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Scaled Propagation Invariant Bessel Beams

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Abstract

We present a new family of Bessel solutions of the paraxial equation. Such solutions keep their form during propagation due to a quadratic phase factor that makes them scaled propagation invariant fields. The Bessel beams we introduce have the particularity that the topological phase is twice the order of the Bessel function and the argument varies quadratically with the radius.

Keywords: Bessel beams; paraxial equation; paraxial Bessel beams; scale propagation beams

It has been shown recently that an adequate quadratic phase produces beams that maintain their shape during propagation [1]. In this manuscript, we produce a kind of Bessel beams with a quadratic argument that keep their form while they propagate, *i.e.*, making them scaled propagation invariant beams. We start by noting that a field of the form

$$E_\mu(r,\theta,z) = \frac{1}{\sqrt{z}} \exp\left(i\frac{kr^2}{4z}\right) J_\mu\left(\frac{kr^2}{4z}\right) \exp(2i\mu\theta), \tag{1}$$

is solution of the paraxial equation

$$\nabla_\perp^2 E(r,\theta,z) + 2ik\frac{\partial E(r,\theta,z)}{\partial z} = 0, \tag{2}$$

in cylindrical coordinates, which can be easily verified by direct inspection. For the field to have real physical meaning, k must be real, and without lost of generality, it will be considered strictly positive, and μ must be an integer or a semi-integer, that can be taken, also without loss of generality, as equal or greater than zero.

The field (1), as the normal Bessel beams, is not square integrable; thus, we look for a generalized version with a multiplicative Gaussian factor. To achieve this goal, we write the paraxial equation, in Cartesian coordinates, as a Schrödinger like equation, $\frac{\partial E(x,y,z)}{\partial z} = \frac{i}{2k}\nabla_\perp^2 E(x,y,z)$, whose formal solution is

$$E(x,y,z) = \exp\left(\frac{i}{2k} z \nabla_\perp^2\right) E(x,y,0), \tag{3}$$

being $E(x,y,0)$ the initial field at $z=0$. In Cartesian coordinates, Eq. (3) becomes

$$E(x,y,z) = \exp\left[-\frac{i}{2k} z\left(\hat{p}_x^2 + \hat{p}_y^2\right)\right] E(x,y,0), \tag{4}$$

where we introduced the operators $\hat{p}_x = -i\frac{\partial}{\partial x}$ and $\hat{p}_y = -i\frac{\partial}{\partial y}$. Now, we write the initial condition as

$$E(x,y,0) = \exp\left[-g\left(x^2+y^2\right)\right] \mathscr{E}(x,y,0) \tag{5}$$

where g is a real constant greater than zero ($g>0$). Substituting this initial condition in Eq. (4), we arrive to

$$E(x,y,z) = \exp\left[-\frac{i}{2k} z\left(\hat{p}_x^2 + \hat{p}_y^2\right)\right] \exp\left[-g\left(x^2+y^2\right)\right] \mathscr{E}(x,y,0). \tag{6}$$

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As next step, we introduce the identity operator $\hat{I}$, written as $\hat{I}=\exp\left[\frac{i}{2k}z\left(\hat{p}_x^2+\hat{p}_y^2\right)\right]\exp\left[-\frac{i}{2k}z\left(\hat{p}_x^2+\hat{p}_y^2\right)\right]$, as follows

$$E(x,y,z)=\exp\left[-\frac{i}{2k}z\left(\hat{p}_x^2+\hat{p}_y^2\right)\right]\exp\left[-g\left(x^2+y^2\right)\right]\exp\left[\frac{i}{2k}z\left(\hat{p}_x^2+\hat{p}_y^2\right)\right]\exp\left[-\frac{i}{2k}z\left(\hat{p}_x^2+\hat{p}_y^2\right)\right]\mathscr{E}(x,y,0).$$

From now on, we use the standard operator techniques of quantum optics to get

$$E(r,\theta,z)=\frac{k}{k+2igz}\exp\left(ar^2\right)\exp\left(2br\hat{p}_r\right)\exp\left(i\frac{1}{2k}\frac{kz}{k+2igz}\nabla_\perp^2\right)\mathscr{E}(r,\theta,0), \tag{7}$$

where

$$a=\frac{gk}{k+2igz}, \qquad b=-\frac{\pi}{4}-\frac{i}{2}\ln\left(i-\frac{2gz}{k}\right), \tag{8}$$

and $\hat{p}_r=-i\frac{\partial}{\partial r}$.

We make the connection with the initial simple solution (1). As the field (1) is solution of the paraxial equation, it must exist an initial condition, let us call it $\mathscr{E}(r,\theta,0)$, from which it evolves; then, the factor

$$\exp\left(i\frac{1}{2k}\frac{kz}{k+2igz}\nabla_\perp^2\right)\mathscr{E}(r,\theta,0)$$

gives

$$\frac{1}{\sqrt{\frac{kz}{k+2igz}}}\exp\left(i\frac{kr^2}{4\frac{kz}{k+2igz}}\right)J_\mu\left(\frac{kr^2}{4\frac{kz}{k+2igz}}\right)\exp(2i\mu\theta),$$

where z has been substituted by $\frac{kz}{k+2igz}$ in (1).

What follows is laborious, but easy and direct, and leads us to

$$E(r,\theta,z)=\exp(2i\mu\theta)\sqrt{\frac{k}{z(k+2igz)}}\exp\left[\frac{kr^2(ik-4gz)}{4z(k+2igz)}\right]J_\mu\left(\frac{k^2r^2}{4kz+8igz^2}\right), \tag{9}$$

that is our new exact solution to the paraxial equation, and which we have baptized as scaled propagation invariant Bessel beams. We remark that the field (9) has a topological charge that doubles the one of the Bessel beams and that its argument varies quadratically with the radius. It is also worth mentioning the Bessel function contained in (9) is modulated by both a quadratic phase function and a Gaussian function. To the best of our knowledge, a field with such characteristics has not been reported in the past, and we believe its propagation properties can be of interest for several applications. Figure 1 shows the magnitude and phase distributions of such a field for a propagation distance z equal to 5 m, $\lambda=633$nm, $g=500$, and $\mu=1/2$. We note that the field changes its scale as it propagates and presents a strong focusing as z approaches zero.

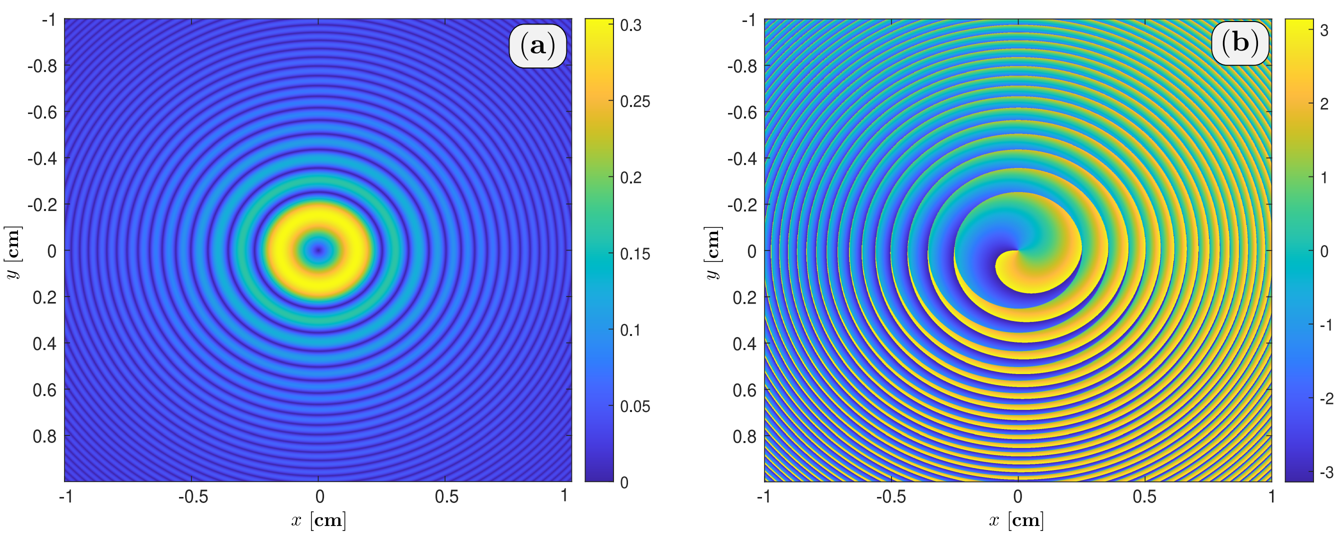

Figure 1: (a) Magnitude and (b) phase of the field described by Eq. (9). The propagation distance is z=5 m, $\lambda=633$nm, $g=500$, and $\mu=1/2$.

We now prove that the introduced beams, (9), are square integrable. Substituting the field intensity of the field (9) in the integral over all space and performing the integral in θ, we get

$$\int_0^\infty r dr \int_0^{2\pi} d\theta \left|E(r,\theta,z)\right|^2 = \frac{2\pi k}{z\sqrt{4g^2z^2+k^2}} \int_0^\infty \exp\left(\frac{gk^2r^2}{4g^2z^2+k^2}\right) \left|J_\mu\left(\frac{k^2r^2}{4kz-8igz^2}\right)\right|^2 r\,dr.$$

We have not been able to compute the integral over r, but we can show it is finite. To do that, using the theorem [2, 3]

Theorem 1. *If ν is real and $\nu \geq -\frac{1}{2}$,*

$$\left|J_\nu(\zeta)\right| \leq \frac{1}{\Gamma(\nu+1)} \left|\frac{\zeta}{2}\right|^\nu \exp\left[\operatorname{Im}(\zeta)\right], \tag{10}$$

where ζ is an arbitrary complex number ,

we have

$$\left|J_\mu\left(\frac{k^2r^2}{4kz+8igz^2}\right)\right| \leq \exp\left[-\frac{gk^2r^2}{2k^2+8g^2z^2}\right] \frac{(kr)^{2\mu}}{8^\mu\Gamma(\mu+1)z^\mu\left(4g^2z^2+k^2\right)^{\mu/2}}. \tag{11}$$

Using elementary calculus, we conclude that

$$\int_0^\infty \exp\left(-\frac{gk^2r^2}{4g^2z^2+k^2}\right) \left|J_\mu\left(\frac{k^2r^2}{4kz+8igz^2}\right)\right|^2 r\,dr \leq \frac{k^{4\mu}}{64^\mu\Gamma^2(\mu+1)z^{2\mu}\left(4g^2z^2+k^2\right)^\mu} \int_0^\infty r^{4\mu+1} \exp\left[-\frac{2gk^2r^2}{k^2+4g^2z^2}\right] dr. \tag{12}$$

The integral in the right side of the previous inequality can be easily done and is finite and positive; thus, the paraxial Besel beams, defined in (9), are square integrable.
To conclude, we have presented a new family of solutions of the paraxial equation, which is essentially formed by a Bessel factor with quadratic dependence in the radius. The fields are structurally stable and are re-scaled with the propagation distance. Thus we suggest to name them as scaled propagation invariant Bessel beams.

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