The Natural Unit Expression of Classical Gravitation

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Abstract

The gravitational constant is equivalent to natural units in its six unit dimensions. Evaluating the classical formulas in each of these dimensions shows that gravitational potentials are proportional to the Planck scale by the ratio of a body's radial density to a radial density limit in the ratio of Planck length to Planck mass.

Keywords: classical gravitation; Planck units; natural units; radial density; Planck scale; proportionality

1 Introduction

The gravitational constant plays an important role in both Newton's and Einstein's formulations of the gravitational interaction. With a single value and six unit dimensions, the constant behaves as a function in the classical formulas, transforming inputs of mass and radius into potentials of energy, velocity, acceleration, force, etc.

While the computational utility of G is incontrovertible, it is not immediately clear why the function works or what physical structure it represents. A more granular look at the formulas in each unit dimension offers additional information about the transformations.

When Max Planck derived the natural units, he assumed that universal constants embody natural units in their unit dimensions [1–4]. As such, the gravitational constant can be expressed as

$$G = \frac{l_P^3}{m_P t_P^2} = \frac{l_P}{m_P} c^2 = 6.67430 \times 10^{-11}\ m^3/kgs^2. \tag{1}$$

An evaluation of the classical formulas in natural units shows that G can be characterized in two parts:

1. The ratio of Planck length to Planck mass gives a computational basis or limit for quantifying the gravitational field generated by mass M and radius r.

2. Two instances of c give a computational basis for quantifying the gravitational field in terms of the momentum and velocity of a test particle or second body.

2 Radial density

Hidden beneath the compound unit dimensions of G, the classical formulas for calculating gravitational potentials compare a body's mass and radius to a Planck scale basis of Planck mass and Planck length.

Two signature inputs into the formulas—mass M in the numerator and radius r in the denominator—produce a dimensionless ratio

$$\frac{l_P}{r}\frac{M}{m_P} \tag{2}$$

which quantifies the gravitational field on a scale of 0 to 1, where 1 is the Planck scale. This ratio between the radial density of a massive body and a Planck scale radial density limit l_p/m_P is the correct scaling factor for quantifying gravitational potentials induced by mass M at distance r. The following evaluation of the formulas employs Equation 2 as a *radial density ratio* for quantifying gravitational potentials in proportion to the Planck scale.

2.1 Gravitational potential energy

The gravitational potential energy of a massive body given by

$$U_g = -\frac{GMm}{r} \tag{3}$$

can be stated according to Equation 1 as

$$U_g = -\left(\frac{l_P}{m_P}c^2\right)\frac{Mm}{r} \tag{4}$$

$$= -\left(\frac{l_P}{r}\frac{M}{m_P}\right)mc^2. \tag{5}$$

Equation 5 illustrates how the *radial density ratio* transforms c^2 into the correct momentum and velocity of a test particle. Setting the gravitational potential energy equal to the particle's kinetic energy

$$\left(\frac{l_P}{r}\frac{M}{m_P}\right)mc^2 = \frac{1}{2}mv^2 \tag{6}$$

gives a translation between the attributes of the massive body and the test particle. The formula illustrates how the radial density of the first body influences the momentum and velocity of the test particle and can be simplified to

$$2\frac{l_P}{r}\frac{M}{m_P} = \frac{v^2}{c^2}. \tag{7}$$

2.2 Velocity

From Equation 7 we may expect to find the escape velocity by taking the square root of twice the *radial density ratio*, and this is how the function works

$$v_e = \sqrt{\frac{2GM}{r}} \tag{8}$$

$$= \sqrt{2\left(\frac{l_P}{m_P}c^2\right)\frac{M}{r}} \tag{9}$$

$$= \sqrt{2\frac{l_P}{r}\frac{M}{m_P}}\;c. \tag{10}$$

The versatility of the gravitational constant is evident in the manner by which the functions produce 1) the correct scaling factor for a given potential and 2) natural units in the correct unit dimensions for each type of potential.

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2.3 Schwarzschild radius

The Schwarzschild radius formula does not quantify the momentum or velocity of a second body, so two superfluous quantities of c in the gravitational constant are removed from the equation

$$R_s = \frac{2GM}{c^2} \tag{11}$$

$$= 2\left(\frac{l_P}{m_P}\cancel{c^2}\right)\frac{M}{\cancel{c^2}} \tag{12}$$

$$= 2\frac{M}{m_P}l_P. \tag{13}$$

Furthermore, re-arranging 13 gives a definition of the Schwarzschild radius as one-half the *radial density ratio*

$$\frac{l_P}{R_s}\frac{M}{m_P} = \frac{1}{2}. \tag{14}$$

This insightful relationship between the Schwarzschild radius and the density limit l_P/m_P is not immediately apparent in the compound unit dimensions of the gravitational constant.

2.4 Acceleration

The gravitational acceleration function combines the *radial density ratio* with the inverse radial distance in natural units l_P/r. These ratios quantify the velocity derivative in proportion to the Planck acceleration

$$g = -\frac{GM}{r^2} \tag{15}$$

$$= -\left(\frac{l_P^3}{m_P t_P^2}\right)\frac{M}{r^2} \tag{16}$$

$$= -\left(\frac{l_P}{r}\frac{M}{m_P}\right)\frac{l_P}{r}a_P. \tag{17}$$

The formula is equivalent to calculating the orbital velocity squared and removing one unit and dimension of distance

$$= -\left(\frac{l_P}{r}\frac{M}{m_P}\right)c^2\frac{1}{r}. \tag{18}$$

2.5 Force

The formula for determining gravitational force

$$F = \frac{GMm}{r^2} \tag{19}$$

combines the mass of a second body with the gravitational acceleration potential generated by the first body in Equation 17 according to $F = ma$

$$F = ma = -\left(\frac{l_P}{r}\frac{M}{m_P}\right)\frac{l_P}{r}ma_P. \tag{20}$$

Gravitational force can also be expressed as a ratio of the Planck force by taking the *radial density* *ratio* of both bodies. Inserting Planck mass in the numerator and denominator gives

$$F = \left(\frac{l_P^3}{m_P t_P^2}\right)\frac{Mm}{r^2}\left(\frac{m_P}{m_P}\right) \tag{21}$$

$$F = \left(\frac{l_P}{r}\frac{M}{m_P}\right)\left(\frac{l_P}{r}\frac{m}{m_P}\right)F_P. \tag{22}$$

The natural unit expression of the formula suggests that, while producing the correct result, formulas using the gravitational constant may omit important structural information.

2.6 Gravitational binding energy

The formula for gravitational binding energy

$$U = \frac{3GM^2}{5R} \tag{23}$$

uses the *radial density ratio* to determine the total energy of a massive body. In the formula, the energy limit Mc^2 is proportionally rescaled to $1/2 Mv^2$ depending on the body's radial density and in agreement with Equation 7

$$U = \frac{3}{5}\left(\frac{l_P}{m_P}c^2\right)\frac{M^2}{R} \tag{24}$$

$$= \frac{3}{5}\left(\frac{l_P}{R}\frac{M}{m_P}\right)Mc^2 \tag{25}$$

$$= \frac{3}{10}Mv^2 \tag{26}$$

Table 1 summarizes the classical gravitational formulas in natural units of length, mass, and time.

Table 1: A summary of certain classical gravitational formulas in natural units.

Physical property	Symbol	Standard formula	Natural Unit Formula
Schwarzschild radius	R_s	$\frac{2GM}{c^2}$	$2\frac{M}{m_P}l_P$
Escape velocity	v_e	$-\sqrt{\frac{2GM}{r}}$	$-\sqrt{2\left(\frac{l_P}{r}\frac{M}{m_P}\right)}c$
Gravitational potential energy	U_g	$-\frac{GMm}{r}$	$-\left(\frac{l_P}{r}\frac{M}{m_P}\right)\frac{m}{m_P}E_P$
Gravitational acceleration	g	$-\frac{GM}{r^2}$	$-\left(\frac{l_P}{r}\frac{M}{m_P}\right)\frac{l_P}{r}a_P$
Gravitational force	F	$\frac{GMm}{r^2}$	$\left(\frac{l_P}{r}\frac{M}{m_P}\right)\left(\frac{l_P}{r}\frac{m}{m_P}\right)F_P$
Gravitational binding energy	U	$-\frac{3GM^2}{5R}$	$-\frac{3}{5}\left(\frac{l_P}{R}\frac{M}{m_P}\right)\frac{M}{m_P}E_P$
Hawking radiation	$k_B T$	$\frac{\hbar c^3}{8\pi GM}$	$\frac{1}{8\pi}\left(\frac{m_P}{M}\right)E_P$

3 Radial density limit

The formulas discussed in Section 2 quantify gravitational potentials in proportion to the ratio of Planck length to Planck mass, making this ratio an important physical constant

$$\frac{l_P}{m_P} = 7.42616 \times 10^{-28}\ m/kg. \tag{27}$$

An interesting property of the limit is how it relates to mass density as a body grows larger. While each natural unit increment of radius accommodates no more than one natural unit of mass, the area and volume of the body continue to grow in accordance with known properties of black holes.

Since the Schwarzschild radius is defined by equation 14 in relation to the radial density limit, the formulas in Table 1 can also be stated in terms of the Schwarzschild radius. Separating l_P/r into two parts

$$\frac{l_P}{r} = \frac{l_P}{R_s}\frac{R_s}{r} \tag{28}$$

and plugging into Equation 2 gives

$$\frac{l_P}{r}\frac{M}{m_P} = \frac{l_P}{R_s}\frac{R_s}{r}\frac{M}{m_P}. \tag{29}$$

From the Schwarzschild radius definition in Equation 14, the formula can be reduced to

$$\frac{l_P}{r}\frac{M}{m_P} = \frac{R_s}{2r}. \tag{30}$$

Table 2 summarizes the classical gravitational potentials in terms of the ratio between one-half the Schwarzschild radius and distance r.

Table 2: Classical gravitational potentials stated in terms of the Schwarzschild radius.

Physical property	Symbol	Simple formula	Natural unit formula
Escape velocity	v_e	$-\sqrt{\frac{R_s}{r}}\,c$	$-\sqrt{\frac{R_s}{r}}\frac{l_P}{t_P}$
Gravitational potential energy	U_g	$-\frac{R_s}{2r}mc^2$	$-\frac{R_s}{2r}\frac{m}{m_P}E_P$
Gravitational acceleration	g	$-\frac{R_s}{2r}\frac{c^2}{r}$	$-\frac{R_s}{2r}\frac{l_P}{r}a_P$
Gravitational force	F	$\frac{R_s}{2r}\frac{mc^2}{r}$	$\frac{R_s}{2r}\frac{l_P}{r}\frac{m}{m_P}F_P$
Gravitational binding energy	U	$-\frac{R_s}{2R}Mc^2$	$-\frac{R_s}{2R}\frac{M}{m_P}E_P$

4 Proportionality

The equations in Section 2 require just two ratios to quantify classical gravitational potentials as a function of radial density: one ratio of radius to the Planck length and a second ratio of mass to the Planck mass. These two ratios offer a simple method for analyzing how the strength of gravity is related to the properties of massive bodies. Tables 3 and 4 simplify this analysis by restating the classical formulas in terms of these ratios where M is the mass of a first body and m_0 is the mass of a neutron test particle

$$\beta_r = \frac{l_P}{r} \tag{31}$$

$$\beta_M = \frac{M}{m_P} \tag{32}$$

$$\beta_m = \frac{m_0}{m_P}. \tag{33}$$

These dimensionless ratios act as coefficients in the formulas, diluting Planck scale gravitational potentials based on the distribution of mass of the two bodies. Table 3 displays the coefficients for several massive bodies [5–7] and a neutron test particle indicating the degree to which gravitational potentials are reduced from the Planck scale.

Table 3: Ratios in the table act as coefficients, collectively diluting Planck scale gravitational potentials on a scale of 0 to 1

Massive body	Body type	β_r $\frac{l_P}{r}$	β_M $\frac{M}{m_P}$	$\beta_r\,\beta_M$ $\frac{l_P}{r}\frac{M}{m_P}$	β_m $\frac{m}{m_P}$
Sagittarius A*	Supermassive BH	1.32×10^{-45}	3.80×10^{44}	0.50	7.70×10^{-20}
GRO J1655-40	Steller mass BH	1.03×10^{-39}	4.85×10^{38}	0.50	7.70×10^{-20}
4U 1820-30	Neutron star	1.78×10^{-39}	1.44×10^{38}	0.26	7.70×10^{-20}
Sirius B	White dwarf	2.76×10^{-42}	9.45×10^{37}	2.61×10^{-4}	7.70×10^{-20}
Sun	Star	2.32×10^{-44}	9.14×10^{37}	2.12×10^{-6}	7.70×10^{-20}
Earth	Planet	2.54×10^{-42}	2.74×10^{32}	6.96×10^{-10}	7.70×10^{-20}

The coefficients in Table3 are plugged into the natural unit formulas in Table 4 to give the correct gravitational potentials as a proportion of the Planck scale.

Table 4: The coefficients in Table 3 quantify gravitational potentials in proportion to the Planck scale.

Massive body	Schwarzschild radius $2\,\beta_M\,\boldsymbol{l_P}$ m	Escape velocity $\sqrt{2\,\beta_r\,\beta_M}\,\boldsymbol{c}$ m/s	Potential energy $\beta_r\,\beta_M\,\beta_m\,\boldsymbol{E_P}$ kgm^2/s^2	Gravitational acceleration $\beta_r\,\beta_M\,\beta_r\,\boldsymbol{a_P}$ m/s^2	Gravitational force $\beta_r\,\beta_M\,\beta_r\,\beta_m\,\boldsymbol{F_P}$ kgm/s^2
Planck scale potential	$\mathbf{1.62\times10^{-35}}$	$\mathbf{-299,792,458}$	$\mathbf{-1.96\times10^{9}}$	$\mathbf{-5.56\times10^{51}}$	$\mathbf{1.21\times10^{44}}$
Sagittarius A*	1.23×10^{10}	$-299,792,458$	-7.53×10^{-11}	-3.66×10^{6}	6.14×10^{-21}
GRO J1655-40	$15,682$	$-299,792,458$	-7.53×10^{-11}	-2.87×10^{12}	4.80×10^{-15}
4U 1820-30	$4,666$	$-214,676,872$	-3.86×10^{-11}	-2.53×10^{12}	4.24×10^{-15}
Sirius B	$3,054$	$-6,850,855$	-3.93×10^{-14}	-4.01×10^{6}	6.72×10^{-21}
Sun	$2,953$	$-617,482$	-3.19×10^{-16}	-273.85	4.59×10^{-25}
Earth	0.0089	$-11,186$	-1.05×10^{-19}	-9.82	1.64×10^{-26}

The formulas in Table 4 show that gravity is not necessarily a weak force but its magnitude is determined by the size, density, and distance between massive bodies. In order for a pair of bodies to generate a force comparable to the Planck force, each of the bodies must have the Planck mass at a distance of one Planck length between them. As the tables demonstrate, these values produce coefficients of 1 yielding the Planck force as a result. The relatively weak strength of gravity in an earth-like environment by comparison is due to the size and density of massive bodies and distances between them.

5 Conclusion

The gravitational constant is a versatile number for quantifying classical gravitational potentials. However, re-stating the constant in natural units provides greater resolution to the formulas and deeper insight into the relationship between the distribution of mass and the gravitational potential it generates.

Natural unit formulas highlight the essential role of the Planck scale in quantifying classical gravitational field strengths. The radial density on one side of the equation and the resulting field potential on the other are both quantified in relation to these natural units.

The radial density limit described by the formulas—the ratio of Planck length to Planck mass—gives a radius of one-half the Schwarzschild radius and permits no more than one natural unit of mass per natural unit of radial distance.

The Planck scale basis in the formulas suggests that gravity may have the same potential as the other forces; however, the formulas give no indication whether the extreme conditions required to produce Planck scale gravitational potentials are physically possible.

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