Article

Not peer-reviewed version

Proof of Two-dimensional Jacobian Conjecture

Yucai Su *

Posted Date: 6 September 2023

doi: 10.20944/preprints202212.0171.v2

Keywords: Jacobian conjecture; Keller maps; injectivity of Keller maps

 doi:10.20944/preprints202212.0171.v2

Article

Proof of Two-Dimensional Jacobian Conjecture

Yucai Su

Department of Mathematics, Jimei University, Xiamen, 361021, China, and School of Mathematical Science, Tongji University, Shanghai, 200092, China yucaisu@jmu.edu.cn

Abstract: Using the local bijectivity of Keller maps, we give a proof of two-dimensional Jacobian conjecture.

Keywords: Jacobian conjecture; Keller maps; injectivity of Keller maps

MSC: 14R15, 14E20, 13B10, 13B25, 17B63

Contents

1. Main theorem

Let $(F, G) \in \mathbb{C}[x, y]^2$ be a pair of two polynomials. The Jacobian problem is to give necessary and sufficient conditions that $\mathbb{C}[F, G] = \mathbb{C}[x, y]$. In geometric terms if

$$\sigma : \mathbb{C}^2 \to \mathbb{C}^2,\ p \mapsto (F(p), G(p)) := (F(a, b), G(a, b)) \quad \text{for} \quad p = (a, b) \in \mathbb{C}^2, \tag{1}$$

then the necessary and sufficient condition is that σ be invertible.

By the chain rule an obvious necessary condition is that the Jacobian determinant is a nonzero constant $J(F, G) := \det \begin{bmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \end{bmatrix} \in \mathbb{C}_{\neq 0}$.

If this condition is verified we call (F, G) a *Jacobian pair* and the corresponding map σ the *Keller map*. The aim of this paper is to give a proof of the well known two-dimensional Jacobian conjecture (cf. [1,5]), formulated by Ott-Heinrich Keller in 1939:

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 6 September 2023 doi:10.20944/preprints202212.0171.v2

Jacobian conjecture (cf. [1,5]): If (F, G) is a Jacobian pair then the Keller map σ is bijective, i.e., F, G are generators of $\mathbb{C}[x, y]$.

It is a well known and standard fact that, in order to prove this conjecture, it is enough to prove that the map σ is injective. This is what we do in

Theorem 1.1. *The Keller map σ is injective. Consequently, the 2-dimensional Jacobian conjecture holds.*

We prove this theorem by contradiction, so we can start with the converse assumption that there exists a Jacobian pair (F, G) such that the corresponding Keller map σ is not injective, i.e., there exist $p_1 = (x_1, y_1), p_2 = (x_2, y_2) \in \mathbb{C}^2$ with

$$\sigma(p_1) = \sigma(p_2), \quad p_1 \neq p_2. \tag{2}$$

For convenience, we denote $(p_1, p_2) = \big((x_1, y_1), (x_2, y_2)\big) \in \mathbb{C}^2 \times \mathbb{C}^2 \cong \mathbb{C}^4$, and

$$V = \big\{(p_1, p_2) = \big((x_1, y_1), (x_2, y_2)\big) \in \mathbb{C}^4 \,\big|\, \sigma(p_1) = \sigma(p_2),\ p_1 \neq p_2\big\}. \tag{3}$$

By assumption (2), $V \neq \emptyset$ (which, to be proven in Lemma 2.20, is in fact a smooth and closed algebraic surface in $\mathbb{C}^4$). We define the *height* (or the *norm*) of $(p_1, p_2) = \big((x_1, y_1), (x_2, y_2)\big) \in V$ to be

$$h_{p_1,p_2} = |x_1| + |y_1| + |x_2| + |y_2|. \tag{4}$$

To prove Theorem 1.1, we need two results. Here is the first one.

Theorem 1.2. *There exists some automorphism ϕ of $\mathbb{C}[x, y]$ such that the Jacobian pair $\big(\phi(F), \phi(G)\big)$, which for convenience is still denoted by (F, G), satisfies the following: for $(p_1, p_2) \in V$ when $h_{p_1,p_2} \to \infty$, we have $|y_1| + |y_2| = o(h_{p_1,p_2})$.*

The reason that the result in this theorem is not symmetric on x, y is because we require (F, G) to satisfy (37), where the variables x and y are not symmetric.

Once we have established Theorem 1.2, we fix the Jacobian pair (F, G) and the variety V satisfying this theorem. Then we define the projection $\pi_1 : V \to \mathbb{C}^2$ by

$$\pi_1 : (p_1, p_2) \quad \mapsto \quad \pi_1(p_1, p_2) := (x_1, x_2) \quad \text{for } (p_1, p_2) = \big((x_1, y_1), (x_2, y_2)\big) \in V. \tag{5}$$

Then the second result can be stated as follows.

Theorem 1.3. *The projection π_1 is proper, finite and surjective.*

We will give the proofs of the above two theorems in section 2. Then finally in section 3, by a rather technical study of the properties so far developed of the variety V, and with no further use of the fact that V arises from the Jacobian problem, we find a contradiction to Theorem 1.3 thus proving the Jacobian conjecture.

Acknowledgements. I would sincerely like to take this opportunity to thank Professor Claudio Procesi, who is a very kind and warmhearted professor, willing to spend more than six months of all his day-and-night's valuable time, selflessly, seriously, and meticulously helping me thoroughly examine every detail of the paper, discussing with me and giving a huge number of useful comments, suggestions and providing simple proofs of Theorem 1.3 and Proposition 3.8. Before that, we do not personally know each other. I have great respect for him and admire his incomparably broad knowledge, mind and thoughts. Anybody who compares the present version with any version before December 2022 can see how much hard work has been done by him. Without his help, the paper cannot be the present form.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 6 September 2023 doi:10.20944/preprints202212.0171.v2

2. Proofs of Theorems 1.2 and 1.3

2.1. Some preparations

For any ring R, we use $R((y^{-1}))$ to denote the ring whose elements have the form $\sum_{i=-\infty}^{\infty} a_i y^i$ with $a_i \in R$ such that $a_i = 0$ when $i \gg 1$. By $R[[y^{-1}]]$ we denote the set of usual power series $\sum_{i=-\infty}^{0} a_i y^i$ with $a_i \in R$. Then $R((y^{-1}))$ is the (algebraic) localization $R((y^{-1})) = R[[y^{-1}]][y]$.

Remark 2.1. (i) An element $\sum_{i=0}^{\infty} a_i y^{-i}$ is invertible in $R[[y^{-1}]]$ if and only if a_0 is invertible in R. An element in $R((y^{-1}))$ can be uniquely written as $y^k \sum_{i=0}^{\infty} a_i y^{-i}$ with $k \in \mathbb{Z}$ and $a_0 \neq 0$ and it is invertible in $R((y^{-1}))$ if and only if a_0 is invertible in R.

(ii) Given $f := \sum_{i=1}^{\infty} a_i y^{-i}$ we can define in $R[[y^{-1}]]$ the *substitution homomorphism* $y^{-1} \mapsto f$. If moreover f is invertible in $R((y^{-1}))$ this extends to a substitution in $R((y^{-1}))$.

Recall the following standard formula, which holds algebraically in the ring $\mathbb{C}[[z]]$ and converges absolutely when $z \in \mathbb{C}$ with $|z| < 1$,

$$(1+z)^{\beta} = \Big(1 + \sum_{j=1}^{\infty} \binom{\beta}{j} z^j\Big), \quad \text{where } \beta \in \mathbb{R}, \ \binom{\beta}{j} := \frac{\beta(\beta-1)\cdots(\beta-(j-1))}{j!}, \tag{6}$$

and in general, we denote the *multi-nomial coefficient* $\binom{k}{\lambda_1, \lambda_2, \ldots, \lambda_i} = \frac{k(k-1)\cdots(k-(\lambda_1+\lambda_2+\cdots+\lambda_i-1))}{\lambda_1!\lambda_2!\cdots\lambda_i!}$.

Remark 2.2. (i) If A is a commutative ring, two formal series $f(z), g(z) \in A[[z]]$ can be composed as if they were functions by

$$f \circ g(z) = \sum_{j=1}^{\infty} a_j \Big(\sum_{i=1}^{\infty} b_i z^i\Big)^j \ \text{ if } f(z) = \sum_{i=1}^{\infty} a_i z^i, \ g(z) = \sum_{i=1}^{\infty} b_i z^i. \tag{7}$$

Moreover $f(z)$ is invertible with respect to $\circ$ if and only if a_1 is invertible in A. We will denote by $f^{\circ -1}$ its inverse in order to avoid confusions.

(ii) In particular if f is invertible we can find $h(z)$ so that $g(z) = f \circ h(z)$ [or $g(z) = h \circ f(z)$].

(iii) Finally the usual *chain rule* holds.

We need some conventions and notations, which, for easy reference, are listed as follows.

Convention 2.3. (1) For $a \in \mathbb{C}$, we write $a = a_{\rm re} + a_{\rm im}\boldsymbol{i}$ for some $a_{\rm re}, a_{\rm im} \in \mathbb{R}$, where $\boldsymbol{i} = \sqrt{-1}$. If a^b appears somewhere, then we always assume $b \in \mathbb{R}$, and in case $a \neq 0$, we interpret a^b as the unique complex number $r^b e^{b\theta\boldsymbol{i}}$ by writing $a = re^{\theta\boldsymbol{i}}$ for some $r \in \mathbb{R}_{>0}$, $0 \le \theta < 2\pi$.

(2) Let $P = \sum_{i \in \mathbb{Z}_{\ge 0}} p_i y^{\alpha - i} = y^{\alpha}(1 + \sum_{i=1}^{\infty} p_i y^{-i}) \in \mathbb{C}[x]((y^{-1}))$ with $\alpha \in \mathbb{Z}$, $p_i \in \mathbb{C}[x]$ and $p_0 = 1$.

(i) Let $\beta \in \mathbb{Q}$ with $\alpha\beta \in \mathbb{Z}$. Then we can uniquely define P^{β} to be an element in $\mathbb{C}[x]((y^{-1}))$ as follows,

$$P^{\beta} = y^{\alpha\beta}\Big(1 + \sum_{j=1}^{\infty} \binom{\beta}{j} \Big(\sum_{i=1}^{\infty} p_i y^{-i}\Big)^j\Big) \in \mathbb{C}[x]((y^{-1})),$$

$$\Big(\sum_{i=1}^{\infty} p_i y^{-i}\Big)^j = \sum_{\sum \lambda_i = j} \binom{j}{\lambda_1, \lambda_2, \ldots} p_1^{\lambda_1} p_2^{\lambda_2} p_3^{\lambda_3} \cdots y^{-\sum i\lambda_i}. \tag{8}$$

(ii) For $Q_1, Q_2 \in \mathbb{C}[x]((y^{-1}))$, we use $P(Q_1, Q_2)$ and $P|_{(x,y)=(Q_1,Q_2)}$ to denote the following element [as long as it is algebraically a well-defined element in $\mathbb{C}[x]((y^{-1}))$],

$$P(Q_1, Q_2) = P|_{(x,y)=(Q_1,Q_2)} = \sum_i p_i(Q_1)Q_2^{\alpha-i}. \tag{9}$$

(iii) If $Q_1, Q_2 \in \mathbb{C}$, we also use (9) to denote a well-defined complex number as long as the series (9) converges absolutely.

(iv) Assume $\alpha \neq 0$. For any $Q = \sum_{i\in\mathbb{Z}_{\geq 0}} q_i y^{\alpha_1 - i} \in \mathbb{C}[x]((y^{-1}))$ with $\alpha_1 \in \mathbb{Z}$, $q_i \in \mathbb{C}[x]$, by comparing coefficients of $y^{\alpha_1 - i}$ in (10) for $i \geq 0$, there exists uniquely $b_i \in \mathbb{C}[x]$ such that we can algebraically write

$$Q = \sum_{i=0}^{\infty} b_i P^{\frac{\alpha_1 - i}{\alpha}}. \tag{10}$$

We call b_i the *coefficient of* $P^{\frac{\alpha_1 - i}{\alpha}}$ *in* Q, and denote by $\mathrm{C_{oeff}}(Q, P^{\frac{\alpha_1 - i}{\alpha}})$. We also use the following notation $\mathrm{C_{oeff}}(Q, x^i y^j)$:

$$\mathrm{C_{oeff}}(Q, x^i y^j) = q_{ij} \quad \text{if } Q \text{ can be written as } Q = \sum_{i,j} q_{ij} x^i y^j \text{ with } q_{ij} \in \mathbb{C}. \tag{11}$$

(3) Let $\varepsilon \to 0$ be a variable. Sometimes we need to consider elements in $\mathbb{C}[x]((y^{-1}))$ which may depend on ε. For any $P \in \mathbb{C}[x]((y^{-1}))$ (or especially in $\mathbb{C}$) which depends on ε, if $P(a,b)$ converges absolutely and $|\varepsilon^{-i} P(a,b)| < \boldsymbol{s}$ for some $i \in \mathbb{Q}_{\geq 0}$ and some fixed $\boldsymbol{s} \in \mathbb{R}_{>0}$, where (a,b) is in some required region, then we use $O(\varepsilon)^i$ to denote P:

$$P = O(\varepsilon)^i. \tag{12}$$

If a, b are some variables depending on another variable $c \to c_0$ (for some $c_0 \in \mathbb{C} \cup \{\infty\}$) such that $\lim_{c\to c_0} \frac{a}{b} = 0$, then we also denote

$$a = o(b). \tag{13}$$

Let $P = \sum_j p_j y^j \in \mathbb{C}[x]((y^{-1}))$, $p_j \in \mathbb{C}[x]$ and $(x_0, y_0) \in \mathbb{C}^2$. If $z_0 = \sum_j |p_j(x_0) y_0^j|$ converges (in particular this requires that $y_0 \neq 0$ if $p_j(x_0) \neq 0$ for some $j < 0$), then z_0 is denoted by $A_{(x_0,y_0)}(P)$ [or by $A_{(y_0)}(P)$ if P is independent of x].

Definition 2.4. (1) Let P be as above and $Q = \sum_i q_i y^i \in \mathbb{C}((y^{-1}))$, $q_i \in \mathbb{R}_{\geq 0}$, $x_0 \in \mathbb{C}$. If $|p_i(x_0)| \leq q_i$ for all possible i, then we say Q is a *controlling function* for P on y at point x_0, and denote

$$P \trianglelefteq_y^{x_0} Q \quad \text{or} \quad Q \trianglerighteq_y^{x_0} P\,, \tag{14}$$

or $P \trianglelefteq_y Q$ when there is no confusion. In particular if P, Q are independent of y then we write $P \trianglelefteq^{x_0} Q$ (thus $a \trianglelefteq b$ for $a, b \in \mathbb{C}$ simply means that $|a| \leq b$ with $b \geq 0$).

(2) An element in $\mathbb{C}((y^{-1}))$ with non-negative coefficients (such as Q above) is called a *controlling function* on y.

(3) If $Q = q_0 y^\alpha + \sum_{j>0} q_j y^{\alpha - j} \in \mathbb{C}((y^{-1}))$ is a controlling function on y with $q_0 > 0$, then we always use the same symbol with subscripts "$_{\rm ign}$" and "$_{\rm neg}$" to denote the elements

$$Q_{\rm ign} = q_0^{-1} \sum_{j>0} q_j y^{-j},$$

$$Q_{\rm neg} := q_0 y^\alpha \Big(1 - q_0^{-1} \sum_{j>0} q_j y^{-j}\Big) = q_0 y^\alpha (1 - Q_{\rm ign}) = 2q_0 y^\alpha - Q. \tag{15}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 6 September 2023 doi:10.20944/preprints202212.0171.v2

We call $Q_{\rm ign}$ the *ignored part* of Q, and $Q_{\rm neg}$ the *negative correspondence of* Q [in sense of (17), where $a, -k$ are nonpositive].

Notice that when Q in (16) is a controlling function then for $k \in \mathbb{R}_{>0}$ we have that both $(1 - q_0^{-1}\sum_{j>0} q_j y^{-j})^{-k}$ and $\frac{(q_0 y^\alpha)^k}{1-kQ_{\rm ign}}$ are controlling functions.

Lemma 2.5. (1) *If*

$$P = p_0 y^\alpha + \sum_{j>0} p_j y^{\alpha - j} \in \mathbb{C}[x]((y^{-1})), \quad Q = q_0 y^\alpha + \sum_{j>0} q_j y^{\alpha-j} \in \mathbb{C}((y^{-1})), \tag{16}$$

with $P \trianglelefteq_y^{x_0} Q$, $x_0 \in \mathbb{C}$ *and* $|p_0(x_0)| = q_0 \in \mathbb{R}_{>0}$, *then for* $a, b, k \in \mathbb{Q}$ *with* $a\alpha, b\alpha, k\alpha \in \mathbb{Z}$,

$$\begin{aligned} &\text{(a) } \frac{\partial P}{\partial y} \trianglelefteq_y^{x_0} \pm \frac{dQ}{dy}, \\ &\text{(b) } P^a \trianglelefteq_y^{x_0} Q_{\rm neg}^a \trianglelefteq_y (q_0 y^\alpha)^{-b} Q_{\rm neg}^{a+b} \text{ for } a, b \in \mathbb{Q}_-, \\ &\text{(c) } Q^k \trianglelefteq_y (q_0 y^\alpha)^{2k} Q_{\rm neg}^{-k} \trianglelefteq_y \begin{cases} \dfrac{(q_0 y^\alpha)^k}{1 - kQ_{\rm ign}} & \text{if } k \in \mathbb{Z}_{\ge 1}, \\ (q_0 y^\alpha)^k \Big(1 + \dfrac{kQ_{\rm ign}}{1 - Q_{\rm ign}}\Big) & \text{if } k \in \mathbb{Q}_{\ge 0} \text{ with } k < 1. \end{cases} \end{aligned} \tag{17}$$

where (17) (a) *holds under the condition: either both* P, Q *are polynomials of* y *(in this case the sign is "+"), or else both are power series of* y^{-1} *(in this case the sign is "−").*

(2) *If* $x_0, y_0 \in \mathbb{C}$ *and* $P_1 \trianglelefteq_y^{x_0} Q_1$, $P_2 \trianglelefteq_y^{x_0} Q_2$, *then*

$$A_{(x_0,y_0)}(P_1 P_2) \le A_{(y_0)}(Q_1) A_{(y_0)}(Q_2) = Q_1(|y_0|) Q_2(|y_0|). \tag{18}$$

Proof. One can see that (2) and (17) (a) are obvious, and (17) (b), (c) are obtained by noting that for $a, b \in \mathbb{Q}_-$ and $i \in \mathbb{Z}_{>0}$, one has

$$\begin{aligned} &(-1)^i \binom{a}{i} = \Big|\binom{a}{i}\Big| \le \Big|\binom{a+b}{i}\Big| = (-1)^i \binom{a+b}{i}, \\ &\binom{k}{i} \le \Big|\binom{-k}{i}\Big| \le \begin{cases} k^i & \text{if } k \in \mathbb{Z}_{\ge 1}, \\ k & \text{if } 0 < k \in \mathbb{Q}_{<1}. \end{cases} \end{aligned} \tag{19}$$

This proves the lemma. □

Throughout the rest of this section, for convenience we regard y^{-1} as the variable z whenever necessary.

Take, where $\tilde f_i \in \mathbb{C}[x]$ with $\tilde f_1 \in \mathbb{C}_{\ne 0}$,

$$\tilde F = \tilde f_1 y^{-1} + \sum_{i=2}^{\infty} \tilde f_i y^{-i} \in \mathbb{C}[x][[y^{-1}]]. \tag{20}$$

Then $\tilde F$ is invertible under the composition "$\circ$" defined in (7) (by regarding y^{-1} as z). We refer $\tilde F^{\circ -1}$ to as the *formal inverse function* of $\tilde F$. Then $\tilde F^{\circ -1}(y^{-1}) \in \mathbb{C}[x][[y^{-1}]]$, and thus we can write

$$\tilde F^{\circ -1}(y^{-1}) = \boldsymbol{b}_1 y^{-1} + \sum_{i=2}^{\infty} \boldsymbol{b}_i y^{-i} \quad \text{for some} \quad \boldsymbol{b}_i \in \mathbb{C}[x]. \tag{21}$$

By definition, we have (noting that y^{-1} is the identity element under the composition " $\circ$ "),

$$y^{-1} = \tilde{F}^{\circ-1} \circ \tilde{F} \overset{(7)}{=} \tilde{F}^{\circ-1}(\tilde{F}) \overset{(21)}{=} \boldsymbol{b}_1 \tilde{F} + \sum_{i=2}^{\infty} \boldsymbol{b}_i \tilde{F}^i. \tag{22}$$

By the notation in Convention 2.3 (2) (iv), we see from (22) that

$$\boldsymbol{b}_i = \mathrm{C}_{\mathrm{oeff}}(y^{-1}, \tilde{F}^i) \in \mathbb{C}[x] \ \text{ for } \ i \geq 1, \tag{23}$$

which can be precisely determined by using (20) to substitute $\tilde{F}$ in (22) and comparing the coefficients of y^{-i} in both sides of (22) such that (we do not need to use the following explicit expression of $\boldsymbol{b}_i$, but we only want to present that $\boldsymbol{b}_i$'s exist),

$$\begin{aligned} \boldsymbol{b}_1 &= \tilde{f}_1^{-1} \in \mathbb{C}_{\neq 0}, \\ \boldsymbol{b}_i &= -\sum_{j=1}^{i-1} \boldsymbol{b}_j \tilde{f}_1^{j-i} \sum_{\ell=0}^{j} \binom{j}{\ell} \sum_{\substack{n \in \mathbb{Z}_{\geq 0},\, \lambda_1, \lambda_2, \ldots, \lambda_n \geq 0 \\ \lambda_1 + 2\lambda_2 + \cdots + n\lambda_n = i-j}} \binom{\ell}{\lambda_1, \lambda_2, \ldots, \lambda_n} \tilde{f}_1^{-\lambda_1 - \lambda_2 - \cdots - \lambda_n} \tilde{f}_2^{\lambda_2} \tilde{f}_3^{\lambda_3} \cdots \tilde{f}_n^{\lambda_n}, \ i \geq 2. \end{aligned} \tag{24}$$

Lemma 2.6. *Let $\hat{a}_i \in \mathbb{R}_{\geq 0}$ with $\hat{a}_1 > 0$, and let*

$$\hat{\boldsymbol{F}} = \hat{a}_1 y^{-1} + \sum_{i=2}^{\infty} \hat{a}_i y^{-i} \in \mathbb{C}[[y^{-1}]] \ \ \text{and} \ \ \hat{\boldsymbol{F}}_{\mathrm{neg}} \overset{(15)}{=} \hat{a}_1 y^{-1} - \sum_{i=2}^{\infty} \hat{a}_i y^{-i} \in \mathbb{C}[[y^{-1}]], \tag{25}$$

be a controlling function on y and its negative correspondence. Write the formal inverse function $\hat{\boldsymbol{F}}_{\mathrm{neg}}^{\circ-1}$ of $\hat{\boldsymbol{F}}_{\mathrm{neg}}$, as

$$\hat{\boldsymbol{F}}_{\mathrm{neg}}^{\circ-1}(y^{-1}) = \hat{\boldsymbol{b}}_1 y^{-1} + \sum_{i=2}^{\infty} \hat{\boldsymbol{b}}_i y^{-i} \ \ \text{for some} \ \ \hat{\boldsymbol{b}}_i \in \mathbb{C}. \tag{26}$$

Then $\hat{\boldsymbol{b}}_1 = \hat{a}_1^{-1}$ and

(1) *$\hat{\boldsymbol{F}}_{\mathrm{neg}}^{\circ-1}(y^{-1})$ is a controlling function on y^{-1} (this means that the formal inverse function of the negative correspondence of a controlling function is a controlling function), i.e., for $i \geq 2$,*

$$\hat{\boldsymbol{b}}_i \geq 0. \tag{27}$$

(2) *If $\tilde{F} \trianglelefteq_y^{x_0} \hat{\boldsymbol{F}}$ with $\tilde{F}$ as in (20) and $|\tilde{f}_1| = \hat{a}_1$, then by regarding $\tilde{F}$ as a variable, we have the following (this means that the formal inverse function of a function can be controlled by the formal inverse function of the negative correspondence of a controlling function that controls the said function),*

$$\begin{aligned} & y^{-1} = \tilde{F}^{\circ-1}(\tilde{F}) \trianglelefteq_{\tilde{F}}^{x_0} \hat{\boldsymbol{F}}_{\mathrm{neg}}^{\circ-1}(\tilde{F}), \ \ i.e., \\ & \boldsymbol{b}_i \overset{(23)}{=} \mathrm{C}_{\mathrm{oeff}}(y^{-1}, \tilde{F}^i) \ \trianglelefteq^{x_0} \ \hat{\boldsymbol{b}}_i = \mathrm{C}_{\mathrm{oeff}}(y^{-1}, \hat{\boldsymbol{F}}_{\mathrm{neg}}^i) \ \ \text{for} \ \ i \geq 2, \end{aligned} \tag{28}$$

where $\boldsymbol{b}_i \trianglelefteq^{x_0} \hat{\boldsymbol{b}}_i$ means that $|\boldsymbol{b}_i(x_0)| \leq \hat{\boldsymbol{b}}_i$. In particular

$$y^{-1} \trianglelefteq_y \hat{\boldsymbol{F}}_{\mathrm{neg}}^{\circ-1}(\hat{\boldsymbol{F}}), \tag{29}$$

where the right side of " $\trianglelefteq_y$ " is regarded as a function of y by using (25) to substitute $\hat{\boldsymbol{F}}$.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 6 September 2023 doi:10.20944/preprints202212.0171.v2

Proof. Note that (1) is a special case of (2) by taking, in (20), $\tilde{f}_1 = \hat{a}_1$ and $\tilde{f}_i = 0$ for $i \geq 2$, i.e., $\tilde{F} = \hat{a}_1 y^{-1}$ [then by definition $\tilde{F}^{\circ -1}(y^{-1})$ is $\hat{a}_1^{-1} y^{-1}$, i.e., $\boldsymbol{b}_1 = \hat{a}_1^{-1}$ and $\boldsymbol{b}_i = 0$ if $i \geq 2$, and so (27) follows from (28)].

Thus we prove (2). Note from (21) that $\boldsymbol{b}_i$ is the coefficient of y^{-i} in the formal inverse function of $\tilde{F}$, or equivalently, the coefficient of $\tilde{F}^i$ in y^{-1} when we regard y^{-1} as a function of $\tilde{F}$ by (22). Thus $i!\boldsymbol{b}_i$ is the constant term of the i-th partial derivative $\frac{\partial^i y^{-1}}{\partial \tilde{F}^i}$. Therefore to prove (28), first we want to prove, for $i \geq 1$,

$$\frac{\partial^i y^{-1}}{\partial \tilde{F}^i} \trianglelefteq_y^{x_0} \frac{d^i y^{-1}}{d\hat{\boldsymbol{F}}_{\text{neg}}^i}, \tag{30}$$

where the left-hand side is understood as that we first use (22) to regard y^{-1} as a function of $\tilde{F}$ (with parameter x) and apply $\frac{\partial^i}{\partial \tilde{F}^i}$ to it, then regard the result as a function of y^{-1} by using (20) to substitute $\tilde{F}$ (and the like for the right-hand side, which does not contain the parameter x). By (17) (a), we have $\frac{\partial \tilde{F}}{\partial y^{-1}} \trianglelefteq_y^{x_0} \frac{d\hat{\boldsymbol{F}}}{dy^{-1}}$ (here we regard y^{-1} as a variable), and thus by (17) (b) and definition (15),

$$\Big(\frac{\partial \tilde{F}}{\partial y^{-1}}\Big)^{-1} \overset{(17)\,(b)}{\trianglelefteq_y^{x_0}} \Big(\Big(\frac{d\hat{\boldsymbol{F}}}{dy^{-1}}\Big)_{\text{neg}}\Big)^{-1} \overset{(15)}{=} \Big(\frac{d\hat{\boldsymbol{F}}_{\text{neg}}}{dy^{-1}}\Big)^{-1}, \tag{31}$$

i.e., $\frac{\partial y^{-1}}{\partial \tilde{F}} \trianglelefteq_y^{x_0} \frac{dy^{-1}}{d\hat{\boldsymbol{F}}_{\text{neg}}}$ and (30) holds for $i = 1$. Inductively, by Lemma 2.5,

$$\begin{aligned} \frac{\partial^i y^{-1}}{\partial \tilde{F}^i} &= \frac{\partial}{\partial \tilde{F}}\Big(\frac{\partial^{i-1} y^{-1}}{\partial \tilde{F}^{i-1}}\Big) = \frac{\partial}{\partial y^{-1}}\Big(\frac{\partial^{i-1} y^{-1}}{\partial \tilde{F}^{i-1}}\Big)\Big(\frac{\partial \tilde{F}}{\partial y^{-1}}\Big)^{-1} \\ &\overset{\text{inductive assumption and (31)}}{\trianglelefteq_y^{x_0}} \frac{d}{dy^{-1}}\Big(\frac{d^{i-1} y^{-1}}{d\hat{\boldsymbol{F}}_{\text{neg}}^{i-1}}\Big)\Big(\frac{d\hat{\boldsymbol{F}}_{\text{neg}}}{dy^{-1}}\Big)^{-1} = \frac{d^i y^{-1}}{d\hat{\boldsymbol{F}}_{\text{neg}}^i}. \end{aligned} \tag{32}$$

This proves (30). Using (30) and noting from (22), (26), we have (noting that $\frac{d^i y^{-1}}{d\hat{\boldsymbol{F}}_{\text{neg}}^i}\Big|_{y^{-1}=0}$ should be understood as the constant term in $\frac{d^i y^{-1}}{d\hat{\boldsymbol{F}}_{\text{neg}}^i}$ when it is written as a series of y^{-1}),

$$\boldsymbol{b}_i \overset{(22)}{=} \frac{1}{i!}\frac{\partial^i y^{-1}}{\partial \tilde{F}^i}\Big|_{\tilde{F}=0} = \frac{1}{i!}\frac{\partial^i y^{-1}}{\partial \tilde{F}^i}\Big|_{y^{-1}=0} \overset{(30)}{\trianglelefteq^{x_0}} \frac{1}{i!}\frac{d^i y^{-1}}{d\hat{\boldsymbol{F}}_{\text{neg}}^i}\Big|_{y^{-1}=0} = \frac{1}{i!}\frac{d^i y^{-1}}{d\hat{\boldsymbol{F}}_{\text{neg}}^i}\Big|_{\hat{\boldsymbol{F}}_{\text{neg}}=0} \overset{(26)}{=} \hat{\boldsymbol{b}}_i. \tag{33}$$

This proves (28).

Note that for any controlling functions Q, P_2, and any $P_1 \in \mathbb{C}[x]((y^{-1}))$ with $P_1 \trianglelefteq_y^{x_0} P_2$, we always have $Q(P_1) \trianglelefteq_y^{x_0} Q(P_2)$. This with the facts that $\tilde{F} \trianglelefteq_y^{x_0} \hat{\boldsymbol{F}}$ and that $\hat{\boldsymbol{F}}_{\text{neg}}^{\circ -1}$ is a controlling function implies

$$\hat{\boldsymbol{F}}_{\text{neg}}^{\circ -1}(\tilde{F}) \trianglelefteq_y^{x_0} \hat{\boldsymbol{F}}_{\text{neg}}^{\circ -1}(\hat{\boldsymbol{F}}). \tag{34}$$

This together with (28) proves

$$y^{-1} \overset{(28)}{\trianglelefteq_{\tilde{F}}^{x_0}} \hat{\boldsymbol{F}}_{\text{neg}}^{\circ -1}(\tilde{F}) \overset{(34)}{\trianglelefteq_y^{x_0}} \hat{\boldsymbol{F}}_{\text{neg}}^{\circ -1}(\hat{\boldsymbol{F}}), \tag{35}$$

i.e., we have (29). □

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 6 September 2023 doi:10.20944/preprints202212.0171.v2

2.2. Proof of Theorem 1.2

First we need to reformulate (F,G) (by applying some automorphism of $\mathbb{C}[x,y]$). Fix a sufficiently large $\ell \in \mathbb{Z}_{>0}$. Applying the following variable change,

$$(x,y) \mapsto (y,\, y^\ell + x), \tag{36}$$

and rescaling F,G, we can assume that F,G have the following forms, for some $m,n \in \mathbb{Z}_{>0}$, $f_{j,k}, g_{j,k} \in \mathbb{C}$,

$$\begin{aligned}
&\text{(i) } F = y^m + F_1, \quad F_1 = \sum_{j=0}^{m-1}\Big(\sum_{k=0}^{m-1-j} f_{j,k}x^k\Big)y^j,\\
&\text{(ii) } G = y^n + G_1, \quad G_1 = \sum_{j=0}^{n-1}\Big(\sum_{k=0}^{n-1-j} g_{j,k}x^k\Big)y^j.
\end{aligned}\tag{37}$$

Note that $\deg F_1 \le m-1$, $\deg G_1 \le n-1$ (where $\deg F_1$ denotes the total degree of F_1). For simplicity, we can also assume $2 \le n$ and $n|m$ (i.e., $\frac{m}{n} \in \mathbb{Z}_{>0}$) by replacing (F,G) by $\big(F + (G+F^k)^k, G+F^k\big)$ for some $k \in \mathbb{Z}_{>0}$ [the reason we assume $n|m$ will be clear in (65), (66) and (79), after all it is a reasonable choice].

Thus the new pair (F,G) is in fact obtained from the original one by applying some automorphism of $\mathbb{C}[x,y]$ and a change of generators in $\mathbb{C}[F,G]$.

In the following we consider F,G as elements in the ring $\mathbb{C}[x]((y^{-1}))$.

By (37), we can rewrite F,G as, where $f_i(x) = \sum_{k=0}^{i-1} f_{m-i,k}x^k$, $g_i(x) = \sum_{k=0}^{i-1} g_{n-i,k}x^k \in \mathbb{C}[x]$, and where in general for any $A \in \mathbb{C}[x]((y^{-1}))$ we use $\deg_x A$, called the *x-degree* of A, to denote the degree of A with respect to variable x,

$$\begin{aligned}
&\text{(i) } F = y^m(1+y^{-m}F_1) = y^m\Big(1+\sum_{j=0}^{m-1}\Big(\sum_{k=0}^{m-1-j} f_{j,k}x^k\Big)y^{j-m}\Big) = y^m\Big(1+\sum_{i=1}^{m} f_i(x)y^{-i}\Big),\\
&\text{(ii) } G = y^n\Big(1+\sum_{j=0}^{n-1}\Big(\sum_{k=0}^{n-1-j} g_{j,k}x^k\Big)y^{j-n}\Big) = y^n\Big(1+\sum_{i=1}^{n} g_i(x)y^{-i}\Big),\\
&\text{(iii) } \deg_x f_i \le i-1 \text{ if } i \le m \text{ and } f_i = 0 \text{ if } i > m,\\
&\text{(iv) } \deg_x g_i \le i-1 \text{ if } i \le n \text{ and } g_i = 0 \text{ if } i > n.
\end{aligned}\tag{38}$$

At this point we want to define a choice of an m-th root for F denoted $F^{\frac{1}{m}}$ by using formulas (6) and (8),

$$F^{\frac{1}{m}} \overset{(8)}{:=} y\Big(1+\sum_{i=1}^{m} f_i(x)y^{-i}\Big)^{\frac{1}{m}} \overset{(6)}{=} y\Big(1+\sum_{j=1}^{\infty}\binom{\frac{1}{m}}{j}\Big(\sum_{i=1}^{m} f_i(x)y^{-i}\Big)^j\Big). \tag{39}$$

This is not just an algebraic formula but setting

$$||F_1|| := \sum_{j=0}^{m-1}\Big(\sum_{k=0}^{m-1-j} |f_{j,k}x^k|\Big)|y|^j, \tag{40}$$

it converges absolutely when $||F_1|| < |y|^m$.

Set $f_0 = g_0 = 1$, and denote the set $A = \frac{n-\mathbb{Z}_{\ge 0}}{m}$. By (38), as in (10) and (22), we can algebraically write,

$$G = \sum_{\alpha\in A} c_\alpha F^\alpha = \sum_{i=0}^{\infty} c_{\frac{n-i}{m}} F^{\frac{n-i}{m}} \quad \text{for some } c_\alpha \in \mathbb{C}[x], \tag{41}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 6 September 2023 doi:10.20944/preprints202212.0171.v2

where as in (24), by comparing the coefficients of y^{n-i}, we can inductively determine $c_{\frac{n-i}{m}} \in \mathbb{C}[x]$ for $i \geq 0$ as follows (again we do not need the explicit expression of $c_{\frac{n-i}{m}}$):

$$c_{\frac{n-i}{m}} = g_i - \sum_{j=1}^{i-1} c_{\frac{n-j}{m}} f_1^{j-i} \sum_{\ell=0}^{j} \binom{j}{\ell} \sum_{\substack{n\in\mathbb{Z}_{\geq 0}, \lambda_1,\lambda_2,\ldots,\lambda_n\geq 0 \\ \lambda_1+2\lambda_2+\cdots+n\lambda_n=i-j}} \binom{\ell}{\lambda_1,\lambda_2,\ldots,\lambda_n} f_1^{-\lambda_1-\lambda_2-\cdots-\lambda_n} f_2^{\lambda_2} f_3^{\lambda_3}\cdots f_n^{\lambda_n}.$$

Remark 2.7. (i) If $U = 1 + \sum_{i=1}^{\infty} u_i y^{-i} \in \mathbb{C}[x][[y^{-1}]]$ with $u_i \in \mathbb{C}[x]$, $\deg_x u_i \leq i$ and $\beta \in \mathbb{R}$ then $U^\beta = 1 + \sum_{i=1}^{\infty} v_i y^{-i}$ for some $v_i \in \mathbb{C}[x]$ with $\deg_x v_i \leq i$.
(ii) Furthermore, if for all i we have $\deg_x u_i < i$ we have $\deg_x v_i < i$.

Proof. This follows from a symbolic computation. Think first the u_i as variables of weight i then, using formula (6) it is enough to prove that, for all $j \in \mathbb{Z}_{>0}$ we have that in $\left(\sum_{i=1}^{\infty} u_i y^{-i}\right)^j$ it is a polynomial in y^{-1} where the coefficient of y^{-a} is a polynomial in the u_i of weight a. This is immediate by writing $u_i y^{-i} = (\lambda^i u_i)(\lambda y)^{-i}$ with λ an auxiliary parameter.

Now by hypothesis the degree of u_i is smaller than or equal to its weight and this property is preserved, the same if it is always smaller than its weight. □

Lemma 2.8. *We have, for some $a_1 \in \mathbb{C}$,*

$$\text{(i) } c_\alpha \in \mathbb{C} \text{ if } \alpha > \frac{-m+1}{m}, \quad \text{(ii) } c_{-1} = 0, \quad \text{(iii) } \deg_x c_{\frac{-m+1-j}{m}} \leq j+1 \text{ if } j \in \mathbb{Z}_{\geq 0},$$
$$\text{(iv) } c_{\frac{-m+1}{m}} = -\frac{J_0}{m}(x + a_1). \tag{42}$$

Proof. First by (41), we have

$$\frac{\partial G}{\partial y} = \sum_{\alpha\in A} \alpha c_\alpha F^{\alpha-1} \frac{\partial F}{\partial y}. \tag{43}$$

Then by (41), we obtain

$$\frac{\partial G}{\partial x} \overset{(41)}{=} \sum_{\alpha\in A} \frac{dc_\alpha}{dx} F^\alpha + \sum_{\alpha\in A} \alpha c_\alpha F^{\alpha-1} \frac{\partial F}{\partial x} \overset{(43)}{=} \sum_{\alpha\in A} \frac{dc_\alpha}{dx} F^\alpha + \frac{\partial G}{\partial y}\left(\frac{\partial F}{\partial y}\right)^{-1} \frac{\partial F}{\partial x}. \tag{44}$$

Equivalently, $\frac{\partial G}{\partial x}\frac{\partial F}{\partial y} = \sum_{\alpha\in A} \frac{dc_\alpha}{dx} F^\alpha \frac{\partial F}{\partial y} + \frac{\partial G}{\partial y}\frac{\partial F}{\partial x}$, i.e.,

$$-J_0 = \sum_{\alpha\in A} \frac{dc_\alpha}{dx} F^\alpha \frac{\partial F}{\partial y}. \tag{45}$$

This gives (i) below, then multiplying (41) with $\frac{\partial F}{\partial y}$ gives (ii) below, i.e.,

$$\text{(i) } J_0 \left(\frac{\partial F}{\partial y}\right)^{-1} \overset{(45)}{=} -\sum_{\alpha\in A} \frac{dc_\alpha}{dx} F^\alpha, \quad \text{(ii) } G\frac{\partial F}{\partial y} \overset{(41)}{=} \frac{\partial}{\partial y}\left(\sum_{-1\neq\alpha\in A} \frac{c_\alpha F^{\alpha+1}}{\alpha+1}\right) + c_{-1}F^{-1}\frac{\partial F}{\partial y}. \tag{46}$$

The y-degree of the left-hand side of (46) (i) is $-m+1$, while that of the right-hand side is $m\alpha_0$, where $\alpha_0 \in A$ is the maximal number with $\frac{dc_{\alpha_0}}{dx} \neq 0$. We obtain (42) (i).

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 6 September 2023 doi:10.20944/preprints202212.0171.v2

Since, for some $\beta_i, \gamma_i \in \mathbb{C}[x]$,

$$\frac{\partial F}{\partial y} \overset{(38)\,(i)}{=} my^{m-1}\Big(1+\sum_{i=1}^{\infty}\beta_i y^{-i}\Big) \implies \Big(\frac{\partial F}{\partial y}\Big)^{-1} = \frac{1}{m}y^{-m+1}\Big(1+\sum_{i=1}^{\infty}\gamma_i y^{-i}\Big), \tag{47}$$

with $(1+\sum_{i=1}^{\infty}\gamma_i y^{-i}) = (1+\sum_{i=1}^{\infty}\beta_i y^{-i})^{-1}$, comparing coefficients of y^{-m+1} in (46) (i) gives that $\frac{dc_{\alpha_0}}{dx} = -m^{-1}J_0$ with $\alpha_0 = \frac{-m+1}{m}$, which implies (42)(iv) for some $a_1 \in \mathbb{C}$.

By comparing coefficients of y^{-1} in (46) (ii), we obtain (42) (ii).

By (38) (iii), (iv), we in particular have

$$\begin{aligned} &\text{(i) } \deg_x C_{\text{oeff}}(F, y^{m-j}) \le j, \qquad \deg_x C_{\text{oeff}}(G, y^{n-j}) \le j, \\ &\text{(ii) } \deg_x C_{\text{oeff}}\Big(\frac{\partial F}{\partial y}, y^{m-1-j}\Big) = \deg_x C_{\text{oeff}}(F, y^{m-j}) \le j \quad \text{for } j \in \mathbb{Z}_{\ge 0}. \end{aligned} \tag{48}$$

Then by Remark 2.7 we have:

$$\deg_x C_{\text{oeff}}\Big(\Big(\frac{\partial F}{\partial y}\Big)^{-1}, y^{-m+1-j}\Big) \le j, \qquad \deg_x C_{\text{oeff}}(F^{\frac{n-j}{m}}, y^{n-j-k}) \le k. \tag{49}$$

From this and (46) (i), we claim that (42) (iii) holds by induction on j by comparing the coefficients of y^{-m+1-j}.

To prove the claim, from Remark 2.7 and formula (39) one has for any $a \in \mathbb{Z}$,

$$F^{\frac{a}{m}} = y^a\Big(1+\sum_{i=1}^{\infty}\ell_{i,a}(x)y^{-i}\Big) \quad \text{for some } \ell_{i,a}(x) \in \mathbb{C}[x] \text{ with } \deg_x \ell_{i,a}(x) \le i. \tag{50}$$

By (47), formula (46) (i) becomes

$$\begin{aligned} \frac{J_0}{m}y^{-m+1}\Big(1+\sum_{i=1}^{\infty}\gamma_i y^{-i}\Big) &\overset{(46)\,(i),(50)}{=} -\sum_{\alpha\in A}\frac{dc_\alpha}{dx}y^{m\alpha}\Big(1+\sum_{i=1}^{\infty}\ell_{i,m\alpha}(x)y^{-i}\Big) \\ &= -\sum_{\alpha\in A}\frac{dc_\alpha}{dx}\Big(y^{m\alpha}+\sum_{i=1}^{\infty}\ell_{i,m\alpha}(x)y^{m\alpha-i}\Big). \end{aligned} \tag{51}$$

The coefficient of y^{-m+1-j} of the left-hand side of the previous formula is $\frac{J_0}{m}\gamma_j$ if $j \ge 1$ of degree $\le j$, it is $\frac{J_0}{m}$ for $j = 0$. Instead, that of the right-hand side of the previous formula is $-\frac{d}{dx}c_{\frac{-m-(j-1)}{m}}$ plus the sum for $\alpha, i \ge 1$ with $m\alpha - i = -m+1-j$ [or $m\alpha = -m-(j-i-1)$] of $-\frac{dc_\alpha}{dx}\ell_{i,m\alpha}(x)$. By what we have proved only $\alpha \le \frac{-m+1}{m}$ contribute that is $-(j-i-1) \le 1$ or $i \le j$. By induction the degree of $-\frac{dc_\alpha}{dx}$ is $\le j-i$ so these terms contribute to a degree $\le j = (j-i)+i$ and the claim follows. The basis for the induction is for $j = 0$ for which we have (42) (iv) degree 1. □

We would like to mention that the following lemma [which in particular implies (107)] is important for us to obtain that the m-th root ω' of unity appearing in the proof of Lemma 2.17 is equal to 1 [cf. (73) (ii) and the arguments after (104)].

Lemma 2.9. *By some reformulation of* (F, G), *we may assume*

$$c_{\frac{4-m}{m}}c_{\frac{3-m}{m}} \ne 0. \tag{52}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 6 September 2023 doi:10.20944/preprints202212.0171.v2

Proof. Fix any $a_0 \in \mathbb{C}$ with $\alpha_0 \neq 3a_1$. We define $\bar F, \bar G$ below (then $\bar F, \bar G$ are still polynomials giving a counterexample to the Jacobian conjecture) such that $\bar F$ have the form (i) below, for some $\bar f_{j,k} \in \mathbb{C}$,

$$
\begin{aligned}
&\text{(i) } \bar F := F(y, y^3 + a_0 y^2 - x) \overset{(37)\,(i)}{=} (y^3 + a_0y^2 - x)^m + \sum_{j=0}^{m-1} \Big(\sum_{k=0}^{m-1-j} f_{j,k} y^k \Big) (y^3 + a_0 y^2 - x)^j \\
&\quad = y^{\bar m} \Big(1 + \sum_{j=1}^{\bar m} \sum_{k=0}^{j-1} \bar f_{j,k} y^{-j} x^k \Big), \\
&\text{(ii) } \bar G = G(y, y^3 + a_0 y^2 - x),
\end{aligned} \tag{53}
$$

where we have used the same symbol with a bar to denote an associated element corresponding to the Jacobian pair $(\bar F, \bar G)$; in particular, $\bar n = 3n$, $\bar m = 3m$.

When (x, y) is set to $(y, y^3 + a_0 y^2 - x) = \big(y, y^3(1 + a_0 y^{-1} - xy^{-3})\big)$, we have that y^{-1} is set to $y^{-3}(1 + a_0 y^{-1} - xy^{-3})^{-1} = y^{-3}(1 - a_0 y^{-1} + \sum_{j=2}^{\infty} t_j(x) y^{-j})$ for some $t_j(x) \in \mathbb{C}[x]$. By Remark 2.7, $\deg_x t_j < j$.

So for $H \in \mathbb{C}[x]((y^{-1}))$ the substitution

$$
H \mapsto H|_{(x,y)=(y, y^3 + a_0 y^2 - x)} := \phi(H), \tag{54}
$$

is a well-defined homomorphism which we denote for simplicity by ϕ. Note from (38) (i) or (53) (i) that $\bar F = \phi(F)$ has the form,

$$
\begin{aligned}
\bar F = \phi(F) &\overset{(38)\,(i),\,(53)\,(i)}{=} (y^3 + a_0 y^2 - x)^m + (\text{terms with } y\text{-degree} \le 3m - 2) \\
&= y^{\bar m} + m a_0 y^{\bar m - 1} + (\text{terms with } y\text{-degree} \le 3m - 2).
\end{aligned} \tag{55}
$$

In particular, since $\phi(F^{\frac{-m-i}{m}})$ is an m-th root of $\phi(F)^{-m-i}$, we have

$$
\phi(F^{\frac{-m-i}{m}}) = \phi(F)^{\frac{-m-i}{m}} = \phi(F)^{\frac{-\bar m - 3i}{\bar m}}. \tag{56}
$$

Thus, we have the following, where the omitted terms are terms with lower y-degrees, or with lower powers of $\phi(F)$,

$$
\begin{aligned}
&\text{(i) } \phi(F)^{\frac{1}{\bar m}} \overset{(8),(55)}{=} y + \frac{a_0}{3} + \cdots \implies \phi(F)^{\frac{h}{\bar m}} = y^h + \frac{a_0}{3} h y^{h-1} + \cdots, \quad \text{for } h \in \mathbb{Z}, \quad \text{thus,} \\
&\text{(ii) } y^h = \phi(F)^{\frac{h}{\bar m}} - \frac{a_0}{3} h \phi(F)^{\frac{h-1}{\bar m}} + \cdots \in \sum_{k=0}^{\infty} \mathbb{C}[x] \bar F^{\frac{h-k}{\bar m}}, \\
&\text{(iii) } \phi\Big(c_{\frac{-m-i}{m}} F^{\frac{-m-i}{m}} \Big) = \sum_{k=0}^{\infty} p_{k,i}(x) \phi(F)^{\frac{2-2i-k-\bar m}{\bar m}} \quad \text{for some } p_{k,i} \in \mathbb{C}[x] \text{ and } i \ge -1,
\end{aligned} \tag{57}
$$

where, by the fact in (42) (iii) that $c_{\frac{-m-i}{m}}$ for $i \ge -1$ has x-degree $\le i + 2$, we see that $\phi(c_{\frac{-m-i}{m}})$ has y-degree $\le i + 2$, and so (iii) has y-degree $\le i + 2 - 3(m + i) = 2 - 2i - \bar m$, and thus we obtain (iii) from (ii).

Note that one can exchange the substitution with the series in the definition of

$$
\begin{aligned}
\bar G := \phi(G) &\overset{(41)}{=} \phi\Big(\sum_{i=0}^{\infty} c_{\frac{n-i}{m}} F^{\frac{n-i}{m}} \Big) = \sum_{i=0}^{\infty} \phi\Big(c_{\frac{n-i}{m}} F^{\frac{n-i}{m}} \Big) = \sum_{i=0}^{\infty} \phi\Big(c_{\frac{-\bar m + (\bar n + \bar m - 3i)}{\bar m}} F^{\frac{-\bar m + (\bar n + \bar m - 3i)}{\bar m}} \Big) \\
&= \sum_{-(\bar n + \bar m) \le i < \infty,\, i \in 3\mathbb{Z}} \phi\Big(c_{\frac{-\bar m + i}{\bar m}} F^{\frac{-\bar m - i}{\bar m}} \Big).
\end{aligned} \tag{58}
$$

Write the right-hand side of (58) as

$$\text{r.h.s. of (58)} \;=\; \sum_{i=0}^{\infty} \bar c_{\frac{\bar n-i}{\bar m}} \bar F^{\frac{\bar n-i}{\bar m}} \;\text{ for some }\; \bar c_{\frac{\bar n-i}{\bar m}} \in \mathbb{C}[x]. \tag{59}$$

Then of course we have the bar version of Lemma 2.8; in particular, $\bar c_{\frac{-\bar m+i}{\bar m}} \in \mathbb{C}$ for $i \geq 2$ by (42) (i).

Claim 2.10. *Only the term with $i = -3$ in the right-hand side of (58) can contribute to $\bar c_{\frac{-\bar m+3}{\bar m}}, \bar c_{\frac{-\bar m+4}{\bar m}}$.*

This can be proven as follows. If $i \leq -6$ then $\frac{-\bar m-i}{\bar m} > \frac{1-m}{m}$ and $c_{\frac{-\bar m-i}{\bar m}}$ is a constant by (42) (i) and thus the correspondent term is equal to $c_{\frac{-\bar m-i}{\bar m}} \bar F^{\frac{-\bar m-i}{\bar m}}$, which cannot contribute; if $i = 0$ the term is zero by (42) (ii); if $i \geq 3$ then by (57) (iii) the term has y-degree $\leq 2 - \frac{2i}{3} - \bar m < -\bar m + 3$, which cannot contribute. Thus the claim follows.

Now consider the term with $i = -3$ in the right-hand side of (58). We prove

$$\begin{aligned}
\phi\Big(c_{\frac{-m+1}{m}} F^{\frac{-m+1}{m}}\Big) &\overset{(42)\,(\text{iv}),(54)}{=} -\frac{J_0}{m}(y+a_1)\phi(F)^{\frac{-\bar m+3}{\bar m}} \\
&\in -\frac{J_0}{m}\Big(\phi(F)^{\frac{4-\bar m}{\bar m}} + (a_1 - \frac{a_0}{3})\phi(F)^{\frac{3-\bar m}{\bar m}} + \sum_{j=0}^{\infty} \mathbb{C}[x]\phi(F)^{\frac{2-\bar m-j}{\bar m}}\Big).
\end{aligned} \tag{60}$$

In fact,

$$y\,\phi(F)^{\frac{-\bar m+3}{\bar m}} \overset{(57)\,(\text{i})}{=} y^{-\bar m+4} + (-\bar m+3)\tfrac{a_0}{3} y^{-\bar m+3} + T_1 \;\text{ for some }\; T_1 \in \sum_{k=-2}^{\infty} \mathbb{C}[x]\phi(F)^{\frac{-\bar m-k}{\bar m}}. \tag{61}$$

From formula (57) we deduce

$$\begin{aligned}
y^{-\bar m+4} &\overset{(57)\,(\text{ii})}{=} \phi(F)^{\frac{-\bar m+4}{\bar m}} - (-\bar m+4)\frac{a_0}{3}\phi(F)^{\frac{-\bar m+3}{\bar m}} + T_2 \;\text{ for some }\; T_2 \in \sum_{k=-2}^{\infty} \mathbb{C}[x]\phi(F)^{\frac{-\bar m-k}{\bar m}}, \\
y^{-\bar m+3} &\overset{(57)\,(\text{ii})}{=} \phi(F)^{\frac{-\bar m+3}{\bar m}} + T_3 \;\text{ for some }\; T_3 \in \sum_{k=-2}^{\infty} \mathbb{C}[x]\phi(F)^{\frac{-\bar m-k}{\bar m}}.
\end{aligned} \tag{62}$$

Thus the part "$\in$" in (60) follows from (61), (62), i.e., we have (60).

By Claim 2.10 and (60), we obtain

$$\bar c_{\frac{-\bar m+4}{\bar m}} = -\frac{J_0}{m} \neq 0, \qquad \bar c_{\frac{-\bar m+3}{\bar m}} = -\frac{J_0}{m}(a_1 - \frac{a_0}{3}) \neq 0. \tag{63}$$

Thus by replacing (F, G) by $(\bar F, \bar G)$ [by (53) we still have (37) after the replacement], we have the lemma. □

From now on, we fix, once and for all, the Jacobian pair (F, G) satisfying (37) and Lemma 2.9 [and obviously, (F, G) is obtained from the original Jacobian pair by applying some automorphism of $\mathbb{C}[x, y]$].

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 6 September 2023 doi:10.20944/preprints202212.0171.v2

In the rest of this section we regard all elements as in the ring

$$\mathbf{R} = \mathbb{C}[x^{\frac{1}{m}}]((y^{-1})), \tag{64}$$

where $x^{\frac{1}{m}}$ is regarded as a parameter such that its m-th power is x.

Now, to be more precise, we slightly generalize notions and notations in Definition 2.4.

Definition 2.11. Let $R = \sum_{i,j} r_{ij} x^i y^j$, $Q = \sum_{i,j} q_{ij} x^i y^j \in \mathbf{R}$ with $r_{ij} \in \mathbb{C}$, $q_{ij} \in \mathbb{R}_{\geq 0}$.

(i) If $|r_{ij}| \leq q_{ij}$ for all possible i, j, then we say R is controlled by Q with respect to x, y, and denote $R \trianglelefteq_{x,y} Q$.

(ii) For $x_0, y_0 \in \mathbb{C}$, if $\sum_{i,j} |r_{ij} x_0^i y_0^j|$ converges, then we say the series R with respect to x, y converges *strongly* when (x, y) is set to (x_0, y_0), and denote $R|_{(x,y)=(x_0,y_0)} = \sum_{i,j} r_{ij} x_0^i y_0^j$.

Note that Lemmas 2.5 and 2.6 can be parallelly generalized.

Denote

$$t = (1 + x^{\frac{1}{m}})^{m-1} \in \mathbf{R}. \tag{65}$$

Throughout the rest of this section, we always use $\mathbf{s}_j \in \mathbb{R}_{>0}$ to denote some fixed number for all possible j; for instance, we can take $\mathbf{s}_0 = \sum_{j,k} |f_{j,k}|$ in (66).

By (38) (iii), we can observe below that every term $x^a y^b$ appearing in F (with non-negative integral a, b) also appears in $\boldsymbol{F}_1$ by using (65) to expand t^j in $\boldsymbol{F}_1$, thus we obtain,

$$\text{(i) } F \overset{(65)}{\trianglelefteq_{x,y}} \boldsymbol{F}_1 := y^m \Big(1 + \mathbf{s}_0 \sum_{j=1}^{m} (ty^{-1})^j\Big) \trianglelefteq_{x,y} \boldsymbol{F} := y^m \Big(1 + \mathbf{s}_0 \sum_{j=1}^{\infty} (ty^{-1})^j\Big), \text{ where}$$

$$\text{(ii) } \boldsymbol{F} = y^m \Big(1 + \frac{\mathbf{s}_0 t y^{-1}}{1 - ty^{-1}}\Big). \tag{66}$$

Remark 2.12. The importance of (66) is that though the structure of the polynomial F may be complicated (and in particular we do not have any information about the coefficient of $x^i y^j$ in F for general $i, j \in \mathbb{Z}_{>0}$), we can always use the controlling function $\boldsymbol{F}$, which has the very simple form in (66) (ii), to control F; consequently, we are able to choose the simple controlling function $\boldsymbol{P}$ in (70) to control P, which allows us to obtain the simple form of the formal inverse function of $\boldsymbol{P}_{\text{neg}}$ in (75); then we can conveniently use Lemma 2.6 to obtain Lemma 2.14, which is the key to obtain Lemma 2.17.

Now by (66) (ii) and definition (15) we have

$$\boldsymbol{F}_{\text{ign}} \overset{(15)}{=} \frac{\mathbf{s}_0 t y^{-1}}{1 - ty^{-1}}. \tag{67}$$

Thus by Lemma 2.5, for $j \in \mathbb{Z}_{\geq 0}$,

$$F^{\pm\frac{j}{m}} \overset{(17)\,(b),\,(c)}{\trianglelefteq_{x,y}} y^{\pm j}\Big(1 - \boldsymbol{F}_{\text{ign}}\Big)^{-\frac{j}{m}} \overset{(67)}{=} y^{\pm j}\Big(\frac{1 - ty^{-1}}{1 - (1 + \mathbf{s}_0)ty^{-1}}\Big)^{\frac{j}{m}}. \tag{68}$$

We use (8) to define

$$P = F^{-\frac{1}{m}}, \tag{69}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 6 September 2023 doi:10.20944/preprints202212.0171.v2

which can be written as the form in (i) below for some $p_j \in \mathbb{C}[x]$, then we use (68) and Lemma 2.5 to obtain,

$$\text{(i) } P \overset{(69)}{:=} F^{-\frac{1}{m}} = y^{-1}\Big(1+\sum_{j=1}^{\infty} p_j y^{-j}\Big) \overset{(68)}{\trianglelefteq_{x,y}} \frac{y^{-1}}{(1-\boldsymbol{F}_{\text{ign}})^{\frac{1}{m}}} \overset{(17)\,(b)}{\trianglelefteq_{x,y}} \boldsymbol{P}, \text{ where,}$$

$$\text{(ii) } \boldsymbol{P} := \frac{y^{-1}}{1-\boldsymbol{F}_{\text{ign}}} \overset{(67)}{=} \frac{y^{-1}(1-ty^{-1})}{1-(1+\boldsymbol{s}_0)ty^{-1}} = y^{-1}\Big(1+\frac{\boldsymbol{s}_0 ty^{-1}}{1-(1+\boldsymbol{s}_0)ty^{-1}}\Big) \overset{(15)}{=} y^{-1}(1+\boldsymbol{P}_{\text{ign}}) \text{ with}$$

$$\text{(iii) } \boldsymbol{P}_{\text{ign}} = \frac{\boldsymbol{s}_0 ty^{-1}}{1-(1+\boldsymbol{s}_0)ty^{-1}}. \tag{70}$$

Thus by definition (15),

$$\boldsymbol{P}_{\text{neg}} = y^{-1}(1-\boldsymbol{P}_{\text{ign}}) \overset{(70)\,(iii)}{=} y^{-1}\Big(1-\frac{\boldsymbol{s}_0 ty^{-1}}{1-(1+\boldsymbol{s}_0)ty^{-1}}\Big) = \frac{y^{-1}(1-(1+2\boldsymbol{s}_0)ty^{-1})}{1-(1+\boldsymbol{s}_0)ty^{-1}}. \tag{71}$$

For convenience, we denote

$$\tilde{c}_j := c_{-\frac{j}{m}} = \tilde{c}_{0,j} + \tilde{c}_{1,j} \text{ with } \tilde{c}_{0,j} \in \mathbb{C},\ \tilde{c}_{1,j} \in x\mathbb{C}[x] \text{ for all possible } j. \tag{72}$$

Then by (42) and Lemma 2.9,

$$\text{(i) } \tilde{c}_{1,j} = 0 \ \text{ for } \ j \le m-2, \qquad \text{(ii) } \tilde{c}_{m-4}\tilde{c}_{m-3} \neq 0. \tag{73}$$

Notation 2.13. We denote $\hat{c}_{1,j} \in x\mathbb{R}_{\ge 0}[x]$ to be the polynomial of x which is obtained from $\tilde{c}_{1,j} \in x\mathbb{C}[x]$ by replacing the coefficient $\mathrm{C}_{\text{oeff}}(\tilde{c}_{1,j}, x^k)$ by its absolute value for all possible j, k.

Lemma 2.14. *We have, for some $\boldsymbol{s}_1, \boldsymbol{s}_2 \in \mathbb{R}_{>0}$,*

$$\text{(i) } G = G_0 + G_1 \ \textit{ with}$$

$$\text{(ii) } G_0 := G|_{(x,P)=(0,P)} \overset{(41),(72)}{=} \sum_{j=-n}^{\infty} \tilde{c}_{0,j} P^j$$
$$\trianglelefteq_{x,P} P^{-n}\Big(1+\frac{\boldsymbol{s}_1 P}{1-\boldsymbol{s}_2 P}\Big) \trianglelefteq_{x,y} \boldsymbol{P}_{\text{neg}}^{-n}\Big(1+\frac{\boldsymbol{s}_1 \boldsymbol{P}}{1-\boldsymbol{s}_2 \boldsymbol{P}}\Big),$$

$$\text{(iii) } G_1 := G - G_0 \overset{(41),(72)}{=} \sum_{j=-n}^{\infty} \tilde{c}_{1,j} P^j$$
$$\trianglelefteq_{x,P} \sum_{j=m-1}^{\infty} x\frac{d\hat{c}_{1,j}}{dx} P^j \trianglelefteq_{x,P} J_0 m^{-1} x P^{m-1}\Big(1-\frac{\boldsymbol{s}_1 tP}{1-\boldsymbol{s}_2 tP}\Big)^{-1}. \tag{74}$$

Proof. Solving y^{-1} from (71) we obtain that the formal inverse function of $\boldsymbol{P}_{\text{neg}}$, which by notations in (22), (26), is $\boldsymbol{P}_{\text{neg}}^{\circ -1}$, is the following [noting that if (71) is regarded as a quadratic equation on variable y^{-1} then we have two solutions and obviously the other solution, when expanded as a series of y^{-1}, contains a nonzero constant term and thus is not the one we require],

$$y^{-1} = \boldsymbol{P}_{\text{neg}}^{\circ -1}(\boldsymbol{P}_{\text{neg}}) := \frac{1+(1+\boldsymbol{s}_0)t\boldsymbol{P}_{\text{neg}} - B}{2(1+2\boldsymbol{s}_0)t}, \ \ B = (1-\beta_+ t\boldsymbol{P}_{\text{neg}})^{\frac{1}{2}}(1-\beta_- t\boldsymbol{P}_{\text{neg}})^{\frac{1}{2}}, \tag{75}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 6 September 2023 doi:10.20944/preprints202212.0171.v2

where we always regard an element $(1+a)^{\frac{1}{2}}$ as the unique element defined by the formula (6), and where $\beta_{\pm} = 1 + 3s_0 \pm 2(s_0 + 2s_0^2)^{\frac{1}{2}} = \big(\sqrt{1+2s_0} \pm \sqrt{s_0}\big)^2 \in \mathbb{R}_{>0}$.

Regarding B defined in (75) as a function of $x, \boldsymbol{P}_{\rm neg}$, by Definition 2.11 (i) and using (17) (with $k = \frac{1}{2}$, $q_0 = 1, \alpha = 0$, and $Q = 1 - \beta_{\pm} t \boldsymbol{P}_{\rm neg}$, and thus $Q_{\rm neg} \overset{(15)}{=} Q$), we obtain, where the second "$\trianglelefteq_{x,\boldsymbol{P}_{\rm neg}}$" is obtained by the fact that $\beta_- < \beta_+$,

$$B \overset{(17)}{\trianglelefteq_{x,\boldsymbol{P}_{\rm neg}}} (1-\beta_+ t\boldsymbol{P}_{\rm neg})^{-\frac{1}{2}}(1-\beta_- t\boldsymbol{P}_{\rm neg})^{-\frac{1}{2}} \trianglelefteq_{x,\boldsymbol{P}_{\rm neg}} (1-\beta_+ t\boldsymbol{P}_{\rm neg})^{-1}. \tag{76}$$

Using this, we can deduce from (75) the following,

$$\boldsymbol{P}_{\rm neg}^{\circ -1}(\boldsymbol{P}_{\rm neg}) \overset{(75),(76)}{\trianglelefteq_{x,\boldsymbol{P}_{\rm neg}}} \boldsymbol{P}_{\rm neg}\Big(1 + \frac{s_3 t \boldsymbol{P}_{\rm neg}}{1 - s_4 t \boldsymbol{P}_{\rm neg}}\Big) \text{ for some } s_3, s_4 \in \mathbb{R}_{>0}. \tag{77}$$

Thus by (17), (28), we have, by choosing sufficiently large $s_1, s_2 \in \mathbb{R}_{>0}$,

$$\begin{aligned} &\text{(i) } y^{-1} \overset{(28)}{\trianglelefteq_{x,P}} \boldsymbol{P}_{\rm neg}^{\circ -1}(P) \overset{(77)}{\trianglelefteq_{x,P}} P\Big(1+\frac{s_3 tP}{1-s_4 tP}\Big) \trianglelefteq_{x,P} P\Big(1+\frac{s_1 tP}{1-s_2 tP}\Big),\\ &\text{(ii) } y \overset{(78)\,(i),(17)}{\trianglelefteq_{x,P}} P^{-1}\Big(1-\frac{s_3 tP}{1-s_4 tP}\Big)^{-1} \trianglelefteq_{x,P} P^{-1}\Big(1+\frac{s_1 tP}{1-s_2 tP}\Big), \end{aligned} \tag{78}$$

where the second "$\trianglelefteq_{x,P}$" in (i) is obtained from (77), and (ii) is obtained from the first two "$\trianglelefteq_{x,P}$" of (i).

By (78), (37) (ii) (with the fact that $n|m$) and (17), (70), we can obtain, for some $s_5, s_6 \in \mathbb{R}_{>0}$,

$$G \overset{(78),(37)}{\trianglelefteq_{x,P}} P^{-n}\Big(1+\frac{s_6 tP}{1-s_5 tP}\Big) \overset{(17),(70)}{\trianglelefteq_{x,y}} \boldsymbol{P}_{\rm neg}^{-n}\Big(1+\frac{s_6 t\boldsymbol{P}}{1-s_5 t\boldsymbol{P}}\Big). \tag{79}$$

From this, (41) and definition of G_0 in (74) (ii), we obtain (note that $t|_{x=0} = 1$),

$$G_0 = G|_{(x,P)=(0,P)} \overset{(79)}{\trianglelefteq_{x,P}} P^{-n}\Big(1+\frac{s_6 (t|_{x=0})P}{1-s_5 (t|_{x=0})P}\Big) \overset{(79)}{\trianglelefteq_{x,y}} \boldsymbol{P}_{\rm neg}^{-n}\Big(1+\frac{s_6 (t|_{x=0})\boldsymbol{P}}{1-s_5 (t|_{x=0})\boldsymbol{P}}\Big), \tag{80}$$

i.e., we have (74) (ii) (by enlarging s_1, s_2).

By (66) and (17) (a), we obtain

$$m^{-1}\big(\frac{\partial F}{\partial y}\big) \trianglelefteq_{x,y} y^{m-1}\Big(1 + s_0 \sum_{j=1}^{m-1} (ty^{-1})^j\Big). \tag{81}$$

Thus we have,

$$\begin{aligned}\sum_{j=-n}^{\infty}\frac{d\tilde c_{1j}}{dx}P^j &\overset{(69),(72)}{=} \sum_{\alpha\in A}\frac{dc_\alpha}{dx}F^\alpha \overset{(46)\,(i)}{=} -J_0\Big(\frac{\partial F}{\partial y}\Big)^{-1}\\ &\overset{(17),(81)}{\trianglelefteq_{x,y}} J_0 m^{-1}y^{-(m-1)}\Big(1-s_0\sum_{j=1}^{m-1}(ty^{-1})^j\Big)^{-1}\\ &\overset{(78)}{\trianglelefteq_{x,P}} J_0 m^{-1}y^{-(m-1)}\Big(1-s_0\sum_{j=1}^{m-1}(ty^{-1})^j\Big)^{-1}\Big|_{y^{-1}=P(1+\frac{s_3tP}{1-s_4tP})}\\ &\trianglelefteq_{x,P} J_0 m^{-1}P^{m-1}\Big(1-\frac{s_1tP}{1-s_2tP}\Big)^{-1},\end{aligned} \tag{82}$$

where the first " $\trianglelefteq_{x,P}$ " is obtained from (78) (i) and the second is obtained by choosing sufficiently large s_1, s_2. Now the first " $\trianglelefteq_{x,P}$ " of (74) (iii) is obvious and the second " $\trianglelefteq_{x,P}$ " is obtained from (82). This proves the lemma. □

The following proposition is a stronger version of Theorem 1.2.

Proposition 2.15. *There exists $s_7 \in \mathbb{R}_{>0}$ such that for any $(p_1, p_2) = \big((x_1, y_1), (x_2, y_2)\big) \in V$ with $h_{p_1,p_2} \ge s_7$, we must have*

$$|y_1| < h_{p_1,p_2}^{\frac{m}{m+1}}, \qquad |y_2| < h_{p_1,p_2}^{\frac{m}{m+1}}. \tag{83}$$

We prove this proposition by contradiction and in several steps. Assume thus that the statement leading to formula (83) does not hold.

Step 1. In this step we prove that, under the previous hypotheses, either for this Jacobian pair (F, G), there exists $(p_{1,i}, p_{2,i}) = \big((x_{1,i}, y_{1,i}), (x_{2,i}, y_{2,i})\big) \in V$ for any $i \in \mathbb{Z}_{>0}$ satisfying,

$$\text{(i) } h_{p_{1,i},p_{2,i}} \ge \tfrac{1}{2}i, \quad x_{1,i} \ne x_{2,i}, \quad \text{(ii) } |y_{1,i}| \ge \tfrac{1}{2}h_{p_{1,i},p_{2,i}}^{\frac{m}{m+1}}, \tag{84}$$

or for an equivalent Jacobian pair $(\tilde F, \tilde G)$ obtained by a simple linear change of coordinates and still satisfying Lemma 2.9, the tilde version of (84) holds.

Proof of Step 1. Assume (83) does not hold, i.e., there exists $(p_{1,i}, p_{2,i}) = \big((x_{1,i}, y_{1,i}), (x_{2,i}, y_{2,i})\big) \in V$ for any $i \in \mathbb{Z}_{>0}$ satisfying $h_{p_{1,i},p_{2,i}} \ge i$, such that

$$\text{(i) } |y_{1,i}| \ge h_{p_{1,i},p_{2,i}}^{\frac{m}{m+1}}, \quad \text{or} \quad \text{(ii) } |y_{2,i}| \ge h_{p_{1,i},p_{2,i}}^{\frac{m}{m+1}}. \tag{85}$$

Since at least one of the conditions in (85) must hold for infinitely many i's, if necessary by switching $p_{1,i}$ and $p_{2,i}$, we can assume (85) (i) holds for infinitely many i. If necessary by replacing the sequence by a subsequence [if the sequence $(p_{1,i}, p_{2,i})$ is replaced by the subsequence (p_{1,i_j}, p_{2,i_j}), then we always have $i_j \ge j$; thus we still have $h_{p_{1,i},p_{2,i}} \ge i$ after the replacement], we may assume (85) (i) holds for all i. Finally if for infinitely many indices i we have $x_{1,i} \ne x_{2,i}$ still passing to a subsequence we may assume that $x_{1,i} \ne x_{2,i}$ holds for all i and the given pair (F, G) satisfies the conditions of Step 1.

Thus we may assume that $x_{1,i} = x_{2,i}$ holds for all i. Take the equivalent Jacobian pair

$$(\tilde F, \tilde G) = \Big(F\big(x + \frac{y}{2}, y\big), G\big(x + \frac{y}{2}, y\big)\Big). \tag{86}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 6 September 2023 doi:10.20944/preprints202212.0171.v2

which corresponds to the change of coordinates

$$(\tilde{x}, \tilde{y}) = \big(x - \frac{y}{2}, y\big). \tag{87}$$

Note that our (F, G) in (86) was obtained, in Lemma 2.9 by the initial choice of (F, G) defined in (53), and the coordinate change $(x, y) \mapsto (y, y^3 + a_0 y^2 - x)$ therefore, when we change x to $x + \frac{y}{2}$ in (86), it amounts to that the tilde version of (F, G) is obtained from the initial (F, G) by mapping $(x, y) \mapsto (y, y^3 + a_0 y^2 - \frac{y}{2} - x)$. Then one can observe that for the tilde version of (F, G), we still have the second equality of (55) and thus we still have (57), Claim 2.10 and (63). Hence, the proof of Lemma 2.9 works again for $(\tilde{F}, \tilde{G})$. This proves that all results before Proposition 2.15 hold for the tilde version Jacobian pair $(\tilde{F}, \tilde{G})$.

Denote the points $(p_{1,i}, p_{2,i})$ in the new coordinates as $(\tilde{p}_{1,i}, \tilde{p}_{2,i})$ below for all $i \in \mathbb{Z}_{>0}$, then (86) together with the fact that $(p_{1,i}, p_{2,i}) \in V$ shows that it is in $\tilde{V}$ (which is the tilde version of V), i.e.,

$$(\tilde{p}_{1,i}, \tilde{p}_{2,i}) = \big((\tilde{x}_{1,i}, \tilde{y}_{1,i}), (\tilde{x}_{2,i}, \tilde{y}_{2,i})\big) \overset{(87)}{=} \Big(\big(x_{1,i} - \frac{y_{1,i}}{2}, y_{1,i}\big), \big(x_{2,i} - \frac{y_{2,i}}{2}, y_{2,i}\big)\Big) \in \tilde{V}. \tag{88}$$

Since by construction $p_{1,i} \neq p_{2,i}$ and the assumption $x_{1,i} = x_{2,i}$ we have thus $\tilde{x}_{1,i} \neq \tilde{x}_{2,i}$ for all i.

Note that every inequality below [except the second inequality in (ii), which follows from (i)] is obtained from one of the following three facts: (1) formula (85) (i) hold and $h_{p_{1,i},p_{2,i}} \geq i$ by noting from arguments after (85); (2) $h_{p_{1,i},p_{2,i}} = |x_{1,i}| + |y_{1,i}| + |x_{2,i}| + |y_{2,i}|$ by definition (4); (3) $\pm(|a| - |b|) \leq |a - b| \leq |a| + |b|$ for all $a, b \in \mathbb{C}$, therefore we have

$$\begin{aligned}
&\text{(i) } \frac{i}{2} \leq \frac{h_{p_{1,i},p_{2,i}}}{2} \leq |x_{1,i}| + \frac{|y_{1,i}|}{2} + |x_{2,i}| + \frac{|y_{2,i}|}{2} \leq \Big|x_{1,i} - \frac{y_{1,i}}{2}\Big| + |y_{1,i}| + \Big|x_{2,i} - \frac{y_{2,i}}{2}\Big| + |y_{2,i}| \\
&\qquad = \tilde{h}_{\tilde{p}_{1,i},\tilde{p}_{2,i}} \leq |x_{1,i}| + \frac{3|y_{1,i}|}{2} + |x_{2,i}| + \frac{3|y_{2,i}|}{2} \leq 2h_{p_{1,i},p_{2,i}}, \\
&\text{(ii) } |\tilde{y}_{1,i}| = |y_{1,i}| \geq h_{p_{1,i},p_{2,i}}^{\frac{m}{m+1}} \overset{(89)\,\text{(i)}}{\geq} \Big(\frac{1}{2}\Big)^{\frac{m}{m+1}} \tilde{h}_{\tilde{p}_{1,i},\tilde{p}_{2,i}}^{\frac{m}{m+1}} \geq \frac{1}{2} \tilde{h}_{\tilde{p}_{1,i},\tilde{p}_{2,i}}^{\frac{m}{m+1}}.
\end{aligned} \tag{89}$$

This proves that the tilde version of (84) holds.

For the purpose of proving Proposition 2.15, to simplify notations, in case the tilde version of (84) holds, we re-denote $(\tilde{F}, \tilde{G})$ as (F, G) until the end of the proof of Proposition 2.15 [note that we abuse notations F, G just for simplicity, but we do not change our real pair (F, G) after the proof of Proposition 2.15, which means that our pair (F, G) is still the pair that satisfies (83)]. In this way, the tilde in all notations is omitted, and so in particular, the conclusion in Step 1 can be uniformly written as the following, for $i \geq 1$,

$$\text{(i) } h_{p_{1,i},p_{2,i}} \geq \tfrac{1}{2} i, \quad \text{(ii) } |y_{1,i}| \geq \tfrac{1}{2} h_{p_{1,i},p_{2,i}}^{\frac{m}{m+1}}, \quad \text{(iii) } x_{1,i} \neq x_{2,i}. \tag{90}$$

Step 2. In this tep we prove that, for the previous sequence of points, we have formula (95) that is $\lim_{i \to \infty} \frac{y_{2,i}}{y_{1,i}} = \omega$, where ω is some m-th root of unity.

Notation 2.16. Before continuing our proof, we need to use the following notations: Let a, b, c be variables such that $c \to \infty$ or $c \to 0$ and a, b are functions of c, we denote

$$\text{(i) } a \sim_c b, \quad \text{(ii) } a \prec_c b, \quad \text{(iii) } a \preceq_c b, \tag{91}$$

 doi:10.20944/preprints202212.0171.v2

which mean respectively, for some fixed $s_1, s_2 \in \mathbb{R}_{>0}$,

$$\text{(i) } s_1 \le \left|\frac{a}{b}\right| \le s_2, \quad \text{(ii) } \lim_{c\to\infty} \frac{a}{b} = 0, \quad \text{(iii) } \left|\frac{a}{b}\right| \le s_1. \tag{92}$$

Then obviously, properties (i), (ii), (iii) are respectively an equivalence, a strict order, an order.

Now for $k = 1, 2$, since $|x_{k,i}| \le h_{p_{1,i},p_{2,i}}$, $|y_{k,i}| \le h_{p_{1,i},p_{2,i}}$, by (37) (i), we have, where (ii) is obtained from (i) and (37) (i),

$$\text{(i) } F_1(x_{k,i}, y_{k,i}) \overset{(37)\,(i)}{\preceq_i} h^{m-1}_{p_{1,i},p_{2,i}} \prec_i h^{\frac{m^2}{m+1}}_{p_{1,i},p_{2,i}} \overset{(90)\,(ii)}{\preceq_i} |y_{1,i}|^m, \qquad \text{(ii) } F(x_{1,i}, y_{1,i}) \sim_i y^m_{1,i}. \tag{93}$$

Thus, where the first equality follows from the fact that $\sigma(p_{1,i}) = \sigma(p_{2,i})$,

$$1 = \frac{F(x_{2,i}, y_{2,i})}{F(x_{1,i}, y_{1,i})} = \lim_{i\to\infty} \frac{F(x_{2,i}, y_{2,i})}{F(x_{1,i}, y_{1,i})} \overset{(37)}{=} \lim_{i\to\infty} \frac{\frac{y^m_{2,i}}{y^m_{1,i}} + \frac{F_1(x_{2,i}, y_{2,i})}{y^m_{1,i}}}{1 + \frac{F_1(x_{1,i}, y_{1,i})}{y^m_{1,i}}} \overset{(93)}{=} \lim_{i\to\infty} \left(\frac{y_{2,i}}{y_{1,i}}\right)^m. \tag{94}$$

Therefore, by replacing the sequence by a subsequence, we have [we would like to remark that ω defined below is not necessarily equal to 1, but we will prove that the m-th root ω' of unity defined in (102) must be 1],

$$\lim_{i\to\infty} \frac{y_{2,i}}{y_{1,i}} = \omega, \quad \text{where } \omega \text{ is some } m\text{-th root of unity.} \tag{95}$$

Step 3. In this step we show that the series appearing in our algeraic identities can be evaluated at the points $p_{1,i}$, $p_{2,i}$ giving rise to numerical estimates.

Denote

$$\varepsilon = h^{-\frac{1}{m(m+1)}}_{p_{1,i},p_{2,i}} \to 0 \ \text{ (when } i \to \infty). \tag{96}$$

Let $a_i \in \mathbb{C}$ with $|a_i| \le h_{p_{1,i},p_{2,i}}$. Then $1 + |a_i|^{\frac{1}{m}} \preceq_i h^{\frac{1}{m}}_{p_{1,i},p_{2,i}}$ so, by (90), (95), we have

$$(1 + |a_i|^{\frac{1}{m}})^{m-1} |y_{k,i}|^{-1} \preceq_i h^{\frac{m-1}{m}}_{p_{1,i},p_{2,i}} |y_{1,i}|^{-1} \preceq_i h^{\frac{m-1}{m} - \frac{m}{m+1}}_{p_{1,i},p_{2,i}} = \varepsilon \text{ for } k = 1, 2. \tag{97}$$

Using the notations in (72) and the results in formula (73) and in formula (74) of Lemma 2.14, we prove the following.

Lemma 2.17. *Let $k = 1, 2$ and $a \in \mathbb{C}$ with $|a| \le h_{p_{1,i},p_{2,i}}$ and $i \gg 1$.*

(i) *In (70) (i), the series $P = F^{-\frac{1}{m}}$ with respect to x, y converges strongly when (x, y) is set to $(a, y_{k,i})$, and*

$$P_{a,k} := P|_{(x,y)=(a,y_{k,i})} = y^{-1}_{k,i}\Big(1 + O(\varepsilon)^1\Big). \tag{98}$$

(ii) *In (78) (ii), the series y with respect to x, P converges strongly when (x, P) is set to $(a, P_{a,k})$, and*

$$Y_{a,k} := y|_{(x,P)=(a,P_{a,k})} = y_{k,i}\Big(1 + O(\varepsilon)^1\Big). \tag{99}$$

(iii) *In* (74)*, the series* G *with respect to* x, P *converges strongly when* (x, P) *is set to* $(a, P_{a,k})$*, and*

$$\text{(a) } A_{a,k,\ell} := \sum_{j=\ell}^{\infty} \tilde{c}_{0j} P_{a,k}^{j} \preceq_i P_{a,k}^{\ell} \sim_i y_{1,k}^{-\ell} \text{ for } \ell \geq -n,$$

$$\text{(b) } B_{a,k} := G_1|_{(x,y)=(a,y_{k,i})} \preceq_i h_{p_{1,i},p_{2,i}} y_{k,i}^{-(m-1)} \prec_i y_{1,i}^{-(m-3)}. \tag{100}$$

(iv) *The series* $\left(\frac{\partial F}{\partial y}\right)^{-1}$ *with respect to* x, P *converges strongly when* (x, P) *is set to* $(a, P_{a,k})$*, and*

$$m\left(\frac{\partial F}{\partial y}\right)^{-1}\Big|_{(x,P)=(a,P_{a,k})} = y_{k,i}^{-(m-1)}\left(1 + O(\varepsilon)^1\right). \tag{101}$$

(v) $P_1 := P_{x_{1,i},1} = P(x_{1,i}, y_{1,i}) = P(x_{2,i}, y_{2,i}) = P_{x_{2,i},2}$.

Proof. From (70) (ii), we see that $\boldsymbol{P}_{\text{ign}}|_{(x,y)=(|a|,|y_{k,i}|)}$ converges to $O(\varepsilon)^1$ by (97), thus (98) follows from (70) (i). Similarly, we obtain (99)–(101) from (78), (74), (82). This proves (i)–(iv).

To prove (v), we have $P(x_{1,i}, y_{1,i})^{-m} \overset{(69)}{=} F(x_{1,i}, y_{1,i}) = F(x_{2,i}, y_{2,i}) \overset{(69)}{=} P(x_{2,i}, y_{2,i})^{-m}$. Thus

$$P(x_{2,i}, y_{2,i}) = \omega' P(x_{1,i}, y_{1,i}), \tag{102}$$

for some m-th root ω' of unity (which may depend on i). Assume there exists $j \leq m-3$ such that

$$\tilde{c}_j \overset{(73)}{=} \tilde{c}_{0j} \neq 0 \quad \text{but} \quad P(x_{1,i}, y_{1,i})^j \neq P(x_{2,i}, y_{2,i})^j \overset{(102)}{=} \omega'^j P(x_{1,i}, y_{1,i})^j. \tag{103}$$

Let $j_0 \leq m-3$ be the minimal such j. Then $\omega'^{j_0} \neq 1$ and $|1 - \omega'^{j_0}| > \delta$ for some fixed $\delta > 0$ (since ω'^{j_0}, which may though depend on i, is an m-th root of unity). By (74) and Lemma 2.17 (iii), we have [note that $\tilde{c}_{0,j_0} \in \mathbb{C}_{\neq 0}$ is a number independent of i and $P(x_{1,i}, y_{1,i}) \sim_i y_{1,i}^{-1}$ by (98)],

$$\begin{aligned} 0 &= G(x_{1,i}, y_{1,i}) - G(x_{2,i}, y_{2,i}) \\ &\overset{(74),(100)}{=} \tilde{c}_{0,j_0}(1 - \omega'^{j_0}) P(x_{1,i}, y_{1,i})^{j_0} + A_{x_{1,i},1,j_0+1} - A_{x_{2,i},2,j_0+1} + B_{x_{1,i},1} - B_{x_{2,i},2} \\ &\overset{(98),(100)}{\sim_i} P(x_{1,i}, y_{1,i})^{j_0} \overset{(98)}{\sim_i} y_{1,i}^{-j_0}, \end{aligned} \tag{104}$$

which is a contradiction. This proves the following crucial fact,

$$\omega'^j = 1 \text{ for all } j \leq m-3 \text{ with } \tilde{c}_j \neq 0. \tag{105}$$

In particular by (73) (ii), $\omega'^{m-4} = 1, \omega'^{m-3} = 1$, which implies that $\omega' = 1$. This proves (v) and the lemma. □

Step 4 (conclusion). Now we can continue our proof of Proposition 2.15. By (41), (74) and Lemma 2.17 (iv), (v), we have the following, where the second equality follows from (41), (72) and Lemma 2.17 (iii), (v), the third equality follows from the fact that $\tilde{c}_j$'s are polynomials in x [the integration path is taken as $x(t) := x_{1,i}(1-t) + tx_{2,i},\ 0 \leq t \leq 1$], the fourth follows from the fact that the series there

 doi:10.20944/preprints202212.0171.v2

converges absolutely and uniformly when $|x| \le h_{p_{1,i},p_{2,i}}$ by Lemma 2.17, and the last two equalities can be observed from (82) (see Remark 2.18 for the reason why we have the fifth equality),

$$\begin{aligned}
0 &= G(x_{1,i}, y_{1,i}) - G(x_{2,i}, y_{2,i}) \overset{(41),(72)}{=} -\sum_{j=-n}^{\infty} \big(\tilde{c}_j(x_{2,i}) - \tilde{c}_j(x_{1,i})\big) P_1^j \\
&= -\sum_{j=-n}^{\infty} \int_{x_{1,i}}^{x_{2,i}} \frac{d\tilde{c}_j}{dx} P_1^j dx = -\int_{x_{1,i}}^{x_{2,i}} \sum_{j=-n}^{\infty} \frac{d\tilde{c}_j}{dx} P_1^j dx \overset{\text{Remark 2.18}}{=} \int_{x_{1,i}}^{x_{2,i}} J_0 \Big(\frac{\partial F}{\partial y}\Big)^{-1}\Big|_{(x,P)=(x,P_1)} dx \\
&\overset{(82)}{=} (x_{2,i} - x_{1,i}) J_0 m^{-1} y_{1,i}^{-(m-1)} \Big(1 + O(\varepsilon)^1\Big) \ne 0,
\end{aligned} \tag{106}$$

which is a contradiction. This proves Proposition 2.15. □

Remark 2.18. Note that the fifth equality of (106) should be understood in this way: first we regard $\big(\frac{\partial F}{\partial y}\big)^{-1}$ as a power series of P with coefficients in $\mathbb{C}[x]$, i.e., $J_0\big(\frac{\partial F}{\partial y}\big)^{-1} = \sum_{j=-n}^{\infty} \frac{dc_j}{dx} P^j$ by (82), then we set P to P_1 (and keep x unchanged with $|x| \le h_{p_{1,i},p_{2,i}}$), and then Lemma 2.17 (iv) shows that the series converges absolutely and uniformly, and so we have the fifth equality.

Proof of Theorem 1.2. It follows immediately from Proposition 2.15. □

Remark 2.19. To prove Theorem 1.1, we always need to make the whole use of the following three facts about (F, G):

(F1) $F \in \mathbb{C}[x, y^{\pm 1}]$, $G \in \mathbb{C}[x]((y^{-1}))$;
(F2) F, G are locally holomorphic functions of x, y on $\mathbb{C}^2$, and so $F, G \in \mathbb{C}[x, y]$;
(F3) $J(F, G)$ is nonzero anywhere in $\mathbb{C}^2$, and so $J(F, G) \in \mathbb{C}_{\ne 0}$.

However, some results may not require all of the above facts; for instance, our approach above shows that Theorem 1.2 holds for any pair (F, G) satisfying (F1) as long as the following holds [of course, when $(F, G) \notin \mathbb{C}[x, y]^2$, the map σ in (1) can be only defined on some subset of $\mathbb{C}^2$, and in case $G \notin \mathbb{C}[x, y^{\pm 1}]$, we need some extra condition on G so that it can be controlled somehow, cf. (74)]:

(i) $J(F, G) \in \mathbb{C}_{\ne 0}$;
(ii) the homogenous part of F with the highest degree contains only one term, i.e., y^m with $m > 1$;
(iii) let c_α be defined in (41), then,

$$\gcd\Big(m, m\alpha \,\Big|\, \alpha \in \frac{1}{m}\mathbb{Z}, \alpha \ge \frac{3-m}{m}, c_\alpha \ne 0\Big) = 1, \tag{107}$$

where the symbol " gcd " denotes the greatest common divisor [cf. (52) and statements after (104), and noting that ω' appeared in (104) is some m-th root of unity].

In particular, Theorem 1.2 holds for the pair $(F, G) = (y^m, y^n + cy^k + xy^{-m+1})$ for any $m, n, k \in \mathbb{Z}$, $c \in \mathbb{C}_{\ne 0}$ with $m > 1$, $n > k \ge 3 - m$ and $\gcd(m, n, k) = 1$, which can be also easily proven directly; in this case, $c_{\frac{n}{m}} = 1$, $c_{\frac{k}{m}} = c \ne 0$. If (107) does not hold, then the m-th root ω' of unity in (104) is not necessarily equal to 1 [if we denote the left-hand side of (107) as d then the proof of Lemma 2.17 (v) shows that $\omega'^d = 1$], and we can easily construct a counter example like this: $(F, G) = (y^6, y^4 + xy^{-5})$; in this case, we can choose, say, $(p_{1i}, p_{2i}) = \big((x_{1i}, y_{1i}), (x_{2i}, y_{2i})\big) = \big((1, i), (-1, -i)\big) \in V$ for $i \in \mathbb{Z}_{>0}$ such that $\omega' = -1$ and Theorem 1.2 does not hold.

2.3. *Proof of Theorem 1.3 provided by Claudio Procesi*

We thank Professor Claudio Procesi for providing us the following simple proof by using a little geometry (for your reference, we also present our original proof of surjectivity of π_1 without using geometry in Appendix A).

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 6 September 2023 doi:10.20944/preprints202212.0171.v2

Lemma 2.20. *The set V defined in (3) is a smooth and closed algebraic surface in $\mathbb{C}^4 = \mathbb{C}^2 \times \mathbb{C}^2$.*

Proof. The map $\phi : \mathbb{C}^4 \to \mathbb{C}^2$,

$$\phi : (p_1, p_2) \mapsto (F(p_1) - F(p_2), G(p_1) - G(p_2)),$$

has Jacobian matrix always of maximal rank so that $\phi^{-1}(0,0) = V \cup D$ is a smooth closed surface where D is the diagonal which is closed. By any argument (the implicit functions theorem, or smoothness) D is also open in $\phi^{-1}(0,0)$ and the lemma follows. □

We need a simple estimate.

Lemma 2.21. *Let $a, b \in \mathbb{R}$, $a, b > 2$. If $b < 2(a+b)^{\frac{m}{m+1}}$ then $b < 2(2a)^m$.*

Proof. We have $ab - a - b = (\frac{a}{2} - 1)b + (\frac{b}{2} - 1)a > 0$, so

$$b < 2(a+b)^{\frac{m}{m+1}} \implies b^{m+1} < 2^{m+1}(a+b)^m < 2^{m+1}(ab)^m \implies b < 2(2a)^m. \qquad \square$$

We apply this to $a = |x_1| + |x_2|$, $b = |y_1| + |y_2|$ with $((x_1, y_1), (x_2, y_2)) \in V$ and to the projection π_1 [defined in (5)] and deduce the following.

Proposition 2.22. *If $A \subset \mathbb{C}^2$ is bounded then also $\pi_1^{-1}(A) \subset V$ is bounded.*

Proof. This follows from Lemma 2.21, and Proposition 2.15. More precisely, fix any $s_8 > s_7$ (which is the number in Proposition 2.15) such that for all $(x_1, x_2) \in A$,

$$a := |x_1| + |x_2| < s_8. \tag{108}$$

Let $(p_1, p_2) = ((x_1, y_1), (x_2, y_2)) \in \pi_1^{-1}(A)$. We claim

$$b := |y_1| + |y_2| < 2(2s_8)^m. \tag{109}$$

Assume conversely that $b \geq 2(2s_8)^m$. Then $h_{p_1,p_2} \overset{(4)}{\geq} b \geq 2(2s_8)^m > s_7$. Thus we can apply Proposition 2.15 to obtain,

$$b = |y_1| + |y_2| < 2h_{p_1,p_2}^{\frac{m}{m+1}} \overset{(4),(108),(109)}{=} 2(a+b)^{\frac{m}{m+1}}. \tag{110}$$

Then by Lemma 2.21, $b < 2(2a)^m \overset{(108)}{<} 2(2s_8)^m$, which is a contradiction with the assumption. Thus we have (109) and the proposition. □

As a consequence we have that the fibres of π_1, being compact affine algebraic varieties, are finite hence the closure, denoted $\overline{\pi_1(V)}$, of $\pi_1(V)$ is a 2-dimensional subvariety of $\mathbb{C}^2$ so it is $\mathbb{C}^2$.

Proof of Theorem 1.3. Now we can proceed the proof of Theorem 1.3 as follows. We already see that π_1 is proper and finite. Now take any point $q \in \mathbb{C}^2 = \overline{\pi_1(V)}$. By the previous discussion there is a sequence of points $\pi_1(p_i)$ for $p_i \in V$ with $\lim_{i\to\infty} \pi_1(p_i) = q$. But the sequence p_i, by the previous proposition, is bounded and V is closed so we can extract a subsequence converging to some $p \in V$ and $\pi_1(p) = q$. □

3. Proof of Theorem 1.1

First we recall after the proof of Lemma 2.9 that we have fixed, once and for all, the Jacobian pair (F, G) satisfying (37) and Lemma 2.9.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 6 September 2023 doi:10.20944/preprints202212.0171.v2

The main purpose of this section is to prove that the projection $\pi_1 : V \to \mathbb{C}^2$ defined in (5) is in fact not surjective (which contradicts Theorem 1.3 and thus proves Theorem 1.1). To help understanding, we first give some explanations.

Throughout the whole section, (though V may have some other properties) we only need to use the following three properties (C1)–(C3) satisfied by V [especially, (C2) will be frequently used], which do not necessarily depend on the fact that V arises from the Jacobian problem. As in (3), an element in V is usually denoted as $(p_1, p_2) = \big((x_1, y_1), (x_2, y_2)\big)$.

(C1) V is a nonempty closed subset of $\mathbb{C}^4$ (by Lemma 2.20);
(C2) $|y_1| + |y_2| = o(h_{p_1,p_2})$ when $h_{p_1,p_2} := |x_1| + |y_1| + |x_2| + |y_2| \to \infty$ (by Theorem 1.2);
(C3) for each $(\tilde p_1, \tilde p_2) \in V$, there exists a neighborhood $\mathcal{O}_{\tilde p_1,\tilde p_2} \subset \mathbb{C}^4$ of $(\tilde p_1, \tilde p_2)$ such that for all $(p_1, p_2) = ((x_1, y_1), (x_2, y_2)) \in V \cap \mathcal{O}_{\tilde p_1,\tilde p_2}$, x_1, y_1 are holomorphic functions of x_2, y_2 (by the local bijectivity of Keller maps).

Note that the local bijectivity of Keller maps also implies that V satisfies,

(C4) there exists a non-constant polynomial $\theta(x_1, y_1, x_2, y_2)$ such that $V \not\subset S_\theta$, where S_θ is the set of zeros of the polynomial θ,

$$S_\theta = \{((\tilde x_1, \tilde y_1), (\tilde x_2, \tilde y_2)) \in \mathbb{C}^4 \,|\, \theta(\tilde x_1, \tilde y_1, \tilde x_2, \tilde y_2) = 0\},$$

and further, for any $(\tilde p_1, \tilde p_2) \in V \backslash S_\theta$, there is some neighborhood $\mathcal{O}_{\tilde p_1,\tilde p_2} \subset \mathbb{C}^4$ of $(\tilde p_1, \tilde p_2)$ such that for all $(p_1, p_2) = ((x_1, y_1), (x_2, y_2)) \in V \cap \mathcal{O}_{\tilde p_1,\tilde p_2}$, x_1, x_2 are holomorphic functions of y_1, y_2.

Remark 3.1. We would like to mention that if we change (C4) to the following slightly stronger form (C4)′, then the proof (attached in Appendix C) becomes very easy by using some geometry [since we already have (C4), if we change (C4)′ to that the Jacobian matrix of π_1 at any $(\tilde p_1, \tilde p_2) \in V \cap S_\theta$ is of maximal rank, then we also get that π_1 is not surjective]:

(C4)′ for any $(\tilde p_1, \tilde p_2) \in V$, there is some neighborhood $\mathcal{O}_{\tilde p_1,\tilde p_2} \subset \mathbb{C}^4$ of $(\tilde p_1, \tilde p_2)$ such that for all $(p_1, p_2) = ((x_1, y_1), (x_2, y_2)) \in V \cap \mathcal{O}_{\tilde p_1,\tilde p_2}$, x_1, x_2 are holomorphic functions of y_1, y_2.

Thus (C4) means that (C4)′ holds for all elements in V except the zeros of θ. Because of Remark 3.1, we do think that it is very reasonable to have that π_1 is not surjective. However since we only have (C4) instead of (C4)′, we cannot treat V in a uniform way. This is why the proof in this section looks so complicated and why we have to consider case by case. As a matter of fact, in this section, we will try to avoid choosing elements of V which are in S_θ [for example the element $(\bar p_1, \bar p_2) \in A_{\boldsymbol{k},\boldsymbol{k}}$ satisfying (115) is an element not in S_θ when $\boldsymbol{k} \gg 1$].

3.1. *Two propositions*

We want to build a proposition, namely, Proposition 3.4, which will immediately imply another proposition, namely, Proposition 3.8, that will give us a contradiction (thus proving Theorem 1.1).

First we need to introduce some notations. For any $k_1, k_2 \in \mathbb{R}_{\geq 0}$, by Theorem 1.3, the following is a nonempty compact subset of V,

$$A_{k_1,k_2} = \big\{(p_1, p_2) = \big((x_1, y_1), (x_2, y_2)\big) \in V \,\big|\, |x_1| = k_1,\ |x_2| = k_2\big\} \neq \emptyset. \tag{111}$$

Thus the following is a well-defined function of $k_1, k_2 \in \mathbb{R}_{\geq 0}$,

$$\Upsilon_{k_1,k_2} = \max\big\{|x_2 + y_2| \,\big|\, (p_1, p_2) = \big((x_1, y_1), (x_2, y_2)\big) \in A_{k_1,k_2}\big\}. \tag{112}$$

We fix some choices of positive numbers satisfying,

$$1 \ll \ell_1 := \delta_1^{-1} \ll \ell := \delta^{-1} \ll \boldsymbol{k} \ll \boldsymbol{n} := \boldsymbol{e}^{-1} \ll \boldsymbol{n}_1 = \boldsymbol{e}_1^{-1} \ll \varepsilon^{-1}. \tag{113}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 6 September 2023 doi:10.20944/preprints202212.0171.v2

Remark 3.2. (i) The choices may depend on the situation we encounter, and further, the choice of a number listed later may depend on all choices of numbers listed earlier; for instance, we may require that $\ell \gg \ell_1^{\ell_1}$, $\boldsymbol{k} \gg \ell^{\ell}$, etc.

(ii) We need to use the following convention: we can regard elements in (113) as parameters to be fixed upon our requirement in the situation we encounter. Sometimes we need to compare some number in (113) with other numbers, in this case, for convenience we may regard some element in (113) as a variable; for instance, if we regard $\boldsymbol{k}$ as a variable, then we need to regard ℓ_1, ℓ as fixed numbers and $\boldsymbol{n}, \boldsymbol{n}_1, \mathcal{E}$ as variables such that we have [cf. (91)],

$$1 \sim_{\boldsymbol{k}} \ell_1 \sim_{\boldsymbol{k}} \ell \prec_{\boldsymbol{k}} \boldsymbol{k} \prec_{\boldsymbol{k}} \boldsymbol{n} \prec_{\boldsymbol{k}} \boldsymbol{n}_1 \prec_{\boldsymbol{k}} \mathcal{E}^{-1}. \tag{114}$$

We fix, once and for all, an element $(\bar p_1, \bar p_2) = \big((\bar x_1, \bar y_1), (\bar x_2, \bar y_2)\big) \in A_{\boldsymbol{k},\boldsymbol{k}}$ satisfying

$$|\bar x_1| = |\bar x_2| = \boldsymbol{k}, \quad |\bar x_2 + \bar y_2| = \Upsilon_{\boldsymbol{k},\boldsymbol{k}}. \tag{115}$$

For any $x_1, x_2, y_1, y_2 \in \mathbb{C}$, we denote,

$$\text{(i) } X_1 = \frac{x_1}{\bar x_1}, \quad \text{(ii) } X_2 = \frac{x_2}{\bar x_2}, \quad \text{(iii) } Z = \frac{x_2 + y_2}{\bar x_2 + \bar y_2}. \tag{116}$$

The reason we define (116) (i)–(iii) is in order that when $(p_1, p_2) = \big((x_1, y_1), (x_2, y_2)\big)$ is close to $(\bar p_1, \bar p_2)$ we have that X_1, X_2, Z are close to 1, which will make our arguments easier.

Definition 3.3. Let $a_{\boldsymbol{k}}, b_{\boldsymbol{k}} > 0$ be the numbers defined in (172). Let $\boldsymbol{l}$ be the small integer bigger than $200(a_{\boldsymbol{k}} + b_{\boldsymbol{k}})$, i.e.,

$$\boldsymbol{l} = 101(a_{\boldsymbol{k}} + b_{\boldsymbol{k}}) + \lambda_0 \in \mathbb{Z}_{>0} \text{ with } 0 < \lambda_0 \le 1, \quad \text{and denote} \quad \boldsymbol{d} = \boldsymbol{l}^{-1}. \tag{117}$$

We will see in (197) that $\boldsymbol{l} \gg 1$. We define,

$$\begin{aligned}
&\text{(i) } A_3 = \alpha_1 Z^{16}, \ \alpha_1 = 1 + \beta_1 \mathcal{E}, \ \beta_1 = \boldsymbol{l}^{11}(1 - 850\boldsymbol{d}), \quad \text{(ii) } A_2 = \frac{\alpha_1^4 A_3^{200} X_2^{10} Z^{10}}{\widetilde X_1^2}, \\
&\text{(iii) } \widetilde X_1 = \frac{\alpha_0 X_1^{100}}{X_2^{20}}, \ \alpha_0 = 1 + \beta_0 \mathcal{E}, \ \beta_0 = \boldsymbol{l}^{10}(a_{\boldsymbol{k}} + b_{\boldsymbol{k}} - 20 + \lambda_0) > 0, \\
&\text{(iv) } A_1 = \frac{\alpha_1^{101} A_3^{101\boldsymbol{l}(1-20\boldsymbol{d})} \widetilde X_1^{2828}}{Z^{101}} \Big(\frac{1}{5} + \frac{4 A_2^{\boldsymbol{l}-1}}{5 A_3^{202\boldsymbol{l}(1-10\boldsymbol{d})} \widetilde X_1^{3636}} \Big), \\
&\text{(v) } B_1 = \frac{A_1 A_3^{101\boldsymbol{l}} \widetilde X_1^{808} Z^{101}}{\alpha_1^{101} A_2^{\boldsymbol{l}-1}}, \quad \text{(vi) } C_1 = \frac{\alpha_1^{202} A_2^{\boldsymbol{l}-1} \widetilde X_1^{2020}}{A_1^2 A_3^{2020} Z^{202}}.
\end{aligned} \tag{118}$$

We will give some explanations after stating our first proposition below.

Proposition 3.4. *There exist some $\kappa_i \in \mathbb{R}_{>0}$ and some $\eta_i \in \mathbb{R}_{\ne 0}$ such that we have the following.*

Denote by V_1 the subset of V consisting of all elements $(p_1, p_2) = \big((x_1, y_1), (x_2, y_2)\big)$ such that its coordinates x_1, x_2, y_2 satisfy,

$$\begin{aligned}
&\text{(a) } 1 < (\kappa_1 |x_2|)^{\eta_1} \le \kappa_2 |x_1|^{\eta_2} \le (\kappa_1 |x_2|)^{\eta_3} < \kappa_3, \\
&\text{(b) } \ell_{p_1,p_2} := \kappa_4 |x_2^{\eta_4}(x_2 + y_2)| + |x_2| + |x_2 + y_2| \ge \kappa_5.
\end{aligned} \tag{119}$$

Denote by V_2 the subset of S_2 consisting of all elements $(p_1, p_2) = ((x_1, y_1), (x_2, y_2))$ such that its coordinates x_1, x_2, y_2 satisfy,

$$\text{(a) } 1 < |A_1|^{\kappa_1} \le |A_2|^{\kappa_2} \le |A_1|^{\kappa_3} < \kappa_4, \quad \text{(b) } \ell_{p_1,p_2} := |A_3 A_1^{\eta_1}| + |x_2| + |x_2 + y_2| \ge \kappa_5,$$
$$\text{(c) } \frac{|B_1|}{1+\boldsymbol{e}_1} < 1 < \frac{1+\boldsymbol{e}^2}{|C_1|}, \quad \text{(d) } \boldsymbol{e}_1|\widetilde{X}_1| < 1 < \frac{\boldsymbol{n}_1}{|X_2|}. \tag{120}$$

Then there exists V_0 which is either equal to V_1 or else equal to V_2 such that V_0 is a nonempty compact subset of $\mathbb{C}^4$, and further when $(p_1, p_2) \in V_0$, we have

$$x_1, x_2, x_2 + y_2 \ne 0, \text{ and } A_3 \ne 0 \text{ in case } V_0 = V_2. \tag{121}$$

Remark 3.5. Denote $\overline{V}_i$ the closure of V_i in $\mathbb{C}^4$ for $i = 1, 2$. Then $\overline{V}_i \subset V$ by (2.20) and obviously by (119), (120), coordinates of elements in them satisfy respectively,

$$\begin{aligned} &\text{(a) } 1 \le (\kappa_1|x_2|)^{\eta_1} \le \kappa_2|x_1|^{\eta_2} \le (\kappa_1|x_2|)^{\eta_3} \le \kappa_3, \\ &\text{(b) } \ell_{p_1,p_2} := \kappa_4|x_2^{\eta_4}(x_2 + y_2)| + |x_2| + |x_2 + y_2| \ge \kappa_5, \end{aligned} \tag{122}$$

and

$$\text{(a) } 1 \le |A_1|^{\kappa_1} \le |A_2|^{\kappa_2} \le |A_1|^{\kappa_3} \le \kappa_4, \quad \text{(b) } \ell_{p_1,p_2} := |A_3 A_1^{\eta_1}| + |x_2| + |x_2 + y_2| \ge \kappa_5,$$
$$\text{(c) } \frac{|B_1|}{1+\boldsymbol{e}_1} \le 1 \le \frac{1+\boldsymbol{e}^2}{|C_1|}, \quad \text{(d) } \boldsymbol{e}_1|\widetilde{X}_1| \le 1 \le \frac{\boldsymbol{n}_1}{|X_2|}. \tag{123}$$

Remark 3.6. (i) We emphasis that the last two terms of ℓ_{p_1,p_2} in (119) (b) or (120) (b) play extremely important roles which guarantee the inequation (125) has solution (q_1, q_2) as required [cf. (135)].

(ii) When we prove Proposition 3.4, the first thing we need to do is to prove $V_0 \ne \emptyset$, which will be done by choosing some suitable element $(p_1, p_2) \in V_0$. We regard such an element as the "initial stage". We will see from our arguments in this section that we observe the following important fact,

Fact 3.7. *The "initial stage" controls globally the growths of $|x_2|$, $|x_2 + y_2|$ on the whole set V_0.*

(iii) We use the strange way in (118) to define A_1, A_2 in order to satisfy Proposition 3.4. More precisely,

(a) Firstly, we will see in the proof of Lemma 3.24 that powers that appear in (118) (i), (iv) play key roles.

(b) Secondly, we need to prove in Lemma 3.21 that $V_2 \ne 0$ by choosing the "initial stage" (p_1, p_2), sufficiently close to $(\bar{p}_1, \bar{p}_2)$, satisfying (120), such that all elements in (118) are $1 + O(\varepsilon)^1$ elements and X_2, Z are of the following form (where $\varepsilon > 0$ is sufficiently small),

$$\text{(i) } X_2 = 1 - \boldsymbol{l}^{10}\varepsilon, \qquad \text{(ii) } Z = 1 + \boldsymbol{l}^{10}\varepsilon. \tag{124}$$

(iv) We would like to point out that because of our choice of the "initial stage" in (124) that $|Z|$ is larger than $|X_2|$, which is the key points to obtain a contradiction when the last inequality of (123) (a) becomes equality as there will exist an inconsistence in (123).

(v) Note that we do not have any information about y_1 in (119), (120). This is because we cannot put too many conditions in the definition of V_0. We can only use Theorem 1.2 to control $|y_1|$ as long as we have some control on $|x_1|, |x_2|$.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 6 September 2023 doi:10.20944/preprints202212.0171.v2

Proposition 3.8. *For any $(\check{p}_1, \check{p}_2) \in V_0$, in every neighborhood of $(\check{p}_1, \check{p}_2)$ there exists $(q_1, q_2) = ((\dot{x}_1, \dot{y}_1), (\dot{x}_2, \dot{y}_2)) \in V_0$ sufficiently close to $(\check{p}_1, \check{p}_2)$ such that*

$$\ell_{q_1,q_2} > \ell_{\check{p}_1,\check{p}_2}. \tag{125}$$

We will first use Proposition 3.4 to give a proof of Proposition 3.8.

3.2. Proof of Proposition 3.8 provided by Claudio Procesi and proof of Theorem 1.1

We thank Professor Claudio Procesi for providing us the material after (134) until the end of proof of Proposition 3.8 (for your reference we also present our original arguments in Appendix B).

Proof of Proposition 3.8. Let $(\check{p}_1, \check{p}_2) = ((\check{x}_1, \check{y}_1), (\check{x}_2, \check{y}_2)) \in V_0$. Note that (121) implies that $\check{x}_1, \check{x}_2, \check{x}_2 + \check{y}_2 \neq 0$. We need to find an element $(q_1, q_2) \in V_0$ which is sufficiently close to $(\check{p}_1, \check{p}_2)$. Thus we can assume $q_1 = (\dot{x}_1, \dot{y}_1)$, $q_2 = (\dot{x}_2, \dot{y}_2)$ such that their coordinates have the following forms, for some sufficiently small $\varepsilon_1 > 0$, and $u, v, s, t \in \mathbb{C}$ (for later convenient use, it is no harm to write the coordinate of $\dot{x}_2 + \dot{y}_2$ instead of $\dot{y}_2$),

$$\dot{x}_1 = \check{x}_1(1 + s\varepsilon_1), \;\; \dot{y}_1 = \check{y}_1 + t\varepsilon_1, \;\; \dot{x}_2 = \check{x}_2(1 + u\varepsilon_1), \;\; \dot{x}_2 + \dot{y}_2 = (\check{x}_2 + \check{y}_2)(1 + v\varepsilon_1). \tag{126}$$

The local bijectivity of Keller maps says that for any $u, v \in \mathbb{C}$ in a bounded set there always exists sufficiently small $\varepsilon_1 > 0$ such that, there exist $s, t \in \mathbb{C}$ with $(q_1, q_2) \in V$, namely, $q_1 \neq q_2$ [which is guaranteed by the fact that (q_1, q_2) is sufficiently close to $(\check{p}_1, \check{p}_2)$ and $\check{p}_1 \neq \check{p}_2$] and

$$(F(q_1), G(q_1)) = (F(q_2), G(q_2)). \tag{127}$$

Since F, G are polynomials, we can always solve s (throughout the paper we do not need to use t so we omit the solution of t) from (126), (127) to obtain that s is a power series of ε_1 such that each coefficient of ε_1^k is a homogenous polynomial of u, v with degree $k + 1$ for all $k \geq 0$; in particular, we can write s as the following form, for some $a, b, \tilde{a}, \tilde{b}, \tilde{c} \in \mathbb{C}$,

$$s = s_1 + (\tilde{a}u^2 + \tilde{b}uv + \tilde{c}v^2)\varepsilon_1 + O(\varepsilon_1)^2, \quad s_1 = au + bv, \quad (a, b) \neq (0, 0). \tag{128}$$

First note that since (q_1, q_2) is sufficiently close to $(\check{p}_1, \check{p}_2)$, we see that (120) (c), (d) are automatically satisfied by (q_1, q_2) in case $V_0 = V_2$. Further when (125) holds for (q_1, q_2), we always have

$$\ell_{q_1,q_2} > \ell_{\check{p}_1,\check{p}_2} \overset{(119)\,(b),\,(120)\,(b)}{\geq} \kappa_5, \tag{129}$$

i.e., (119) (b) [or respectively (120) (b)] automatically holds for (q_1, q_2). Thus in order for (q_1, q_2) to be in V_0 and to satisfy (125), we only need to require (119) (a) [respectively (120) (a)] and (125) to be satisfied by (q_1, q_2).

Observe from the first strict inequality of (119) (a) or (120) (a) that two equalities cannot simultaneously occur in the second and third inequalities of (119) (a) or (120) (a). Therefore, we only need to consider the following two possible cases.

Case 1: Assume that, for $(\check{p}_1, \check{p}_2)$, all inequalities of (119) (a) [*or respectively* (120) (a)] *are strict inequalities.*

Then the same must hold for (q_1, q_2) (since ε_1 is infinitesimal), i.e., (q_1, q_2) is automatically in V_0 if (125) holds. Thus in this case we only need to consider one inequation, i.e., (125). Therefore we see that this case is easier than the case we encounter below (as we always have no need to worry about E_1 appeared below).

Case 2: Assume that, for $(\check{p}_1, \check{p}_2)$, equality occurs in either the second or else the third inequality of (119) (a) [*or respectively* (120) (a)].

Accordingly, we need to consider two inequations for (q_1, q_2): one is the inequation for (q_1, q_2) corresponding to the part of (119) (a) [or respectively (120) (a)], where equality occurs for $(\check{p}_1, \check{p}_2)$; another is (125).

First assume we have case (119) and the second equality of (119) (a) holds for $(\check{p}_1, \check{p}_2)$ (the case for the third equality is similar), i.e., $\kappa_2\kappa_1^{-\eta_1}|\check{x}_1^{\eta_2}\check{x}_2^{-\eta_1}| = 1$. Then the two inequations we need to consider for (q_1, q_2) are

$$(\kappa_1|\dot{x}_2|)^{\eta_1} \le \kappa_2|\dot{x}_1|^{\eta_2} \text{ and } \ell_{q_1,q_2} := \kappa_4|\dot{x}_2|^{\eta_4}|\dot{x}_2 + \dot{y}_2| + |\dot{x}_2| + |\dot{x}_2 + \dot{y}_2| > \ell_{\check{p}_1,\check{p}_2}, \tag{130}$$

which, by (119) (b), (126), can be rewritten as the following forms,

$$\begin{aligned}
&E_1 := \kappa_2\kappa_1^{-\eta_1}|\dot{x}_1^{\eta_2}\dot{x}_2^{-\eta_1}| - 1 \overset{(126)}{=} |(1+s\varepsilon_1)^{\eta_2}(1+u\varepsilon_1)^{-\eta_1}| - 1 \ge 0,\\
&E_2 := (\ell_{q_1,q_2} - \ell_{\check{p}_1,\check{p}_2})|\check{x}_2|^{-1}\\
&\overset{(119)\,(b)}{=} \kappa_4|\dot{x}_2^{\eta_4}(\dot{x}_2+\dot{y}_2)\check{x}_2^{-1}| + |\dot{x}_2\check{x}_2^{-1}| + |(\dot{x}_2+\dot{y}_2)\check{x}_2^{-1}| - \Big(\kappa_5|\check{x}_2^{\eta_4}(\check{x}_2+\check{y}_2)\check{x}_2^{-1}| + 1 + |(\check{x}_2+\check{y}_2)\check{x}_2^{-1}|\Big)\\
&\overset{(126)}{=} \kappa_1'|(1+u\varepsilon_1)^{\eta_4}(1+v\varepsilon_1)| + |1+u\varepsilon_1| + \kappa_2'|1+v\varepsilon_1| - (\kappa_1' + 1 + \kappa_2') > 0,
\end{aligned} \tag{131}$$

where, $\kappa_1' = \kappa_4|\check{x}_2^{\eta_4}(\check{x}_2+\check{y}_2)\check{x}_2^{-1}|$, $\kappa_2' = |(\check{x}_2+\check{y}_2)\check{x}_2^{-1}|$.

Using (128), and expanding each element inside the absolute value symbols in E_1 and E_2 in (131) as a power series of ε_1 (such that every coefficient of ε_1^k is a homogenous polynomial of u, v with degree k for $k \ge 1$), we see that (131) can be always rewritten as the following forms, where $f(u\varepsilon_1, v\varepsilon_1)$, $g(u\varepsilon_1, v\varepsilon_1)$ are two holomorphic functions of u, v (when ε_1 is regarded as fixed) in some neighbourhood of $(0, 0)$ and vanishing at $(0, 0)$,

$$\begin{aligned}
&\text{(i) } E_1 := \big|1 + f(u\varepsilon_1, v\varepsilon_1)\big| - 1 \ge 0,\\
&\text{(ii) } E_2 := \kappa_1'\big|1 + g(u\varepsilon_1, v\varepsilon_1)\big| + |1+u\varepsilon_1| + \kappa_2'|1+v\varepsilon_1| - (\kappa_1' + 1 + \kappa_2') > 0.
\end{aligned} \tag{132}$$

For the case (120), since $A_i \ne 0$ for $i = 1, 2, 3$ by (120), (121), and $\varepsilon_1 > 0$ is sufficiently small, we can always write $A_i|_{(p_1,p_2)=(q_1,q_2)}$ as, for some $\alpha_{1,i}, \alpha_{2,i} \in \mathbb{C}$,

$$A_i|_{(p_1,p_2)=(q_1,q_2)} = A_i|_{(p_1,p_2)=(\check{p}_1,\check{p}_2)}\Big(1 + (\alpha_{1,i}u + \alpha_{2,i}v)\varepsilon_1 + O(\varepsilon_1)^2\Big). \tag{133}$$

Thus we can also write the two inequations we need to consider as the forms in (132).

First observe that for any $\alpha = \alpha_{\rm re} + \alpha_{\rm im}\boldsymbol{i} \in \mathbb{C}$ [recall Convention 2.3 (1)], we have,

$$|1 + \alpha\varepsilon_1| = \sqrt{(1+\alpha_{\rm re}\varepsilon_1)^2 + (a_{\rm im}\varepsilon_1)^2} = 1 + \alpha_{\rm re}\varepsilon_1 + \frac{(\alpha_{\rm im})^2}{2}\varepsilon_1^2 + O(\varepsilon_1)^3. \tag{134}$$

Therefore the coefficient of the ε_1 term of E_i in (132) has the form $\ell_i(u, v)_{\rm re}$ for some linear homogeneous form $\ell_i(u, v)$ in u, v for $i = 1, 2$. If these two forms are linearly independent of course we can choose u, v so that the two forms take positive real values (and the proof of Proposition 3.8 is then completed), otherwise there exist two complex numbers α, β not both zero so that setting $u := \alpha z$, $v := \beta z$ we can have that $\ell_i(\alpha z, \beta z) = 0$, $i = 1, 2$.

Therefore we consider the later case and assume for instance that α is nonzero and set $w := \alpha z$. Then by (134) the ε_1^2 coefficient in (132) (ii) is of the form $A(w)$ below for some $\gamma \in \mathbb{C}$ [as mentioned in Remark 3.6 (i), here is the reason why we need the last two terms of ℓ_{p_1,p_2}, which guarantee that we

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 6 September 2023 doi:10.20944/preprints202212.0171.v2

always have the last two non-negative terms of $A(w)$ and at least one of both is positive (noting that in general it is possible that $\alpha \neq 0,\ \beta = 0$ or $\alpha = 0,\ \beta \neq 0$)],

$$E_2 = A(w)\varepsilon_1^2 + O(\varepsilon_1)^3, \quad A(w) := (\gamma w^2)_{\rm re} + \frac{(w_{\rm im})^2}{2} + \kappa_2' \frac{\big((\beta\alpha^{-1}w)_{\rm im}\big)^2}{2}. \tag{135}$$

Further E_1 is either zero or of the form $(cw^k)_{\rm re}\varepsilon_1^k + O(\varepsilon_1)^{k+1}$ for some $c \in \mathbb{C}_{\neq 0}$, $k \geq 2$. Assume we have the later case as otherwise $E_1 \geq 0$ holds trivially. Namely,

$$E_1 = (cw^k)_{\rm re}\varepsilon_1^k + O(\varepsilon_1)^{k+1}. \tag{136}$$

If $\gamma = 0$ then we can easily choose $w \in \mathbb{C}$ with $w_{\rm im} \neq 0$ and $(cw^k)_{\rm re} > 0$ so that $E_1 > 0$ and $E_2 > 0$ by (135) and (136).

Thus assume $\gamma \neq 0$. We may take $w \in \mathbb{C}$ with $|w| = 1$, $w = e^{\theta i}$ for $\theta \in \mathbb{R}$ with $0 \leq \theta < 2\pi$.

Notice now that for any integer $h_1 \geq 1$ and any nonzero constant complex number h_2 the two sets where $(h_2 w^{h_1})_{\rm re} > 0$ respectively $(h_2 w^{h_1})_{\rm im} > 0$ are two open sets of the circle $|w| = 1$ of arclength π.

Lemma 3.9. *The set where $A(w) > 0$ is an open set of the circle $|w| = 1$ of arclength $\geq \pi + \delta$ for some $\delta > 0$ so there is a nonempty open set of the circle $|w| = 1$ of arclength $\geq \delta$ where both inequalities $A(w) > 0$ and $(cw^k)_{\rm re} > 0$ hold.*

Proof. The set where $(\gamma w^2)_{\rm re} \leq 0$ is formed by two opposite arcs of total arclength $\frac{\pi}{2}$ so at least two of the 4 endpoints of these arcs, where $(\gamma w^2)_{\rm re} = 0$, are different from ± 1 where $w_{\rm im} = 0$, thus in a neighbourhood of these two points $(\gamma w^2)_{\rm re} + \frac{(w_{\rm im})^2}{2} > 0$. □

Note that we can always choose sufficiently small $\varepsilon_1 > 0$ such that $0 < \varepsilon_1 \ll \delta$. This proves Proposition 3.8. □

Proof of Theorem 1.1. Now we use Proposition 3.8 to prove Theorem 1.1. Proposition 3.8 (which says that the continuous function ℓ_{p_1,p_2} does not have the maximal value on the nonempty compact set V_0) immediately gives a contradiction, which proves the first statement of Theorem 1.1. The second statement follows from [2,4]. □

3.3. Proof of Proposition 3.4

It remains to prove Proposition 3.4. We would like to mention that because we require V_0 to be closed, it seems to us that in general there is no way to define a system of inequations (119) or (120) satisfying that V_0 is closed. We need some "extra fact". By considering all possible different cases, each of which provides us some different "extra fact", we will be able to achieve the goal.

Before proceeding our proof of Proposition 3.4, let us give some further explanations. We aim to construct a nonempty subset V_0 satisfying, say for example, (119). To do this, usually we can start with some given element

$$(\tilde{p}_1, \tilde{p}_2) = \big((\tilde{x}_1, \tilde{y}_1), (\tilde{x}_2, \tilde{y}_2)\big) \in V,$$

and of course we may choose some different $(\tilde{p}_1, \tilde{p}_2)$ for a different case. Look at the condition (119): if, for example, we take

$$\kappa_0 = 1, \quad \kappa_1 = |\tilde{x}_2|^{-1}, \quad \kappa_2 = |\tilde{x}_1|^{-\eta_2},$$

then (122) (a) can be satisfied by $(\tilde{p}_1, \tilde{p}_2)$ by choosing some suitable $\kappa_3 > 1$.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 6 September 2023 doi:10.20944/preprints202212.0171.v2

We always need to choose κ_5 in (119) (b) to be as big as possible otherwise V_0 can possibly contain some extra elements which we may be unable to control somehow. In order to do this, we should choose $(\tilde p_1, \tilde p_2) \in V$ such that $|\tilde x_2 + \tilde y_2|$ is maximal (i.e., $\Upsilon_{|\tilde x_1|,|\tilde x_2|}$). In this case we can choose κ_5 to be

$$\beta_0 := \kappa_4 |\tilde x_2|^{\eta_4} \Upsilon_{|\tilde x_1|,|\tilde x_2|} + |\tilde x_2| + \Upsilon_{|\tilde x_1|,|\tilde x_2|}.$$

In this way, we may have that $(\tilde p_1, \tilde p_2) \in \overline{V}_0$ (the closure of V_0). However then, equality can occur in the first inequality of (122) (a) and thus (119) (a) is not satisfied [i.e., $(\tilde p_1, \tilde p_2) \notin V_0$]. To avoid this, we have to enlarge κ_5 [this is why we choose α in (143) (b), respectively (159) (b), to be as in (142), respectively (158)]. Then $(\tilde p_1, \tilde p_2)$ cannot be in V_0. This means that we cannot choose the given element $(\tilde p_1, \tilde p_2)$ as the "initial stage". We have to find some other element as the "initial stage", which is the reason why we require some "extra fact".

First we need a lemma.

Lemma 3.10. *The Υ_{k_1,k_2} defined in (112) is a strictly increasing function of $k_1 \in \mathbb{R}_{\ge 0}$ when $k_2 \in \mathbb{R}_{\ge 0}$ is fixed, i.e.,*

$$\Upsilon_{k_1',k_2} > \Upsilon_{k_1,k_2} \quad \text{if } k_1' > k_1 \ge 0,\ k_2 \ge 0. \tag{137}$$

Consequently, $\Upsilon_{k_1,k_2} > 0$ for any $k_1 \in \mathbb{R}_{>0}$, $k_2 \in \mathbb{R}_{\ge 0}$.

Proof. For any $k_1' > 0$, let

$$\begin{aligned} \beta &:= \max\{\Upsilon_{k_1,k_2} \,|\, 0 \le k_1 \le k_1'\} \\ &\overset{(112)}{=} \max\{\, |x_2 + y_2| \,\big|\, (p_1, p_2) = \big((x_1, y_1), (x_2, y_2)\big) \in V,\ 0 \le |x_1| \le k_1',\ |x_2| = k_2 \,\} \ge 0. \end{aligned} \tag{138}$$

Assume conversely that there exists $k_1 < k_1'$ with $k_1 \ge 0$ such that $\Upsilon_{k_1,k_2} = \beta$. We will use the local bijectivity of Keller maps to obtain a contradiction. Let

$$(\tilde p_1, \tilde p_2) = \big((\tilde x_1, \tilde y_1), (\tilde x_2, \tilde y_2)\big) \in V \quad \text{with} \quad |\tilde x_1| = k_1,\ |\tilde x_2| = k_2,\ |\tilde x_2 + \tilde y_2| = \beta. \tag{139}$$

Using x_2, y_2 as local coordinates in V we choose $(q_1, q_2) \in V$ sufficiently close to $(\tilde p_1, \tilde p_2)$, satisfying, for some $v \in \mathbb{C}$ with $|v|$ being sufficiently small,

$$q_1 = (\dot x_1, \dot y_1),\ \ q_2 := (\dot x_2, \dot y_2) = (\tilde x_2, \tilde y_2 + v), \tag{140}$$

such that

$$|\dot x_2 + \dot y_2| = |\tilde x_2 + \tilde y_2 + v| > |\tilde x_2 + \tilde y_2|. \tag{141}$$

Since $|\tilde x_1| = k_1 < k_1'$, and locally x_1 is a holomorphic, hence a continuous function of (x_2, y_2), we have $|\dot x_1| < k_1'$ when $|v| > 0$ is sufficiently small. This means that we can choose $(q_1, q_2) \in V$ with

$$|\dot x_1| < k_1', \quad |\dot x_2| = k_2, \ \text{ but } \ |\dot x_2 + \dot y_2| > \beta,$$

which is a contradiction with the definition of β in (138). This proves the first assertion of the lemma. Then for any $k_1 > 0$, $k_2 \ge 0$, we have

$$\Upsilon_{k_1,k_2} \overset{(137)}{>} \Upsilon_{\frac{k_1}{2},k_2} \ge 0,$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 6 September 2023 doi:10.20944/preprints202212.0171.v2

proving the second. □

Proof of Proposition 3.4. Now we proceed the proof of Proposition 3.4 case by case.

Remark 3.11. In the first five cases, we are able to define ℓ_{p_1,p_2} such that we can choose the "initial stage", which controls the growths of $|x_2|$, $|x_2+y_2|$ as mentioned in Fact 3.7, to guarantee that $|x_2+y_2|$ can grow faster (or decrease slower) than $|x_1|+|x_2|$ when $|x_1|$ or $|x_2|$ goes to infinity [see for example (193)]; thus by Theorem 1.2, $|x_2|$ cannot go too far from the correspondent value of the "initial stage", which allow us to choose some κ_4 in (119) (a) so that for any element in $\overline{V}_0$, equality cannot occur in the last inequality of (122) (a).

Case 1: Assume Υ_{k_1,k_2} is not a weakly increasing function of k_2, i.e., $\Upsilon_{k_1,k_2'} > \Upsilon_{k_1,k_2}$ for some $k_1>0$, $k_2 > k_2' > 0$.

This provides us the following "extra fact":

1) There exists a $\delta > 0$ such that

$$(k_2^{-1}k_2')^{\delta}\Upsilon_{k_1,k_2'} > \Upsilon_{k_1,k_2}$$

since strict inequality holds when $\delta = 0$.

2) Once such δ is fixed, there is a constant $\boldsymbol{k}_0$ so that for $\boldsymbol{k} > \boldsymbol{k}_0$ we have:

$$\alpha := (k_2^{-1}k_2')^{\delta}\Upsilon_{k_1,k_2'} + (k_2'+\Upsilon_{k_1,k_2'})\boldsymbol{k}^{-1} > \Upsilon_{k_1,k_2} + (k_2+\Upsilon_{k_1,k_2})\boldsymbol{k}^{-1}. \tag{142}$$

We will see in (150) what is the role the above "extra fact" plays.

We define $V_1 = V_1(\boldsymbol{k})$, depending on the parameter $\boldsymbol{k} > \boldsymbol{k}_0$, to be the subset of V consisting of all elements $(p_1,p_2) = ((x_1,y_1),(x_2,y_2))$ such that its coordinates x_1, x_2, y_2 satisfy,

$$\begin{aligned} &\text{(a) } 1 < (k_2|x_2|^{-1})^{\boldsymbol{k}^{-1}} \le k_1^{-1}|x_1| \le (k_2|x_2|^{-1})^{\boldsymbol{k}^{-1}+\boldsymbol{k}^{-2}} < \boldsymbol{k}^{\boldsymbol{k}^{-1}+\boldsymbol{k}^{-2}}, \\ &\text{(b) } \ell_{p_1,p_2} := (k_2^{-1}|x_2|)^{\delta}|x_2+y_2| + (|x_2|+|x_2+y_2|)\boldsymbol{k}^{-1} \ge \alpha. \end{aligned} \tag{143}$$

Remark 3.12. (i) As mentioned in Remark 3.2 (ii), we treat $\boldsymbol{k}$ as a parameter which will be fixed upon our requirement in the course of the proof.

(ii) We put $\boldsymbol{k}^{-1}$ in (143) (b) in order for the element $(|x_2|+|x_2+y_2|)\boldsymbol{k}^{-1}$ to be sufficiently smaller than α when (143) holds.

(iii) The reason we put the factor $(k_2^{-1}|x_2|)^{\delta}$ in (143) (b) is to guarantee that when the last strict inequality of (143) (a) becomes an equality, $|x_2|$ will decrease, but $|x_2+y_2|$ will grow.

(iv) Note that the number in the last term of (143) (a) cannot be too big, otherwise for an element in $\overline{V}_1$, when the last strict inequality of (143) (a) becomes equality, $|x_1|$ will grow faster than $|x_2+y_2|$ and then we cannot apply Theorem 1.2.

First, multiplying (143) (b) by $\boldsymbol{k}$ and re-denoting $\boldsymbol{k}\ell_{p_1,p_2}$ as ℓ_{p_1,p_2}, we see that (143) can be rewritten in the form (119).

Remark 3.13. For a point in V_1, by (143) (a), we have $1 < (k_2|x_2|^{-1})^{\boldsymbol{k}^{-1}}$, $(k_2|x_2|^{-1})^{\boldsymbol{k}^{-1}+\boldsymbol{k}^{-2}} < \boldsymbol{k}^{\boldsymbol{k}^{-1}+\boldsymbol{k}^{-2}}$, which implies (ii) below,

$$\text{(i) } k_1 \le |x_1| < \boldsymbol{k}^{\boldsymbol{k}^{-1}+\boldsymbol{k}^{-2}}k_1, \quad \text{(ii) } k_2\boldsymbol{k}^{-1} < |x_2| < k_2. \tag{144}$$

where (i) is obtained from the part $1 < k_1^{-1}|x_1| < \boldsymbol{k}^{\boldsymbol{k}^{-1}+\boldsymbol{k}^{-2}}$ in (143) (a).

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 6 September 2023 doi:10.20944/preprints202212.0171.v2

Now we prove that $V_1(\boldsymbol{k}) \neq \emptyset$ (for all $\boldsymbol{k} > 1$) by choosing some suitable "initial stage" $(\check{p}_1, \check{p}_2)$ mentioned in Remark 3.6 (ii). By hypothesis $k_2 k_2'^{-1} > 1$, so denote

$$\check{k}_1 := (k_2 k_2'^{-1})^{\boldsymbol{k}^{-1}} k_1 > k_1. \tag{145}$$

By definition, there exists,

$$(\check{p}_1, \check{p}_2) = ((\check{x}_1, \check{y}_1), (\check{x}_2, \check{y}_2)) \in V \quad \text{with } |\check{x}_1| = \check{k}_1, \ |\check{x}_2| = k_2', \ |\check{x}_2 + \check{y}_2| = \Upsilon_{\check{k}_1, k_2'}. \tag{146}$$

When (p_1, p_2) is set to $(\check{p}_1, \check{p}_2)$, we denote the middle three terms in (143) (a) by T_1, T_2, T_3 respectively. Then we have the following three facts:

(a) $T_1 := (k_2 |\check{x}_2|^{-1})^{\boldsymbol{k}^{-1}} \overset{(146)}{=} (k_2 k_2'^{-1})^{\boldsymbol{k}^{-1}} \overset{(145)}{>} 1$ by (146) and (145) respectively;

(b) $T_2 := k_1^{-1} |\check{x}_1| \overset{(146),(145)}{=} (k_2 k_2'^{-1})^{\boldsymbol{k}^{-1}} \overset{(a)}{=} T_1$ by (146), (145) and (a) respectively;

(c) $T_3 := (k_2 |\check{x}_2|^{-1})^{\boldsymbol{k}^{-1} + \boldsymbol{k}^{-2}} \overset{(a)}{=} T_1^{1+\boldsymbol{k}^{-1}} > T_1 \overset{(b)}{=} T_2$ by (a) and (b) respectively.

Thus we obtain

$$1 < T_1 = T_2 < T_3 < \boldsymbol{k}, \tag{147}$$

i.e., (143) (a) is satisfied by $(\check{p}_1, \check{p}_2)$.

As for (143) (b) recall that, from Lemma 3.10,we have $\Upsilon_{\check{k}_1, k_2'} > \Upsilon_{k_1, k_2'}$. Using (146) and the definition of α given by (142), we obtain:

$$\begin{aligned}(k_2^{-1}|\check{x}_2|)^{\delta} |\check{x}_2 + \check{y}_2| + (|\check{x}_2| + |\check{x}_2 + \check{y}_2|)\boldsymbol{k}^{-1} &\overset{(146)}{=} (k_2^{-1} k_2')^{\delta} \Upsilon_{\check{k}_1, k_2'} + (k_2' + \Upsilon_{\check{k}_1, k_2'})\boldsymbol{k}^{-1} \\ &\overset{(137),(145)}{>} (k_2^{-1} k_2')^{\delta} \Upsilon_{k_1, k_2'} + (k_2' + \Upsilon_{k_1, k_2'})\boldsymbol{k}^{-1} \overset{(142)}{=} \alpha,\end{aligned} \tag{148}$$

namely, (143) (b) is satisfied by $(\check{p}_1, \check{p}_2)$. Hence we see that the "initial stage" $(\check{p}_1, \check{p}_2)$ is in V_1. We take $V_0 = V_1 \neq \emptyset$.

Observe that

$$\lim_{\boldsymbol{k} \to \infty} \boldsymbol{k}^{\boldsymbol{k}^{-1} + \boldsymbol{k}^{-2}} = 1.$$

So we can take $\boldsymbol{k}_0$ so that if $\boldsymbol{k} > \boldsymbol{k}_0$ we have $\boldsymbol{k}^{\boldsymbol{k}^{-1} + \boldsymbol{k}^{-2}} < 2$. Thus by (144), we obtain, when $\boldsymbol{k} > \boldsymbol{k}_0$ and $(p_1, p_2) = ((x_1, y_1), (x_2, y_2)) \in V_1$,

$$\text{(i) } k_1 \leq |x_1| \leq k_1 \boldsymbol{k}^{\boldsymbol{k}^{-1} + \boldsymbol{k}^{-2}} < 2k_1, \quad \text{(ii) } k_2 \boldsymbol{k}^{-1} \leq |x_2| \leq k_2. \tag{149}$$

In particular, we have $x_1, x_2 \neq 0$. Assume (143) holds with $x_2 + y_2 = 0$. Then (143) (b) shows that $|x_2| \geq \alpha \boldsymbol{k} > k_2$ by (142) (when $\boldsymbol{k}$ is sufficiently larger than $\boldsymbol{k}$ as $\Upsilon_{k_1, k_2} > 0$), a contradiction with (149) (2). Thus we have (121) [we do not have A_3 in the present case]. Further, V_0 is bounded by (149) and Proposition 2.22.

Now we want to prove that V_0 is closed. Thus we let $(p_1, p_2) \in \overline{V}_0$. By Remark 3.5, first assume that the first strict inequality of (143) (a) becomes an equality for (p_1, p_2). Then we immediately obtain that $|x_2| = k_2$, $|x_1| = k_1$.

For this point thus we must have

$$\ell_{p_1, p_2} := |x_2 + y_2| + (k_2 + |x_2 + y_2|)\boldsymbol{k}^{-1} \geq \alpha$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 6 September 2023 doi:10.20944/preprints202212.0171.v2

By definition (112), we have $\gamma_{k_1,k_2} \geq |x_2 + y_2|$, which gives

$$\gamma_{k_1,k_2} + (k_2 + \gamma_{k_1,k_2})\boldsymbol{k}^{-1} \overset{(112)}{\geq} \ell_{p_1,p_2} \geq \alpha. \tag{150}$$

The above is a contradiction with the "extra fact" (142), which proves that the first strict inequality of (143) (a) is satisfied by (p_1, p_2).

This also implies that the second and third inequalities cannot be both equalities.

Continuing, now assume the last strict inequality of (143) (a) becomes an equality for (p_1, p_2), i.e., $k_2|x_2|^{-1} = \boldsymbol{k}$, we denote $\beta = \frac{1}{2}(k_2^{-1}k_2')^{\delta}\gamma_{k_1,k_2'} > 0$. Then by (142),

$$2\beta < \alpha \quad \text{or} \quad \alpha - \beta > \beta. \tag{151}$$

First assume

$$\alpha - (|x_2| + |x_2 + y_2|)\boldsymbol{k}^{-1} \geq \beta. \tag{152}$$

Then by (143) (b),

$$|x_2 + y_2| \overset{(143)\,(b),\,(152)}{\geq} \beta(k_2^{-1}|x_2|)^{-\delta} = \beta\boldsymbol{k}^{\delta}. \tag{153}$$

Now assume

$$\begin{aligned} &\alpha - (|x_2| + |x_2 + y_2|)\boldsymbol{k}^{-1} < \beta \iff |x_2| + |x_2 + y_2| \overset{(154)}{>} (\alpha - \beta)\boldsymbol{k}, \implies \\ &|x_2 + y_2| > \beta\boldsymbol{k} - k_2\boldsymbol{k}^{-1}. \end{aligned} \tag{154}$$

Again for $\boldsymbol{k}$ sufficiently large we have $\beta\boldsymbol{k} - k_2\boldsymbol{k}^{-1} > \beta\boldsymbol{k}^{\delta}$ and also $\beta\boldsymbol{k} - 2k_2\boldsymbol{k}^{-1} > \beta\boldsymbol{k}^{\delta}$.

Now then $|y_2| \geq |x_2 + y_2| - |x_2| \geq \beta\boldsymbol{k} - 2k_2\boldsymbol{k}^{-1} \geq \beta\boldsymbol{k}^{\delta}$, which, together with the definition of h_{p_1,p_2} in (4) implies

$$h_{p_1,p_2} \overset{(4)}{\geq} |y_2| > \beta\boldsymbol{k}^{\delta}. \tag{155}$$

Note from (113), Remark 3.2 (ii) and Remark 3.12 that we can choose $\boldsymbol{k}$ as large as we wish. When $\boldsymbol{k}$ tends to infinity also $h_{p_1,p_2} \to \infty$ by (155). Thus we can apply Theorem 1.2 to obtain that $h_{p_1,p_2} \sim_{\boldsymbol{k}} |x_1| + |x_2|$, which with (149) shows that $h_{p_1,p_2} \sim_{\boldsymbol{k}} 1$, a contradiction with (155). This proves that (143) is satisfied by (p_1, p_2). By definition, $(p_1, p_2) \in V_0$. Thus $\overline{V}_0 = V_0$.

The above shows that with $V_1 = V_1(\boldsymbol{k})$ being defined by (143), we have, for $\boldsymbol{k}$ large, Proposition 3.4. Case 1 is now completed.

Thus from now on, we can assume that γ_{k_1,k_2} is a weakly increasing function of $k_2 \in \mathbb{R}_{>0}$ when $k_1 \in \mathbb{R}_{>0}$ is fixed, i.e.,

$$\gamma_{k_1,k_2} \geq \gamma_{k_1,k_2'} \quad \text{if } k_1 > 0,\ k_2 > k_2' > 0. \tag{156}$$

Case 2: Assume $\gamma_{\bar{k}^{1+\delta'}k_1,\bar{k}k_2} \geq \bar{k}\gamma_{k_1,k_2}$ *for some* $\delta' \in \mathbb{R}_{\geq 0}$, $\bar{k}, k_1, k_2 \in \mathbb{R}_{>0}$ *with* $\bar{k} > 1$, $\delta' < \frac{1}{m}$.

By choosing δ'' with $\delta' < \delta'' < \frac{1}{m}$ and by Lemma 3.10 [or (137)], we have

$$\gamma_{\bar{k}^{1+\delta''}k_1,\bar{k}k_2} \overset{(137)}{>} \gamma_{\bar{k}^{1+\delta'}k_1,\bar{k}k_2} \geq \bar{k}\gamma_{k_1,k_2}. \tag{157}$$

This, as in Case 1, provides us the following "extra fact" with $\delta, \boldsymbol{k}, \boldsymbol{k}$ satisfying (113) [here we require that $\ell \gg \max\{\bar{k}^{1+\delta''}k_1, \bar{k}k_2\}$, cf. Remark 3.2 (i)],

$$\alpha := (\bar{k}k_2)^{-(1+\delta)}\Upsilon_{\bar{k}^{1+\delta''}k_1,\bar{k}k_2} + (\bar{k}k_2 + \Upsilon_{\bar{k}^{1+\delta''}k_1,\bar{k}k_2})\boldsymbol{n}^{-1} > k_2^{-(1+\delta)}\Upsilon_{k_1,k_2} + (k_2 + \Upsilon_{k_1,k_2})\boldsymbol{n}^{-1}. \quad (158)$$

We define V_1 to be the subset of V consisting of all elements $(p_1, p_2) = ((x_1, y_1), (x_2, y_2))$ such that its coordinates x_1, x_2, y_2 satisfy (cf. Remark 3.12),

$$\text{(a) } 1 < (k_2^{-1}|x_2|)^{1+\delta''-\boldsymbol{k}^{-1}} \le k_1^{-1}|x_1| \le (k_2^{-1}|x_2|)^{1+\delta''+\boldsymbol{k}^{-1}} < \boldsymbol{k}^{1+\delta''+\boldsymbol{k}^{-1}},$$
$$\text{(b) } \ell_{p_1,p_2} := \frac{|x_2+y_2|}{|x_2|^{1+\delta}} + (|x_2| + |x_2+y_2|)\boldsymbol{n}^{-1} \ge \alpha. \quad (159)$$

As in Case 1, we can rewrite (159) as the form in (119).

Remark 3.14. The reason we put the power $1+\delta$, which is bigger than 1, in the denominator of (159) (b), is to ensure that $|x_2+y_2|$ will grows faster than $|x_2|$ when the last strict inequality of (159) (a) becomes equality for an element in $\overline{V}_1$.

Now we prove that $V_1 \neq \emptyset$ by choosing some suitable "initial stage" $(\check{p}_1, \check{p}_2)$ mentioned in Remark 3.6 (ii). By definition, there exists

$$(\check{p}_1, \check{p}_2) = ((\check{x}_1, \check{y}_1), (\check{x}_2, \check{y}_2)) \in V \text{ with } |\check{x}_1| = \bar{k}^{1+\delta''}k_1,\ |\check{x}_2| = \bar{k}k_2, |\check{x}_2+\check{y}_2| = \Upsilon_{\bar{k}^{1+\delta''}k_1,\bar{k}k_2}. \quad (160)$$

When (p_1, p_2) is set to $(\check{p}_1, \check{p}_2)$, we denote the middle three terms in (159) (a) by T_1, T_2, T_3 respectively. Then using (160) and definition of α in (158), one can verify

$$\text{(a) } 1 < T_1 := (k_2^{-1}|\check{x}_2|)^{1+\delta''-\boldsymbol{k}^{-1}} \overset{(160)}{=} \bar{k}^{1+\delta''-\boldsymbol{k}^{-1}} < T_2 := k_1^{-1}|\check{x}_1| \overset{(160)}{=} \bar{k}^{1+\delta''}$$
$$< T_3 := (k_2^{-1}|\check{x}_2|)^{1+\delta''+\boldsymbol{k}^{-1}} \overset{(160)}{=} \bar{k}^{1+\delta''+\boldsymbol{k}^{-1}} < \boldsymbol{k}^{1+\delta''+\boldsymbol{k}^{-1}},$$
$$\text{(b) } \frac{|\check{x}_2+\check{y}_2|}{|\check{x}_2|^{1+\delta}} + (|\check{x}_2| + |\check{x}_2+\check{y}_2|)\boldsymbol{n}^{-1} \overset{(160)}{=} \frac{\Upsilon_{\bar{k}^{1+\delta''}k_1,\bar{k}k_2}}{(\bar{k}k_2)^{1+\delta}} + (\bar{k}k_2 + \Upsilon_{\bar{k}^{1+\delta''}k_1,\bar{k}k_2})\boldsymbol{n}^{-1} \overset{(158)}{=} \alpha, \quad (161)$$

i.e., (159) is satisfied by $(\check{p}_1, \check{p}_2)$, namely, the "initial stage" $(\check{p}_1, \check{p}_2)$ is in V_1. We take $V_0 = V_1 \neq \emptyset$.

Exactly similar to Case 1, we can obtain that V_0 is a bounded set and (121) holds.

Let $(p_1, p_2) \in \overline{V}_0$. Assume the first strict inequality of (159) (a) becomes equality for (p_1, p_2). Then we obtain that $|x_2| = k_2$, $|x_1| = k_1$. We have, where the first inequality follows from the definition of Υ_{k_1,k_2} in (112), while the second from (159) (b),

$$k_2^{-(1+\delta)}\Upsilon_{k_1,k_2} + (k_2 + \Upsilon_{k_1,k_2})\boldsymbol{n}^{-1} \overset{(112)}{\ge} \frac{|x_2+y_2|}{|x_2|^{1+\delta}} + (|x_2| + |x_2+y_2|)\boldsymbol{n}^{-1} \overset{(159)\,(b)}{\ge} \alpha, \quad (162)$$

which is a contradiction with the "extra fact" (158).

Assume the last strict inequality of (159) (a) becomes equality for (p_1, p_2). Then using (91), (113) [cf. Remark 3.2 (ii) and Remark 3.12], one obtains, where (ii) is obtained by the second and third inequalities of (159) (a),

$$\text{(i) } |x_2| = k_2\boldsymbol{k} \sim_{\boldsymbol{k}} \boldsymbol{k}; \quad \text{(ii) } |x_1| = k_1(k_2^{-1}|x_2|)^{1+\delta''+O(\boldsymbol{k}^{-1})^1} \overset{(159)\,(a)}{\sim_{\boldsymbol{k}}} \boldsymbol{k}^{1+\delta''}. \quad (163)$$

Note from Theorem 1.2 or (83) that

$$h_{p_1,p_2} \overset{(4),(83)}{\sim_{\boldsymbol{k}}} |x_1| + |x_2| \overset{(163)}{\sim_{\boldsymbol{k}}} \boldsymbol{k}^{1+\delta''} \text{ when } \boldsymbol{k} \gg 1, \tag{164}$$

but by (159) (b), as in Case 1, we have

$$|x_2 + y_2| \overset{(159)\,(b)}{\succeq_{\boldsymbol{k}}} |x_2|^{1+\delta} \overset{(163)}{\sim_{\boldsymbol{k}}} \boldsymbol{k}^{1+\delta} \overset{(163)}{\succ_{\boldsymbol{k}}} |x_2|, \tag{165}$$

and thus (note that here is the only place we make use of the fact that $\delta'' < \frac{1}{m}$),

$$|y_2| \overset{(165)}{\sim_{\boldsymbol{k}}} |x_2 + y_2| \overset{(165)}{\succ_{\boldsymbol{k}}} \boldsymbol{k} \succ_{\boldsymbol{k}} \boldsymbol{k}^{\frac{(1+\delta'')m}{1+m}} \overset{(164)}{\sim_{\boldsymbol{k}}} h_{p_1,p_2}^{\frac{m}{m+1}}, \tag{166}$$

a contradiction with (83).

This proves that (159) is satisfied by (p_1, p_2), i.e., $(p_1, p_2) \in V_0$, and so V_0 is closed. Proposition 3.4 holds. Case 2 is now completed.

Hence from now on, we can assume

$$\Upsilon_{\bar{k}^{1+\delta'}k_1,\bar{k}k_2} < \bar{k}\Upsilon_{k_1,k_2} \text{ for any } \delta' \in \mathbb{R}_{\geq 0},\ \bar{k}, k_1, k_2 \in \mathbb{R}_{>0} \text{ with } \bar{k} > 1,\ \delta' < \tfrac{1}{m}. \tag{167}$$

Then we can prove

Lemma 3.15. *For any $k_1, k_2 \in \mathbb{R}_{>0}$, we have $\Upsilon_{k_1,k_2} > k_2$.*

Proof. Assume $\Upsilon_{k_1,k_2} \leq k_2$ for some $k_1, k_2 \in \mathbb{R}_{>0}$. Choosing $k_1' \in \mathbb{R}_{>0}$ with $k_1' < k_1$, by Lemma 3.10, we obtain that $\Upsilon_{k_1',k_2} < \Upsilon_{k_1,k_2} \leq k_2$. Then $\alpha := \frac{\Upsilon_{k_1',k_2}}{k_2} < 1$. By (167), we have $\Upsilon_{\boldsymbol{k}k_1',\boldsymbol{k}k_2} < \boldsymbol{k}\Upsilon_{k_1',k_2} = \boldsymbol{k}k_2\alpha$ for all $\boldsymbol{k} \gg 1$. Let

$$(\tilde{p}_1, \tilde{p}_2) \in V \quad \text{with} \quad |\tilde{x}_1| = \boldsymbol{k}k_1', \quad |\tilde{x}_2| = \boldsymbol{k}k_2, \quad |\tilde{x}_2 + \tilde{y}_2| = \Upsilon_{\boldsymbol{k}k_1',\boldsymbol{k}k_2} < \boldsymbol{k}k_2\alpha. \tag{168}$$

By Proposition 2.15 or (83), we have [using notation (91)]

$$h_{\tilde{p}_1,\tilde{p}_2} \overset{(4),(83)}{\sim_{\boldsymbol{k}}} |\tilde{x}_1| + |\tilde{x}_2| \overset{(168)}{\sim_{\boldsymbol{k}}} \boldsymbol{k}, \tag{169}$$

but then

$$|\tilde{y}_2| \geq |\tilde{x}_2| - |\tilde{x}_2 + \tilde{y}_2| \overset{(168)}{>} (1-\alpha)\boldsymbol{k}k_2 \overset{(169)}{>} h_{\tilde{p}_1,\tilde{p}_2}^{\frac{m}{m+1}} \text{ (when } \boldsymbol{k} \gg 1), \tag{170}$$

which is a contradiction with (83). □

Recall that we fix some choices of positive numbers satisfying (113) (cf. Remark 3.2) and that the element $(\bar{p}_1, \bar{p}_2)$ satisfies (115). We choose $(q_1, q_2) = ((\dot{x}_1, \dot{y}_1), (\dot{x}_2, \dot{y}_2)) \in V$ sufficiently close to $(\bar{p}_1, \bar{p}_2)$ such that its coordinates have the form as in (126), i.e.,

$$\dot{x}_1 = \bar{x}_1(1 + s\varepsilon), \ \ \dot{y}_1 = \bar{y}_1 + t\varepsilon, \ \ \dot{x}_2 = \bar{x}_2(1 + u\varepsilon), \ \ \dot{x}_2 + \dot{y}_2 = (\bar{x}_2 + \bar{y}_2)(1 + v\varepsilon). \tag{171}$$

and as in (128), we can solve from (127) to obtain, for some $a_{\boldsymbol{k}}, b_{\boldsymbol{k}} \in \mathbb{C}$ [here for later convenience, we re-denote a, b in (128) as $-a_{\boldsymbol{k}}, b_{\boldsymbol{k}}$],

$$s = -a_{\boldsymbol{k}} u + b_{\boldsymbol{k}} v + O(\varepsilon)^1, \quad (a_{\boldsymbol{k}}, b_{\boldsymbol{k}}) \neq (0,0). \tag{172}$$

We will see that the numbers $a_{\boldsymbol{k}}, b_{\boldsymbol{k}}$ play crucial roles in our definition of V_0. First we prove

Lemma 3.16. *We have $a_{\boldsymbol{k}} \geq 0$, $b_{\boldsymbol{k}} \geq 0$.*

Proof. Assume $a_{\boldsymbol{k}\,\mathrm{im}} \neq 0$ or $a_{\boldsymbol{k}\,\mathrm{re}} < 0$ [cf. Convention 2.3 (1)]. Then we can choose $u, v \in \mathbb{C}$ in the following way: we always choose $u_{\mathrm{re}} < 0$, $v_{\mathrm{re}} > 0$ so that (174) (ii), (iii) hold; if $a_{\boldsymbol{k}\,\mathrm{im}} \neq 0$, we can choose $u_{\mathrm{im}} \neq 0$ with $a_{\boldsymbol{k}\,\mathrm{im}} u_{\mathrm{im}}$ being sufficiently large to guarantee (173) (iii) holds; if $a_{\boldsymbol{k}\,\mathrm{re}} < 0$, we can simply take $u_{\mathrm{im}} = v_{\mathrm{im}} = 0$, $v_{\mathrm{re}} = 1$ and $u_{\mathrm{re}} \ll -1$ to guarantee (173) (iii) [thus (174) (i) holds], i.e.,

$$\text{(i) } u_{\mathrm{re}} < 0, \;\; \text{(ii) } v_{\mathrm{re}} > 0, \;\; \text{and such that } \; \text{(iii) } s_{\mathrm{re}} \overset{(172)}{=} (-a_{\boldsymbol{k}} u + b_{\boldsymbol{k}} v)_{\mathrm{re}} + O(\varepsilon)^1 < 0. \tag{173}$$

Then by (171) and (173), we have

$$\begin{aligned} &\text{(i) } 0 < k_1 := |\dot{x}_1| = \boldsymbol{k}|1 + s\varepsilon| \overset{(173)\,(iii)}{<} \boldsymbol{k}, \qquad \text{(ii) } 0 < k_2 := |\dot{x}_2| = \boldsymbol{k}|1 + u\varepsilon| < \boldsymbol{k}, \\ &\text{(iii) } |\dot{x}_2 + \dot{y}_2| = \Upsilon_{\boldsymbol{k},\boldsymbol{k}}|1 + v\varepsilon| > \Upsilon_{\boldsymbol{k},\boldsymbol{k}}. \end{aligned} \tag{174}$$

This means that $0 < k_1 < \boldsymbol{k}$ and $0 < k_2 < \boldsymbol{k}$ with the following [where the first inequality follows from definition (112), the second from (174) (iii)],

$$\Upsilon_{k_1,k_2} = \Upsilon_{|\dot{x}_1|,|\dot{x}_2|} \overset{(112)}{\geq} |\dot{x}_2 + \dot{y}_2| \overset{(174)\,(iii)}{>} \Upsilon_{\boldsymbol{k},\boldsymbol{k}}, \tag{175}$$

which is a contradiction with the fact obtained from Lemma 3.10 and (156) that Υ_{k_1,k_2} is an increasing function on both variables. Thus $a_{\boldsymbol{k}} \geq 0$.

Similarly, if $b_{\boldsymbol{k}\,\mathrm{im}} \neq 0$ or $b_{\boldsymbol{k}\,\mathrm{re}} < 0$, then we can choose $u, v \in \mathbb{C}$ in the following way: again we always choose $u_{\mathrm{re}} < 0$, $v_{\mathrm{re}} > 0$ so that (174) (ii), (iii) hold; if $b_{\boldsymbol{k}\,\mathrm{im}} \neq 0$, we can choose $v_{\mathrm{im}} \neq 0$ with $-b_{\boldsymbol{k}\,\mathrm{im}} v_{\mathrm{im}}$ being sufficiently large to guarantee (173) (iii) holds; if $b_{\boldsymbol{k}\,\mathrm{re}} < 0$, we can simply take $u_{\mathrm{im}} = v_{\mathrm{im}} = 0$, $v_{\mathrm{re}} \gg 1$ and $u_{\mathrm{re}} = -1$ to guarantee (173) (iii) holds. □

Case 3: Assume for any $\boldsymbol{s} \in \mathbb{R}_{>0}$ there exists $\boldsymbol{k} > \boldsymbol{s}$ such that $b_{\boldsymbol{k}} < 1 + \frac{1}{m} + a_{\boldsymbol{k}}$.

Then we can choose sufficiently small $\delta' > 0$ [which can depend on $\boldsymbol{k}$ but is independent of $\boldsymbol{n}$ when $\boldsymbol{n} \gg \boldsymbol{k}$, cf. notation (113) and Remark 3.2 (i)] such that we have the following "extra fact" (note that the following holds when $\delta' = 0$ thus holds when $\delta' > 0$ is sufficiently small),

$$(1 + \delta')b_{\boldsymbol{k}} < 1 + \tfrac{1}{m} - \delta' + a_{\boldsymbol{k}}. \tag{176}$$

We define V_1 to be the subset of V consisting of all elements $(p_1, p_2) = ((x_1, y_1), (x_2, y_2))$ such that its coordinates x_1, x_2, y_2 satisfy (cf. Remark 3.12),

$$\begin{aligned} &\text{(i) } 1 < (\boldsymbol{k}^{-1}|x_2|)^{1+\frac{1}{m}-\delta'-\boldsymbol{n}^{-1}} \leq \boldsymbol{k}^{-1}|x_1| \leq (\boldsymbol{k}^{-1}|x_2|)^{1+\frac{1}{m}} < \boldsymbol{n}^{1+\frac{1}{m}}, \\ &\text{(ii) } \ell_{p_1,p_2} := \frac{\Upsilon_{\boldsymbol{k},\boldsymbol{k}}^{-1}|x_2 + y_2|}{(\boldsymbol{k}^{-1}|x_2|)^{1+\delta'}} + (|x_2| + |x_2 + y_2|)\varepsilon^3 \geq 1 + \varepsilon^2. \end{aligned} \tag{177}$$

Similar to (159), we put the power $1+\delta'$, which is bigger than 1, in the denominator of (177) (ii) is to ensure that $|x_2+y_2|$ will grows faster than $|x_2|$ when equality occurs in the last strict inequality of (177) (ii) for an element in $\overline{V}_1$.

We will prove $V_1 \neq \emptyset$, but firstly, as in Cases 1 and 2, we can rewrite (177) as the form in (119), and V_1 is bounded such that $x_1, x_2 \neq 0$ if $(p_1,p_2) \in V_1$. Secondly, we explain the reason we put $\varepsilon^2, \varepsilon^3$ in (177) (ii) is that we want to guarantee the following

$$(|x_2| + |x_2+y_2|)\varepsilon^3 < \varepsilon^2 \text{ when } (p_1,p_2) \in V_1. \tag{178}$$

which is obtained by Theorem 1.2 and the facts from (177) (a) that $|x_1|, |x_2| < \boldsymbol{n}^2$ and $\varepsilon^{-1} \gg \boldsymbol{n}$. Then (177) (ii) with (178) gives

$$|x_2+y_2| \overset{(177)\,(ii),(178)}{>} \Upsilon_{\boldsymbol{k},\boldsymbol{k}}\big(\boldsymbol{k}^{-1}|x_2|\big)^{1+\delta'}, \tag{179}$$

which is also satisfied by any element in $\overline{V}_0$ by Remark 3.5. In particular, we have (121).

Now we prove that $V_1 \neq \emptyset$ by choosing some suitable "initial stage" (q_1,q_2) mentioned in Remark 3.6 (ii). With (q_1,q_2), sufficiently close to $(\bar{p}_1,\bar{p}_2)$, being defined in (171), we want to choose suitable u,v such that (177) is satisfied by (q_1,q_2), i.e.,

$$\begin{aligned} &\text{(i) } 1 < |1+u\varepsilon|^{1+\frac{1}{m}-\delta'-\boldsymbol{n}^{-1}} \le |1+s\varepsilon| \le |1+u\varepsilon|^{1+\frac{1}{m}} < \boldsymbol{n}^{1+\frac{1}{m}},\\ &\text{(ii) } \frac{\big|1+v\varepsilon\big|}{|1+u\varepsilon|^{1+\delta'}} + O(\varepsilon)^3 \ge 1+\varepsilon^2. \end{aligned} \tag{180}$$

We take $u,v \in \mathbb{R}_{>0}$ such that

$$u = 1, \quad v = \frac{a_{\boldsymbol{k}} + 1 + \frac{1}{m} - \delta'}{b_{\boldsymbol{k}}}, \quad \text{then } s \overset{(172)}{=} 1 + \frac{1}{m} - \delta' + O(\varepsilon)^1, \tag{181}$$

where the last equation is obtained from (172). Then the coefficients of ε in the middle three terms of (180) (i) are respectively

$$1+\frac{1}{m}-\delta'-\boldsymbol{n}^{-1}, \quad 1+\frac{1}{m}-\delta', \quad 1+\frac{1}{m}, \tag{182}$$

i.e., all inequalities in (180) (i) are strict inequalities by recalling from notation (113) that we have complete freedom in choosing ε with $0 < \varepsilon \ll \boldsymbol{n}^{-1}$ independently of all other choices of the parameters. Further, the coefficient of ε^1 in the left hand-side of (180) (ii) is

$$v - (1+\delta')u \overset{(181)}{=} \frac{a_{\boldsymbol{k}} + 1 + \frac{1}{m} - \delta'}{b_{\boldsymbol{k}}} - (1+\delta'), \tag{183}$$

which is bigger than 0 by the "extra fact" (176). Thus the "initial stage" (q_1,q_2) is in V_1. We take $V_0 = V_1 \neq \emptyset$.

Next we let $(p_1,p_2) \in \overline{V}_0$. Assume the first strict inequality of (177) (i) becomes equality for (p_1,p_2). Then $|x_1| = |x_2| = \boldsymbol{k}$, but by (179) (which also holds for elements in $\overline{V}_0$), we have $|x_2+y_2| > \Upsilon_{\boldsymbol{k},\boldsymbol{k}}$, a contradiction with the definition of $\Upsilon_{\boldsymbol{k},\boldsymbol{k}}$ in (112).

Assume the last strict inequality of (177) (i) becomes equality for (p_1,p_2). Then we obtain, when $\boldsymbol{n} \gg \boldsymbol{k}$ [using (91), (114), cf. Remark 3.2 (ii)],

$$|x_2| = \boldsymbol{k}\boldsymbol{n} \sim_{\boldsymbol{n}} \boldsymbol{n}, \quad |x_1| \le \boldsymbol{k}\boldsymbol{n}^{1+\frac{1}{m}} \sim_{\boldsymbol{n}} \boldsymbol{n}^{1+\frac{1}{m}}, \tag{184}$$

but by (179),

$$|x_2+y_2| \overset{(179)}{\succeq_{\boldsymbol{n}}} |x_2|^{1+\delta'} \overset{(184)}{\sim_{\boldsymbol{n}}} \boldsymbol{n}^{1+\delta'} \overset{(184)}{\succ_{\boldsymbol{n}}} |x_2|. \tag{185}$$

Thus

$$|y_2| \overset{(185)}{\sim_{\boldsymbol{n}}} |x_2+y_2| \overset{(185)}{\succ_{\boldsymbol{n}}} \boldsymbol{n}^{1+\delta'}. \tag{186}$$

By (184) and (83), we have

$$h_{p_1,p_2} \overset{(4),(83)}{\sim_{\boldsymbol{n}}} |x_1|+|x_2| \overset{(184)}{\preceq_{\boldsymbol{n}}} \boldsymbol{n}^{1+\frac{1}{m}} \overset{(186)}{\prec_{\boldsymbol{n}}} |y_2|^{\frac{m+1}{m}}, \tag{187}$$

a contradiction with (83).

This proves that (177) is satisfied by (p_1,p_2), i.e., $(p_1,p_2)\in V_0$, and so V_0 is closed. Proposition 3.4 holds. Case 3 is now completed.

Hence from now on, we can assume

$$b_{\boldsymbol{k}} \ge 1+\frac{1}{m}+a_{\boldsymbol{k}} \ \text{ when } \boldsymbol{k} \gg \ell. \tag{188}$$

Case 4: Assume there exists a fixed but sufficiently small $\delta' \in \mathbb{R}_{>0}$ such that for any $\boldsymbol{s}\in\mathbb{R}_{>0}$ there exists $\boldsymbol{k}>\boldsymbol{s}$ satisfying $(1-\delta')b_{\boldsymbol{k}} > 1+a_{\boldsymbol{k}}$ (the "extra fact").

First we remark that the reason we require δ' to be independent of $\boldsymbol{k}$ is to guarantee that we have (195). We define V_1 to be the subset of V consisting of all elements $(p_1,p_2)=((x_1,y_1),(x_2,y_2))$ such that its coordinates x_1,x_2,y_2 satisfy (cf. Remark 3.12),

$$\begin{aligned}
&\text{(i) } 1 < \left(\boldsymbol{k}^{-1}|x_2|\right)^{-1+\boldsymbol{k}^{-1}} \le \left(\boldsymbol{k}^{-1}|x_1|\right)^{-1} \le \left(\boldsymbol{k}^{-1}|x_2|\right)^{-1-\boldsymbol{k}^{-1}} < (1-\delta')^{-1-\boldsymbol{k}^{-1}},\\
&\text{(ii) } \ell_{p_1,p_2} := \frac{\Upsilon_{\boldsymbol{k},\boldsymbol{k}}^{-1}|x_2+y_2|}{\left(\boldsymbol{k}^{-1}|x_2|\right)^{1-\delta'}} + (|x_2|+|x_2+y_2|)\varepsilon^3 \ge 1+\varepsilon^2.
\end{aligned} \tag{189}$$

As in the previous case, we have (119), (121) and V_1 is bounded.

Remark 3.17. (i) In contrary to (177), in the denominator of (189) (ii), we put the power $1-\delta'$, which is smaller than 1 and is a number independent of $\boldsymbol{k}$ [that is important otherwise we cannot obtain (195)], is to ensure that $|x_2+y_2|$ will decrease slower than $|x_2|$ when the last strict inequality of (189) (i) becomes equality for an element in $\overline{V}_1$.

(ii) Note that the number in the last term of (189) (i) cannot be too big otherwise for an element in $\overline{V}_1$, when the last strict inequality of (189) (i) becomes equality, $|x_2|$ will fall to a too small number and then we cannot use Theorem 1.2 or (83) (for example we cannot choose the number to be $\boldsymbol{k}$).

With (q_1,q_2) being defined in (171), we want to choose suitable u,v such that (189) (i), (ii) hold for (q_1,q_2), i.e.,

$$\begin{aligned}
&\text{(i) } 1 < |1+u\varepsilon|^{-1+\boldsymbol{k}^{-1}} \le |1+s\varepsilon|^{-1} \le |1+u\varepsilon|^{-1-\boldsymbol{k}^{-1}} < (1-\delta')^{-1-\boldsymbol{k}^{-1}},\\
&\text{(ii) } \frac{\left|1+v\varepsilon\right|}{|1+u\varepsilon|^{1-\delta'}} + O(\varepsilon)^3 \ge 1+\varepsilon^2.
\end{aligned} \tag{190}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 6 September 2023 doi:10.20944/preprints202212.0171.v2

Take $u, v \in \mathbb{R}_{<0}$ such that [the last equation is obtained from (172)],

$$u = -1, \quad v = -\frac{1+a_k}{b_k}, \quad \text{and} \quad s \overset{(172)}{=} -1 + O(\varepsilon)^1. \tag{191}$$

Then the coefficients of middle three terms in (190) (i) are respectively $1 - \boldsymbol{k}^{-1}$, 1, $1 + \boldsymbol{k}^{-1}$, i.e., all inequalities in (190) (i) are strict inequalities. Further, the coefficient of ε^1 in the left hand-side of (190) (ii) is $1 - \delta' - \frac{1+a_k}{b_k}$, which is positive by the "extra fact". Thus $(q_1, q_2) \in V_1$. We take $V_0 = V_1 \neq \emptyset$.

Similarly to (179), we obtain from (189) (ii) the following (which also holds for any element in $\overline{V}_0$),

$$|x_2 + y_2| > \Upsilon_{\boldsymbol{k},\boldsymbol{k}} (\boldsymbol{k}^{-1}|x_2|)^{1-\delta'}. \tag{192}$$

Let $(p_1, p_2) \in \overline{V}_0$. Assume the last strict inequality of (189) (i) becomes equality. Then we obtain that $|x_2| = (1-\delta')\boldsymbol{k}$, $|x_1| \le \boldsymbol{k}$, and by (192) and Lemma 3.15, we have

$$|x_2 + y_2| \overset{(192)}{>} (1-\delta')^{1-\delta'} \Upsilon_{\boldsymbol{k},\boldsymbol{k}} \overset{\text{Lemma 3.15}}{>} \Big(1 - \delta' + \delta'^2 + O(\delta')^3\Big)\boldsymbol{k}. \tag{193}$$

As before we have

$$h_{p_1,p_2} \overset{(4),(83)}{\sim_{\boldsymbol{k}}} |x_1| + |x_2| \sim_{\boldsymbol{k}} \boldsymbol{k}, \tag{194}$$

and we obtain

$$|y_2| \ge |x_2 + y_2| - |x_2| \overset{(193)}{>} (\delta'^2 + O(\delta')^3)\boldsymbol{k} \sim_{\boldsymbol{k}} \boldsymbol{k} \succ_{\boldsymbol{k}} \boldsymbol{k}^{\frac{m}{m+1}} \overset{(194)}{\sim_{\boldsymbol{k}}} h_{p_1,p_2}^{\frac{m}{m+1}}, \tag{195}$$

a contradiction with Theorem 1.2 or (83).

Assume the first strict inequality of (189) (i), becomes equality for (p_1, p_2). Then $|x_1| = |x_2| = \boldsymbol{k}$, but by (192), $|x_2 + y_2| > \Upsilon_{\boldsymbol{k},\boldsymbol{k}}$, a contradiction with definition (112).

This proves that (189) is satisfied by (p_1, p_2), i.e., $(p_1, p_2) \in V_0$, and so V_0 is closed. Proposition 3.4 holds. Case 4 is now completed.

Hence from now on, we assume

$$(1-\delta')b_k \le 1 + a_k \text{ for any fixed sufficiently small } \delta' > 0 \text{ and all } \boldsymbol{k} \gg \ell. \tag{196}$$

This with (188) shows that $a_k \ge \frac{(1-\delta')(1+\frac{1}{m})}{\delta'}$. Since $\delta' > 0$ is arbitrarily sufficiently small number, we see that $a_k > 0$ (thus also $b_k > 0$) is unbounded, i.e.,

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} b_k = \infty, \quad \text{and} \quad \lim_{k\to\infty} \frac{a_k}{b_k} = 1. \tag{197}$$

Case 5: Assume there exist some fixed λ, $\lambda_2 \in \mathbb{R}_{>0}$ with $\lambda < 1$ such that whenever $\boldsymbol{k} \gg 1$ there exist $k_1, k_2, \hat{k} \in \mathbb{R}_{>0}$ (which can depend on $\boldsymbol{k}$) with $k_1, k_2, \hat{k} < 1$ and $(\hat{p}_1, \hat{p}_2) = ((\hat{x}_1, \hat{y}_1), (\hat{x}_2, \hat{y}_2)) \in V$ satisfying the following,

$$\text{(i) } |\hat{x}_1| = k_1\boldsymbol{k}, \quad |\hat{x}_2| = k_2\boldsymbol{k}, \quad |\hat{x}_2 + \hat{y}_2| = \hat{k}\Upsilon_{\boldsymbol{k},\boldsymbol{k}},$$
$$\text{(ii) } \lambda \le k_2 < 1, \quad \text{(iii) } 0 < k_1 < 1, \quad \text{(iv) } 0 < \hat{k} < 1, \quad \text{(v) } k_2 < \hat{k}^{1+\lambda_2}. \tag{198}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 6 September 2023 doi:10.20944/preprints202212.0171.v2

First we remark that the reason we require λ_2 to be independent of $\boldsymbol{k}$ is to guarantee that we have (206) and the reason we require λ to be independent of $\boldsymbol{k}$ is to guarantee that $\hat{x}_2$ is large enough for us to apply Theorem 1.2 [cf. (199)].

Fix any sufficiently small $\delta > 0$ (independent of $\boldsymbol{k}$) such that $\delta \ll \min\{\lambda, \lambda_2\}$. First assume $k_2 \le 1 - \delta$. Then we have, when $\boldsymbol{k} \gg 1$,

$$\text{(i) } |\hat{x}_2| = k_2\boldsymbol{k} \overset{(198)\,\text{(ii)}}{>} \lambda\boldsymbol{k} \sim_{\boldsymbol{k}} \boldsymbol{k}, \quad \text{(ii) } |\hat{x}_1| = k_1\boldsymbol{k} \overset{(198)\,\text{(iii)}}{<} \boldsymbol{k} \sim_{\boldsymbol{k}} \boldsymbol{k},$$
$$\text{(iii) } |\hat{y}_2| \ge |\hat{x}_2 + \hat{y}_2| - |\hat{x}_2| \overset{(198)}{\ge} k_2^{\frac{1}{1+\lambda_2}}\Upsilon_{\boldsymbol{k},k} - k_2\boldsymbol{k} \overset{\text{Lemma 3.15}}{>} (k_2^{\frac{1}{1+\lambda_2}} - k_2)\boldsymbol{k}$$
$$\sim_{\boldsymbol{k}} \boldsymbol{k} \overset{(199)\text{(i)},\text{(ii)},(4),(83)}{\sim_{\boldsymbol{k}}} h_{\hat{p}_1,\hat{p}_2}, \tag{199}$$

where the strict inequality in (iii) follows from the fact in Lemma 3.15 that $\Upsilon_{\boldsymbol{k},k} > \boldsymbol{k}$, and where the first " $\sim_{\boldsymbol{k}}$ " in (199) (iii) follows from the fact that $\min\{x^{\frac{1}{1+\lambda_2}} - x \,|\, \lambda \le x \le 1-\delta\}$ is a fixed positive number (i.e., independent of $\boldsymbol{k}$), and the last " $\sim_{\boldsymbol{k}}$ " follows from (199) (i), (ii) and (4), (83). We obtain from (199) a contradiction with (83). Thus

$$k_2 > 1 - \delta. \tag{200}$$

Since $\hat{k} \ne 1$, we can always write, for some $\lambda'_1, \lambda'_2 \in \mathbb{R}_{>0}$,

$$\text{(i) } k_1 = \hat{k}^{\lambda'_1}, \quad \text{(ii) } k_2 = \hat{k}^{\lambda'_2}, \quad \text{then} \quad \text{(iii) } \lambda'_2 \overset{(198)\,\text{(v)}}{>} 1 + \lambda_2, \tag{201}$$

where the inequation follows from (198) (v).

Let $\varepsilon_0 \in \mathbb{R}_{>0}$ such that $\varepsilon_0^{-1} \gg \boldsymbol{k}$. Now we define $V_0 = V_1$ to be the subset of V consisting of elements $(p_1, p_2) = ((x_1, y_1), (x_2, y_2))$ satisfying, where $\log(\cdot)$ is the natural logarithm function,

$$\text{(i) } 1 < (\boldsymbol{k}^{-1}|x_2|)^{-\frac{\lambda'_1}{\lambda'_2}+\varepsilon_0} \le (\boldsymbol{k}^{-1}|x_1|)^{-1} \le (\boldsymbol{k}^{-1}|x_2|)^{-\frac{\lambda'_1}{\lambda'_2}-\varepsilon_0} < (1-\delta)^{-\frac{\lambda'_1}{\lambda'_2}-\varepsilon_0}, \tag{202}$$
$$\text{(ii) } \frac{\Upsilon_{\boldsymbol{k},k}^{-1}|x_2 + y_2| + \varepsilon_0^2}{(\boldsymbol{k}^{-1}|x_2|)^{\lambda_2'^{-1}(1+\varepsilon_0)}} \ge \kappa_2, \quad \text{(iii) } \kappa_2 = \frac{\hat{k} + \varepsilon_0^2}{\hat{k}^{1+\varepsilon_0}} = 1 - \log(\hat{k})\varepsilon_0 + O(\varepsilon_0)^2 \overset{(198)\,\text{(iv)}}{>} 1 + \varepsilon_0^2,$$

where the inequality in (202) (iii) follows by noting that $\log(\hat{k}) < 0$ since $0 < \hat{k} < 1$ by (198) (iv). Then as before we can rewrite (202) as the form in (119) and (121) holds and V_0 is bounded. Similar to (189), in the denominator of (202) (ii), we put the power $\lambda_2'^{-1}(1+\varepsilon_0)$, which by (201) (iii) is, up to $O(\varepsilon_0)^1$, smaller than $\frac{1}{1+\lambda_2}$ (that is a number independent of $\boldsymbol{k}$) is to ensure that $|x_2 + y_2|$ will decrease slower than $|x_2|$ when the last strict inequality of (202) (i) becomes equality for an element in $\overline{V}_1$.

When we set (p_1, p_2) to be $(\hat{p}_1, \hat{p}_2)$ in (202), the middle three terms in (202) (i) are respectively

$$T_1 := k_2^{-\frac{\lambda'_1}{\lambda'_2}+\varepsilon_0}, \quad T_2 := k_1^{-1} \overset{(201)}{=} \hat{k}^{-\lambda'_1} \overset{(201)}{=} k_2^{-\frac{\lambda'_1}{\lambda'_2}}, \quad T_3 = k_2^{-\frac{\lambda'_1}{\lambda'_2}-\varepsilon_0}. \tag{203}$$

Since $1 - \delta < k_2 < 1$ by (198) (ii), (200), we see that (202) (i) holds for $(\hat{p}_1, \hat{p}_2)$. Further the left-hand side of (202) (ii) is $\frac{\hat{k}+\varepsilon_0^2}{\hat{k}^{1+\varepsilon_0}} = \kappa_2$, i.e., (202) (ii) holds for $(\hat{p}_1, \hat{p}_2)$. Thus $(\hat{p}_1, \hat{p}_2) \in V_0$.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 6 September 2023 doi:10.20944/preprints202212.0171.v2

Let $(p_1, p_2) \in \overline{V}_0$. Assume the last strict inequality of (202) (i) becomes equality for (p_1, p_2). Then

$$\text{(i) } |x_1| \le \boldsymbol{k}, \quad \text{(ii) } |x_2| = (1-\delta)\boldsymbol{k}, \tag{204}$$

thus by (83),

$$h_{p_1,p_2} \overset{(4),(83)}{\sim_{\boldsymbol{k}}} |x_1| + |x_2| \sim_{\boldsymbol{k}} |x_2| \overset{(83)}{\sim_{\boldsymbol{k}}} |x_2 + y_2| \overset{(204)\,\text{(ii)}}{\sim_{\boldsymbol{k}}} \boldsymbol{k}. \tag{205}$$

By (202) (ii), (iii), (204) (ii), we have the first inequality below; the second inequality follows from the inequation in (201) (iii); the third inequality follows from Lemma 3.15, thus we obtain, where we conduct computations up to $O(\varepsilon_0)^1$ so ε_0 in (202) is omitted,

$$\begin{aligned} |x_2 + y_2| &\overset{(202)\,\text{(ii)},\,\text{(iii)},\,(204)\,\text{(ii)}}{>} (1-\delta)^{\lambda_2^{l'-1}} \Upsilon_{\boldsymbol{k},\boldsymbol{k}} \overset{(201)\,\text{(iii)}}{>} (1-\delta)^{\frac{1}{1+\lambda_2}} \Upsilon_{\boldsymbol{k},\boldsymbol{k}} \\ &\overset{\text{Lemma 3.15}}{\ge} \Big(1 - \frac{\delta}{1+\lambda_2} + O(\delta)^2\Big)\boldsymbol{k} \\ &\overset{(207)}{=} \Big(\big(1+\alpha\delta + O(\delta)^2\big)(1-\delta)\Big)\boldsymbol{k} = \big(1+\alpha\delta+O(\delta)^2\big)|x_2|, \end{aligned} \tag{206}$$

where

$$\alpha = \frac{\lambda_2}{1+\lambda_2} > 0, \text{ which is a number independent of } \boldsymbol{k}. \tag{207}$$

Using arguments as in (193), (195), we obtain a contradiction.

Assume the first strict inequality of (202) (i) becomes equality for (p_1, p_2). Then $|x_1| = |x_2| = \boldsymbol{k}$, and by (202) (ii), (iii), $|x_2 + y_2| \ge (\kappa_2 - \varepsilon_0^2)\Upsilon_{\boldsymbol{k},\boldsymbol{k}} > \Upsilon_{\boldsymbol{k},\boldsymbol{k}}$, a contradiction with the definition of $\Upsilon_{\boldsymbol{k},\boldsymbol{k}}$ in (112).

This proves that (202) is satisfied by (p_1, p_2), i.e., $(p_1, p_2) \in V_0$, and so V_0 is closed. Proposition 3.4 holds. Case 5 is now completed.

Thus from now on we assume that Case 5 does not occur.

Case 6: The remaining case.

In this case, unfortunately, we are not able to use (119) to define V_0. We have to define V_0 in a more complicated way. The reason is below:

Remark 3.18. Because $b_{\boldsymbol{k}}$ is too big [by (188)], we are unable to choose suitable "initial stage" (q_1, q_2) mentioned in Remark 3.6 (ii) such that u, v, s satisfy (172), and (q_1, q_2), defined in (171), satisfies (177) if we define V_0 as in (177). To see this, observe that in order for (177) (ii) to hold, we have to choose v to be bigger than u but then (172) shows that s may possibly be too big, which implies that $|x_1|$ may grow too fast by Fact 3.7. Since we do not have any information about y_1 in our definition of V_0, when $|x_1| \gg |x_2|$ we can not apply Theorem 1.2 to obtain a contradiction if the last strict inequality of (159) (i) becomes equality for an element in $\overline{V}_0$. Therefore in order to able to obtain a contradiction, we have to find some other way to control the growth of $|x_1|$.

Recall notations (113), (116) and (118) and Remark 3.2; in particular, we will frequently use the fact that $0 < \varepsilon \ll \boldsymbol{e}_1 = \boldsymbol{n}_1^{-1} \ll \boldsymbol{e} = \boldsymbol{n}^{-1} \ll \boldsymbol{k}^{-1} \ll 1$ and $\boldsymbol{e} \ll \boldsymbol{d}$ or $\boldsymbol{n} \gg \boldsymbol{l}$ [recall (117)].

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 6 September 2023 doi:10.20944/preprints202212.0171.v2

Definition 3.19. With A_1, A_2, A_3 being defined in (118), and with a_1, a_2, a_3 to be defined in (218) in the proof of Lemma 3.21 (at this point we only need to use the facts that $a_1, a_2 > 0$, but do not require the explicit values of a_1, a_2), we take $V_0 = V_2$ with (120) (a), (b) being specified as follows,

$$
\begin{aligned}
&\text{(i) } 1 < |A_1|^{a_2 a_1^{-1} - e_1^3} \le |A_2| \le |A_1|^{a_2 a_1^{-1} + e_1^3} < \boldsymbol{n}^{a_2 a_1^{-1} + e_1^3}, \\
&\text{(ii) } \ell_{p_1,p_2} := |A_3 A_1^{-a_3 a_1^{-1} + e_1^3}| + (|x_2| + |x_2 + y_2|)\varepsilon^3 \ge 1 + \varepsilon^2.
\end{aligned} \tag{208}
$$

Remark 3.20. The reason we put $\pm \boldsymbol{e}_1^3$ in the powers of the second and fourth terms of (208) (i) is to control $|A_2|$ such that when $(p_1, p_2) \in \overline{V}_0$, we have [we will see in (218) that $a_2 a_1^{-1} = 2\boldsymbol{d} + \frac{2130\boldsymbol{d}^2}{101} + O(\boldsymbol{d})^3$],

$$
|A_2| \overset{(208)\,\text{(i)}}{=} |A_1|^{a_2 a_1^{-1} + O(e_1)^3} \overset{(210)\,\text{(a)}}{=} |A_1|^{a_2 a_1^{-1}} + O(\boldsymbol{e}_1)^3 = |A_1|^{2\boldsymbol{d} + \frac{2130\boldsymbol{d}^2}{101} + O(\boldsymbol{d})^3}. \tag{209}
$$

Further, condition (208) (i) implies that when $(p_1, p_2) \in \overline{V}_0$, we have,

$$
\text{(a) } 1 \le |A_1| \le \boldsymbol{n}, \qquad \text{(b) } 1 \le |A_2| \le |A_1|^{a_2 a_1^{-1} + O(e_1)^3} < \boldsymbol{n}^{2\boldsymbol{d} + \frac{2130\boldsymbol{d}^2}{101} + O(\boldsymbol{d})^3}. \tag{210}
$$

Now we divide the proof of Proposition 3.4 into three lemmas.

Lemma 3.21. *The set V_0 is nonempty.*

Proof. First at this point it may be worth recalling that we have fixed $(\bar{p}_1, \bar{p}_2) = ((\bar{x}_1, \bar{y}_1), (\bar{x}_2, \bar{y}_2)) \in A_{\boldsymbol{k},\boldsymbol{k}} \subset V$ satisfying (115), i.e.,

$$
|\bar{x}_1| = |\bar{x}_2| = \boldsymbol{k}, \quad |\bar{x}_2 + \bar{y}_2| = \gamma_{\boldsymbol{k},\boldsymbol{k}}. \tag{211}
$$

We need to choose a suitable "initial stage" (q_1, q_2) as mentioned in Remark 3.6 (ii). To do this, as already mentioned in (171), we choose $(q_1, q_2) = ((\dot{x}_1, \dot{y}_1), (\dot{x}_2, \dot{y}_2)) \in V_0$ to be sufficiently close to $(\bar{p}_1, \bar{p}_2)$ such that (171) holds, i.e.,

$$
\dot{x}_1 = \bar{x}_1(1 + s\varepsilon), \quad \dot{y}_1 = \bar{y}_1 + t\varepsilon, \quad \dot{x}_2 = \bar{x}_2(1 + u\varepsilon), \quad \dot{x}_2 + \dot{y}_2 = (\bar{x}_2 + \bar{y}_2)(1 + v\varepsilon), \tag{212}
$$

and further, s is determined by (172), i.e.,

$$
s = s_0 + O(\varepsilon)^1, \quad s_0 = -a_{\boldsymbol{k}} u + b_{\boldsymbol{k}} v, \tag{213}
$$

where $a_{\boldsymbol{k}}, b_{\boldsymbol{k}}$ satisfy (197). We want to take suitable u, v such that (208) is satisfied by (q_1, q_2), defined in (171). Note from (116) and (212) that setting (p_1, p_2) to (q_1, q_2) corresponds to that X_1, X_2, Z are respectively set to,

$$
\text{(i) } X_1 \overset{(116),(212)}{=} 1 + s\varepsilon, \quad \text{(ii) } X_2 \overset{(116),(212)}{=} 1 + u\varepsilon, \quad \text{(iii) } Z \overset{(116),(212)}{=} 1 + v\varepsilon. \tag{214}
$$

At this point, we also want to remind from notation (113) that we have complete freedom in choosing ε with $0 < \varepsilon \ll \boldsymbol{n}_1^{-1}$ independently of all other choices of the parameters. Further, it may be worth presenting the following obvious facts, for any $a, b \in \mathbb{C}$ which are independent of ε,

$$
(1 + a\varepsilon)^{-1} = 1 - a\varepsilon + O(\varepsilon)^2, \qquad (1 + a\varepsilon)(1 + b\varepsilon) = 1 + (a + b)\varepsilon + O(\varepsilon)^2. \tag{215}
$$

Recall from (117) that $\boldsymbol{l} = 101(a_k + b_k) + \lambda_0 \in \mathbb{Z}$ with $\lambda_0 \in \mathbb{R}$ satisfying $0 < \lambda_0 \le 1$. Now we take ,

$$\text{(i) } u = -\boldsymbol{l}^{10}, \quad \text{(ii) } v = \boldsymbol{l}^{10}, \quad \text{(iii) } s_0 \overset{(213)}{=} \boldsymbol{l}^{10}(a_k + b_k). \tag{216}$$

So [the following is the reason we define β_0 in (118) (iii)],

$$\begin{aligned} \beta_0 + 100 s_0 - 20u &\overset{(118)\,\text{(iii)}}{=} \boldsymbol{l}^{10}(a_k + b_k - 20 + \lambda_0) + 100\boldsymbol{l}^{10}(a_k + b_k) + 20\boldsymbol{l}^{10} \\ &= \Big(101(a_k + b_k) + \lambda_0\Big)\boldsymbol{l}^{10} = \boldsymbol{l}^{11}. \end{aligned} \tag{217}$$

Then by expanding in ε, up to $O(\varepsilon)^2$, we have, for $i = 1, 2, 3$,

$$\begin{aligned} &\text{(i) } \widetilde{X}_1 = 1 + \tilde{s}\varepsilon, \quad \tilde{s} = \beta_0 + 100 s_0 - 20u \overset{(217)}{=} \boldsymbol{l}^{11}, \qquad \text{(ii) } A_i = 1 + a_i\varepsilon + O(\varepsilon)^2, \\ &\text{(iii) } a_3 \overset{(118)\,\text{(i)}}{=} \beta_1 + 16v = \boldsymbol{l}^{11}(1 - 834\boldsymbol{d}) \text{ [recall from (118) (i) that } \beta_1 = \boldsymbol{l}^{11}(1 - 850\boldsymbol{d})\text{]}, \\ &\text{(iv) } a_2 \overset{(118)\,\text{(ii)}}{=} 4\beta_1 + 200a_3 + 10u + 10v - 2\tilde{s} = 2\boldsymbol{l}^{11}(101 - 85100\boldsymbol{d}), \\ &\text{(v) } a_1 = 101\Big(\beta_1 + \boldsymbol{l}(1 - 20\boldsymbol{d})a_3 + 28\tilde{s} - v + \frac{4}{5}\Big(\frac{(\boldsymbol{l}-1)a_2}{101} - 2\boldsymbol{l}(1 - 10\boldsymbol{d})a_3 - 36\tilde{S}\Big)\Big) \\ &\qquad = \boldsymbol{l}^{12}(101 - 86165\boldsymbol{d} + 387145\boldsymbol{d}^2), \end{aligned} \tag{218}$$

where (218) (v) is precisely computed, up to $O(\varepsilon)^2$, by recalling,

$$\begin{aligned} A_1 &= \frac{\alpha_1^{101} A_3^{101\boldsymbol{l}(1-20\boldsymbol{d})} \widetilde{X}_1^{2828}}{Z^{101}} \Big(\frac{1}{5} + \frac{4A_2^{\boldsymbol{l}-1}}{5A_3^{202\boldsymbol{l}(1-10\boldsymbol{d})} \widetilde{X}_1^{3636}}\Big) \\ &= \frac{(1 + \beta_1\varepsilon)^{101}(1 + a_3\varepsilon)^{101\boldsymbol{l}(1-20\boldsymbol{d})}(1 + \tilde{s}\varepsilon)^{2828}}{(1 + v\varepsilon)^{101}} \Big(\frac{1}{5} + \frac{4(1 + a_2\varepsilon)^{\boldsymbol{l}-1}}{5(1 + a_3\varepsilon)^{202\boldsymbol{l}(1-10\boldsymbol{d})}(1 + \tilde{s}\varepsilon)^{3636}}\Big) \\ &= \Big(1 + 101(\beta_1 + \boldsymbol{l}(1 - 20\boldsymbol{d})a_3 + 28\tilde{s} - v)\varepsilon\Big)\Big(1 + \frac{404}{5}\Big(\frac{(\boldsymbol{l}-1)a_2}{101} - 2\boldsymbol{l}(1 - 10\boldsymbol{d})a_3 - 36\tilde{s}\Big)\varepsilon\Big) \\ &= 1 + a_1\varepsilon. \end{aligned} \tag{219}$$

Similarly, we can compute (218) (ii), (iii). We remark that, as mentioned in Remark 3.18 (ii), we define v to be larger than u, which allows us to obtain a contradiction if equality occurs in the last inequality of (208) (i) for an element in $\overline{V}_0$.

Now we take (p_1, p_2) to be (q_1, q_2). By (218) we can easily see that the coefficients of ε^1 in the middle three terms of (208) (i) are respectively,

$$(a_2 a_1^{-1} - \boldsymbol{e}_1^3)a_1 = a_2 - a_1\boldsymbol{e}_1^3, \quad a_2, \quad (a_2 a_1^{-1} + \boldsymbol{e}_1^3)a_1 = a_2 + a_1\boldsymbol{e}_1^3.$$

We see that all inequalities in (208) (i) are strict inequalities when (p_1, p_2) is set to (q_1, q_2), i.e., (208) (i) is satisfied by (q_1, q_2).

When (p_1, p_2) is set to (q_1, q_2), one can observe from (211)–(214) that $(|x_2| + |x_2 + y_2|)\varepsilon^3$ is an $O(\varepsilon)^3$ element [by notation (113)], and then from (208) (ii), (211), (212), we see that ℓ_{q_1,q_2} is a $1 + O(\varepsilon)^1$ element such that the coefficient of ε^1 in ℓ_{q_1,q_2} is, by (218), $a_3 - (a_3 a_1^{-1} - \boldsymbol{e}_1^3)a_1 = a_1\boldsymbol{e}_1^3 > 0$, which means that the inequality in (208) (ii) is a strict inequality.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 6 September 2023 doi:10.20944/preprints202212.0171.v2

When (p_1, p_2) is set to (q_1, q_2), one can easily observe from (214), (215) that X_1, X_2 are all $1 + O(\varepsilon)^1$ elements, thus all strict inequalities in (120) (c), (d) hold.

Hence the "initial stage" $(q_1, q_2) \in V_0$. □

By Remark 3.5, we see that elements of $\overline{V}_0 \subset V$ satisfy (123) (c), (d) and

$$\text{(i) } 1 \le |A_1|^{a_2 a_1^{-1} - e_1^3} \le |A_2| \le |A_1|^{a_2 a_1^{-1} + e_1^3} \le n^{a_2 a_1^{-1} + e_1^3},$$
$$\text{(ii) } \ell_{p_1,p_2} := |A_3 A_1^{-a_3 a_1^{-1} + e_1^3}| + (|x_2| + |x_2 + y_2|)\varepsilon^3 \ge 1 + \varepsilon^2. \tag{220}$$

For convenience, we denote

$$T_2 := \Big\{ \tilde{x}_1 = |\widetilde{X}_1|,\ x_1 = |X_1|,\ x_2 = |X_2|,\ z = |Z|,\ a_i = |A_i|,\ i = 1, 2, 3 \Big\}. \tag{221}$$

Lemma 3.22. *When $(p_1, p_2) \in \overline{V}_0$, we have the following,*

$$(1)\ a_3 > a_1^{a_3 a_1^{-1} - e_1^3} = a_1^{\frac{d}{101} + \frac{1931 d^2}{10201} + O(d)^3},\ (2)\ \frac{|B_1|}{1+e_1} < 1 < \frac{1+e_1^2}{|C_1|},\ (3)\ e_1 < \tilde{x}_1, x_2 < n_1. \tag{222}$$

Proof. First we have,

$$\text{(i) } x_1 \overset{(118)\,(iii)}{=} \alpha_0^{-\frac{1}{100}} x_2^{\frac{1}{5}} \tilde{x}_1^{\frac{1}{100}} \overset{(118)\,(iii)}{<} x_2^{\frac{1}{5}} \tilde{x}_1^{\frac{1}{100}} \overset{(123)\,(c)}{<} n_1,$$
$$\text{(ii) } |x_1| \overset{(115),(116)}{=} k x_1 \overset{(223)\,(i)}{<} k n_1,$$
$$\text{(iii) } |x_2| \overset{(115),(116)}{=} k x_2 \le k n_1. \tag{223}$$

Then by Theorem 1.2, we must have $|y_1| \le 2 k n_1$. Thus $(|x_2| + |x_2 + y_2|)\varepsilon^3 < \varepsilon^2$, which with (220) (ii) implies (222) (1).

The proof of (222) (2) is a little trick, first we give a remark.

Remark 3.23. We will use the condition in the last inequality of (123) (c) that $|C_1| \le 1 + e_1^2$ to prove that equality cannot occur in the first inequality of (123) (c), i.e., $|B_1| < 1 + e_1$. Then we will use the condition in the first inequality of (123) (c) that $|B_1| \le 1 + e_1$ to prove equality cannot occur in the last inequality of (208) (iv), i.e., $|C_1| < 1 + e_1^2$. Note that this is no problem as we only use the defining conditions on V_0.

To prove (222) (1), recall from (118) (iv) that $A_1 = \frac{\alpha_1^{101} A_3^{101 l (1-20d)} \widetilde{X}_1^{2828}}{Z^{101}} \Big(\frac{1}{5} + \frac{4 A_2^{l-1}}{5 A_3^{202 l (1-10d)} \widetilde{X}_1^{3636}} \Big)$, which can be rewritten as,

$$\text{(i) } a + b + c = 0 \ \text{ with}$$
$$\text{(ii) } a = \frac{4 \alpha_1^{101} A_2^{l(1-d)}}{5 A_1 A_3^{101 l} \widetilde{X}_1^{808} Z^{101}}, \quad \text{(iii) } b = -1, \quad \text{(iv) } c = \frac{\alpha_1^{101} A_3^{101 l (1-20d)} \widetilde{X}_1^{2828}}{5 A_1 Z^{101}}. \tag{224}$$

Therefore [the following is obtained by first regarding (224) (i) as the equation $a\lambda^2 + b\lambda + c = 0$ and then solving λ and then setting $\lambda = 1$],

$$\Big(1 + \frac{b}{2a}\Big(1 + (1 - 4acb^{-2})^{\frac{1}{2}}\Big)\Big)\Big(1 + \frac{b}{2a}\Big(1 - (1 - 4acb^{-2})^{\frac{1}{2}}\Big)\Big) \overset{(224)\,(i)}{=} 0, \tag{225}$$

where,

$$\text{(i)}\ \frac{b}{2a} \overset{(224)\,\text{(ii)},\,\text{(iii)},\,(118)\,\text{(v)}}{=} -\frac{5B_1}{8},$$
$$\text{(ii)}\ 4acb^{-2} \overset{(224)\,\text{(ii)}-\text{(iv)},\,(118)\,\text{(vi)}}{=} \frac{16C_1}{25}, \tag{226}$$

and where we always choose $(1-4acb^{-2})^{\frac{1}{2}}$ to be the unique element defined by (6),

$$(1-4acb^{-2})^{\frac{1}{2}} = \left(1-\frac{16C_1}{25}\right)^{\frac{1}{2}} = 1+\sum_{i=1}^{\infty}\binom{\frac{1}{2}}{i}\left(-\frac{16C_1}{25}\right)^i, \tag{227}$$

which converges absolutely by the fact from (123) (c) that $\frac{16|C_1|}{25} \leq \frac{16(1+e_1^2)}{25} < 1$. Therefore, we obtain from (225), (226) that either

$$\text{(i)}\ 1 = B_1D_1, \quad \text{(ii)}\ D_1 := \frac{5}{8}\left(1+\left(1-\frac{16C_1}{25}\right)^{\frac{1}{2}}\right) \overset{(227)}{=} \frac{5}{8}\left(2+\sum_{i=1}^{\infty}\binom{\frac{1}{2}}{i}\left(-\frac{16C_1}{25}\right)^i\right), \tag{228}$$

or else

$$1 = \frac{5B_1}{8}\left(1-\left(1-\frac{16C_1}{25}\right)^{\frac{1}{2}}\right). \tag{229}$$

Assume we have the later case. Then, where the first inequality follows from the fact that $(-1)^{i+1}\binom{\frac{1}{2}}{i}$ is positive for all $i \geq 1$,

$$\begin{aligned} |B_1|^{-1} &\overset{(229)}{=} \frac{5}{8}\left|1-\left(1-\frac{16C_1}{25}\right)^{\frac{1}{2}}\right| \overset{(227)}{=} \frac{5}{8}\left|-\sum_{i=1}^{\infty}\binom{\frac{1}{2}}{i}\left(\frac{16C_1}{25}\right)^i\right| \\ &\leq \frac{5}{8}\sum_{i=1}^{\infty}(-1)^{i+1}\binom{\frac{1}{2}}{i}\left(\frac{16|C_1|}{25}\right)^i \overset{(123)\,\text{(c)}}{\leq} \frac{5}{8}\sum_{i=1}^{\infty}(-1)^{i+1}\binom{\frac{1}{2}}{i}\left(\frac{16(1+e_1^2)}{25}\right)^i \\ &\overset{(6)}{=} \frac{5}{8}\left(1-\left(1-\frac{16(1+e_1^2)}{25}\right)^{\frac{1}{2}}\right) = \frac{1}{4}+\frac{e_1^2}{3}+O(e_1)^4 < (1+e_1)^{-1}, \end{aligned} \tag{230}$$

which is a contradiction with the first inequality of (123) (c). This proves that we can only have (228). Exactly similar to the evaluation in (230), we can deduce from the right-hand side of (228) (ii) the following,

$$\begin{aligned} |D_1| &\overset{(228)\,\text{(ii)}}{\geq} \frac{5}{8}\left(2-\sum_{i=1}^{\infty}(-1)^{i+1}\binom{\frac{1}{2}}{i}\left(\frac{16|C_1|}{25}\right)^i\right) \\ &\overset{(6)}{=} \frac{5}{8}\left(1+\left(1-\frac{16(1+e_1^2)}{25}\right)^{\frac{1}{2}}\right) = 1-\frac{e_1^2}{3}+O(e_1)^4. \end{aligned} \tag{231}$$

Thus we obtain from (228) (i) the following

$$|B_1| = |D_1|^{-1} \overset{(231)}{\leq} 1+\frac{e_1^2}{3}+O(e_1)^4, \tag{232}$$

which in particular gives the first inequality of (222) (2).

Using (118) (v), (iv) and (232), up to $O(e_1)^3$ [then $\alpha_1 = 1 + O(\varepsilon)^1$ can be omitted, and so $z = a_3^{\frac{1}{16}}$ by (118) (i)], we have [observing from (218) that $a_2 a_1^{-1} = 2d + \frac{2130d^2}{101} + O(d)^3$, $a_3 a_1^{-1} = \frac{d}{101} + \frac{1931d^2}{10201} + O(d)^3$, one can see that the power of a_1 in the last equality is negative],

$$\begin{aligned}
|C_1| &\overset{(232)}{\leq} \Big(1+\frac{e_1^2}{3}\Big)^{\frac{5}{2}} \frac{|C_1|}{|B_1|^{\frac{5}{2}}} = \Big(1+\frac{5e_1^2}{6}\Big) \frac{a_2^{l-1}\tilde{x}_1^{2020} a_1^{-2} a_3^{-2020-\frac{101}{8}}}{\big(a_1 a_3^{101l+\frac{101}{16}} \tilde{x}_1^{808} a_2^{-l+1}\big)^{\frac{5}{2}}} \\
&= \Big(1+\frac{5e_1^2}{6}\Big) a_1^{-\frac{9}{2}} a_2^{\frac{7l(1-d)}{2}} a_3^{-101(\frac{5l}{2}+\frac{649}{32})} \\
&\overset{(209),(222)\,(1)}{\leq} \Big(1+\frac{5e_1^2}{6}\Big) a_1^{-\frac{9}{2}+\frac{7l(1-d)a_2}{2a_1}-101(\frac{5l}{2}+\frac{649}{32})a_3 a_1^{-1}} \\
&= \Big(1+\frac{5e_1^2}{6}\Big) a_1^{-\frac{4093d}{3232}+O(d)^2} \overset{(210)\,(a)}{\leq} 1+\frac{5e_1^2}{6} < 1+e_1^2 \ \ (\text{up to } O(e_1)^3).
\end{aligned} \tag{233}$$

This proves (222) (2).

Now note from (118) that we have the following

$$\text{(i) } 1 \overset{(118)\,(i),\,(ii)}{=} a_2^{-1} a_3^{\frac{1605}{8}} \tilde{x}_1^{-2} x_2^{10} + O(e_1)^2, \tag{234}$$

$$\text{(ii) } (1+e_1)^{-1} \overset{(222)\,(2)}{<} |B_1|^{-1} = a_1^{-1} a_2^{l-1} a_3^{-101(l+\frac{1}{16})} \tilde{x}_1^{-808} + O(e_1)^2 \leq \frac{1}{8}\big(5+\sqrt{41}\big) + O(e_1)^2,$$

where the last inequality of (ii) is obtained from (228) by noting the following

$$|D_1| \leq \frac{5}{8}\Big(1+\big(1+\frac{16(1+e_1^2)}{25}\big)^{\frac{1}{2}}\Big) = \frac{1}{8}\big(5+\sqrt{41}\big) + O(e_1)^2. \tag{235}$$

Thus (234) (ii) shows that, up to $O(e_1)^1$, we have

$$\begin{aligned}
\tilde{x}_1 &= \eta_1 a_1^{-\frac{1}{808}} a_2^{\frac{l-1}{808}} a_3^{-\frac{l(16+d)}{128}} \overset{(209)}{=} \eta_1 a_1^{\frac{1}{808}+\frac{241d}{10201}+O(d)^2} a_3^{-\frac{l(16+d)}{128}} \\
&\overset{(222)\,(1)}{\leq} \eta_1 a_1^{-\frac{149d}{1305728}+O(d)^2} \ \ (\text{up to } O(e_1)^1),
\end{aligned} \tag{236}$$

for some $\eta_1 \in \mathbb{R}$ with $\frac{1}{2} < \eta_1 < 2$ by noting that $1 < \big(\frac{1}{8}(5+\sqrt{41})\big)^{\frac{1}{808}} < 2$. Using this in (234) (i), we obtain, up to $O(e_1)^1$,

$$\begin{aligned}
x_2 &= \eta_1^{\frac{1}{5}} a_2^{\frac{1}{10}} a_3^{-\frac{321}{16}} \tilde{x}_1^{\frac{1}{5}} \overset{(209),(236)}{=} \eta_1^{\frac{1}{5}} a_1^{\frac{1}{4040}+\frac{10442d}{51005}+O(d)^2} a_3^{-\frac{l}{40}-\frac{12841}{640}+O(d)^1} \\
&\overset{(222)\,(1)}{\leq} \eta_1^{\frac{1}{5}} a_1^{\frac{8739d}{6528640}+O(d)^2} \ \ (\text{up to } O(e_1)^1).
\end{aligned} \tag{237}$$

Now (210) with (236), (237) shows that the last inequality of (222) (3) holds for $\tilde{x}_1, x_2$.

Now assume

$$\tilde{x}_1 \leq e_1. \tag{238}$$

Then, up to $O(e_1)^1$ [recall from (113) that $n_1 = e_1^{-1} \gg n$ and we can also require that $n_1^d \gg n$, therefore $\tilde{x}_1^{-\frac{d}{2}} \ge n_1^{\frac{d}{2}} \gg n$, which gives (i) below],

$$\text{(i) } z \overset{(118)\,(i)}{=} a_3^{\frac{1}{16}} \overset{(236)}{=} \left(\tilde{x}\eta_1^{-1} a_1^{-\frac{1}{808}+O(d)^1}\right)^{-\frac{d}{2}+O(d)^2} \overset{(210)\,(a),(238)}{\gg} 1,$$
$$\text{(ii) } x_2 \overset{(237)}{=} \eta_1^{\frac{1}{5}} a_2^{\frac{1}{10}} a_3^{-\frac{321}{16}} \tilde{x}_1^{\frac{1}{5}} \overset{(210)\,(b),(238),(239)\,(i)}{\ll} 1,$$
$$\text{(iii) } x_1 \overset{(223)\,(i)}{<} x_2^{\frac{1}{5}} \tilde{x}_1^{\frac{1}{100}} \overset{(238),(239)\,(ii)}{\ll} 1. \tag{239}$$

We obtain [we will frequently apply the following; we remark that as long as $x_1, x_2 < z$ and $z \ge 1$, we can obtain the following; note that we require $z > 1$ to apply (167), however in our case here, the third inequality is an equality when $z = 1$],

$$\Upsilon_{|x_1|,|x_2|} = \Upsilon_{x_1 k, x_2 k} \overset{(137)}{<} \Upsilon_{zk, x_2 k} \overset{(156)}{\le} \Upsilon_{zk,zk} \overset{(167)}{\le} z\Upsilon_{k,k} \overset{(115),(116)}{=} |x_2 + y_2| \overset{(112)}{\le} \Upsilon_{|x_1|,|x_2|}, \tag{240}$$

which is a contradiction. Thus the assumption (238) is wrong, i.e., the first inequality of (222) (3) holds for $\tilde{x}_1$. Similarly, if $x_2 \le e_1$, exactly as the above, we can use (236), (237) to obtain (239) and (240), which is again a contradiction. This proves (222) (3).

This completes the proof of Lemma 3.22. □

Lemma 3.24. *We have $\overline{V}_0 = V_0$.*

Proof. Let $(p_1, p_2) \in \overline{V}_0$. We already see from (222) that all strict inequalities in (120) (c), (d) hold.

Assume equality occurs in the first inequality of (123) (a). Using notations in (221), we have

$$\text{(i) } a_1 = a_2 = 1, \qquad \text{(ii) } a_3 \overset{(222)\,(1)}{>} 1. \tag{241}$$

We claim

$$a = 1 + O(e_1)^1 \text{ for } a \in T_3 := \{\tilde{x}_1, x_2, z, a_3, \alpha_1\}, \tag{242}$$

which obviously holds for α_1 by (118) (i). To see it holds for $\tilde{x}_1$, first by the first inequality of (234) (ii) and (241), we obtain that $\tilde{x}_1 \le 1 + O(e_1)^1$, more precisely,

$$\tilde{x}_1 \overset{(234)\,(ii)}{<} (1+e_1)^{\frac{1}{808}} < 1 + e_1 \text{ (up to } O(e_1)^2). \tag{243}$$

To prove $\tilde{x}_1$ is a $1 + O(e_1)^1$ element, assume, say,

$$\tilde{x}_1 < 1 - 6e_1 < 1. \tag{244}$$

Then (234) (i) with (241), (244) shows

$$x_2 \le 1 - e_1 < 1. \tag{245}$$

If $z \ge 1$, then this together with (244), (245) implies that we can obtain a contradiction as in (240) [cf. the remark before (240) when $z = 1$].

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 6 September 2023 doi:10.20944/preprints202212.0171.v2

Thus $z < 1$. We want to verify if (198) holds. To do this, we use notation there by denoting $\hat{k} = z$ and so $|x_2 + y_2| = \hat{k}\gamma_{k,k}$ by (115), (116), and $k_1 := x_1$ with $|x_1| = k_1 k$, and $k_2 := x_2$ with $|x_2| = k_2 k$, and further we can obtain

$$\begin{aligned}
&\text{(i) } \hat{k} = z \overset{(118)\,\text{(i)},\,(241)\,\text{(ii)}}{=} 1 + O(\varepsilon)^1, \\
&\text{(ii) } k_2 = x_2 \overset{(245)}{<} 1 - e_1 \overset{(246)\,\text{(i)}}{<} \hat{k}^{1+\lambda_2} \ \text{ with, say, } \lambda_2 = 1, \\
&\text{(iii) } k_1 = x_1 \overset{(223)\,\text{(i)}}{<} x_2^{\frac{1}{5}} \tilde{x}_1^{\frac{1}{100}} \overset{(244),\,(245)}{<} 1.
\end{aligned} \tag{246}$$

We must have $k_2 \ge \lambda := \frac{1}{2}$, otherwise by (246) (i), as in (199) (iii), we can obtain a contradiction [in fact in many cases we have proved that x_2 cannot differ from z by a number which is independent of k]. Thus we see that (198) holds, which contradicts the assumption that Case 5 does not occur. This proves that (242) holds for $\tilde{x}_1$.

Using (241), (234) (i) and the fact that $\tilde{x}_1 = 1 + O(e_1)^1$, we see that $x_2 \le 1 + O(e_1)^1$. If, say, $x_2 < \tilde{x}_1(1 - 10000e_1)$, then this with (223) (i), (243) (234) (i), (118) (i) shows that $z > 1$ and $z > x_1, x_2$, and we obtain a contradiction again as in (240). Thus x_2 is also a $1 + O(e_1)^1$ element. Then by (234) (i), (118) (i), a_3, z are $1 + O(e_1)^1$ elements too, i.e., we have (242).

Now we need the precise definitions in (118). We have

$$\begin{aligned}
&\text{(i) } 1 < a_3 = \gamma_3 := \alpha_1 z^{16}, \\
&\text{(ii) } 1 = a_2^{-\frac{1}{2}} = \alpha_1^{-2} a_3^{-100} x_1 x_2^{-5} z^{-5} \overset{(241)\,\text{(ii)}}{<} \gamma_2 := \alpha_1^{-2} x_1 x_2^{-5} z^{-5}, \\
&\text{(iii) } 1 = a_1 \le \gamma_{10} := \alpha_1^{101} a_3^{101l(1-20d)} \tilde{x}_1^{2828} z^{-101} \left(\frac{1}{5} + \frac{4a_3^{-202l(1-10d)} \tilde{x}_1^{-3636}}{5} \right) \\
&\qquad < \gamma_0 := \alpha_1^{101} \tilde{x}_1^{2828} z^{-101} \left(\frac{1}{5} + \frac{4\tilde{x}_1^{-3636}}{5} \right),
\end{aligned} \tag{247}$$

where the strict inequality in (iii) is obtained by noting that γ_{10} is locally a strict decreasing function on a_3 (as $\frac{\partial \gamma_{10}}{\partial a_3}|_{(\alpha_1, a_3, \tilde{x}_1, z) = (1,1,1,1)} < -\frac{303l}{5} < 0$), and $a_3 > 1$ and we have (242). Now regarding γ_0 as a local function on $\alpha_1, \tilde{x}_1, z$, at point $(1, 1, 1)$, we have

$$\frac{\partial \gamma_0}{\partial \alpha_1} = 101, \quad \frac{\partial \gamma_0}{\partial \tilde{x}_1} = -\frac{404}{5}, \quad \frac{\partial \gamma_0}{\partial z} = -101. \tag{248}$$

This suggests us to denote (where $s = -1$ if $\tilde{x}_1 \ge 1$ and $s = 1$ otherwise),

$$\text{(i) } \gamma_1 := \alpha_1^{101} \tilde{x}_1^{-(\frac{404}{5} + s d^{20})} z^{101}, \quad \text{(ii) } \tilde{\gamma}_1 = \gamma_1^{-1} \gamma_0. \tag{249}$$

Then $\tilde{\gamma}_1$ is locally a strict decreasing function on $\tilde{x}_1$ if $\tilde{x}_1 \ge 1$ or a strict increasing function on $\tilde{x}_1$ if $\tilde{x}_1 < 1$ (noting that $0 < e_1 \ll d^{20}$). Thus in any case, we obtain that $\tilde{\gamma}_1 < \tilde{\gamma}_1|_{\tilde{x}_1 = 1} = 1$. This with (247) (iii), (249) (ii) proves

$$1 < \gamma_1 = \alpha_1^{101} \tilde{x}_1^{-\frac{404}{5} + O(d)^{20}} z^{101}. \tag{250}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 6 September 2023 doi:10.20944/preprints202212.0171.v2

We have,

$$
\begin{aligned}
&\text{(i) } 1 < (\gamma_1^2\gamma_2^{101})^{\frac{5}{101}} \overset{(250),(247)\,\text{(ii)}}{=} \gamma_{12} := \tilde{x}_1^{-3+O(d)^{20}}(x_2^5 z^7)^{-5},\\
&\text{(ii) } 1 < \gamma_2\gamma_3^2 \overset{(247)\,\text{(i),(ii)}}{=} \gamma_{23} := x_1 x_2^{-5} z^{27},\\
&\text{(iii) } 1 < \gamma_{12}\gamma_{23}^{3+O(d)^{20}} = x_2^{-40+O(d)^{20}} z^{46+O(d)^{20}}, \quad \Longrightarrow \quad \text{(iv) } x_2 < z^{\frac{23+O(d)^{20}}{20}},\\
&\text{(v) } 1 < (\gamma_{12}\gamma_2^{3+O(d)^{20}})^{\frac{1}{10}} = \alpha_1^{-\frac{3}{5}} x_2^{-4+O(d)^{20}} z^{-5+O(d)^{20}} \overset{(118)\,\text{(i)}}{<} x_2^{-4+O(d)^{20}} z^{-5+O(d)^{20}},\\
&\text{(vi) } x_2 \overset{(251)\,\text{(iv),(v)}}{<} 1.\\
&\text{(vii) } x_1 \overset{(223)\,\text{(i)}}{<} x_2^{\frac{1}{5}}\tilde{x}_1^{\frac{1}{100}} \overset{(251)\,\text{(i)}}{<} x_2^{\frac{1}{5}}\Big((x_2^5 z^7)^{-\frac{5}{3}+O(d)^{20}}\Big)^{\frac{1}{100}} \overset{(251)\,\text{(v)}}{<} x_2^{\frac{1}{5}-\frac{1}{12}-\frac{7}{60}\times(-\frac{4}{5})+O(d)^{20}}\\
&\qquad = x_2^{\frac{21}{100}+O(d)^{20}} \overset{(251)\,\text{(vi)}}{<} 1.
\end{aligned} \tag{251}
$$

Now if $z \geq 1$, then by the facts in (251) (vi), (vii) that $x_1 < 1$, $x_2 < 1$, we obtain a contradiction as in (240).

Hence $z < 1$. Then z is a $1 + O(\varepsilon)^1$ element as in (246) (i), and we have (246) (iii) by (251) (vii). By (251) (iv), $x_2 < z^{1+\frac{3+O(d)^{20}}{20}} < z^{1+\lambda_2}$ with $\lambda_2 = \frac{1}{10}$. The same statement after (246) shows that we still have $x_2 \geq \lambda := \frac{1}{2}$, i.e., (198) holds, which again contradicts the assumption that Case 5 does not occur.

This proves that the first strict inequality of (208) (i) holds for (p_1, p_2).

Next assume in (220) (i), equality occurs in the last inequality, i.e., $|A_1| = n$. Then

$$
\begin{aligned}
&\text{(i) } z \overset{(118)\,\text{(i)}}{=} a_3^{\frac{1}{16}} + O(e)^2 \overset{(222)\,(1)}{\geq} n^{\frac{d}{101}+O(d)^2} \gg 1,\\
&\text{(ii) } x_2 \overset{(237)}{\leq} \eta_1^{\frac{1}{5}} a_1^{\frac{8739d}{6528640}+O(d)^2} \overset{(252)\,\text{(i)}}{<} z,\\
&\text{(iii) } \tilde{x}_1 \overset{(236)}{\leq} \eta_1 n^{-\frac{149d}{1305728}+O(d)^2} \overset{(252)\,\text{(i)}}{\ll} z,\\
&\text{(iv) } x_1 \overset{(223)\,\text{(i)}}{<} x_2^{\frac{1}{5}}\tilde{x}_1^{\frac{1}{100}} \overset{(252)\,\text{(i)}-\text{(iii)}}{<} z.
\end{aligned} \tag{252}
$$

By (252) (i), (ii), (iv), we obtain a contradiction as in (240). This proves that the last strict inequality of (208) (i) holds for (p_1, p_2).

The above with (220) shows that (208) is satisfied by (p_1, p_2), which implies that $(p_1, p_2) \in V_0$. This proves that $\overline{V}_0 = V_0$. □

Now Proposition 3.4 follows from Lemmas 3.21–3.24 together with the following facts. When $(p_1, p_2) \in V_0$, by (222) (3), x_1, x_2 are bounded, thus V_0 is bounded by Proposition 2.22. Further by (222) (1), (3), (118) (i), (116), we see that $x_1, x_2, x_2 + y_2, A_3 \neq 0$, i.e., (121) holds. This completes the proof.
□

Appendix A Another proof of surjectivity of π_1

To see our original idea on how to obtain the surjectivity of π_1, for your reference, below we would like to present our original proof of surjectivity of π_1 by simply using fundamental mathematical methods.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 6 September 2023 doi:10.20944/preprints202212.0171.v2

Assume conversely π_1 is not surjective. Fix any $(\xi_1, \xi_2) \in \mathbb{C}^2 \backslash \pi_1(V)$, and define

$$L_{p_1,p_2} = |x_1 - \xi_1|^2 + |x_2 - \xi_2|^2 \text{ for } (p_1, p_2) = ((x_1, y_1), (x_2, y_2)) \in V. \quad \text{(A1)}$$

Then $L_{p_1,p_2} > 0$ for $(p_1, p_2) \in V$. Fix any $(\bar{p}_1, \bar{p}_2) \in V$, and denote

$$V_3 = \{(p_1, p_2) \in V \,|\, L_{p_1,p_2} \le L_{\bar{p}_1,\bar{p}_2}\}. \quad \text{(A2)}$$

Then $V_3 \ne \emptyset$. By (A1) and Theorem 1.2, V_3 is compact in $\mathbb{C}^4$. Then the following proposition (which says that the continuous function L_{p_1,p_2} does not have the minimal value on the compact set V_3) immediately gives a contradiction, which proves the surjectivity of π_1.

Proposition A.1. *For any $(p_1, p_2) \in V_3$, in every neighborhood of (p_1, p_2) there exists $(q_1, q_2) = ((\dot{x}_1, \dot{y}_1), (\dot{x}_2, \dot{y}_2)) \in V_3$ such that*

$$L_{q_1,q_2} < L_{p_1,p_2}. \quad \text{(A3)}$$

Proof. Let $(p_1, p_2) = ((x_1, y_1), (x_2, y_2)) \in V$ and set (and define G_1, G_2 similarly)

$$F_1 = F(x_1 + \alpha_1 x, y_1 + y) - F(x_1, y_1), \quad F_2 = F(x_2 + \alpha_2 x, y_2 + y) - F(x_2, y_2), \text{ where} \quad \text{(A4)}$$

$$\alpha_1 = \begin{cases} 1 & \text{if } x_1 = \xi_1, \\ x_1 - \xi_1 & \text{else,} \end{cases} \qquad \alpha_2 = \begin{cases} 1 & \text{if } x_2 = \xi_2, \\ x_2 - \xi_2 & \text{else.} \end{cases} \quad \text{(A5)}$$

Define q_1, q_2 accordingly [cf. (140) and (126)],

$$q_1 := (\dot{x}_1, \dot{y}_1) = (x_1 + \alpha_1 s\varepsilon, y_1 + t\varepsilon), \qquad q_2 := (\dot{x}_2, \dot{y}_2) = (x_2 + \alpha_2 u\varepsilon, y_2 + v\varepsilon). \quad \text{(A6)}$$

We need to choose (q_1, q_2) to be in V [then it is automatically in V_3 when (A3) holds].

Let A_1 be the unique invertible 2×2 complex matrix such that for Jacobian pair

$$(\bar{F}_i, \bar{G}_i) := (F_i, G_i)A_1 \text{ for } i = 1, 2, \quad \text{(A7)}$$

the linear parts of $\bar{F}_1, \bar{G}_1, \bar{F}_2, \bar{G}_2$, denoted as $\bar{F}_1^{\text{lin}}, \bar{G}_1^{\text{lin}}, \bar{F}_2^{\text{lin}}, \bar{G}_1^{\text{lin}}$, are of the following forms,

$$\bar{F}_1^{\text{lin}} = x, \quad \bar{G}_1^{\text{lin}} = y, \qquad \bar{F}_2^{\text{lin}} = ax + by, \quad \bar{G}_2^{\text{lin}} = cx + dy, \quad \text{(A8)}$$

for some $a, b, c, d \in \mathbb{C}$ with $ad - bc = 1$. Note from (A4),(A6) and (A7) that the equation (127) is equivalent to the equation

$$\big(\bar{F}_1(s\varepsilon, t\varepsilon), \bar{G}_1(s\varepsilon, t\varepsilon)\big) = \big(\bar{F}_2(u\varepsilon, v\varepsilon), \bar{G}_2(u\varepsilon, v\varepsilon)\big). \quad \text{(A9)}$$

Then we can solve from (A9) to obtain

$$s = au + bv + O(\varepsilon)^1. \quad \text{(A10)}$$

Note that $(x_1, x_2) \ne (\xi_1, \xi_2)$ by the assumption. First suppose $x_1 \ne \xi_1$, $x_2 \ne \xi_2$ (then $\alpha_1 = x_1 - \xi_1$, $\alpha_2 = x_2 - \xi_2$). In this case, by (A1),(A3), we need to choose u, v such that,

$$C_1 := \beta_1|1 + s\varepsilon|^2 + \beta_2|1 + u\varepsilon|^2 - (\beta_1 + \beta_2) < 0, \quad \text{(A11)}$$

where $\beta_1 = |x_1 - \xi_1|^2 > 0$, $\beta_2 = |x_2 - \xi_2|^2 > 0$. Using (A10) in (A11), we immediately see (by comparing the coefficients of ε^1) that if $b \ne 0$ or $a \ne -\beta_1\beta_2^{-1}$, then we have a solution for (A11). Thus assume

$$b = 0, \qquad a = -\beta_2\beta_1^{-1} \in \mathbb{R}_{\ne 0}. \quad \text{(A12)}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 6 September 2023 doi:10.20944/preprints202212.0171.v2

Then we can write

$$\text{(i) } \bar F_1 = \sum_{i\ge 2} a_i y^i + x\Big(1+\sum_{i\ge 1}\hat a_i y^i\Big)+\cdots,\quad \bar F_2 = \sum_{i\ge 2} b_i z^i + ax\Big(1+\sum_{i\ge 1}\hat b_i z^i\Big)+\cdots,$$
$$\text{(ii) } \bar G_1 = y+\sum_{i\ge 2} c_i y^i + x\sum_{i\ge 1} c_i' y^i+\cdots,\quad \bar G_2 = z+\sum_{i\ge 2} d_i z^i + x\sum_{i\ge 1} d_i' z^i+\cdots,\ z=cx+dy, \tag{A13}$$

for some $a_i, \hat a_i, b_i, \hat b_i, c_i, c_i', d_i, d_i' \in \mathbb{C}$, and where we regard $\bar F_2, \bar G_2$ as polynomials on x, z and we omit terms with x-degree ≥ 2.

Lemma A.2. *There exists some $i \ge 2$ such that $(a_i, c_i) \ne (b_i, d_i)$.*

Proof. Denote $(\bar F, \bar G) = (F, G)A_1$, and we use the same symbols with a bar to denote the associated elements corresponding to the pair $(\bar F, \bar G)$. Then by definition of Keller maps, we have

$$\bar\sigma(p) = \big(\bar F(p), \bar G(p)\big) = \big(F(p), G(p)\big)A_1 = \sigma(p)A_1 \text{ for } p\in\mathbb{C}^2. \tag{A14}$$

Assume $(a_i, c_i) = (b_i, d_i)$ for $i \ge 2$. Then by the bar version of (A4), we obtain (and the like for $\bar G$),

$$\bar F(x_1, y_1+\boldsymbol{k}) = \bar F_1|_{(x,y)=(0,\boldsymbol{k})} = \bar F_2|_{(x,z)=(0,\boldsymbol{k})} = \bar F(x_2, y_2+d^{-1}\boldsymbol{k}), \tag{A15}$$

i.e., $\bar\sigma(\hat p_1) = \bar\sigma(\hat p_2)$ and so $\sigma(\hat p_1) = \sigma(\hat p_2)$ by (A14) for all $\boldsymbol{k} \gg 1$, where

$$\hat p_1 := (\hat x_1, \hat y_1) = (x_1, y_1+\boldsymbol{k}),\quad \hat p_2 := (\hat x_2, \hat y_2) = (x_2, y_2+d^{-1}\boldsymbol{k}). \tag{A16}$$

Since $p_1 \ne p_2$, we have $\hat p_1 \ne \hat p_2$ when $\boldsymbol{k} \gg 1$, i.e., $(\hat p_1, \hat p_2) \in V$. Then (A16) gives that $h_{\hat p_1,\hat p_2} \sim_k \boldsymbol{k}$ and $|\hat y_2| \sim_k \boldsymbol{k} \succ_k h_{\hat p_1,\hat p_2}^{\frac{m}{m+1}}$, a contradiction with (83). □

Lemma A.3. *Let $i_0 \ge 2$ be the minimal number satisfying Lemma A.2. We can assume*

$$\text{(i) } a_i = 0,\ i=2,...,2i_0,\ \text{(ii) } b_i = 0,\ i=2,...,2i_0,\ \text{(iii) } c_i = d_i,\ i=2,...,i_0-1,\ c_{i_0}\ne d_{i_0}. \tag{A17}$$

Proof. To solve s from (A9), we can replace $\bar F_j$ by $\bar F_j + \sum_{i=2}^{2i_0}\beta_i \bar G_j^i$ for some $\beta_i \in \mathbb{C}$ and $j = 1, 2$ (observe that i_0 is still the minimal number satisfying Lemma A.2 after the replacement by considering either $c_{i_0} \ne d_{i_0}$ or $c_{i_0} = d_{i_0}$, $a_{i_0} \ne b_{i_0}$), thanks to the term y in $\bar G_1$, we can then suppose (A17) (i) holds. Assume $b_k \ne 0$ for some $k \le 2i_0$. Take minimal such $k \ge 2$. Setting [noting from (A13) that this amounts to setting $x = u\varepsilon = \check u\varepsilon^k$, $z = w\varepsilon$ in $\bar F_2$, $\bar G_2$, and setting $x = s\varepsilon$, $y = t\varepsilon$ in $\bar F_1$, $\bar G_1$, and letting $\bar F_1 = \bar F_2$, $\bar G_1 = \bar G_2$ to solve s, t],

$$u = \check u\varepsilon^{k-1},\qquad v = d^{-1}(w-cu), \tag{A18}$$

and regarding $\check u, w$ as new variables, by (A13) (i), we have

$$\bar F_2|_{(x,z)=(\check u\varepsilon^k, w\varepsilon)} = (b_k w^k + a\check u)\varepsilon^k + O(\varepsilon)^{k+1} = \bar F_1(s\varepsilon, t\varepsilon) = O(\varepsilon)^{k+1} + s\varepsilon\Big(1+O(t\varepsilon)^1\Big) + O(s\varepsilon)^2. \tag{A19}$$

This shows that we have $s\varepsilon = O(\varepsilon)^k$ and the right-hand side of (A19) becomes $s\varepsilon + O(\varepsilon)^{k+1}$. Hence,

$$s = (b_k w^k + a\check u)\varepsilon^{k-1} + O(\varepsilon)^k. \tag{A20}$$

Using this, (A12) and the first equation of (A18) in (A11), we obtain by choosing $\check{u} = 0$ and $w \in \mathbb{C}$ with $(b_k w^k)_{\rm re} < 0$,

$$C_0 := \beta_1 \Big| 1 + (-\beta_1^{-1}\beta_2 \check{u} + b_k w^k)\varepsilon^k + O(\varepsilon)^{k+1} \Big|^2 + \beta_2 |1 + \check{u}\varepsilon^k|^2 - (\beta_1 + \beta_2)$$
$$= 2\beta_1 (b_k w^k)_{\rm re} \varepsilon^k + O(\varepsilon)^{k+1} < 0.$$

This proves Proposition A.1 in this case. Therefore, we can assume (A17) (ii) holds. Then we have (A17) (iii) by Lemma A.2. □

Lemma A.4. *We have* $\hat{a}_i = \hat{b}_i$ *for* $1 \le i \le i_0 - 2$ *and* $\kappa_0 := \hat{b}_{i_0-1} - \hat{a}_{i_0-1} = i_0(c_{i_0} - d_{i_0}) \neq 0$.

Proof. For $1 \le i \le i_0 - 1$, by (A13) we have,

$$\hat{a}_i + \sum_{1 \le j < i} (j+1)\hat{a}_{j-i} c_{j+1} + (i+1)c_{i+1} = \mathrm{C}_{\rm oeff}\big(J(\bar{F}_1, \bar{G}_1), x^0 y^i\big) = 0,$$
$$\hat{b}_i + \sum_{1 \le j < i} (j+1)\hat{b}_{j-i} d_{j+1} + (i+1)d_{i+1} = \frac{\mathrm{C}_{\rm oeff}\big(J(\bar{F}_2, \bar{G}_2), x^0 y^i\big)}{ad} = 0. \tag{A21}$$

Induction on i for $1 \le i \le i_0$, we obtain the lemma by (A17) (iii). □

Now we set,

$$u = u_1 \boldsymbol{i} \varepsilon^{i_0 - 1}, \qquad v = d^{-1}(w - cu), \tag{A22}$$

for $u_1 \in \mathbb{R}_{\neq 0}$. As in (A19), one can see that $s\varepsilon = O(\varepsilon)^{i_0}$. Thus by (A13) (ii), we have

$$\bar{G}_1(s\varepsilon, t\varepsilon) = t\varepsilon + \sum_{i=2}^{i_0} c_i (t\varepsilon)^i + O(\varepsilon)^{i_0+1} = \bar{G}_2|_{(x,z)=(u_1 \boldsymbol{i}\varepsilon^{i_0}, w\varepsilon)} = w\varepsilon + \sum_{i=2}^{i_0} d_i (w\varepsilon)^i + O(\varepsilon)^{i_0+1}. \tag{A23}$$

Hence $t\varepsilon = w\varepsilon + O(\varepsilon)^{i_0}$. Using this and (A13) (i), we obtain,

$$\bar{F}_1(s\varepsilon, t\varepsilon) = s\varepsilon \Big(1 + \sum_{i=1}^{i_0-1} \hat{a}_i (t\varepsilon)^i\Big) + O(\varepsilon)^{2i_0}$$
$$= \bar{F}_2|_{(x,z)=(u_1 \boldsymbol{i}\varepsilon^{i_0}, w\varepsilon)} = a u_1 \boldsymbol{i} \varepsilon^{i_0} \Big(1 + \sum_{i=1}^{i_0-1} \hat{b}_i (w\varepsilon)^i\Big) + O(\varepsilon)^{2i_0}, \tag{A24}$$

which with Lemma A.4 gives that $s\varepsilon = a u_1 \boldsymbol{i} \varepsilon^{i_0}(1 + \kappa_1 w^{i_0-1}\varepsilon^{i_0-1}) + O(\varepsilon)^{2i_0}$. Thus C_0 defined in (A11) becomes,

$$C_0 = \beta_1 \Big| 1 - \beta_2 \beta_1^{-1} u_1 \boldsymbol{i} \varepsilon^{i_0}(1 + \kappa_1 w^{i_0-1}\varepsilon^{i_0-1}) + O(\varepsilon)^{2i_0} \Big|^2 + \beta_2 |1 + u_1 \boldsymbol{i} \varepsilon^{i_0}|^2 - \beta_1 - \beta_2$$
$$= 2\beta_2 u_1 (\kappa_1 w^{i_0-1})_{\rm im} \varepsilon^{2i_0-1} + O(\varepsilon)^{2i_0}, \tag{A25}$$

which is negative if we choose $u_1 = 1$ and $w \in \mathbb{C}$ with $(\kappa_1 w^{i_0-1})_{\rm im} < 0$. This proves Proposition A.1 in case $\beta_1 > 0$, $\beta_2 > 0$.

Now if $x_1 = \xi_1$ (then $x_2 \neq \xi_2$), then the first term of C_0 in (A11) becomes $|s\varepsilon|^2 = O(\varepsilon)^2$ and we can easily choose any u with $u_{\rm re} < 0$ to satisfy that $C_0 < 0$. Similarly, if $x_2 = \xi_2$ (then $x_1 \neq \xi_1$), then the second term of C_0 in (A11) becomes $|u\varepsilon|^2 = O(\varepsilon)^2$ and using (A10), we can easily choose u, v with $(au)_{\rm re} < 0$ and $v = 0$ (in case $a \neq 0$) or with $u = 0$ and $(bv)_{\rm re} < 0$ (in case $b \neq 0$) to satisfy that $C_0 < 0$. This proves Proposition A.1. □

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 6 September 2023 doi:10.20944/preprints202212.0171.v2

Appendix B Another proof of Proposition 3.8

To see our original idea on how to obtain Proposition 3.8, for your reference, below we wish to present our original proof of Proposition 3.8.

Another proof of Proposition 3.8. We can more precisely rewrite (132) as follows, for some $\tilde\alpha_i, \tilde\beta_i \in \mathbb{C}$,

$$\begin{aligned}&\text{(i) } C_1 := \big|1 + (\tilde\alpha_1 u + \tilde\alpha_2 v)\varepsilon_1 + (\tilde\beta_1 u^2 + \tilde\beta_2 uv + \tilde\beta_3 v^2)\varepsilon_1^2\big| - 1 + O(\varepsilon_1)^3 \ge 0,\\ &\text{(ii) } C_2 := \kappa_1'\big|1 + (\tilde\alpha_3 u + \tilde\alpha_4 v)\varepsilon_1 + (\tilde\beta_4 u^2 + \tilde\beta_5 uv + \tilde\beta_6 v^2)\varepsilon_1^2\big| + |1 + u\varepsilon_1|\\ &\qquad + \kappa_2'|1 + v\varepsilon_1| - (\kappa_1' + 1 + \kappa_2') + O(\varepsilon_1)^3 > 0.\end{aligned} \tag{A26}$$

First assume $\tilde\alpha_1 \ne 0$. Looking at (A26) (i), we can take

$$u = -\tilde\alpha_1^{-1}\tilde\alpha_2 v + (\beta_1 v^2 + \beta_2 w)\varepsilon_1 \text{ for some } \beta_i, w \in \mathbb{C} \text{ with } w_{\rm re} > 0, \tag{A27}$$

so that C_1 has the form [the fact is that we can use (A26) (i), (A27) to solve β_1, β_2 precisely from the first equality below],

$$C_1 = |1 + w\varepsilon_1^2| - 1 + O(\varepsilon_1)^3 = w_{\rm re}\varepsilon_1^2 + O(\varepsilon_1)^3 > 0, \tag{A28}$$

i.e., (A26) (i) holds. Using (A27) in (A26) (ii), we see that C_2 becomes the following form, for some $\tilde\alpha_i \in \mathbb{C}$,

$$\begin{aligned}C_2 = {}&\kappa_1'\big|1 + \tilde\alpha_5 v\varepsilon_1 + (\tilde\alpha_6 v^2 + \tilde\alpha_7 w)\varepsilon_1^2\big| + \big|1 + \tilde\alpha_8 v\varepsilon_1 + (\tilde\alpha_9 v^2 + \tilde\alpha_{10} w)\varepsilon_1^2\big|\\ &+ \kappa_2'|1 + v\varepsilon_1| - (\kappa_1' + 1 + \kappa_2') + O(\varepsilon_1)^3 > 0.\end{aligned} \tag{A29}$$

Observe that for any $\alpha = \alpha_{\rm re} + \alpha_{\rm im}\boldsymbol{i} \in \mathbb{C}$ [recall Convention 2.3 (1)], we have,

$$\begin{aligned}&\text{(i) } |1 + \alpha\varepsilon_1| = \sqrt{(1 + \alpha_{\rm re}\varepsilon_1)^2 + (a_{\rm im}\varepsilon_1)^2} = 1 + \alpha_{\rm re}\varepsilon_1 + \frac{(\alpha_{\rm im})^2}{2}\varepsilon_1^2 + O(\varepsilon_1)^3,\\ &\text{(ii) } A_0 := |1 + \alpha v^2\varepsilon_1^2| = 1 + \Big(\alpha_{\rm re}\big((v_{\rm re})^2 - (v_{\rm im})^2\big) - 2\alpha_{\rm im}v_{\rm re}v_{\rm im}\Big)\varepsilon_1^2 + O(\varepsilon_1)^3.\end{aligned} \tag{A30}$$

We see that C_2 is an $O(\varepsilon_1)^1$ element and we can easily compute

$$\mathrm{C_{oeff}}(C_2, \varepsilon_1) = \kappa_1'(\tilde\alpha_5 v)_{\rm re} + (\tilde\alpha_8 v)_{\rm re} + \kappa_2' v_{\rm re} = (c_0 v)_{\rm re}, \ \text{ where } \ c_0 = \kappa_1'\tilde\alpha_5 + \tilde\alpha_8 + \kappa_2'. \tag{A31}$$

Thus if $c_0 \ne 0$, we can always choose $v \in \mathbb{C}$ with $(c_0 v)_{\rm re} > 0$ to satisfy (A29).

Hence assume $c_0 = 0$. Then C_2 in (A29) becomes an $O(\varepsilon_1)^2$ element. Our purpose is to compute $\tilde\beta$ defined in (A34) below. First using notation (11), one can observe from (A30) (ii) the facts that $\mathrm{C_{oeff}}\big(A_0, (v_{\rm re})^2\varepsilon_1^2\big) = \alpha_{\rm re}$ and $\mathrm{C_{oeff}}\big(A_0, (v_{\rm im})^2\varepsilon_1^2\big) = -\alpha_{\rm re}$, which imply the following important fact,

$$\mathrm{C_{oeff}}\big(A_0, (v_{\rm re})^2\varepsilon_1^2\big) + \mathrm{C_{oeff}}\big(A_0, (v_{\rm im})^2\varepsilon_1^2\big) = \alpha_{\rm re} - \alpha_{\rm re} = 0. \tag{A32}$$

From this and (A29), (A30) (ii), one can observe that $\tilde\alpha_6$, $\tilde\alpha_7$, $\tilde\alpha_9$, $\tilde\alpha_{10}$ do not contribute to $\tilde\beta$ defined in (A34). Then by (A30) (i) and noting that $\big((\tilde\alpha_5 v)_{\rm im}\big)^2 = (\tilde\alpha_{5\,\rm re}v_{\rm im} + \tilde\alpha_{5\,\rm im}v_{\rm re})^2$, we obtain

$$\begin{aligned}\mathrm{C_{oeff}}(2C_2, \varepsilon_1^2) = {}&\big(\kappa_1'(\tilde\alpha_{5\,\rm re})^2 + (\tilde\alpha_{8\,\rm re})^2 + \kappa_2'\big)(v_{\rm im})^2\\ &+ \big(\kappa_1'(\tilde\alpha_{5\,\rm im})^2 + (\tilde\alpha_{8\,\rm im})^2\big)(v_{\rm re})^2 + 2(\kappa_1'\tilde\alpha_{5\,\rm re}\tilde\alpha_{5\,\rm im} + \tilde\alpha_{8\,\rm re}\tilde\alpha_{8\,\rm im})v_{\rm re}v_{\rm im} + \cdots,\end{aligned} \tag{A33}$$

where $\cdots$ means terms contributed by $\tilde\alpha_6$, $\tilde\alpha_7$, $\tilde\alpha_9$, $\tilde\alpha_{10}$, which do not contribute the following, therefore, we obtain the following crucial fact,

$$\begin{aligned}\tilde\beta &:= \mathrm{C_{oeff}}\big(2C_2, (v_{\rm re})^2\varepsilon_1^2\big) + \mathrm{C_{oeff}}\big(2C_2, (v_{\rm im})^2\varepsilon_1^2\big)\\ &= \kappa_1'\big((\tilde\alpha_{5\,\rm re})^2 + (\tilde\alpha_{5\,\rm im})^2\big) + (\tilde\alpha_{8\,\rm re})^2 + (\tilde\alpha_{8\,\rm im})^2 + \kappa_2' \ge \kappa_2' > 0. \end{aligned} \tag{A34}$$

This will ensure us to achieve our goal: we can choose v with $(v_{\rm re})^2$ being sufficiently larger than $(v_{\rm im})^2$ if $\mathrm{C_{oeff}}\big(C_2, (v_{\rm re})^2\varepsilon_1^2\big) > 0$ or with $(v_{\rm im})^2$ being sufficiently larger than $(v_{\rm re})^2$ if $\mathrm{C_{oeff}}\big(C_2, (v_{\rm im})^2\varepsilon_1^2\big) > 0$, to guarantee that (A29) holds (when w is fixed). This proves Proposition 3.8 for the case that $\tilde\alpha_1 \ne 0$.

Now assume $\tilde\alpha_1 = 0$. By the symmetry of u and v in (A26), we may also assume $\tilde\alpha_2 = 0$. Then we have one of the following,

$$\text{(i) } C_1 = 0, \quad \text{or} \quad \text{(ii) } C_1 = |1 + g(u,v)\varepsilon_1^k| - 1 + O(\varepsilon_1)^{k+1}, \tag{A35}$$

for some nonzero homogeneous polynomial $g(u,v)$ of u, v with degree $k \in \mathbb{Z}_{\ge 2}$, and we assume we have (A35) (ii) as (A26) (i) holds trivially for case (A35) (i).

In case $c_1 := \kappa_1'\tilde\alpha_3 + 1 \ne 0$ [see (A26) (ii)], we can solve the problem as follows: Let $\alpha \in \mathbb{C}$ be determined later, and take $v = \alpha u$. Then by (A30) (i), as in (A31), we can compute from (A26) (ii), (A35) (ii),

$$\begin{aligned}&\text{(i) } \mathrm{C_{oeff}}(C_2, \varepsilon_1) = \Big(\big(\kappa_1'(\tilde\alpha_3 + \tilde\alpha_4\alpha) + 1 + \kappa_2'\alpha\big)u\Big)_{\rm re} = (\tilde c_1 u)_{\rm re} \ \text{ with } \ \tilde c_1 = c_1 + (\kappa_1'\tilde\alpha_4 + \kappa_2')\alpha,\\ &\text{(ii) } \mathrm{C_{oeff}}(C_1, \varepsilon_1^k) = \big(g_1(\alpha)u^k\big)_{\rm re} \ \text{ with } \ g_0(\alpha) = g(1,\alpha). \end{aligned} \tag{A36}$$

Note that $g_0(\alpha)$ is a nonzero polynomial of α with degree $\le k$. We can always $\alpha \in \mathbb{C}$ such that $\tilde c_1 \ne 0$ and $g_0(\alpha) \ne 0$. Then we choose $u \in \mathbb{C}$ with $(\tilde c_1 u)_{\rm re} > 0$ so that (A26) (ii) holds by (A36) (i), and further $\big(g_0(\alpha)u^k\big)_{\rm re} > 0$ (this can be always done since $k \ge 2$), i.e., $C_1 > 0$ by (A35) (ii), (A36) (ii).

If $c_2 := \kappa_1'\tilde\alpha_4 + \kappa_2' \ne 0$, we can solve the problem symmetrically.

Now assume $c_1 = c_2 = 0$, i.e.,

$$\tilde\alpha_3 = -\kappa_1'^{-1} \in \mathbb{R}, \quad \tilde\alpha_4 = -\kappa_2'\kappa_1'^{-1} \in \mathbb{R}. \tag{A37}$$

Then C_2 is an $O(\varepsilon_1)^2$ element. One can easily compute as in (A34),

$$\mathrm{C_{oeff}}\big(2C_2, (v_{\rm re})^2\varepsilon_1^2\big) + \mathrm{C_{oeff}}\big(2C_2, (v_{\rm im})^2\varepsilon_1^2\big) = \kappa_1'\alpha_4^2 + \kappa_2' = \kappa_2' + \kappa_2'^2\kappa_1'^{-1} > 0. \tag{A38}$$

If $g(u,v)$ does not depend on v [i.e., $g(u,v) = b''u^k$ for some $b'' \in \mathbb{C}_{\ne 0}$], then we can first choose $u \in \mathbb{C}$ to satisfy that $g(u,v)_{\rm re} = (b''u^k)_{\rm re} > 0$ then choose $v \in \mathbb{C}$ with $(v_{\rm re})^2$ being sufficiently larger than $(v_{\rm im})^2$ if $\mathrm{C_{oeff}}\big(C_2, (v_{\rm re})^2\varepsilon_1^2\big) > 0$ or with $(v_{\rm im})^2$ being sufficiently larger than $(v_{\rm re})^2$ if $\mathrm{C_{oeff}}\big(C_2, (v_{\rm im})^2\varepsilon_1^2\big) > 0$, to guarantee that $\mathrm{C_{oeff}}\big(C_2, \varepsilon_1^2\big) > 0$, i.e., (A26) (ii) holds.

Thus assume $g(u,v)$ depend on v. We set $v = \alpha u$ with $\alpha, u \in \mathbb{C}$ being determined later. Then (A35) (ii) and (A26) (ii) become the following forms, for some $\tilde\alpha_{11} \in \mathbb{C}$, and the non-constant polynomial $g_0(\alpha) := g(1,\alpha)$ of α with degree $\le k$ [where the number $\tilde\alpha := -\kappa_1'^{-1}(1+\alpha)$ in C_2 is obtained from $\tilde\alpha_3 u + \tilde\alpha_4 v$ using (A37), and $\tilde\alpha_{11} = \tilde\beta_4 + \alpha\tilde\beta_5 + \alpha^2\tilde\beta_6$],

$$\begin{aligned}&\text{(i) } C_1 = |1 + g_0(\alpha)u^k\varepsilon_1^k| - 1 + O(\varepsilon_1)^{k+1},\\ &\text{(ii) } C_2 = \kappa_1'|1 + \tilde\alpha u\varepsilon_1 + \tilde\alpha_{11}u^2\varepsilon_1^2| + |1 + u\varepsilon_1| + \kappa_2'|1 + \alpha u\varepsilon_1| - (\kappa_1' + 1 + \kappa_2') + O(\varepsilon_1)^3 > 0. \end{aligned} \tag{A39}$$

Since $g_0(\alpha)$ is a non-constant polynomial of α, we can choose $\alpha \in \mathbb{C}$ satisfying (i) below, then we can choose $\theta, u \in \mathbb{C}$ satisfying, where $\delta > 0$ is sufficiently small,

$$\begin{aligned}
&\text{(i) } g_0(\alpha) = 1 \text{ if } k \equiv 0,1,2,7 \ (\mathrm{mod}\, 8), \text{ and } \ g_0(\alpha) = -1 \text{ else},\\
&\text{(ii) } \theta = 1 \text{ if } (\tilde{\alpha}_{11})_{\rm re} \le 0, \text{ and } \theta = -1 \text{ else},\\
&\text{(iii) } u = \sqrt{2} e^{\frac{\theta(1-\delta)\pi \boldsymbol{i}}{4}},\\
&\text{(iv) } u_{\rm re} = 1 + O(\delta)^1, \ \ u_{\rm im} = \theta + O(\delta)^1, \ \ (u_{\rm re})^2 = (u_{\rm im})^2 + O(\delta)^1 = 1 + O(\delta)^1,
\end{aligned} \tag{A40}$$

where (iv) simply follows from (iii). We can compute from (A39) to obtain the following, where (i) follows from (A40) (i) and the facts that $\theta = \pm 1$ and $\delta > 0$ is sufficiently small, while (ii) is obtained, as in (A33), by using (A30), (A40), (iv) and the fact from (A40) (ii) that $\theta(\tilde{\alpha}_{11})_{\rm re} \ge 0$,

$$\begin{aligned}
&\text{(i) } \mathrm{C_{oeff}}(C_1, \varepsilon_1^k) = \sqrt{2}\Big(g_0(\alpha) e^{\frac{k\theta(1-\delta)\pi \boldsymbol{i}}{4}}\Big)_{\rm re} = \sqrt{2} g_0(\alpha) \cos\Big(\frac{k(1-\delta)\pi \boldsymbol{i}}{4}\Big) > 0,\\
&\text{(ii) } \mathrm{C_{oeff}}(C_2, \varepsilon_1^2) = \frac{1}{2}\Big(\kappa_1'\big((\tilde{\alpha}_{\rm re})^2 + (\tilde{\alpha}_{\rm im})^2\big) + 1 + \kappa_2'\big((\alpha_{\rm re})^2 + (\alpha_{\rm im})^2\big)\Big) - 2\theta(\tilde{\alpha}_{11})_{\rm re} + O(\delta)^1\\
&\qquad \ge \frac{1}{2} + O(\delta)^1 > 0.
\end{aligned} \tag{A41}$$

Thus $C_1 > 0$ and $C_2 > 0$ by (A41). This proves Proposition 3.8. □

Appendix C A proposition provided by Bin Xu

We wish to thank professor Bin Xu from University of Science and Technology of China, who provides us the following material.

Proposition C.1. *Assume that V satisfies* (C1)–(C3), (C4)′ *stated in the beginning of section* 3*. Then the projection $\pi_1 : V \to \mathbb{C}^2$, $(p_1, p_2) \mapsto (x_1, x_2)$ for $(p_1, p_2) \in V$, is not surjective.*

For a proof, we need the following lemma in page 7 of [3].

Lemma C.2. *Let X, Y be two Hausdorff, locally compact topological spaces, and let $\pi : X \to Y$ be both surjective and a local homeomorphism (i.e. for any $\alpha \in X$, there exists an open neighbourhood X_α of α such that $Y_\alpha = \pi(X_\alpha)$ is open in Y and $\pi|_{X_\alpha}$ is a homeomorphism onto Y_α). Then π is a finite covering (i.e. for any $y_0 \in Y$, there is an open neighbourhood Y_{y_0} of y_0 such that $\pi^{-1}(Y_{y_0})$ is a disjoint union $\cup_{j=1}^n X_j$ of open sets X_j of X with the property that $\pi|_{X_j}$ is a homeomorphism onto Y_{y_0} for all $1 \le j \le n$) if and only if it is proper (i.e. for any compact set $K \subset Y$, the inverse image $\pi^{-1}(K)$ is compact in X).*

Proof of Proposition C.1. Assume that π_1 is surjective. By (C3), (C4)′, π_1 is a local homeomorphism. By (C2), it is proper (see also Proposition 2.22). Hence, by Lemma C.2, π_1 is a covering map. Since $\mathbb{C}^2$ is simply connected and V is connected, π_1 is a holomorphic homeomorphism. Then there exist two holomorphic functions ϕ_1, ϕ_2 on $\mathbb{C}^2$ such that

$$V = \Big\{\Big(\big(x_1, \phi_1(x_1, x_2)\big), \big(x_2, \phi_2(x_1, x_2)\big)\Big) \,\Big|\, (x_1, x_2) \in \mathbb{C}^2\Big\}.$$

By using both (C2) and the Taylor developments of ϕ_1 and ϕ_2 on $\mathbb{C}^2$, we see that ϕ_1, ϕ_2 are constant, which contradicts that in V, x_1, y_1 are locally expressed by holomorphic functions on x_2, y_2. □

 doi:10.20944/preprints202212.0171.v2

1. H. Bass, E.H. Connell, D. Wright, The Jacobian conjecture: reduction of degree and formal expansion of the inverse, *Bull. Amer. Math. Soc.* **7** (1982), 287–330.
2. A. Bialynicki-Birula, M. Rosenlicht, Injective morphisms of real algebraic varieties, *Proc. Amer. Math. Soc.* **13** (1962), 200–203.
3. R. Narasimhan, *Compact Riemann Surfaces*, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 1992.
4. D.J. Newman, One-one polynomial maps, *Proc. Amer. Math. Soc.* **11** (1960), 867–870.
5. A. van den Essen, *Polynomial automorphisms and the Jacobian conjecture,* Progress in Mathematics **190**, Birkhäuser Verlag, Basel, 2000.