,t)$ is differentiable in the domain $(z,t) \in (-1,1) \times (0,1)$ and has continuous partial derivatives there, whereas the function $z(t)$ is continuously differentiable on the interval $t \in (0,1)$ and takes values from the interval $\left(0, \frac{\sqrt{3}}{3}\right)$. So,

$$\sup_{0<t<1} H(t)(1-t^2)^{3/2} = \sup_{0<t<1} h(t),$$

where $h(t) = M(z(t), t)(1-t^2)^{3/2}$, $t \in (0,1)$. It is obvious that h is continuously differentiable on the interval $t \in (0,1)$.

Find the stationary points of the function h on the interval $t \in (0,1)$. We have

$$h'(t) = (1-t^2)^{3/2}(M(z(t),t))'_t - 3t\sqrt{1-t^2}M(z(t),t),$$

$$(M(z(t),t))'_t = M'_z(z,t)\big|_{z=z(t)} \cdot z'(t) + M'_t(z,t)\big|_{z=z(t)} = M'_t(z,t)\big|_{z=z(t)}.$$

For $t \in [t_0, 1)$ we have

$$\tfrac{1}{3}h'(t) = \frac{(1-2t^2)\sqrt{1-z^2(t)} - 2tz(t)\sqrt{1-t^2}}{z^2(t)+1}.$$

In the domain $(z,t) \in \left(0, \frac{\sqrt{3}}{3}\right) \times [t_0, 1)$ the equation

$$(1-2t^2)\sqrt{1-z^2} - 2tz\sqrt{1-t^2} = 0$$

is satisfied only by the couples $(1-2t^2, t)$ for $t \in \left[t_0, \frac{\sqrt{2}}{2}\right)$, whence with the account of that $z(t) = 1-2t^2$ only for $t = t_0$ (see Lemma 4 **(d)**), we obtain that $t = t_0$ is a stationary point of the function h.

For $t \in (0, t_0]$ we have

$$\tfrac{1}{3}h'(t) = \frac{-2t^3 + t + z^2(t)(4t^2-3)t + z(t)(1-4t^2)\sqrt{1-t^2}\sqrt{1-z^2(t)}}{\sqrt{1-t^2}(z^2(t)+1)}.$$

Find the solutions to the equation

$$z\left(1-4t^2\right)\sqrt{1-t^2}\sqrt{1-z^2} = t(2t^2-1-z^2(4t^2-3))$$

in the domain $(z,t) \in (0, \frac{\sqrt{3}}{3}) \times (0, t_0]$. If a point (z,t) satisfies this equation, it also satisfies the equation

$$z^2(1-4t^2)^2(1-t^2)(1-z^2) = t^2(2t^2-1-z^2(4t^2-3))^2,$$

which is equivalent to

$$(t-z)(t+z)(z-2t^2+1)(z+2t^2-1) = 0,$$

therefore, the original equation can be satisfied only by the points

$$(t,t), \quad (-t,t), \quad (1-2t^2, t), \quad (-1+2t^2, t), \quad 0 < t < 1.$$

By direct calculations we make sure that in the domain $(z,t) \in (0, \frac{\sqrt{3}}{3}) \times (0, t_0]$ the original equation is satisfied only by the couples $(1-2t^2, t)$ for $t \in \left(\sqrt{\frac{3-\sqrt{3}}{6}}, t_0\right]$ and the couple

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$\left(\frac{1}{2},\frac{1}{2}\right)$. Since $z(t) = 1-2t^2$ only for $t = t_0$, we obtain that $t = t_0$ is the stationary point of the function h. It remains to show that $t = \frac{1}{2}$ is not a stationary point of the function h. For this purpose it suffices to show that $z\left(\frac{1}{2}\right) \neq \frac{1}{2}$. Recall (see Theorem 1), that $z(t)$ turns the equation

$$g(z) := \frac{4z\sqrt{1-z^2}}{3z^2-1} = \frac{4t^2-3}{3t\sqrt{1-t^2}}$$

into identity on the interval $t \in (0, t_0)$. By direct verification we make sure that $(z,t) = \left(\frac{1}{2},\frac{1}{2}\right)$ is not a root of this equation.

Thus, the function h has a unique stationary point $t = t_0$ on the interval $t \in (0,1)$, moreover,

$$h(t_0) = \frac{\sqrt{1-t_0^2}}{1-2t_0^2+2t_0^4} = \frac{\sqrt{17+7\sqrt{7}}}{4} = 1.489971\ldots .$$

Also note that

$$\lim_{t\to 0} h(t) = h(0) = 1 < h(t_0),$$

$$\lim_{t\to 1} h(t) = \frac{(1-t^2)^{3/2}(b(t)\sqrt{1-z^2(t)}+2z)}{z^2(t)+1} =$$

$$= \lim_{t\to 1} \frac{t\sqrt{1-z^2(t)}(3-2t^2)+2z(t)(1-t^2)^{\frac{3}{2}}}{z^2(t)+1} = 1 < h(t_0),$$

therefore the point t_0 is point of global maximum of the function h on the interval $(0,1)$, whereas the function h increases on the interval $[0,t_0]$ and decreases on the interval $[t_0,1)$. The fact that the maximum is attained on the two-point distribution follows from Theorem 1.

The upper bound $H(t)(1-t^2)^{3/2} \leqslant \hat{H}(t)(1-t^2)^{3/2}$, $0 \leqslant t \leqslant t_0$, obviously follows from Theorem 1. Prove the equivalence of the left-hand and right-hand sides of this inequality as $t \to 0$. From the proof of Theorem 1 (see (35)) we have

$$H(t) = 1 + \frac{3t}{1-t^2}\cdot\frac{2+o(1)}{4\sqrt{2}+o(1)} = \hat{H}(t), \quad t\to 0,$$

whence with the account of the asymptotics $(1-t^2)^\alpha = 1+o(t)$, $t \to 0$, it follows that

$$H(t)(1-t^2)^{3/2} = (1-t^2)^{3/2} + 3t\sqrt{1-t^2}\frac{2+o(1)}{4\sqrt{2}+o(1)} =$$

$$= 1+o(t)+3t(1+o(t))(2+o(1))\left(\frac{1}{4\sqrt{2}}+o(1)\right) =$$

$$= 1+o(t)+3t\left(\frac{2}{4\sqrt{2}}+o(1)\right) = 1+\frac{3\sqrt{2}}{4}t+o(t) = \hat{H}(t)(1-t^2)^{3/2},$$

and hence,

$$\lim_{t\to 0+}\frac{\hat{H}(t)(1-t^2)^{3/2}-1}{H(t)(1-t^2)^{3/2}-1} = \lim_{t\to 0+}\frac{\frac{3\sqrt{2}}{4}t+o(t)}{\frac{3\sqrt{2}}{4}t+o(t)} = 1.$$

It is obvious that the function

$$\hat{H}(t)(1-t^2)^{3/2} = \frac{\sqrt{1-t^2}(5t+6\sqrt{2})}{2(3\sqrt{2}-2t)} =: s(t)$$

is continuous on $[0, t_0]$ by virtue of continuity of $\hat{H}$. Prove that $\hat{H}(t)(1-t^2)^{3/2}$ monotonically increases on $(0, t_0)$. We have

$$s'(t) = \frac{10t^3 - 30\sqrt{2}t^2 - 36t + 27\sqrt{2}}{2\sqrt{1-t^2}(2t - 3\sqrt{2})^2}.$$

With the account of the positiveness of the denominator for $t \in (0, t_0)$ it suffices to prove that the numerator of $s'(t)$ is positive, that is,

$$s_1(t) := 10t^3 - 30\sqrt{2}t^2 - 36t + 27\sqrt{2} > 0, \quad t \in (0, t_0).$$

Since $5t^2 - 6 < -1$ for all $t \in (0, 1)$, we have

$$s_1'(t) = 6(5t^2 - 10\sqrt{2}t - 6) < 6(-1 - 10\sqrt{2}t) < 0, \quad t \in (0, t_0),$$

therefore, $s_1(t)$ decreases on the interval $t \in (0, t_0)$ and hence, for all $t \in (0, t_0)$

$$s_1(t) = s_1(t_0) = 1.4442\ldots > 0. \quad \square$$

Proof of Theorem 3. According to the Berry–Esseen inequality (1), the following estimate in terms of the non-central Lyapunov ratio holds:

$$\Delta_\lambda(X) \leqslant C_1 \cdot \frac{L_1(X)}{\sqrt{\lambda}}, \quad \lambda > 0.$$

From Theorem 2 it follows that for any $\mathscr{L}(X) \in \mathscr{P}$ with $\mathbb{E}X/\sqrt{\mathbb{E}X^2} = t \in (-1, 1)$

$$\frac{L_1(X)}{L_0(X)} = \frac{\mathbb{E}|X|^3}{\mathbb{E}|X - \mathbb{E}X|^3}\left(\frac{\mathbb{D}X}{\mathbb{E}X^2}\right)^{3/2} = \frac{\mathbb{E}|X|^3(1-t^2)^{3/2}}{\mathbb{E}|X - \mathbb{E}X|^3} \leqslant H(t)\Big(1-t^2\Big)^{3/2} \leqslant \frac{\sqrt{17+7\sqrt{7}}}{4},$$

and hence,

$$\Delta_\lambda(X) \leqslant C_1 \cdot H(t)\Big(1-t^2\Big)^{3/2}\frac{L_0(X)}{\sqrt{\lambda}} \leqslant \frac{\sqrt{17+7\sqrt{7}}}{4}C_1 \cdot \frac{L_0(X)}{\sqrt{\lambda}},$$

that is, iequality (17) holds with $C_0(t) = C_1 \cdot H(t)(1-t^2)^{3/2} \leqslant \frac{\sqrt{17+7\sqrt{7}}}{4}C_1$. The estimate $C_1 \leqslant 0.3031$ was obtained in [18, Theorem 4].

In Theorem 2 it was also shown that $H(t)(1-t^2)^{3/2}$ monotonically increases for $0 \leqslant t \leqslant t_0$ and monotonically decreases for $t_0 \leqslant t < 1$. Therefore,

$$C_0(t) \leqslant C_0(t \wedge t_0), \quad 0 \leqslant t < 1,$$

and the function $C_0(t \wedge t_0)$ does not decrease for $0 \leqslant t < 1$. Hence, for $|\mathbb{E}X|/\sqrt{\mathbb{E}X^2} = s \leqslant t$ in accordance with what has just been proven we have

$$\Delta_\lambda(X) \leqslant C_0(s)\frac{L_0(X)}{\sqrt{\lambda}} \leqslant C_0(s \wedge t_0) \cdot \frac{L_0(X)}{\sqrt{\lambda}} \leqslant C_0(t \wedge t_0) \cdot \frac{L_0(X)}{\sqrt{\lambda}}.$$

Finally, the upper bound of $C_0(t)$ for $0 \leqslant t \leqslant t_0$ declared in the formulation of the theorem, trivially follows from the inequality $H(t) \leqslant \hat{H}(t)$ obtained in Theorem 1 with the account of the particular upper bound for the constant $C_1 \leqslant 0.3031$:

$$C_0(t) \leqslant C_1 \cdot \hat{H}(t)(1-t^2)^{3/2} \leqslant 0.3031 \cdot \frac{\sqrt{1-t^2}(5t + 6\sqrt{2})}{2(3\sqrt{2} - 2t)}, \quad 0 \leqslant t \leqslant t_0.$$

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The monotonicity of this upper bound follows from that of the function $\hat{H}(t)(1-t^2)^{3/2}$ proved in Theorem 2. □

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Abbreviations

The following abbreviations are used in this manuscript:

i.i.d.	independent and identically distributed
r.v.	random variable

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