# On one-point time on an oriented set

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**Abstract**. The notion of oriented set is the basic elementary concept of the theory of changeable sets. The main motivation for the introduction of changeable sets was the sixth Hilbert problem, that is, the problem of mathematically rigorous formulation of the fundamentals of theoretical physics. In the present paper the necessary and sufficient condition of the existence of one-point time on an oriented set is established. From the intuitive point of view, one-point time is the time associated with the evolution of a system consisting of only one object (for example, from one material point). Namely, it is proven that the one-point time exists on the oriented set if and only if this oriented set is a quasi-chain. Also, using the obtained result, the problem of describing all possible images of linearly ordered sets is solved. This problem naturally arises in the theory of ordered sets.

**Keywords**: Oriented sets, changeable sets, time, ordered sets, axiom of choice

## 1 Introductory remarks

The subject of this article is closely related to the theory of changeable sets. The main motivation for building this theory was the sixth Hilbert problem, that is, the problem of mathematically rigorous formulation of the fundamentals of theoretical physics. This problem was posed in 1900, but it remains very relevant today [1]. From an intuitive point of view changeable sets are the sets of objects, which can be in process of continuous transformations. In particular, these objects can change their properties, appear or disappear, break down into several parts or, conversely, unite into a single unit. Moreover, the evolution of a changeable set or it's components may depend of the area of observation or reference frame. The problem of constructing the mathematical theory of changeable sets (that is the "sets" possessing the properties listed above) was emerged in particular in the papers [2–6]. On the mathematically strict level the theory of changeable sets was developed in the papers [7–10] etc. The most complete and systematic presentation of this theory can be found in the preprint [11].

The notion of oriented set is the basic most elementary concept of the theory of changeable sets. Oriented sets were introduced in [7,8] as most simple abstract models of the collections of evolving objects in the framework of one (fixed) reference frame (see also [11, Section 1]). Moreover, in the above-mentioned papers it was introduced the concept of time on oriented sets. As well in the article [8, Theorem 4.1] the sufficient condition of existence of one-point time for oriented sets is established (see also [11, Theorem 1.3.1]). Note that from the intuitive point of view, one-point time should be understood as the time associated with the evolution of a system consisting of only one object (for example, from one material point). Emphasize that Theorem 4.1 from [8] gives only sufficient condition for existence of one-point time. That is why in the paper [11, Problem 1.3.1] the problem of detection necessary and sufficient condition for existence of one-point time on oriented set is posed. Below in this paper the solution of the above problem will be presented. Namely, it will be specified the properties for oriented set to be able to define the one-point time on it. Using

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the obtained result, the problem of describing all possible images of linearly ordered sets will be solved. Such problem naturally arises in the theory of ordered sets.

## 2 On Oriented Sets and One-point Time

**Definition 1.** *Let, $M$ be any non-empty set ($M \neq \emptyset$).*

*An arbitrary reflexive binary relation $\Leftarrow$ on $M$ (that is a relation satisfying $\forall x \in M \; x \Leftarrow x$) we name an **orientation**, and the pair $\mathcal{M} = (M, \Leftarrow)$ we call an **oriented set**. In this case the set $M$ is named the **basic** set or the set of all **elementary states** of the oriented set $\mathcal{M}$ and it is denoted by $\mathfrak{Bs}(\mathcal{M})$. The relation $\Leftarrow$ we name the **directing relation of changes (transformations)** of $\mathcal{M}$, and denote it by $\underset{\mathcal{M}}{\leftarrow}$.*

In the case where the oriented set $\mathcal{M}$ is known in advance, the char $\mathcal{M}$ in the notation $\underset{\mathcal{M}}{\leftarrow}$ will be released, and we will use notation $\leftarrow$ instead. For the elements $x, y \in \mathfrak{Bs}(\mathcal{M})$ the record $y \leftarrow x$ should be understood as "the elementary state $y$ is the result of transformations (or the transformation offspring) of the elementary state $x$".

Let $\mathcal{M}$ be an oriented set.

**Definition 2.** *The non-empty subset $N \subseteq \mathfrak{Bs}(\mathcal{M})$ will be referred to as **transitive** in $\mathcal{M}$ if for any $x, y, z \in N$ such, that $z \leftarrow y$ and $y \leftarrow x$ we have $z \leftarrow x$.*

*The transitive subset $L \subseteq \mathfrak{Bs}(\mathcal{M})$ will be referred to as **chain** in $\mathcal{M}$ if for any $x, y \in L$ at least one of the relations $y \leftarrow x$ or $x \leftarrow y$ is true.*

*Oriented set $\mathcal{M}$ will be called a **chain oriented set** if the set $\mathfrak{Bs}(\mathcal{M})$ is the chain of $\mathcal{M}$, that is if the relation $\leftarrow$ if transitive on $\mathfrak{Bs}(\mathcal{M})$ and for any $x, y \in \mathfrak{Bs}(\mathcal{M})$ at least one of the conditions $x \leftarrow y$ or $y \leftarrow x$ is satisfied (note that in this case the oriented set $\mathcal{M}$ is a linearly quasi-ordered set).*

Recall that linearly ordered set is an ordered pair of kind $\mathbb{T} = (\mathbf{T}, \leq)$ with reflexive, asymmetric and transitive binary relation $\leq$ on $\mathbf{T}$ satisfying the following additional condition:

**(LnO)** for every $t, \tau \in \mathbf{T}$ it is performed at least one of the correlations $t \leq \tau$ or $\tau \leq t$.

**Definition 3.** *Let $\mathcal{M}$ be an oriented set and $\mathbb{T} = (\mathbf{T}, \leq)$ be a linearly ordered set. A mapping $\psi : \mathbf{T} \to 2^{\mathfrak{Bs}(\mathcal{M})}$ is referred to as **time** on $\mathcal{M}$ if the following conditions are satisfied:*

1. *For any elementary state $x \in \mathfrak{Bs}(\mathcal{M})$ there exists an element $t \in \mathbf{T}$ such that $x \in \psi(t)$.*
2. *If $x_1, x_2 \in \mathfrak{Bs}(\mathcal{M})$, $x_2 \leftarrow x_1$ and $x_1 \neq x_2$, then there exist elements $t_1, t_2 \in \mathbf{T}$ such that $x_1 \in \psi(t_1)$, $x_2 \in \psi(t_2)$ and $t_1 < t_2$ (this means that there is a temporal separateness of successive unequal elementary states).*

*In this case:*

- *The elements $t \in \mathbf{T}$ we call the moments of time*
- *The pair $\mathcal{H} = (\mathbb{T}, \psi) = ((\mathbf{T}, \leq), \psi)$ we name by **chronologization** of $\mathcal{M}$*

We say that an oriented set $\mathcal{M}$ ***can be chronologized*** if there exists at least one chronologization of $\mathcal{M}$. It turns out that any oriented set can be chronologized. To make sure this we may consider any linearly ordered set $\mathbb{T} = (\mathbf{T}, \leq)$, which contains at least two elements and put:

$$\psi(t) := \mathfrak{Bs}(\mathcal{M}), \quad t \in \mathbf{T}.$$

It is easy to verify that the conditions of Definition 3 for this function $\psi(\cdot)$ are satisfied. More non-trivial methods to chronologize an oriented set were considered, in particular, in [8].

**Definition 4.** *Let $\mathcal{M}$ be an oriented set and $\mathbb{T} = (\mathbf{T}, \leq)$ be a linearly ordered set.*

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1. *The time $\psi : \mathbf{T} \to 2^{\mathfrak{Bs}(\mathcal{M})}$ will be called* ***quasi one-point*** *if for every $t \in \mathbf{T}$ the set $\psi(t)$ is a singleton.*

2. *The time $\psi$ will be called* ***one-point*** *if the following conditions are satisfied:*

   *(a) The time $\psi$ is quasi one-point;*

   *(b) for any $x_1, x_2 \in \mathfrak{Bs}(\mathcal{M})$ the conditions $x_1 \in \psi(t_1)$, $x_2 \in \psi(t_2)$ and $t_1 \leq t_2$, lead to $x_2 \leftarrow x_1$.*

We say that an oriented set $\mathcal{M}$ can be chronologized quasi one-point / one-point if there exists at least one chronologization $\mathcal{H} = ((\mathbf{T}, \leq), \psi)$ of $\mathcal{M}$ with quasi one-point /one-point time $\psi$ (correspondingly). In this case we name the chronologization $\mathcal{H}$ as quasi one-point /one-point (correspondingly).

***Example*** 1. Let us consider the arbitrary mapping $f : \mathcal{I} \to \mathbb{R}^d$ $(d \in \mathbb{N})$, where $\mathcal{I} \subseteq \mathbb{R}$ is some connected subset of Real axis $\mathbb{R}$. This mapping can be interpreted as equation of motion of single material point in the space $\mathbb{R}^d$. This mapping $f$ generates the oriented set $\mathcal{M}_f = \left(\mathfrak{Bs}(\mathcal{M}_f), \underset{\mathcal{M}_f}{\leftarrow}\right)$, where $\mathfrak{Bs}(\mathcal{M}_f) = \mathfrak{R}(f) = \{f(t) \,|\, t \in \mathcal{I}\} \subseteq \mathbb{R}^d$ and for $x, y \in \mathfrak{Bs}(\mathcal{M}_f)$, the correlation $y \underset{\mathcal{M}_f}{\leftarrow} x$ is valid if and only if there exist $t_1, t_2 \in \mathcal{I}$ such, that $x = f(t_1)$, $y = f(t_2)$ and $t_1 \leq t_2$. It is easy to verify, that the following mapping is a one-point time on $\mathcal{M}_f$:

$$\psi_f(t) = \{f(t)\} \subseteq \mathfrak{Bs}(\mathcal{M}), \quad t \in \mathcal{I}.$$

Example 1 makes clear the definition of one-point time. It is evident, that *any one-point time is quasi one-point*. Examples contained in the paper [8] show that the inverse statement is not true in the general case (see also [11, Example 1.3.2]).

**Theorem 1** (ZF+LO, [8])**.** *Any oriented set can be quasi one-point chronologized.*

Note that proof of Theorem 1 can be found also in [11, Theorem 1.3.2].

*Remark* 1. Proof of Theorem 1 uses the Linear Ordering principle (LO) in addition to Zermelo–Fraenkel axiomatic system (ZF). This principle asserts that any set can be linearly ordered. It is evident that the above principle follows from the famous well-ordering Zermelo's theorem, and therefore, from the axiom of choice (AC). But it is known that LO-principle also follows from Ultrafilter Theorem of Tarski and, moreover, it is logically weaker than this theorem and therefore than the axiom of choice [12, pages 17,18]. On the relationship between LO and AC see, also, [13].

**Theorem 2** ( [8])**.** *Any chain oriented set can be one-point chronologized.*

Note that the proof of Theorem 2 can be found also in [11]. It turns out that Theorem 2 is not reversible. And the next example demonstrates the existence of non-chain oriented sets, which can be one-point chronologized.

***Example*** 2. Consider the function $\boldsymbol{f}_0 : [0, 2\pi] \to \mathbb{R}^2$, defined by the formula:

$$\boldsymbol{f}_0(t) = (\cos t, \sin t) \qquad (t \in [0, 2\pi]).$$

According to Example 1, using this function, we may construct the oriented set $\mathcal{M}_{\boldsymbol{f}_0}$. This oriented set can be one-point chronologized by mens of the time:

$$\psi_{\boldsymbol{f}_0}(t) = \{\boldsymbol{f}_0(t)\} \qquad (t \in [0, 2\pi]).$$

At the same time, this oriented set is not a chain, because the binary relation $\underset{\mathcal{M}_{\boldsymbol{f}_0}}{\longleftarrow}$ is not transitive on $\mathfrak{Bs}\left(\mathcal{M}_{\boldsymbol{f}_0}\right)$. Indeed, consider the points: $\mathrm{x}_1 := (0, -1) = \boldsymbol{f}_0\left(\frac{3}{2}\pi\right)$, $\mathrm{x}_2 := (1, 0) = \boldsymbol{f}_0(0) = \boldsymbol{f}_0(2\pi)$, $\mathrm{x}_3 := (0, 1) = \boldsymbol{f}_0\left(\frac{\pi}{2}\right)$. For these points we have: $\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3 \in \mathfrak{R}(\boldsymbol{f}_0) = \mathfrak{Bs}\left(\mathcal{M}_{\boldsymbol{f}_0}\right)$ and $\mathrm{x}_2 \underset{\mathcal{M}_{\boldsymbol{f}_0}}{\longleftarrow} \mathrm{x}_1$, $\mathrm{x}_3 \underset{\mathcal{M}_{\boldsymbol{f}_0}}{\longleftarrow} \mathrm{x}_2$ but $\mathrm{x}_3 \underset{\mathcal{M}_{\boldsymbol{f}_0}}{\not\longleftarrow} \mathrm{x}_1$.

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The above facts generate the following problem:

**Problem 1.** *Find necessary and sufficient conditions of existence of one-point chronologization for oriented set.*

Note that Problem 1 was also posed in [11, Problem 1.3.1]. The main aim of the present paper is to give the solution of Problem 1.

## 3 Quasi-chain Oriented Sets and Formulation of Main Theorem

**Notation 1.** *On any oriented set $\mathcal{M}$ we introduce the following additional binary relation:*

- *For every $x, y \in \mathfrak{Bs}(\mathcal{M})$ we note $y \overset{+}{\underset{\mathcal{M}}{\leftarrow}} x$ if and only if:*

$$y \underset{\mathcal{M}}{\leftarrow} x \qquad \text{and} \qquad x \underset{\mathcal{M}}{\not\leftarrow} y.$$

- *In the cases where it does not lead to misunderstanding we use the notation $y \overset{+}{\leftarrow} x$ instead of the record $y \overset{+}{\underset{\mathcal{M}}{\leftarrow}} x$.*

**Notation 2.** *Let $\mathbf{M}$ be an arbitrary set and $\mathsf{R}_1, \mathsf{R}_2, \ldots, \mathsf{R}_n \subseteq \mathbf{M}^2$ $(n \in \mathbb{N})$ be any binary relations on $\mathbf{M}$. Further for $x_0, \ldots, x_n \in \mathbf{M}$ we use the abbreviated notation:*

$$x_0 \mathsf{R}_1 x_1 \mathsf{R}_2 x_2 \ldots x_{n-1} \mathsf{R}_n x_n$$

*for indication the fact that:*

$$(x_0 \mathsf{R}_1 x_1) \,\&\, (x_1 \mathsf{R}_2 x_2) \,\&\, \ldots \,\&\, (x_{n-1} \mathsf{R}_n x_n).$$

**Assertion 1.** *Let $\mathcal{M}$ be an oriented set, $\mathbb{T} = (\mathbf{T}, \leq)$ be a linearly ordered set and $\psi : \mathbf{T} \to 2^{\mathfrak{Bs}(\mathcal{M})}$ be one-point time on $\mathcal{M}$. Then for any $x_1, x_2 \in \mathfrak{Bs}(\mathcal{M})$ the conditions:*

$$x_1 \in \psi(t_1), \; x_2 \in \psi(t_2) \;\; \text{and} \;\; x_2 \overset{+}{\leftarrow} x_1$$

*lead to the inequality:*

$$t_1 < t_2.$$

*Proof.* Indeed, suppose that $\mathcal{M}$ is an oriented set, $\mathbb{T} = (\mathbf{T}, \leq)$ is a linearly ordered set and $\psi : \mathbf{T} \to 2^{\mathfrak{Bs}(\mathcal{M})}$ is an one-point time on $\mathcal{M}$. Let the elements $x_1, x_2 \in \mathfrak{Bs}(\mathcal{M})$ be such that $x_1 \in \psi(t_1)$, $x_2 \in \psi(t_2)$ and $x_2 \overset{+}{\leftarrow} x_1$. Assume the contrary: $t_2 \leq t_1$. Then, according to Definition 4 (item 2), from the conditions $x_1 \in \psi(t_1)$, $x_2 \in \psi(t_2)$ and $t_2 \leq t_1$ it follows that $x_1 \leftarrow x_2$. But the last correlation is in contradiction to the condition $x_2 \overset{+}{\leftarrow} x_1$. Hence the assumption about $t_2 \leq t_1$ is false. Therefore we have $t_1 < t_2$. □

**Definition 5.** *The oriented set $\mathcal{M}$ is called* ***quasi-chain*** *if and only if the following conditions are satisfied:*

**(QL1)** *For any $x_1, x_2 \in \mathfrak{Bs}(\mathcal{M})$ it holds at least one from the correlations $x_2 \leftarrow x_1$ or $x_1 \leftarrow x_2$.*

**(QL2)** *For every $x_0, x_1, x_2, x_3 \in \mathfrak{Bs}(\mathcal{M})$ the condition $x_3 \overset{+}{\leftarrow} x_2 \leftarrow x_1 \overset{+}{\leftarrow} x_0$ ensures the correlation $x_3 \overset{+}{\leftarrow} x_0$ (quasi-transitivity).*

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*Remark* 2. It is easy to prove that the transitivity of the binary relation $\leftarrow$ on the oriented set $\mathcal{M}$ implies its quasi-transitivity. It turns out that the inverse statement in general is not valid. Example 2 shows that there exist the oriented set $\mathcal{M} = \mathcal{M}_{\boldsymbol{f}_0}$ such that the relation $\underset{\mathcal{M}}{\leftarrow}$ is quasi-transitive but not transitive. So quasi-chain oriented set need not be chain.

The main result of this paper is the following theorem.

**Theorem 3** (ZF+AC)**.** *The oriented $\mathcal{M}$ set can be one-point chronologized if and only if it is a quasi-chain.*

*Remark* 3. We emphasize that proof of the necessity for Theorem 3 does not require the axiom of choice (AC). This axiom is needed only for the proof of sufficiency of the condition, pointed out in Theorem 3.

The proof of Theorem 3 is divided into two main lemmas. Lemma 1 in the next section assures the necessity for Theorem 3, whereas Lemma 3 (see below) provides the sufficiency.

## 4 Proof of Necessity for Theorem 3

The main aim of this section is to prove the following lemma, which ensures the necessity for Theorem 3.

**Lemma 1.** *If the oriented set $\mathcal{M}$ can be one-point chronologized then it is a quasi-chain.*

*Proof.* Let $\mathbb{T} = (\mathbf{T}, \leq)$ be a linearly ordered set and $\psi : \mathbf{T} \to 2^{\mathfrak{Bs}(\mathcal{M})}$ be an one-point time on the oriented set $\mathcal{M}$.

$\Rrightarrow$ **1.** First we will validate the condition **(QL1)**. Chose any $x_1, x_2 \in \mathfrak{Bs}(\mathcal{M})$. By Definition 3 the time points $t_1, t_2 \in \mathbf{T}$ must exist such, that $x_1 \in \psi(t_1)$, $x_2 \in \psi(t_2)$. Since $\mathbb{T} = (\mathbf{T}, \leq)$ is a linearly ordered set then for $t_1, t_2 \in \mathbf{T}$ at least one of the inequalities must be fulfilled $t_1 \leq t_2$ or $t_2 \leq t_1$. In Accordance with Definition 4, in the case $t_1 \leq t_2$ we obtain $x_2 \leftarrow x_1$. Similarly in the case $t_2 \leq t_1$ we deduce $x_1 \leftarrow x_2$.

$\Rrightarrow$ **2.** Now we validate the condition **(QL2)**. Consider any elements $x_0, x_1, x_2, x_3 \in \mathfrak{Bs}(\mathcal{M})$ such, that $x_3 \overset{+}{\leftarrow} x_2 \leftarrow x_1 \overset{+}{\leftarrow} x_0$. Consider any $t_0, t_3 \in \mathbf{T}$ such, that $x_0 \in \psi(t_0)$, $x_3 \in \psi(t_3)$ (by Definition 3 such $t_0, t_3$ exist). Since $x_2 \leftarrow x_1$, then, according to Definition 3 the time points $t_1, t_2 \in \mathbf{T}$ must exist such that $x_1 \in \psi(t_1)$, $x_2 \in \psi(t_2)$ and $t_1 \leq t_2$. Taking into account the correlations $x_0 \in \psi(t_0)$, $x_1 \in \psi(t_1)$ and $x_1 \overset{+}{\leftarrow} x_0$, as well as Assertion 1, we obtain, $t_0 < t_1$. Similarly from the correlations $x_2 \in \psi(t_2)$, $x_3 \in \psi(t_3)$ and $x_3 \overset{+}{\leftarrow} x_2$ we deduce $t_2 < t_3$. Therefore the following inequalities are performed:

$$t_0 < t_1 \leq t_2 < t_3.$$

That is why $t_0 < t_3$. Thus we have:

$$\forall\, t_0, t_3 \in \mathbf{T}\ \left( (x_0 \in \psi(t_0)) \,\&\, (x_3 \in \psi(t_3)) \Rightarrow (t_0 < t_3) \right). \tag{1}$$

In accordance with the statement, proven in the item 1, at least one from the correlations $x_0 \leftarrow x_3$ or $x_3 \leftarrow x_0$ must hold. Assume, that $x_0 \leftarrow x_3$. Then, by Definition 3 the elements $\widetilde{t}_0, \widetilde{t}_3 \in \mathbf{T}$ must exist such that $x_0 \in \psi\left(\widetilde{t}_0\right)$, $x_3 \in \psi\left(\widetilde{t}_3\right)$ and $\widetilde{t}_3 \leq \widetilde{t}_0$. But the last inequality is in a contradiction to (1). Hence, the correlation $x_0 \leftarrow x_3$ is impossible. Thus the only possible one it remains the correlation $x_3 \overset{+}{\leftarrow} x_0$, that it was necessary to prove. □

The proof of the sufficiency for Theorem 3 is much more complicated. First of all we need to work out some auxiliary technical results for this purpose. This work will be done in the next section.

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# 5 Some Auxiliary Technical Results

## 5.1 Some Additional Properties of Quasi-chain Oriented Sets

**Assertion 2.** *Let, $\mathcal{M}$ be a quasi-chain oriented set and $x_0, x_1, x_2, x_3 \in \mathfrak{Bs}(\mathcal{M})$ be arbitrary elementary states of $\mathcal{M}$. Then the following properties are performed:*

**(QL3)** *If $x_3 \leftarrow x_2 \overset{+}{\leftarrow} x_1 \leftarrow x_0$ then $x_3 \leftarrow x_0$.*

**(QL4)** *If $x_3 \overset{+}{\leftarrow} x_2 \leftarrow x_1$ then $x_3 \leftarrow x_1$.*

**(QL5)** *If $x_3 \leftarrow x_2 \overset{+}{\leftarrow} x_1$ then $x_3 \leftarrow x_1$.*

**(QL6)** *If $x_3 \overset{+}{\leftarrow} x_2 \overset{+}{\leftarrow} x_1$ then $x_3 \overset{+}{\leftarrow} x_1$.*

*Proof.* The proofs of the properties (QL3)–(QL6) are listed below.

⇛ **(QL3).** Let $x_0, x_1, x_2, x_3 \in \mathfrak{Bs}(\mathcal{M})$ and $x_3 \leftarrow x_2 \overset{+}{\leftarrow} x_1 \leftarrow x_0$. Assume that $x_3 \not\leftarrow x_0$. Then, taking into account the fact that the oriented set $\mathcal{M}$ is quasi-chain, we get $x_0 \overset{+}{\leftarrow} x_3$. Thus, we have, $x_0 \overset{+}{\leftarrow} x_3 \leftarrow x_2 \overset{+}{\leftarrow} x_1$. Hence, by Definition 5 (condition (QL2)) we get, $x_0 \overset{+}{\leftarrow} x_1$, which is in acontradiction to the correlation $x_1 \leftarrow x_0$. Therefore assumption about $x_3 \not\leftarrow x_0$ is false. So we have $x_3 \leftarrow x_0$.

⇛ **(QL4).** Suppose that $x_1, x_2, x_3 \in \mathfrak{Bs}(\mathcal{M})$ and $x_3 \overset{+}{\leftarrow} x_2 \leftarrow x_1$. Then, by Definition 1, we have, $x_3 \leftarrow x_3 \overset{+}{\leftarrow} x_2 \leftarrow x_1$. Thence, using Property (QL3), we obtain $x_3 \leftarrow x_1$.

⇛ **(QL5).** If we assume that $x_3 \leftarrow x_2 \overset{+}{\leftarrow} x_1$, then we will have $x_3 \leftarrow x_2 \overset{+}{\leftarrow} x_1 \leftarrow x_1$. Thence, applying Property (QL3), we obtain $x_3 \leftarrow x_1$.

⇛ **(QL6).** If we suppose that $x_3 \overset{+}{\leftarrow} x_2 \overset{+}{\leftarrow} x_1$, then we will deliver $x_3 \overset{+}{\leftarrow} x_2 \leftarrow x_2 \overset{+}{\leftarrow} x_1$. Thence, by Definition 5 (condition (QL2)), we deduce $x_3 \overset{+}{\leftarrow} x_1$. □

**Notation 3.** *For any oriented set $\mathcal{M}$ we introduce the following additional binary relations:*

⇛ *For any $x, y \in \mathfrak{Bs}(\mathcal{M})$ we will note $y \overset{+-}{\underset{\mathcal{M}}{\leftarrow}} x$ if and only if there exists the element $\widetilde{x} \in \mathfrak{Bs}(\mathcal{M})$ such that $y \overset{+}{\leftarrow} \widetilde{x} \leftarrow x$.*

⇛ *For arbitrary $x, y \in \mathfrak{Bs}(\mathcal{M})$ we will write $y \overset{-+}{\underset{\mathcal{M}}{\leftarrow}} x$ if and only if there exists the element $\widetilde{x} \in \mathfrak{Bs}(\mathcal{M})$ such that $y \leftarrow \widetilde{x} \overset{+}{\leftarrow} x$.*

*Further in the cases, where the oriented set $\mathcal{M}$ is known in advance, we will use the notations $y \overset{+-}{\leftarrow} x$ and $y \overset{-+}{\leftarrow} x$ instead of $y \overset{+-}{\underset{\mathcal{M}}{\leftarrow}} x$ and $y \overset{-+}{\underset{\mathcal{M}}{\leftarrow}} x$ (correspondingly).*

**Assertion 3.** *Let, $\mathcal{M}$ be a quasi-chain oriented set and $x_1, x_2, x_3 \in \mathfrak{Bs}(\mathcal{M})$ be arbitrary elementary states of $\mathcal{M}$. Then the following properties are holding:*

**(QL7)** *If $x_3 \leftarrow x_2 \overset{+-}{\leftarrow} x_1$ then $x_3 \leftarrow x_1$.*

**(QL8)** *If $x_3 \overset{-+}{\leftarrow} x_2 \leftarrow x_1$ then $x_3 \leftarrow x_1$.*

**(QL9)** *If $x_3 \overset{+}{\leftarrow} x_2 \overset{-+}{\leftarrow} x_1$ then $x_3 \overset{+}{\leftarrow} x_1$.*

**(QL10)** *If $x_3 \overset{+-}{\leftarrow} x_2 \overset{+}{\leftarrow} x_1$ then $x_3 \overset{+}{\leftarrow} x_1$.*

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**(QL11)** *If* $x_2 \overset{+-}{\leftarrow} x_1$ *or* $x_2 \overset{-+}{\leftarrow} x_1$ *then* $x_2 \leftarrow x_1$.

**(QL12)** *If* $x_2 \overset{+}{\leftarrow} x_1$ *then* $x_2 \overset{+-}{\leftarrow} x_1$ *and* $x_2 \overset{-+}{\leftarrow} x_1$.

**(QL13)** *If* $x_3 \overset{-+}{\leftarrow} x_2 \overset{+-}{\leftarrow} x_1$ *then* $x_3 \leftarrow x_1$.

**(QL14)** *If* $x_3 \overset{+-}{\leftarrow} x_2 \overset{+-}{\leftarrow} x_1$ *then* $x_3 \overset{+-}{\leftarrow} x_1$.

**(QL15)** *If* $x_3 \overset{-+}{\leftarrow} x_2 \overset{-+}{\leftarrow} x_1$ *then* $x_3 \overset{-+}{\leftarrow} x_1$.

**(QL16)** *If* $x_3 \overset{+}{\leftarrow} x_2 \overset{+-}{\leftarrow} x_1$ *then* $x_3 \overset{+-}{\leftarrow} x_1$.

**(QL17)** *If* $x_3 \overset{-+}{\leftarrow} x_2 \overset{+}{\leftarrow} x_1$ *then* $x_3 \overset{-+}{\leftarrow} x_1$.

*Proof.* Properties **(QL7)–(QL10)** follow from Assertion 2 and Definition 5. In particular (QL7) and (QL8) follow from (QL3), as well as (QL9) and (QL10) follow from (QL2).

Property **(QL11)** follows from properties **(QL4)** and **(QL5)** of Assertion 2.

Now we are going to prove **(QL12)**. If $x_2 \overset{+}{\leftarrow} x_1$ then by Definition 1, we obtain, $x_2 \overset{+}{\leftarrow} x_1 \leftarrow x_1$ and $x_2 \leftarrow x_2 \overset{+}{\leftarrow} x_1$. So, according to Notation 3, we have $x_2 \overset{+-}{\leftarrow} x_1$ and $x_2 \overset{-+}{\leftarrow} x_1$.

**(QL13)** follows from (QL6) and (QL3).

**(QL14)–(QL15)** are caused by inequality (QL2) or (QL3).

**(QL16)–(QL17)** foloow from (QL6). □

### 5.2 Finite-repeating Time on Oriented Sets

**Definition 6.** *Let $\mathbb{T} = (\mathbf{T}, \leq)$ be a linearly ordered set and $\mathcal{M}$ be an oriented set.*

⇛ *The time $\psi : \mathbf{T} \to 2^{\mathfrak{Bs}(\mathcal{M})}$ will be named as **finite-repeating** if and only if for every $x \in \mathfrak{Bs}(\mathcal{M})$ the following condition is fulfilled:*

$$\mathbf{card}\left(\left\{t \in \mathbf{T} \,\middle|\, x \in \psi(t)\right\}\right) < \aleph_0$$

*(where $\mathbf{card}(\mathbf{M})$ is the cardinality of a set $\mathbf{M}$). Moreover, the number:*

$$\mathsf{Rp}_x(\psi) = \mathbf{card}\left(\left\{t \in \mathbf{T} \,\middle|\, x \in \psi(t)\right\}\right)$$

*will be refered to as **repeatability** of the time $\psi$ relatively the element $x \in \mathfrak{Bs}(\mathcal{M})$.*

⇛ *We name the time $\psi$ as **bounded-repeating** if and only if the time $\psi$ is finite-repeating and there exists the number $K \in \mathbb{N}$ such that the inequality $\mathsf{Rp}_x(\psi) < K$ is performed for each $x \in \mathfrak{Bs}(\mathcal{M})$. In this case the number:*

$$\mathsf{Rp}(\psi) = \max_{x \in \mathfrak{Bs}(\mathcal{M})} \mathsf{Rp}_x(\psi)$$

*is named as **maximal repeatability** of the time $\psi$.*

⇛ *Let $n \in \mathbb{N}$. The time $\psi$ is named as $n$**-repeating**, if and only if the time $\psi$ is finite-repeating and*

$$\forall x \in \mathfrak{Bs}(\mathcal{M})\ (\mathsf{Rp}_x(\psi) = n).$$

**Notation 4.** *Let $\psi : \mathbf{T} \to 2^{\mathfrak{Bs}(\mathcal{M})}$ be a **finite-repeating** time on the oriented set $\mathcal{M}$. For every $x \in \mathfrak{Bs}(\mathcal{M})$ we note:*

$$\widehat{\psi}^{+}(x) := \max\left(\left\{t \in \mathbf{T} \,\middle|\, x \in \psi(t)\right\}\right);$$
$$\widehat{\psi}^{-}(x) := \min\left(\left\{t \in \mathbf{T} \,\middle|\, x \in \psi(t)\right\}\right),$$

*where maximum and minimum should be understood it the sense of the linearly ordered set $\mathbb{T} = (\mathbf{T}, \leq)$.*

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**Assertion 4.** *Let $\mathbb{T} = (\mathbf{T}, \leq)$ be a linearly ordered set and $\psi : \mathbf{T} \to 2^{\mathfrak{Bs}(\mathcal{M})}$ be a finite-repeating one-point time on the oriented set $\mathcal{M}$. Then for any $x, x_1, x_2 \in \mathfrak{Bs}(\mathcal{M})$ the following properties are holding:*

**(FR1)** $\widehat{\psi}^-(x) \leq \widehat{\psi}^+(x)$. *If, in addition,* $\mathsf{Rp}_x(\psi) \geq 2$ *then* $\widehat{\psi}^-(x) < \widehat{\psi}^+(x)$.

**(FR2)** *The correlation $x_2 \leftarrow x_1$ is true if and only if $\widehat{\psi}^-(x_1) \leq \widehat{\psi}^+(x_2)$. If, in addition, $x_1 \neq x_2$ then $x_2 \leftarrow x_1$ if and only if $\widehat{\psi}^-(x_1) < \widehat{\psi}^+(x_2)$.*

**(FR3)** *$x_2 \overset{+}{\leftarrow} x_1$ if and only if $\widehat{\psi}^+(x_1) < \widehat{\psi}^-(x_2)$.*

**(FR4)** *If $x_2 \overset{+-}{\leftarrow} x_1$ then $\widehat{\psi}^-(x_1) < \widehat{\psi}^-(x_2)$.*

**(FR5)** *If $x_2 \overset{-+}{\leftarrow} x_1$ then $\widehat{\psi}^+(x_1) < \widehat{\psi}^+(x_2)$.*

**(FR6)** *If, in addition, the time $\psi$ is $n$-repeating with $n \geq 2$ then $x_2 \leftarrow x_1$ if and only if $\widehat{\psi}^-(x_1) < \widehat{\psi}^+(x_2)$.*

*Proof.* $\Rrightarrow$ **(FR1)**: Let $x \in \mathfrak{Bs}(\mathcal{M})$. Then according to Notation 4, we have $\widehat{\psi}^-(x) = \min\left(\left\{t \in \mathbf{T} \,\middle|\, x \in \psi(t)\right\}\right) \leq \max\left(\left\{t \in \mathbf{T} \,\middle|\, x \in \psi(t)\right\}\right) = \widehat{\psi}^+(x)$. If, in addition, $\mathsf{Rp}_x(\psi) \geq 2$ then the set $\left\{t \in \mathbf{T} \,\middle|\, x \in \psi(t)\right\}$ contains at least two elements. So minimum of this set is less then maximum.

$\Rrightarrow$ **(FR2)**: Suppose that $x_1, x_2 \in \mathfrak{Bs}(\mathcal{M})$ and $x_2 \leftarrow x_1$. Then in the case $x_1 = x_2$ we have the inequality $\widehat{\psi}^-(x_1) \leq \widehat{\psi}^+(x_2)$ according to Property (FR1). Hence we will consider that $x_1 \neq x_2$. Since $x_2 \leftarrow x_1$ and $x_1 \neq x_2$, then, by Definition 3, the time points $t_1, t_2 \in \mathbf{T}$ exist such that $x_1 \in \psi(t_1)$, $x_2 \in \psi(t_2)$ and $t_1 < t_2$. Therefore:

$$\widehat{\psi}^-(x_1) = \min\left(\left\{t \in \mathbf{T} \,\middle|\, x_1 \in \psi(t)\right\}\right) \leq t_1 < t_2 \leq \max\left(\left\{t \in \mathbf{T} \,\middle|\, x_2 \in \psi(t)\right\}\right) = \widehat{\psi}^+(x_2).$$

So, for every $x_1, x_2 \in \mathfrak{Bs}(\mathcal{M})$ it is performed the following implication:

$$(x_2 \leftarrow x_1) \,\&\, (x_1 \neq x_2) \Rightarrow \left(\widehat{\psi}^-(x_1) < \widehat{\psi}^+(x_2)\right). \tag{2}$$

Thus, in the both cases for any $x_1, x_2 \in \mathfrak{Bs}(\mathcal{M})$ we have the implication:

$$(x_2 \leftarrow x_1) \Rightarrow \left(\widehat{\psi}^-(x_1) \leq \widehat{\psi}^+(x_2)\right). \tag{3}$$

Conversely, suppose that $\widehat{\psi}^-(x_1) \leq \widehat{\psi}^+(x_2)$. Put:

$$\widehat{t}_1 := \widehat{\psi}^-(x_1), \qquad \widehat{t}_2 := \widehat{\psi}^+(x_2).$$

Then in accordance with Notation 4, we have, $x_1 \in \psi\left(\widehat{t}_1\right)$, $x_2 \in \psi\left(\widehat{t}_2\right)$ and $\widehat{t}_1 \leq \widehat{t}_2$. Hence, by Definition 4, we deduce $x_2 \leftarrow x_1$. Thus for every $x_1, x_2 \in \mathfrak{Bs}(\mathcal{M})$ we have the implication:

$$\left(\widehat{\psi}^-(x_1) \leq \widehat{\psi}^+(x_2)\right) \Rightarrow (x_2 \leftarrow x_1) \tag{4}$$

The implications (3) and (4) assure the desired equivalence: $(x_2 \leftarrow x_1) \Leftrightarrow \left(\widehat{\psi}^-(x_1) \leq \widehat{\psi}^+(x_2)\right)$.

If we assume that, in addition, $x_1 \neq x_2$ then from (2) and (4) we deliver the equivalence: $(x_2 \leftarrow x_1) \Leftrightarrow \left(\widehat{\psi}^-(x_1) < \widehat{\psi}^+(x_2)\right)$.

$\Rrightarrow$ **(FR3)**: Let $x_2 \overset{+}{\leftarrow} x_1$. Assume that $\widehat{\psi}^-(x_2) \leq \widehat{\psi}^+(x_1)$. Then according to Property (FR2), we obtain the correlation $x_1 \leftarrow x_2$, which contradicts to $x_2 \overset{+}{\leftarrow} x_1$. Therefore, $\widehat{\psi}^+(x_1) < \widehat{\psi}^-(x_2)$.

Conversely, suppose that $\widehat{\psi}^{+}(x_1) < \widehat{\psi}^{-}(x_2)$. Then, applying Property (FR1), we deliver $\widehat{\psi}^{-}(x_1) \leq \widehat{\psi}^{+}(x_1) < \widehat{\psi}^{-}(x_2) \leq \widehat{\psi}^{+}(x_2)$. Hence, according to Property (FR2), we obtain $x_2 \leftarrow x_1$. Assume that the condition $x_1 \leftarrow x_2$ also is performed. Then by Property (FR2), we get the inequality $\widehat{\psi}^{-}(x_2) \leq \widehat{\psi}^{+}(x_1)$, which contradicts to the inequality $\widehat{\psi}^{+}(x_1) < \widehat{\psi}^{-}(x_2)$. That is the assumption about $x_1 \leftarrow x_2$ is wrong. That is why we have $x_2 \overset{+}{\leftarrow} x_1$.

$\Rrightarrow$ **(FR4)**: Let $x_1, x_2 \in \mathfrak{Bs}(\mathcal{M})$ and $x_2 \overset{+-}{\leftarrow} x_1$. Then there exists the element $\widetilde{x} \in \mathfrak{Bs}(\mathcal{M})$ such, that $x_2 \overset{+}{\leftarrow} \widetilde{x} \leftarrow x_1$. Therefore, using Properties (FR2) and (FR3), we obtain $\widehat{\psi}^{-}(x_1) \leq \widehat{\psi}^{+}(\widetilde{x}) < \widehat{\psi}^{-}(x_2)$, that is $\widehat{\psi}^{-}(x_1) < \widehat{\psi}^{-}(x_2)$.

$\Rrightarrow$ **(FR5)**: Let $x_1, x_2 \in \mathfrak{Bs}(\mathcal{M})$ and $x_2 \overset{-+}{\leftarrow} x_1$. Then there exists the element $\widetilde{x} \in \mathfrak{Bs}(\mathcal{M})$ such, that $x_2 \leftarrow \widetilde{x} \overset{+}{\leftarrow} x_1$. Hence, using Properties (FR2) and (FR3), we get, $\widehat{\psi}^{+}(x_1) < \widehat{\psi}^{-}(\widetilde{x}) \leq \widehat{\psi}^{+}(x_2)$.

$\Rrightarrow$ **(FR6)**: In the case $x_1 \neq x_2$ Property (FR6) follows from Property (FR2). In the case $x_1 = x_2$ this property follows from Property (FR1). □

### 5.3 Strict Supremum and Strict Infimum in Linearly Ordered Sets

**Definition 7.** *Let $\mathbb{T} = (\mathbf{T}, \leq)$ be a linearly ordered set and $\mathcal{U} \subseteq \mathbf{T}$ be nonempty subset of $\mathbf{T}$.*

$\Rrightarrow$ **1)** *Element $\tau \in \mathbf{T}$ is said to be strict **upper bound** (**lower bound**) of the set $\mathcal{U}$ if and only if for each element $t \in \mathcal{U}$ it is valid the inequality $t < \tau$ $(\tau < t)$ correspondingly.*

$\Rrightarrow$ **2)** *Element $\tau \in \mathbf{T}$ is named by **strict supremum** (**strict infimum**) of the set u $\mathcal{U}$ if and only if:*

- *$\tau$ is strict upper (strict lower) bound of the set $\mathcal{U}$.*
- *For every strict upper (strict lower) bound $\widetilde{\tau}$ of the set $\mathcal{U}$ it is perfrmed the inequality $\tau \leq \widetilde{\tau}$ $(\widetilde{\tau} \leq \tau)$ correspondingly.*

Directly from Definition 7 it follows the following assertion:

*If strict supremum (strict infimum) of the subset $\mathcal{U} \subseteq \mathbf{T}$ exists then it is unique.*

Indeed, assume that $\tau, \tau'$ are two strict supremums of the set $\mathcal{U}$. Then, by Definition 7 both of the inequalities $\tau \leq \tau'$ and $\tau' \leq \tau$ must be fulfilled. Hence $\tau = \tau'$.

$\Rrightarrow$ For strict supremum of the set $\mathcal{U} \subseteq \mathbf{T}$ (if it exists) we will use the notation:

$$\sup\nolimits^{*}_{\mathbb{T}} \mathcal{U}.$$

$\Rrightarrow$ For strict infimum of the set $\mathcal{U} \subseteq \mathbf{T}$ (if it exists) we use the notation:

$$\inf\nolimits^{*}_{\mathbb{T}} \mathcal{U}.$$

**Notation 5.** *Let $\mathbb{T}_1 = (\mathbf{T}_1, \leq_1)$, $\mathbb{T}_2 = (\mathbf{T}_2, \leq_2)$ be linearly ordered sets. We will write $\mathbb{T}_1 \sqsubseteq \mathbb{T}_2$ if and only if the following conditions are fulfilled:*

*1) $\mathbf{T}_1 \subseteq \mathbf{T}_2$;*

*2) For arbitrary $t, \tau \in \mathbf{T}_1$ the inequality $t \leq_2 \tau$ is performed if and only if $t \leq_1 \tau$.*

**Assertion 5.** *Let $\mathbb{T}_1 = (\mathbf{T}_1, \leq_1)$ be a linearly ordered set, $\mathcal{U} \subseteq \mathbf{T}_1$ be any nonempty subset of $\mathbf{T}_1$ and $t_0$ be an arbitrary element such, that $t_0 \notin \mathbf{T}_1$. Then there exists the linearly ordered set $\mathbb{T} = (\mathbf{T}, \leq)$, satisfying the following conditions:*

1. $\mathbb{T}_1 \sqsubseteq \mathbb{T}$;
2. $\mathbf{T} = \mathbf{T}_1 \cup \{t_0\}$;
3. $\sup^*_{\mathbb{T}} \mathcal{U} = t_0$.

*Proof.* We put:

$$\mathbf{T}_1^- := \left\{\tau \in \mathbf{T}_1 \middle|\, \exists\, t \in \mathcal{U}\ (\tau \leq_1 t)\right\}; \tag{5}$$

$$\mathbf{T}_1^+ := \left\{\tau \in \mathbf{T}_1 \middle|\, \nexists\, t \in \mathcal{U}\ (\tau \leq_1 t)\right\} = \left\{\tau \in \mathbf{T}_1 \middle|\, \forall\, t \in \mathcal{U}\ (t <_1 \tau)\right\} =$$
$$= \left\{\tau \in \mathbf{T}_1 \middle|\, \tau \text{ is strict upper bound of the set } \mathcal{U} \text{ in } \mathbb{T}_1\right\}, \tag{6}$$

where $<_1$ is the strict linear order, generated by the non-strict order $\leq_1$. It is easy to verify that the sets $\mathbf{T}_1^-$ and $\mathbf{T}_1^+$ possess the following properties:

$$\mathbf{T}_1^+ \cap \mathbf{T}_1^- = \emptyset; \tag{7}$$
$$\mathbf{T}_1^+ \cup \mathbf{T}_1^- = \mathbf{T}_1; \tag{8}$$
$$\forall\, t^+ \in \mathbf{T}_1^+\ \forall\, t^- \in \mathbf{T}_1^- \quad \left(t^- <_1 t^+\right). \tag{9}$$

Properties (7)–(9) stipulate the following properties:

$$\text{If} \quad t \in \mathbf{T}_1^+ \quad \text{and} \quad t \leq_1 \widetilde{t} \quad \text{then} \quad \widetilde{t} \in \mathbf{T}_1^+; \tag{10}$$
$$\text{If} \quad t \in \mathbf{T}_1^- \quad \text{and} \quad \widetilde{t} \leq_1 t \quad \text{then} \quad \widetilde{t} \in \mathbf{T}_1^-. \tag{11}$$

Denote:

$$\mathbf{T} := \mathbf{T}_1 \cup \{t_0\}. \tag{12}$$

On the set $\mathbf{T}$ we introduce the order relation. Namely, for every $t, t' \in \mathbf{T}_1$ we write $t \leq t'$ if and only if one of the following conditions is satisfied:

$$t, t' \in \mathbf{T}_1 \quad \text{and} \quad t \leq_1 t'; \tag{13}$$
$$t = t_0 \quad \text{and} \quad t' \in \mathbf{T}_1^+; \tag{14}$$
$$t \in \mathbf{T}_1^- \quad \text{and} \quad t' = t_0; \tag{15}$$
$$t = t' = t_0. \tag{16}$$

It is not hard to prove that $\mathbb{T} = (\mathbf{T}, \leq)$ is the linearly ordered set.

The formula (12) leads to the correlation $\mathbf{T}_1 \subseteq \mathbf{T}$, whereas from the formulas (13)–(16) it follows that for any $t, \tau \in \mathbf{T}_1$ the condition $t \leq \tau$ is fulfilled if and only if $t \leq_1 \tau$. Hence, corresponding to Notation 5, we have:

$$\mathbb{T}_1 \sqsubseteq \mathbb{T}. \tag{17}$$

Now our aim is to prove that $\sup^*_{\mathbb{T}} \mathcal{U} = t_0$. From the formula (5) it follows the inclusion $\mathcal{U} \subseteq \mathbf{T}_1^-$. Hence, according to the formula (15), for every $\tau \in \mathcal{U}$ we get $\tau \leq t_0$. Since $\mathcal{U} \subseteq \mathbf{T}_1$ and $t_0 \notin \mathbf{T}_1$, then we have $\tau \neq t_0$ for each $\tau \in \mathcal{U}$. Therefore:

$$\forall\, \tau \in \mathcal{U}\ \ (\tau < t_0) \tag{18}$$

Assume that for some element $t_0' \in \mathbf{T}$ such that $t_0' \neq t_0$ it is performed the condition:

$$\forall\, \tau \in \mathcal{U}\ \ (\tau < t_0'). \tag{19}$$

Then the inequality $t_0' \neq t_0$ ensures the correlation $t_0' \in \mathbf{T}_1$. So, according to (13), we obtain, $\forall\, \tau \in \mathcal{U}\ \ (\tau <_1 t_0')$. This means that $t_0'$ is a strict upper bound of $\mathcal{U}$ in $\mathbb{T}_1$. Hence in accordance with (6), we get $t_0' \in \mathbf{T}_0^+$. Thence, by the condition (14), we deduce $t_0 \leq t_0'$. Thus the condition (18) is performed and, moreover, if some element $t_0' \in \mathbf{T}$ satisfies the condition (19) then we have $t_0 \leq t_0'$. That is why $t_0$ is strict supremum of the set $\mathcal{U}$ in $\mathbb{T}$, ie:

$$\sup^*_{\mathbb{T}} \mathcal{U} = t_0. \tag{20}$$

From the formulas (17), (12) and (20) it follows that the linearly ordered set $\mathbb{T}$ satisfies conditions $1, 2, 3$ of Assertion 5. ☐

Using the Duality principle (see [14, page 14]) we deduce the following assertion from Assertion 5:

**Assertion 6.** *Let $\mathbb{T}_1 = (\mathbf{T}_1, \leq_1)$ be a linearly ordered set, $\mathcal{U} \subseteq \mathbf{T}_1$ be any nonempty subset of $\mathbf{T}_1$ and $t_0$ be an arbitrary element such, that $t_0 \notin \mathbf{T}_1$. Then there exists the linearly ordered set $\mathbb{T} = (\mathbf{T}, \leq)$, satisfying the following conditions:*

1. $\mathbb{T}_1 \sqsubseteq \mathbb{T}$;
2. $\mathbf{T} = \mathbf{T}_1 \cup \{t_0\}$;
3. $\inf^*_{\mathbb{T}} \mathcal{U} = t_0$.

Applying assertions 5 and 6 successively we obtain the following assertion:

**Assertion 7.** *Let $\mathbb{T}_1 = (\mathbf{T}_1, \leq_1)$ be a linearly ordered set, $\mathcal{U}_1, \mathcal{U}_2 \subseteq \mathbf{T}_1$ be arbitrary two nonempty subsets of $\mathbf{T}_1$ such, that*

$$\forall \tau_1 \in \mathcal{U}_1 \, \forall \tau_2 \in \mathcal{U}_2 \quad (\, \tau_1 <_1 \tau_2 \,), \tag{21}$$

*and $t_0^{(1)}$, $t_0^{(2)}$ be any two elements such, that $t_0^{(1)}, t_0^{(2)} \notin \mathbf{T}_1$ and $t_0^{(1)} \neq t_0^{(2)}$. Then there exists the linearly ordered set $\mathbb{T} = (\mathbf{T}, \leq)$, possessing the following properties:*

1. $\mathbb{T}_1 \sqsubseteq \mathbb{T}$;
2. $\mathbf{T} = \mathbf{T}_1 \cup \left\{t_0^{(1)}, t_0^{(2)}\right\}$;
3. $\sup^*_{\mathbb{T}} \mathcal{U}_1 = t_0^{(1)}, \quad \inf^*_{\mathbb{T}} \mathcal{U}_2 = t_0^{(2)}$.

*Proof.* According to Assertion 5, the linearly ordered set $\mathbb{T}^* = (\mathbf{T}^*, \leq^*)$ exists such that:

1.1) $\mathbb{T}_1 \sqsubseteq \mathbb{T}^*$;

1.2) $\mathbf{T}^* = \mathbf{T}_1 \cup \left\{t_0^{(1)}\right\}$;

1.3) $\sup^*_{\mathbb{T}^*} (\mathcal{U}_1) = t_0^{(1)}$.

Taking into account that $\mathbb{T}_1 \sqsubseteq \mathbb{T}^*$ as well as condition (21), we see that for every elements $\tau_1 \in \mathcal{U}_1$ and $\tau_2 \in \mathcal{U}_2$ the inequality $\tau_1 <^* \tau_2$ holds, where $<^*$ is the strict linear order, generated by the non-strict order $\leq^*$. Hence any element $\tau_2 \in \mathcal{U}_2$ is a strict upper bound of the set $\mathcal{U}_1$ in the linearly ordered set $\mathbb{T}^*$. And since $t_0^{(1)} = \sup^*_{\mathbb{T}^*} (\mathcal{U}_1)$, we have the inequality:

$$t_0^{(1)} \leq^* \tau_2 \quad (\forall \tau_2 \in \mathcal{U}_2)\,.$$

But since $t_0^{(1)} \notin \mathbf{T}_1$ and $\mathcal{U}_2 \subseteq \mathbf{T}_1$, the equality $t_0^{(1)} = \tau_2$ is impossible for each $\tau_2 \in \mathcal{U}_2$. Therefore:

$$t_0^{(1)} <^* \tau_2 \quad (\forall \tau_2 \in \mathcal{U}_2)\,. \tag{22}$$

In accordance with Assertion 6 the linearly ordered set $\mathbb{T} = (\mathbf{T}, \leq)$ exists such that:

2.1) $\mathbb{T}^* \sqsubseteq \mathbb{T}$;

2.2) $\mathbf{T} = \mathbf{T}^* \cup \left\{t_0^{(2)}\right\} = \mathbf{T}_1 \cup \left\{t_0^{(1)}, t_0^{(2)}\right\}$;

2.3) $\inf^*_{\mathbb{T}} (\mathcal{U}_2) = t_0^{(2)}$.

To complete the proof, it is necessary to make sure that $\sup^*_{\mathbb{T}} (\mathcal{U}_1) = t_0^{(1)}$. By condition of the assertion we have $\mathbb{T}^* \sqsubseteq \mathbb{T}$. So in the correlation (22) the sign $<^*$ may be replaced by $<$. Therefore from the formula (22) as well as equality $\inf^*_{\mathbb{T}} (\mathcal{U}_2) = t_0^{(2)}$ it follows the

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inequality $t_0^{(1)} \leq t_0^{(2)}$. But, by condition of the assertion we have $t_0^{(1)} \neq t_0^{(2)}$. Thus we obtain the strict inequality:

$$t_0^{(1)} < t_0^{(2)} \tag{23}$$

Since $\sup^*_{\mathbb{T}^*} (\mathcal{U}_1) = t_0^{(1)}$, then for every $\tau_1 \in \mathcal{U}_1$ we have the inequality $\tau_1 <^* t_0^{(1)}$. Thence, taking into account the correlation $\mathbb{T}^* \sqsubseteq \mathbb{T}$, we deliver:

$$\tau_1 < t_0^{(1)} \qquad (\forall\, \tau_1 \in \mathcal{U}_1)\,.$$

Hence $t_0^{(1)}$ is a strict upper bound of the set $\mathcal{U}_1$ relatively the linearly ordered set $\mathbb{T}$ as well.

Let $\widetilde{t}_0$ be other element of $\mathbf{T}$ such that:

$$\tau_1 < \widetilde{t}_0 \qquad (\forall\, \tau_1 \in \mathcal{U}_1)\,. \tag{24}$$

If we assume that $\widetilde{t}_0 \in \mathbf{T}^*$, then, taking into account the equality $t_0^{(1)} = \sup^*_{\mathbb{T}^*} (\mathcal{U}_1)$, we obtain the inequality $t_0^{(1)} \leq^* \widetilde{t}_0$, and therefore (since $\mathbb{T}^* \sqsubseteq \mathbb{T}$) we get the inequality:

$$t_0^{(1)} \leq \widetilde{t}_0. \tag{25}$$

So it remains to consider only the case $\widetilde{t}_0 = t_0^{(2)}$. But in this case the inequality (25) follows from (23). Thus the inequality (25) holds for each element $\widetilde{t}_0 \in \mathbf{T}$, satisfying condition (24). Consequently:

$$t_0^{(1)} = \sup^*_{\mathbb{T}} (\mathcal{U}_1)\,.$$

□

### 5.4 Evolutionary maximums and evolutionary minimums of oriented sets

**Definition 8.** ⇛ *We name an element $x^* \in \mathfrak{Bs}(\mathcal{M})$ by* ***evolutionary maximum*** *of the oriented set $\mathcal{M}$ if and only if for any element $x \in \mathfrak{Bs}(\mathcal{M})$ the following condition is performed:*

$$x = x^* \quad or \quad x^* \overset{+}{\leftarrow} x.$$

⇛ *We name an element $x_* \in \mathfrak{Bs}(\mathcal{M})$ as* ***evolutionary minimum*** *of the oriented set $\mathcal{M}$, if and only if for every element $x \in \mathfrak{Bs}(\mathcal{M})$ the following condition holds:*

$$x = x_* \quad or \quad x \overset{+}{\leftarrow} x_*.$$

**Assertion 8.** *If evolutionary maximum (evolutionary minimum) of the oriented set $\mathcal{M}$ exists then it is unique.*

*Proof.* Let $x^*$ and $x^{**}$ be two evolutionary maximums of the oriented set $\mathcal{M}$. Suppose that $x^* \neq x^{**}$. Then by Definition 8, we obtain $x^{**} \overset{+}{\leftarrow} x^*$ and $x^* \overset{+}{\leftarrow} x^{**}$, which is impossible. Hence, $x^* = x^{**}$. Similarly it can be proven the uniqueness of evolutionary minimum. □

**Notation 6.** *If $x^*$ is an evolutionary maximum ($x_*$ is an evolutionary minimum) of the oriented set $\mathcal{M}$ we will write:*

$$x^* = \max{}^*(\mathcal{M}) \qquad (x_* = \min{}^*(\mathcal{M}))$$

*correspondingly.*

**Definition 9.**

⇛ *An oriented set $\mathcal{M}$ will be named* ***evolutionary bounded*** *if and only if $\max^*(\mathcal{M})$ and $\min^*(\mathcal{M})$ exist.*

⇛ *An oriented set $\mathcal{M}$ will be named* ***strictly evolutionary bounded*** *if and only if it is evolutionary bounded and* $\mathbf{card}\,(\mathfrak{Bs}(\mathcal{M})) \geq 2$.

**Assertion 9.** *If the oriented set $\mathcal{M}$ is strictly evolutionary bounded then* $\max^*(\mathcal{M}) \neq \min^*(\mathcal{M})$ *and, moreover,* $\max^*(\mathcal{M}) \overset{+}{\leftarrow} \min^*(\mathcal{M})$.

*Proof.* Let $\mathcal{M}$ be strictly evolutionary bounded oriented set. Assume that $\max^*(\mathcal{M}) = \min^*(\mathcal{M}) = x^*$. Since the oriented set $\mathcal{M}$ is strictly evolutionary bounded, we have, $\mathbf{card}\,(\mathfrak{Bs}(\mathcal{M})) \geq 2$ (by Definition 9). Hence there exist an element $x \in \mathfrak{Bs}(\mathcal{M})$ such that $x \neq x^*$. Then, using equality $\max^*(\mathcal{M}) = \min^*(\mathcal{M}) = x^*$ as well as Definition 8, we conclude that $x \overset{+}{\leftarrow} x^*$ and $x^* \overset{+}{\leftarrow} x$, which is impossible. Thus $\max^*(\mathcal{M}) \neq \min^*(\mathcal{M})$ and, by Definition 8, we deduce that $\max^*(\mathcal{M}) \overset{+}{\leftarrow} \min^*(\mathcal{M})$. □

### 5.5 Partial one-point chronologizations of oriented sets

In this section it will be introduced one more auxiliary technical concept necessary to formulate the main lemma, needed for the proof of main result.

**Notation 7.** *Let $\mathcal{M}$ be an oriented set and $\mathsf{M}_1 \subseteq \mathfrak{Bs}(\mathcal{M})$ be arbitrary nonempty subset of $\mathfrak{Bs}(\mathcal{M})$. Henceforth the record $\mathcal{M}_{\restriction \mathsf{M}_1}$ will mean the oriented set, satisfying the following conditions:*

- $\mathfrak{Bs}\left(\mathcal{M}_{\restriction \mathsf{M}_1}\right) = \mathsf{M}_1$;
- *For every $x, y \in \mathfrak{Bs}\left(\mathcal{M}_{\restriction \mathsf{M}_1}\right) = \mathsf{M}_1$ the correlation $y \underset{\mathcal{M}_{\restriction \mathsf{M}_1}}{\longleftarrow} x$ is performed if and only if $y \underset{\mathcal{M}}{\leftarrow} x$.*

*Further we will adhere to the following convention:*

***Convention***. *In the cases, where we simultaneously deal with the oriented set $\mathcal{M}$ as well as some its "oriented subset" $\mathcal{M}_{\restriction \mathsf{M}_1}$, for $x, y \in \mathsf{M}_1$ the records $y \overset{+-}{\leftarrow} x$ and $y \overset{-+}{\leftarrow} x$ always will mean that $y \underset{\mathcal{M}}{\overset{+-}{\leftarrow}} x$ and $y \underset{\mathcal{M}}{\overset{-+}{\leftarrow}} x$ (but not $y \underset{\mathcal{M}_{\restriction \mathsf{M}_1}}{\overset{+-}{\longleftarrow}} x$ or $y \underset{\mathcal{M}_{\restriction \mathsf{M}_1}}{\overset{-+}{\longleftarrow}} x$).*

It turns out that it is simpler to prove the existence of exactly 2-repeating one-point time on every quasi-chain oriented set. This fact serves as a motivation for the following auxiliary technical definition.

**Definition 10.** *Let $\mathcal{M}$ be any oriented set and $\mathsf{N} \subseteq \mathfrak{Bs}(\mathcal{M})$ be arbitrary nonempty subset of $\mathfrak{Bs}(\mathcal{M})$.*

*By **partial** 2**-repeating one-point chronologization** of the oriented set $\mathcal{M}$ with respect to the subset $\mathsf{N}$ we will understand the ordered pair $(\mathbb{T}, \psi)$, which satisfies the following conditions:*

1. *$\mathbb{T} = (\mathbf{T}, \leq)$ is a linearly ordered set.*
2. *$\psi$ is a mapping of type $\psi : \mathbf{T} \to 2^{\mathfrak{Bs}(\mathcal{M}_{\restriction \mathsf{N}})}$, moreover it is one-point 2-repeating time on the oriented set $\mathcal{M}_{\restriction \mathsf{N}}$.*
3. *For every $x, y \in \mathfrak{Bs}\left(\mathcal{M}_{\restriction \mathsf{N}}\right) = \mathsf{N}$ the following implications are performed:*

   **3.1** $\left(y \overset{+-}{\leftarrow} x\right) \Rightarrow \left(\widehat{\psi}^-(x) < \widehat{\psi}^-(y)\right)$;

   **3.2** $\left(y \overset{-+}{\leftarrow} x\right) \Rightarrow \left(\widehat{\psi}^+(x) < \widehat{\psi}^+(y)\right)$.

*Remark* 4. According to the convention, accepted in Notation 7, the conditions 3.1 and 3.2 of Definition 10 are essential and they do not follow from Assertion 4 (items (FR4) and (FR5)). Indeed, by this convention the records $y \overset{+-}{\leftarrow} x$ and $y \overset{-+}{\leftarrow} x$ must be understood as $y \underset{\mathcal{M}}{\overset{+-}{\leftarrow}} x$ and $y \underset{\mathcal{M}}{\overset{-+}{\leftarrow}} x$ (but not $y \underset{\mathcal{M}_{\restriction \mathsf{N}}}{\overset{+-}{\longleftarrow}} x$ or $y \underset{\mathcal{M}_{\restriction \mathsf{N}}}{\overset{-+}{\longleftarrow}} x$).

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*Remark* 5. Further for conciseness we will use the term **"partial 2-chronologization"** instead of "partial 2-repeating one-point chronologization".

## 6 The main lemma

The next main lemma is needed for the proof of sufficiency for Theorem 3.

**Lemma 2.** *Suppose the following:*

1. *An oriented set* $\mathcal{M}$ *is strictly evolutionary bounded and quasi-chain.*
2. $(\mathbb{T}_1, \psi_1) = ((\mathbf{T}_1, \leq_1), \psi_1)$ *is a partial* 2*-chronologization of* $\mathcal{M}$ *relatively a subset* $\mathsf{N} \subseteq \mathfrak{Bs}(\mathcal{M})$ *such that* $\min^*(\mathcal{M}), \max^*(\mathcal{M}) \in \mathsf{N}$.
3. $x_0 \in \mathfrak{Bs}(\mathcal{M}) \setminus \mathsf{N}$.
4. $t_0, t_0'$ *are arbitrary elements such that* $t_0, t_0' \notin \mathbf{T}_1$ *and* $t_0 \neq t_0'$.

*Then there exists the partial* 2*-chronologization* $(\mathbb{T}, \psi) = ((\mathbf{T}, \leq), \psi)$ *of the oriented set* $\mathcal{M}$ *relatively the subset* $\mathsf{N} \cup \{x_0\}$ *satisfying the following conditions:*

**a)** $\mathbb{T}_1 \sqsubseteq \mathbb{T}$.

**b)** $\mathbf{T} = \mathbf{T}_1 \cup \{t_0, t_0'\}$.

**c)** $\forall t \in \mathbf{T}_1 \ (\psi(t) = \psi_1(t))$.

*Proof.* $\boxed{1.}$ Introduce the following notations:

$$\mathsf{N}_+ := \left\{ x \in \mathsf{N} \mid x \overset{\mp}{\leftleftarrows} x_0 \right\}, \tag{26}$$

$$\mathsf{N}_- := \left\{ x \in \mathsf{N} \mid x_0 \overset{\pm}{\leftleftarrows} x \right\}, \tag{27}$$

$$x^* := \max^*(\mathcal{M}); \quad x_* := \min^*(\mathcal{M}).$$

Since the oriented set $\mathcal{M}$ is strictly evolutionary bounded then, by Assertion 9, we get $x^* \neq x_*$.

According to condition of the lemma, we have $x^*, x_* \in \mathsf{N}$. So, since $x_0 \notin \mathsf{N}$ then by Definition 8, we deduce: $x^* \overset{+}{\leftarrow} x_0 \overset{+}{\leftarrow} x_*$. Hence, in accordance with Assertion 3 (property (QL12)), we get: $x^* \overset{\mp}{\leftleftarrows} x_0 \overset{\pm}{\leftleftarrows} x_*$. Thus, we have, $x_* \in \mathsf{N}_-$ and $x^* \in \mathsf{N}_+$. Therefore the sets $\mathsf{N}_-$ and $\mathsf{N}_+$ are nonempty.

For every $x \in \mathsf{N}$ we put:

$$\widehat{\psi}^{+}_{1,x_0}(x) := \begin{cases} \widehat{\psi}^{-}_{1}(x), & x \overset{+}{\leftarrow} x_0 \\ \widehat{\psi}^{+}_{1}(x), & x \overset{+}{\not\leftarrow} x_0 \end{cases} \tag{28}$$

$$\widehat{\psi}^{-}_{1,x_0}(x) := \begin{cases} \widehat{\psi}^{+}_{1}(x), & x_0 \overset{+}{\leftarrow} x \\ \widehat{\psi}^{-}_{1}(x), & x_0 \overset{+}{\not\leftarrow} x, \end{cases} \tag{29}$$

where for $x_1, x_2 \in \mathfrak{Bs}(\mathcal{M})$ the record $x_2 \overset{+}{\not\leftarrow} x_1$ means that the correlation $x_2 \overset{+}{\leftarrow} x_1$ is not performed.

Now we are going to prove the following property of the functions $\widehat{\psi}^{+}_{1,x_0}$ and $\widehat{\psi}^{-}_{1,x_0}$:

**($\psi x_0$1)** *If* $x_1 \in \mathsf{N}_-$ *and* $x_2 \in \mathsf{N}_+$ *then* $\widehat{\psi}^-_{1,x_0}(x_1) <_1 \widehat{\psi}^+_{1,x_0}(x_2)$.

*Indeed*, consider any elements $x_1 \in \mathsf{N}_-$ and $x_2 \in \mathsf{N}_+$.

$\Rrightarrow$ In the case $x_2 \overset{+}{\not\leftarrow} x_0 \overset{+}{\not\leftarrow} x_1$, according to (26), (27), we get $x_2 \overset{-+}{\leftarrow} x_0 \overset{+-}{\leftarrow} x_1$. Hence, by Property (QL13) (see Assertion 3), we obtain, $x_2 \leftarrow x_1$. So, taking into account that the time $\psi_1$ is 2-repeating and using the formulas (28), (29) as well as Assertion 4 (Property (FR6)), we deliver:

$$\widehat{\psi}^-_{1,x_0}(x_1) = \widehat{\psi}^-_1(x_1) <_1 \widehat{\psi}^+_1(x_2) = \widehat{\psi}^+_{1,x_0}(x_2).$$

$\Rrightarrow$ In the case $x_2 \overset{+}{\leftarrow} x_0 \overset{+}{\not\leftarrow} x_1$, according to (27), we get, $x_2 \overset{+}{\leftarrow} x_0 \overset{+-}{\leftarrow} x_1$. Hence, by Property (QL16) (see Assertion 3), we obtain, $x_2 \overset{+-}{\leftarrow} x_1$. Thence (taking into account that $(\mathbb{T}_1, \psi_1)$ is a partial 2-chronologization of the oriented set $\mathcal{M}$ relatively $\mathsf{N}$), by Definition 10 (item 3.1), we have $\widehat{\psi}^-_1(x_1) <_1 \widehat{\psi}^-_1(x_2)$. So, according to (28), (29) we obtain:

$$\widehat{\psi}^-_{1,x_0}(x_1) = \widehat{\psi}^-_1(x_1) <_1 \widehat{\psi}^-_1(x_2) = \widehat{\psi}^+_{1,x_0}(x_2).$$

$\Rrightarrow$ In the case $x_2 \overset{+}{\not\leftarrow} x_0 \overset{+}{\leftarrow} x_1$, according to (26) we have, $x_2 \overset{-+}{\leftarrow} x_0 \overset{+}{\leftarrow} x_1$. Hence, by Property (QL17) (see Assertion 3), we obtain, $x_2 \overset{-+}{\leftarrow} x_1$. Thence, by Definition 10 (item 3.2), we obtain $\widehat{\psi}^+_1(x_1) <_1 \widehat{\psi}^+_1(x_2)$. So, in accordance with (28), (29), we get:

$$\widehat{\psi}^-_{1,x_0}(x_1) = \widehat{\psi}^+_1(x_1) <_1 \widehat{\psi}^+_1(x_2) = \widehat{\psi}^+_{1,x_0}(x_2).$$

$\Rrightarrow$ In the case $x_2 \overset{+}{\leftarrow} x_0 \overset{+}{\leftarrow} x_1$, according to Property (QL6) (see Assertion 2), we deduce $x_2 \overset{+}{\leftarrow} x_1$. Hence, by Assertion 4 (Property (FR3)), we obtain $\widehat{\psi}^+_1(x_1) <_1 \widehat{\psi}^-_1(x_2)$. So, in accordance with (28), (29), we conclude:

$$\widehat{\psi}^-_{1,x_0}(x_1) = \widehat{\psi}^+_1(x_1) <_1 \widehat{\psi}^-_1(x_2) = \widehat{\psi}^+_{1,x_0}(x_2).$$

Thus, in all possible cases the property have been proven.

Denote:

$$\mathbf{T}^+_1 := \left\{ \widehat{\psi}^+_{1,x_0}(x) \,\middle|\, x \in \mathsf{N}_+ \right\}; \qquad (30)$$

$$\mathbf{T}^-_1 := \left\{ \widehat{\psi}^-_{1,x_0}(x) \,\middle|\, x \in \mathsf{N}_- \right\}. \qquad (31)$$

From the Property ($\psi x_0$1) it follows that:

$$\forall\, t_1 \in \mathbf{T}^-_1 \ \forall\, t_2 \in \mathbf{T}^+_1 \ \ (t_1 <_1 t_2). \qquad (32)$$

Formula (32) together with Assertion 7 ensure the existence of the linearly ordered set $\mathbb{T} = (\mathbf{T}, \leq)$, possessing the following properties:

**(T1)** $\mathbb{T}_1 \sqsubseteq \mathbb{T}$;

**(T2)** $\mathbf{T} = \mathbf{T}_1 \cup \{t_0, t'_0\}$;

**(T3)** $\sup^*_{\mathbb{T}}\left(\mathbf{T}^-_1\right) = t_0$, $\ \inf^*_{\mathbb{T}}\left(\mathbf{T}^+_1\right) = t'_0$.

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From the formula (32) as well as Condition (T1) it follows that $\forall\, t_1 \in \mathbf{T}_1^- \ \forall\, t_2 \in \mathbf{T}_1^+ \ (t_1 < t_2)$. Therefore from Condition (T3) the inequality $t_0 \leq t_0'$ is deduced. But, by conditions of the lemma, we have $t_0 \neq t_0'$. Consequently:

$$t_0 < t_0'. \tag{33}$$

Below we establish some additional properties of the functions $\widehat{\psi}_1^+$ and $\widehat{\psi}_1^-$, namely the properties $(\psi x_0 2)$–$(\psi x_0 5)$ (see further).

**$(\psi x_0 2)$** *If $x \in \mathsf{N}$ and $x \leftarrow x_0$ then $t_0 < \widehat{\psi}_1^+(x)$.*

*Indeed*, let $x \in \mathsf{N}$ and $x \leftarrow x_0$. Let us prove that for an arbitrary element $x_1 \in \mathsf{N}_-$ the following inequality holds:

$$\widehat{\psi}_{1,x_0}^-(x_1) < \widehat{\psi}_1^+(x)\,. \tag{34}$$

$\Rrightarrow$ In the case $x_0 \overset{+}{\not\leftarrow} x_1$ according to (27) we get $x \leftarrow x_0 \overset{+-}{\leftarrow} x_1$. So, by Property (QL7) (see Assertion 3), we have $x \leftarrow x_1$. Consequently, taking into account the fact that the time $\psi$ is 2-repeating, using formula (29) and Property (FR6) from Assertion 4, we obtain, $\widehat{\psi}_{1,x_0}^-(x_1) = \widehat{\psi}_1^-(x_1) <_1 \widehat{\psi}_1^+(x)$. Thence, since $\mathbb{T}_1 \sqsubseteq \mathbb{T}$ (by condition (T1)), we get the inequality (34).

$\Rrightarrow$ In the case $x_0 \overset{+}{\leftarrow} x_1$ we have $x \leftarrow x_0 \overset{+}{\leftarrow} x_1$, ie $x \overset{-+}{\leftarrow} x_1$. Hence, applying the formula (29) as well as Definition 10 (item 3.2), we get, $\widehat{\psi}_{1,x_0}^-(x_1) = \widehat{\psi}_1^+(x_1) <_1 \widehat{\psi}_1^+(x)$. Thence, since $\mathbb{T}_1 \sqsubseteq \mathbb{T}$ (by condition (T1)), we get the inequality (34).

Therefore in the both cases the inequality (34) had been proven. Since the inequality (34) holds for each element $x_1 \in \mathsf{N}_-$, then, by the formula (31), we have:

$$t < \widehat{\psi}_1^+(x) \qquad \left(\forall t \in \mathbf{T}_1^-\right).$$

Hence the element $\widehat{\psi}_1^+(x) \in \mathbf{T}_1 \subseteq \mathbf{T}$ is a strict upper bound of the set $\mathbf{T}_1^-$, where, by condition (T3), $\sup^*_{\mathbb{T}}\left(\mathbf{T}_1^-\right) = t_0$. Thence, by Definition 7, (item 2), we obtain the inequality $t_0 \leq \widehat{\psi}_1^+(x)$. But, since $t_0 \notin \mathbf{T}_1$ (by conditions of the lemma) and $\widehat{\psi}_1^+(x) \in \mathbf{T}_1$, then the equality $t_0 = \widehat{\psi}_1^+(x)$ is impossible. Thus, we obtain the desired strict inequality $t_0 < \widehat{\psi}_1^+(x)$. The property have been proven.

**$(\psi x_0 3)$** *If $x \in \mathsf{N}$ and $x_0 \leftarrow x$ then $\widehat{\psi}_1^-(x) < t_0'$.*

*Indeed*, let $x \in \mathsf{N}$ and $x_0 \leftarrow x$. Let us prove that for an arbitrary element $x_1 \in \mathsf{N}_+$ the following inequality is fulfilled:

$$\widehat{\psi}_1^-(x) < \widehat{\psi}_{1,x_0}^+(x_1) \tag{35}$$

$\Rrightarrow$ In the case $x_1 \overset{+}{\not\leftarrow} x_0$ according to (26) we get $x_1 \overset{-+}{\leftarrow} x_0 \leftarrow x$. So, by Property (QL8) (see Assertion 3), we have, $x_1 \leftarrow x$. Consequently, taking into account the fact that the time $\psi$ is 2-repeating, using Property (FR6) from Assertion 4 and formula (28), we obtain, $\widehat{\psi}_1^-(x) <_1 \widehat{\psi}_1^+(x_1) = \widehat{\psi}_{1,x_0}^+(x_1)$. Thence, since $\mathbb{T}_1 \sqsubseteq \mathbb{T}$ (by condition (T1)), we get the inequality (35).

$\Rrightarrow$ In the case $x_1 \overset{+}{\leftarrow} x_0$ we have $x_1 \overset{+}{\leftarrow} x_0 \leftarrow x$, that is $x_1 \overset{+-}{\leftarrow} x$. Hence, taking into account Definition 10 (item 3.1), as well as the formula (28), we obtain, $\widehat{\psi}_1^-(x) <_1 \widehat{\psi}_1^-(x_1) = \widehat{\psi}_{1,x_0}^+(x_1)$. Thence, since $\mathbb{T}_1 \sqsubseteq \mathbb{T}$ (by condition (T1)), we get the inequality (35).

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Therefore, in the both cases the inequality (35) have been proven. Since the inequality (35) is valid for arbitrary element $x_1 \in \mathsf{N}_+$, then, according to the formula (30), we have:

$$\widehat{\psi}_1^-(x) < t \qquad \left(\forall t \in \mathbf{T}_1^+\right).$$

Hence the element $\widehat{\psi}_1^-(x) \in \mathbf{T}_1 \subseteq \mathbf{T}$ is a strict lower bound of the set $\mathbf{T}_1^+$. Thence, taking into account the condition (T3) (namely the equality $\inf_{\mathbb{T}}^* \left(\mathbf{T}_1^+\right) = t_0'$), we obtain the inequality $\widehat{\psi}_1^-(x) \leq t_0'$. But since $t_0' \notin \mathbf{T}_1$ (by condition of the lemma) and $\widehat{\psi}_1^-(x) \in \mathbf{T}_1$, then the equality $t_0' = \widehat{\psi}_1^-(x)$ is impossible. So we deliver the desired strict inequality $\widehat{\psi}_1^-(x) < t_0'$. The property have been proven.

**($\psi x_0$4)** *If $x \in \mathsf{N}$ and $x \overset{+-}{\leftarrow} x_0$ then $t_0 < \widehat{\psi}_1^-(x)$.*

*Indeed*, let $x \in \mathsf{N}$ and $x \overset{+-}{\leftarrow} x_0$. Let us prove that for an arbitrary element $x_1 \in \mathsf{N}_-$ the following inequality is fulfilled:

$$\widehat{\psi}_{1,x_0}^-(x_1) < \widehat{\psi}_1^-(x). \tag{36}$$

Below we consider two cases: $x_0 \overset{+}{\not\leftarrow} x_1$ and $x_0 \overset{+}{\leftarrow} x_1$.

- $\Rrightarrow$ Case $x_0 \overset{+}{\not\leftarrow} x_1$. In this case according to (27) we have, $x \overset{+-}{\leftarrow} x_0 \overset{+-}{\leftarrow} x_1$. Thence, by Assertion 3 (property (QL14)), we obtain, $x \overset{+-}{\leftarrow} x_1$ (where $x_1, x \in \mathsf{N}$). Hence, by Definition 10 (item 3.1), we get $\widehat{\psi}_1^-(x_1) <_1 \widehat{\psi}_1^-(x)$. Therefore, taking into account the fact that $x_0 \overset{+}{\not\leftarrow} x_1$, as well as the formula (29), we deduce, $\widehat{\psi}_{1,x_0}^-(x_1) = \widehat{\psi}_1^-(x_1) <_1 \widehat{\psi}_1^-(x)$. Thence, since $\mathbb{T}_1 \sqsubseteq \mathbb{T}$ (by condition (T1)), we get the inequality (36).
- $\Rrightarrow$ Case $x_0 \overset{+}{\leftarrow} x_1$. In this case we have, $x \overset{+-}{\leftarrow} x_0 \overset{+}{\leftarrow} x_1$. That is, by Assertion 3 (property (QL10)), we deliver $x \overset{+}{\leftarrow} x_1$. So, by Assertion 4 (property (FR3)), we have, $\widehat{\psi}_1^+(x_1) <_1 \widehat{\psi}_1^-(x)$. Hence, since, $x_0 \overset{+}{\leftarrow} x_1$, then according to the formula (29), we have, $\widehat{\psi}_{1,x_0}^-(x_1) = \widehat{\psi}_1^+(x_1) <_1 \widehat{\psi}_1^-(x)$. Thence, since $\mathbb{T}_1 \sqsubseteq \mathbb{T}$ (by condition (T1)), we get the inequality (36).

Therefore, in the both cases the inequality (36) have been proven. Since the inequality (36) is valid for arbitrary element $x_1 \in \mathsf{N}_-$, then, according to the formula (31), we obtain:

$$t < \widehat{\psi}_1^-(x) \qquad \left(\forall t \in \mathbf{T}_1^-\right).$$

Hence the element $\widehat{\psi}_1^-(x) \in \mathbf{T}_1 \subseteq \mathbf{T}$ is a strict upper bound of the set $\mathbf{T}_1^-$. Thence, taking into account the condition (T3) (namely the equality $\sup_{\mathbb{T}}^* \left(\mathbf{T}_1^-\right) = t_0$), we obtain the inequality $t_0 \leq \widehat{\psi}_1^-(x)$. But since $t_0 \notin \mathbf{T}_1$ (by condition of the lemma) and $\widehat{\psi}_1^-(x) \in \mathbf{T}_1$, then the equality $t_0 = \widehat{\psi}_1^-(x)$ is impossible. So we deliver the desired strict inequality $t_0 < \widehat{\psi}_1^-(x)$. The property have been proven.

**($\psi x_0$5)** *If $x \in \mathsf{N}$ and $x_0 \overset{-+}{\leftarrow} x$ then $\widehat{\psi}_1^+(x) < t_0'$.*

*Indeed*, let, $x \in \mathsf{N}$ and $x_0 \overset{-+}{\leftarrow} x$. Let us prove that for an arbitrary $x_1 \in \mathsf{N}_+$ the following inequality is valid:

$$\widehat{\psi}_1^+(x) < \widehat{\psi}_{1,x_0}^+(x_1). \tag{37}$$

Below we consider two cases: $x_1 \overset{+}{\not\leftarrow} x_0$ and $x_1 \overset{+}{\leftarrow} x_0$.

⇛ Case $x_1 \overset{+}{\not\leftarrow} x_0$. In this case according to (26), we have, $x_1 \overset{+}{\rightleftarrows} x_0 \overset{+}{\rightleftarrows} x$. Thence, by Assertion 3 (property (QL15)), we obtain, $x_1 \overset{+}{\rightleftarrows} x$ (where $x, x_1 \in \mathsf{N}$). Hence, by Definition 10 (item 3.2), we get, $\widehat{\psi}_1^+(x) <_1 \widehat{\psi}_1^+(x_1)$. That is why, taking into account the specifics of the case (ie $x_1 \overset{+}{\not\leftarrow} x_0$) as well as the formula (28), we deduce, $\widehat{\psi}_1^+(x) <_1 \widehat{\psi}_1^+(x_1) = \widehat{\psi}_{1,x_0}^+(x_1)$. Thence, since $\mathbb{T}_1 \sqsubseteq \mathbb{T}$ (by condition (T1)), we get the inequality (37).

⇛ Case $x_1 \overset{+}{\leftarrow} x_0$. In this case we have, $x_1 \overset{+}{\leftarrow} x_0 \overset{+}{\rightleftarrows} x$. So, by Assertion 3 (property (QL9)), we deliver $x_1 \overset{+}{\leftarrow} x$ (where $x, x_1 \in \mathsf{N}$). Therefore, by Assertion 4 (property (FR3)), we have, $\widehat{\psi}_1^+(x) <_1 \widehat{\psi}_1^-(x_1)$. That is why, taking into account the specifics of the case (ie $x_1 \overset{+}{\leftarrow} x_0$) as well as the formula (28), we deduce, $\widehat{\psi}_1^+(x) <_1 \widehat{\psi}_1^-(x_1) = \widehat{\psi}_{1,x_0}^+(x_1)$. Thence, since $\mathbb{T}_1 \sqsubseteq \mathbb{T}$ (by condition (T1)), we get the inequality (37).

Thus, in the both cases the inequality (37) have been proven. Since the inequality (37) is valid for arbitrary element $x_1 \in \mathsf{N}_+$, then, according to the formula (30), we obtain:

$$\widehat{\psi}_1^+(x) < t \qquad \left(\forall t \in \mathbf{T}_1^+\right).$$

Hence the element $\widehat{\psi}_1^+(x) \in \mathbf{T}_1 \subseteq \mathbf{T}$ is a strict lower bound of the set $\mathbf{T}_1^+$. Thence, taking into account the condition (T3), (namely the equality $\inf^*_{\mathbb{T}}\left(\mathbf{T}_1^+\right) = t_0'$), we obtain the inequality $\widehat{\psi}_1^+(x) \leq t_0'$. But since $t_0' \notin \mathbf{T}_1$ (by condition of the lemma) and $\widehat{\psi}_1^+(x) \in \mathbf{T}_1$, then the equality $t_0' = \widehat{\psi}_1^+(x)$ is impossible. So we deliver the desired strict inequality $\widehat{\psi}_1^+(x) < t_0'$. The property have been proven.

For each $t \in \mathbf{T}$ we put:

$$\psi(t) := \begin{cases} \psi_1(t), & t \in \mathbf{T}_1 \\ \{x_0\}, & t \in \{t_0, t_0'\}. \end{cases} \tag{38}$$

$\boxed{\mathbf{2.}}$ Now we are going to prove that the mapping $\psi$ is a time on the oriented set $\mathcal{M}_{\upharpoonright \mathsf{N}\cup\{x_0\}} = (\mathsf{N} \cup \{x_0\}, \leftarrow)$ (where the symbol $\leftarrow$ in reality means the restriction of the relation $\underset{\mathcal{M}}{\leftarrow}$ into the set $\mathsf{N} \cup \{x_0\} \subseteq \mathfrak{Bs}(\mathcal{M})$).

**2.1)** From the formula (38) and the fact that $\psi_1$ is a time on the oriented set $\mathcal{M}_{\upharpoonright \mathsf{N}} = (\mathsf{N}, \leftarrow)$ we conclude that $\forall x \in \mathsf{N} \cup \{x_0\}\, \exists t \in \mathbf{T}\ (x \in \psi(t))$ (where $\mathsf{N} \cup \{x_0\} = \mathfrak{Bs}\left(\mathcal{M}_{\upharpoonright \mathsf{N}\cup\{x_0\}}\right)$). So the first condition of Definition 3 is satisfied for the mapping $\psi$.

**2.2)** Let $x_1, x_2 \in \mathsf{N} \cup \{x_0\}$, $x_2 \leftarrow x_1$ and $x_1 \neq x_2$.

**2.2.1.** First we consider the case, where $x_1, x_2 \in \mathsf{N}$. From the fact that $\psi_1$ is a time on the oriented set $\mathcal{M}_{\upharpoonright \mathsf{N}} = (\mathsf{N}, \leftarrow)$ it follows the existence the time points $t_1, t_2 \in \mathbf{T}_1$ such that $x_1 \in \psi_1(t_1)$, $x_2 \in \psi(t_2)$ and $t_1 <_1 t_2$. Thence, using the formula (38) and condition (T1) ($\mathbb{T}_1 \sqsubseteq \mathbb{T}$) we obtain that $t_1, t_2 \in \mathbf{T}$, $x_1 \in \psi(t_1)$, $x_2 \in \psi(t_2)$, $t_1 < t_2$. Thus it only remains to consider the cases where $x_1 = x_0$ or $x_2 = x_0$. We consider these cases below.

**2.2.2.** Case $x_1 = x_0$. In this case we have, $x_2 \leftarrow x_1 = x_0$ and $x_2 \neq x_1 = x_0$. And from the inequality $x_2 \neq x_0$ if follows that $x_2 \in \mathsf{N}$. Thence, taking into account the property ($\psi x_0 2$), we obtain the inequality, $t_0 < \widehat{\psi}_1^+(x_2)$. Denote: $t_1 := t_0$, $t_2 := \widehat{\psi}_1^+(x_2)$. Then, applying the formula (38) as well as Notation 4, we obtain:

$$\begin{aligned} &x_1 = x_0 \in \{x_0\} = \psi(t_0) = \psi(t_1); \\ &x_2 \in \psi_1(t_2) = \psi(t_2) \quad \text{and} \\ &t_1 < t_2. \end{aligned}$$

**2.2.3.** Case $x_2 = x_0$. In this case we have, $x_0 = x_2 \leftarrow x_1$ and $x_1 \neq x_2 = x_0$. From the inequality $x_1 \neq x_0$ if follows that $x_1 \in \mathsf{N}$. Thence, taking into account the property $(\psi x_0 3)$, we obtain the inequality, $\widehat{\psi}_1^-(x_1) < t_0'$. Denote: $t_1 := \widehat{\psi}_1^-(x_1)$, $t_2 := t_0'$. Then, applying the formula (38) as well as Notation 4, we deduce:

$$\begin{aligned} & x_1 \in \psi_1(t_1) = \psi(t_1); \\ & x_2 = x_0 \in \{x_0\} = \psi(t_0') = \psi(t_2); \\ \text{and} \quad & t_1 < t_2. \end{aligned}$$

Therefore in all possible cases we have proven that, for every $x_1, x_2 \in \mathsf{N} \cup \{x_0\}$ such that $x_2 \leftarrow x_1$ and $x_1 \neq x_2$ there exist time points $t_1, t_2 \in \mathbf{T}$ such, that $x_1 \in \psi(t_1)$, $x_2 \in \psi(t_2)$ and $t_1 < t_2$. So, the second condition of Definition 3 is also satisfied for the mapping $\psi$. Thus, the mapping $\psi$ is a time on the oriented set $\mathcal{M}_{\upharpoonright \mathsf{N} \cup \{x_0\}} = (\mathsf{N} \cup \{x_0\}, \leftarrow)$.

$\boxed{3.}$ The next aim is to prove that the time $\psi$ is one-point.

**3.1)** Since the time $\psi_1$ is one-point, then from the formula (38) it follows that for each $t \in \mathbf{T}$ the set $\psi(t)$ is a singleton. Therefore, by Definition 4, ***the time $\psi$ is quasi one-point***.

**3.2)** Let $t_1, t_2 \in \mathbf{T}$, $x_1, x_2 \in \mathsf{N} \cup \{x_0\}$, $t_1 \leq t_2$, $x_1 \in \psi(t_1)$, $x_2 \in \psi(t_2)$. Let us prove that in this case $x_2 \leftarrow x_1$.

**3.2.1.** In the case $t_1 = t_2$ using the fact that (by item 3.1)) $\psi(t)$ is a singleton set, we obtain that $x_1 = x_2$. So, by Definition 1 in this case we have the desired correlation $x_2 \leftarrow x_1$. Hence, *further we consider that $t_1 < t_2$*.

**3.2.2.** In the case $t_1, t_2 \in \mathbf{T}_1$, taking into account the formula (38) as well as condition (T1) (ie $\mathbb{T}_1 \sqsubseteq \mathbb{T}$), we get:

$$\psi(t_1) = \psi_1(t_1), \qquad \psi(t_2) = \psi_1(t_2), \qquad t_1 <_1 t_2.$$

And since (by conditions of the lemma) the time $\psi_1$ is one-point, according to Definition 4 we obtain $x_2 \leftarrow x_1$.

Thus it remains to consider only two cases $t_1 \in \{t_0, t_0'\}$ and $t_2 \in \{t_0, t_0'\}$ under additional condition $t_1 < t_2$.

**3.2.3.** Case $t_1 \in \{t_0, t_0'\}$ $(t_1 < t_2)$. In this case, taking into account the inequality (33) as well as the formula (38), we have:

$$t_0 \leq t_1 < t_2, \qquad x_1 \in \psi(t_1) = \psi(t_0), \qquad x_2 \in \psi(t_2). \tag{39}$$

From the condition $x_1 \in \psi(t_0)$, according to formula (38), it follows that $x_1 = x_0$. Therefore it needs to prove that $x_2 \leftarrow x_0$. Assume the contrary, ie $x_2 \not\leftarrow x_0$. Then, taking into account that the oriented set $\mathcal{M}$ is quasi-chain, by Definition 5, we obtain $x_0 \overset{+}{\leftarrow} x_2$. So, the equality $x_0 = x_2$ is impossible, that is $x_2 \neq x_0$. Hence $x_2 \in \mathsf{N}$. Thus, using Assertion 3 (property (QL12)), we obtain:

$$x_2 \in \mathsf{N}_- = \left\{ x \in \mathsf{N} \,\middle|\, x_0 \overset{+-}{\leftarrow} x \right\}.$$

Since $x_0 \overset{+}{\leftarrow} x_2$ then, in accordance with formulas (29) and (31), we have, $\widehat{\psi}_1^+(x_2) = \widehat{\psi}_{1,x_0}^-(x_2) \in \mathbf{T}_1^-$. Thence we deliver:

$$\widehat{\psi}_1^+(x_2) < \sup\nolimits_{\mathbb{T}}^* \left(\mathbf{T}_1^-\right) = t_0. \tag{40}$$

Since $x_2 \in \mathsf{N}$ (ie $x_2 \neq x_0$) and (according to (39)), $x_2 \in \psi(t_2)$, then taking into account the fact that $\psi(t_0) = \psi(t_0') = \{x_0\}$, we get $t_2 \notin \{t_0, t_0'\}$. Hence, $t_2 \in \mathbf{T}_1$. So, by formula (38), we have $\psi(t_2) = \psi_1(t_2)$, that is (according to (39)) $x_2 \in \psi_1(t_2)$. Thence, $t_2 \leq_1 \widehat{\psi}_1^+(x_2)$. And, taking into account condition (T1) ($\mathbb{T}_1 \sqsubseteq \mathbb{T}$), as well as the inequality (40) we obtain:

$$t_2 \leq \widehat{\psi}_1^+(x_2) < t_0.$$

The last inequality contradicts to conditions (39). Hence, the assumption, made before is wrong. That is why we obtain, $x_2 \leftarrow x_0 = x_1$ (ie $x_2 \leftarrow x_1$). Which was to be proven.

**3.2.4.** Now we consider the case $t_2 \in \{t_0, t_0'\}$ $(t_1 < t_2)$. In this case, taking into account the inequality (33) as well as the formula (38), we have:

$$t_1 < t_2 \leq t_0', \qquad x_1 \in \psi\left(t_1\right), \qquad x_2 \in \psi\left(t_2\right) = \psi\left(t_0'\right). \tag{41}$$

Let us prove that $x_2 \leftarrow x_1$. From the condition $x_2 \in \psi\left(t_0'\right)$, according to formula (38), it follows that $x_2 = x_0$. Therefore it needs to prove that $x_0 \leftarrow x_1$. Assume the contrary, ie $x_0 \not\leftarrow x_1$. Then, taking into account that the oriented set $\mathcal{M}$ is quasi-chain, we have $x_1 \overset{+}{\leftarrow} x_0$. Therefore, the equality $x_0 = x_1$ is impossible, that is $x_1 \neq x_0$. Hence $x_1 \in \mathsf{N}$. Thus, using Assertion 3 (property (QL12)) we deduce:

$$x_1 \in \mathsf{N}_+ = \left\{ x \in \mathsf{N} \,\middle|\, x \overset{-+}{\leftarrow} x_0 \right\}.$$

Since $x_1 \overset{+}{\leftarrow} x_0$ then, in accordance with formulas (28) and (30), we have, $\widehat{\psi}_1^-\left(x_1\right) = \widehat{\psi}_{1,x_0}^+\left(x_1\right) \in \mathbf{T}_1^+$. Thence we deliver:

$$t_0' = \inf\nolimits_{\mathbb{T}}^{*}\left(\mathbf{T}_1^+\right) < \widehat{\psi}_1^-\left(x_1\right). \tag{42}$$

Since $x_1 \in \mathsf{N}$ (ie $x_1 \neq x_0$) and (according to (41)), $x_1 \in \psi\left(t_1\right)$, then taking into account the fact that $\psi\left(t_0\right) = \psi\left(t_0'\right) = \{x_0\}$, we get $t_1 \notin \{t_0, t_0'\}$. Hence, $t_1 \in \mathbf{T}_1$. So, by formula (38), we have $\psi\left(t_1\right) = \psi_1\left(t_1\right)$, that is (according to (41)) $x_1 \in \psi_1\left(t_1\right)$. Thence, $\widehat{\psi}_1^-\left(x_1\right) \leq_1 t_1$. And, taking into account condition (T1) ($\mathbb{T}_1 \sqsubseteq \mathbb{T}$), as well as the inequality (42) we obtain:

$$t_0' < \widehat{\psi}_1^-\left(x_1\right) \leq t_1.$$

The last inequality contradicts to conditions (41). Hence, the assumption, made before is wrong. That is why we obtain, $x_2 = x_0 \leftarrow x_1$. Which was to be proven.

From the items **3.2.1**–**3.2.4** it follows that in all possible cases the conditions $t_1, t_2 \in \mathbf{T}$, $x_1, x_2 \in \mathsf{N} \cup \{x_0\}$, $t_1 \leq t_2$, $x_1 \in \psi\left(t_1\right)$, $x_2 \in \psi\left(t_2\right)$ stipulate the correlation $x_2 \leftarrow x_1$.

By Definition 4, from the items **3.1)**, **3.2)** it follows that the time $\psi$ is one-point on the oriented set $\mathcal{M}_{\restriction \mathsf{N} \cup \{x_0\}} = \left(\mathsf{N} \cup \{x_0\}, \leftarrow\right)$.

$\boxed{\mathbf{4.}}$ Since the time $\psi_1$ is 2-repeating, the equality (38) ensures that the time $\psi$ also is 2-repeating.

$\boxed{\mathbf{5.}}$ Now we are going to prove that for any $x, y \in \mathsf{N} \cup \{x_0\} = \mathfrak{Bs}\left(\mathcal{M}_{\restriction \mathsf{N} \cup \{x_0\}}\right)$ the condition $y \overset{+-}{\leftarrow} x$ involves the inequality $\widehat{\psi}^-(x) < \widehat{\psi}^-(y)$. Hence, consider any $x, y \in \mathsf{N} \cup \{x_0\}$ such that $y \overset{+-}{\leftarrow} x$.

**5.1)** In the case, where $x, y \in \mathsf{N}$ the formula (38) delivers the equalities $\widehat{\psi}^-(x) = \widehat{\psi}_1^-(x)$, $\widehat{\psi}^-(y) = \widehat{\psi}_1^-(y)$. That is why in this case the desired implication follows from the fact that $(\mathbb{T}_1, \psi_1)$ is a partial 2-chronologization of $\mathcal{M}$ relatively the subset $\mathsf{N} \subseteq \mathfrak{Bs}(\mathcal{M})$. Therefore it remains to consider only the cases where $x = x_0$ or $y = x_0$.

**5.2)** Consider the case $x = x_0$. Since we have $y \overset{+-}{\leftarrow} x$, then the equality $y = x$ is impossible (by definition of relations $\overset{+}{\leftarrow}$ and $\overset{+-}{\leftarrow}$ (see notations 1 and 3)). Hence $y \neq x$, that is $y \neq x_0$. Thus, we have $y \in \mathsf{N}$ and $y \overset{+-}{\leftarrow} x_0$. Thence, using property ($\psi x_0 4$), we obtain $t_0 < \widehat{\psi}_1^-(y)$. Since $y \in \mathsf{N}$ then the formula (38) gives $\widehat{\psi}^-(y) = \widehat{\psi}_1^-(y)$. Thus, using the formula (38), we obtain:

$$\widehat{\psi}^-(x) = \widehat{\psi}^-(x_0) = t_0 < \widehat{\psi}_1^-(y) = \widehat{\psi}^-(y).$$

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**5.3)** Now we consider the case where $y = x_0$. Since $y \overset{+-}{\longleftarrow} x$, then we have $x_0 \overset{+-}{\longleftarrow} x$. Consequently, $x \neq x_0$, that is $x \in \mathsf{N}$. Thus, we have, $x_0 \overset{+-}{\longleftarrow} x$, where $x \in \mathsf{N}$. Thence it follows that $x \in \mathsf{N}_-$ (by formula (27)). That is why, applying (38) and (29), we obtain:

$$\widehat{\psi}^- (x) = \widehat{\psi}_1^- (x) \leq_1 \widehat{\psi}_{1,x_0}^- (x) .$$

Hence, taking into account that $\mathbb{T}_1 \sqsubseteq \mathbb{T}$ we deduce, $\widehat{\psi}^- (x) \leq \widehat{\psi}_{1,x_0}^- (x)$, where $\widehat{\psi}_{1,x_0}^- (x) \in \left\{ \widehat{\psi}_{1,x_0}^- (\widetilde{x}) \mid \widetilde{x} \in \mathsf{N}_- \right\} = \mathbf{T}_1^-$. Therefore, according to condition (T3) as well as Definition 7, we obtain:

$$\widehat{\psi}^- (x) \leq \widehat{\psi}_{1,x_0}^- (x) < \sup\nolimits_{\mathbb{T}}^* \left( \mathbf{T}_1^- \right) = t_0 = \widehat{\psi}^- (x_0) = \widehat{\psi}^- (y) .$$

Results, established in the items 5.1)–5.3) ensure that (in all possible cases) condition $y \overset{+-}{\longleftarrow} x$ leads to the inequality $\widehat{\psi}^- (x) < \widehat{\psi}^- (y)$ (for arbitrary $x, y \in \mathsf{N} \cup \{x_0\}$), which was to be proven.

$\boxed{6.}$ Let us prove that for any $x, y \in \mathsf{N} \cup \{x_0\}$ the condition $y \overset{-+}{\longleftarrow} x$ stipulates the inequality $\widehat{\psi}^+ (x) < \widehat{\psi}^+ (y)$. So, consider any $x, y \in \mathsf{N} \cup \{x_0\}$ such that $y \overset{+-}{\longleftarrow} x$.

**6.1)** In the case, where $x, y \in \mathsf{N}$ the formula (38) leads to the equalities $\widehat{\psi}^- (x) = \widehat{\psi}_1^- (x)$, $\widehat{\psi}^- (y) = \widehat{\psi}_1^- (y)$. That is why in this case the desired implication follows from the fact that $(\mathbb{T}_1, \psi_1)$ is a partial 2-chronologization of $\mathcal{M}$ relatively the subset $\mathsf{N} \subseteq \mathfrak{Bs}(\mathcal{M})$. Therefore it remains to consider only the cases where $x = x_0$ or $y = x_0$.

**6.2)** Consider the case $x = x_0$. Since we have $y \overset{-+}{\longleftarrow} x$, then the equality $y = x$ is impossible (by definition of relations $\overset{+}{\longleftarrow}$ and $\overset{-+}{\longleftarrow}$). Hence $y \neq x$, that is $y \neq x_0$. Thus, we have $y \in \mathsf{N}$ and $y \overset{-+}{\longleftarrow} x_0$. Thence it follows that $y \in \mathsf{N}_+$ (by formula (26)). That is why, applying (28) and (38), we obtain:

$$\widehat{\psi}_{1,x_0}^+ (y) \leq_1 \widehat{\psi}_1^+ (y) = \widehat{\psi}^+ (y) .$$

Hence, taking into account that $\mathbb{T}_1 \sqsubseteq \mathbb{T}$, we deduce, $\widehat{\psi}_{1,x_0}^+ (y) \leq \widehat{\psi}^+ (y)$, where $\widehat{\psi}_{1,x_0}^+ (y) \in \left\{ \widehat{\psi}_{1,x_0}^+ (\widetilde{x}) \mid \widetilde{x} \in \mathsf{N}_+ \right\} = \mathbf{T}_1^+$. Therefore, according to condition (T3), we obtain:

$$\widehat{\psi}^+ (x_0) = t_0' = \inf\nolimits_{\mathbb{T}}^* \left( \mathbf{T}_1^+ \right) < \widehat{\psi}_{1,x_0}^+ (y) \leq \widehat{\psi}^+ (y) .$$

And taking into account that $x = x_0$ we have the desired inequality $\widehat{\psi}^+ (x) < \widehat{\psi}^+ (y)$.

**6.3)** Now we consider the case where $y = x_0$. Since we have $y \overset{-+}{\longleftarrow} x$, then the equality $y = x$ is impossible (by definition of relations $\overset{+}{\longleftarrow}$ and $\overset{-+}{\longleftarrow}$). Hence, $x \neq y$, that is $x \neq x_0$. Thus, we have $x \in \mathsf{N}$ and $x_0 = y \overset{-+}{\longleftarrow} x$. Thence, using property $(\psi x_0 5)$, we get the inequality $\widehat{\psi}_1^+ (x) < t_0'$. That is why, applying (38), we obtain:

$$\widehat{\psi}_1^+ (x) < t_0' = \widehat{\psi}^+ (x_0) = \widehat{\psi}^+ (y) . \tag{43}$$

Taking into account that $x \in \mathsf{N}$, we deduce, $\widehat{\psi}^+ (x) = \widehat{\psi}_1^+ (x)$ (by formula (38)). Hence, from the inequality (43) we deliver the desired inequality, $\widehat{\psi}^+ (x) < \widehat{\psi}^+ (y)$.

Thus, in all possible cases the conditions $x, y \in \mathsf{N} \cup \{x_0\}$ and $y \overset{-+}{\longleftarrow} x$ lead to the inequality $\widehat{\psi}^+ (x) < \widehat{\psi}^+ (y)$.

$\Rrightarrow$ Conditions (T1), (T2), equality (38), as well as results, established in the items 2-6 of the present proof assure that $(\mathbb{T}, \psi)$ is partial 2-chronologization of the oriented set $\mathcal{M}$ relatively the subset $\mathsf{N} \cup \{x_0\}$, satisfying the additional conditions a)-c) of this lemma. $\square$

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# 7 Proof of Sufficiency for Theorem 3

The next lemma ensures the sufficiency for Theorem 3.

**Lemma 3.** *If the oriented set $\mathcal{M}$ is a quasi-chain then it can be one-point chronologized. Moreover there exists the chronologization $\mathcal{H} = ((\mathbf{T}, \leq), \psi)$ of $\mathcal{M}$ with 2-repeating one-point time $\psi$.*

*Proof.* Let $\mathcal{M}$ be quasi-chain oriented set.

**I.** First we prove the lemma under the following additional assumption:

**Assumption** $^*$ *Oriented set $\mathcal{M}$ is strictly evolutionary bounded.*

So, suppose that Assumption $^*$ holds. Then there exist $\min^*(\mathcal{M})$ and $\max^*(\mathcal{M})$. Consider an arbitrary set $\mathcal{T}$, satisfying the following conditions:

$$1)\ \mathbf{card}(\mathcal{T}) \geq \aleph_0, \qquad 2)\ \mathbf{card}(\mathcal{T}) > \mathbf{card}(\mathfrak{Bs}(\mathcal{M})).$$

**I.1.** Introduce the set $\mathscr{H}$ of all ordered pairs of kind $\mathbf{h} = (\mathbb{T}_{\mathbf{h}}, \psi_{\mathbf{h}}) = ((\mathbf{T}_{\mathbf{h}}, \leq_{\mathbf{h}}), \psi_{\mathbf{h}})$ satisfying the following conditions:

**$1^0$.** $\mathbf{h}$ is partial 2-chronologization of the oriented set $\mathcal{M}$ relatively a some subset $\mathsf{N}_{\mathbf{h}} \subseteq \mathfrak{Bs}(\mathcal{M})$ such, that $\min^*(\mathcal{M}), \max^*(\mathcal{M}) \in \mathsf{N}_{\mathbf{h}}$.

**$2^0$.** $\mathbf{T}_{\mathbf{h}} \subseteq \mathcal{T}$.

Note that from the item $1^0$ it readily follows that for each chronologization $\mathbf{h} = (\mathbb{T}_{\mathbf{h}}, \psi_{\mathbf{h}}) = ((\mathbf{T}_{\mathbf{h}}, \leq_{\mathbf{h}}), \psi_{\mathbf{h}}) \in \mathscr{H}$ the set $\mathsf{N}_{\mathbf{h}}$ can be expressed by the formula:

$$\mathsf{N}_{\mathbf{h}} = \bigcup_{t \in \mathbf{T}_{\mathbf{h}}} \psi_{\mathbf{h}}(t). \tag{44}$$

Let us prove that the set $\mathscr{H}$ is nonempty. Chose arbitrary four elements $\tau_1, \tau_2, \tau_3, \tau_4 \in \mathcal{T}$ (where $\tau_i \neq \tau_j$ for $i \neq j$). Denote:

$$\begin{aligned} \boldsymbol{x}_- &:= \min^*(\mathcal{M}), & \boldsymbol{x}_+ &:= \max^*(\mathcal{M}) \\ \mathsf{N}_{\mathbf{h}_0} &:= \{\boldsymbol{x}_-, \boldsymbol{x}_+\}, & \mathbf{T}_{\mathbf{h}_0} &:= \{\tau_1, \tau_2, \tau_3, \tau_4\}. \end{aligned}$$

Further for $\tau_i, \tau_j \in \mathbf{T}_{\mathbf{h}_0}$ $(i, j \in \overline{1,4})$ we consider that $\tau_i \leq_{\mathbf{h}_0} \tau_j$ if and only if $i \leq j$ (where $\leq$ is the standard order on the set of natural numbers). It is obviously that the ordered pair:

$$\mathbb{T}_{\mathbf{h}_0} = (\mathbf{T}_{\mathbf{h}_0}, \leq_{\mathbf{h}_0})$$

is a linearly ordered set. Denote:

$$\psi_{\mathbf{h}_0}(t) := \begin{cases} \{x_-\}, & t \in \{\tau_1, \tau_2\} \\ \{x_+\}, & t \in \{\tau_3, \tau_4\} \end{cases} \qquad (t \in \mathbf{T}_{\mathbf{h}_0}).$$

It is not hard to verify that $\psi_{\mathbf{h}_0}$ is an 2-repeating one-point time on the oriented set $\mathcal{M}_{\restriction \mathsf{N}_{\mathbf{h}_0}}$, moreover $\{\min^*(\mathcal{M}), \max^*(\mathcal{M})\} = \{\boldsymbol{x}_-, \boldsymbol{x}_+\} = \mathsf{N}_{\mathbf{h}_0}$ and $\mathbf{T}_{\mathbf{h}_0} = \{\tau_1, \tau_2, \tau_3, \tau_4\} \subseteq \mathcal{T}$. Hence, $\mathbf{h}_0 \in \mathscr{H}$. That is why $\mathscr{H} \neq \emptyset$, which was to be proven.

We introduce the following binary relation on the set $\mathscr{H}$:

($\mathcal{H}o$) For any chronologizations $\mathbf{h} = (\mathbb{T}_{\mathbf{h}}, \psi_{\mathbf{h}}) = ((\mathbf{T}_{\mathbf{h}}, \leq_{\mathbf{h}}), \psi_{\mathbf{h}}) \in \mathscr{H}$ and $\mathbf{H} = (\mathbb{T}_{\mathbf{H}}, \psi_{\mathbf{H}}) = ((\mathbf{T}_{\mathbf{H}}, \leq_{\mathbf{H}}), \psi_{\mathbf{H}}) \in \mathscr{H}$ we write $\mathbf{h} \leq^{\mathscr{H}} \mathbf{H}$ if and only if the following conditions are satisfied:

$\mathcal{H}o$**1.** $\mathbb{T}_{\mathbf{h}} \sqsubseteq \mathbb{T}_{\mathbf{H}}$ (in particular from the last correlation it follows that $\mathbf{T}_{\mathbf{h}} \subseteq \mathbf{T}_{\mathbf{H}}$).

$\mathcal{H}o$**2.** $\forall\, t \in \mathbf{T_h}\ (\psi_{\mathbf{h}}(t) = \psi_{\mathbf{H}}(t))$ (ie $\psi_{\mathbf{h}} \subseteq \psi_{\mathbf{H}}$).

It is easy to verify that the binary relation $\le_{\mathscr{H}}$ is a partial order (that is reflexive, asymmetric and transitive relation) on the set of chronologizations $\mathscr{H}$.

Now we are going to prove that in the ordered set $\left(\mathscr{H}, \le^{\mathscr{H}}\right)$ every chain has an upper bound.

Let $\mathscr{L} \subseteq \mathscr{H}$ ($\mathscr{L} \neq \emptyset$) be any chain of the ordered set $\left(\mathscr{H}, \le^{\mathscr{H}}\right)$. Below we construct some partial 2-chronologization $\widetilde{\mathbf{H}} = \left(\mathbb{T}_{\widetilde{\mathbf{H}}}, \psi_{\widetilde{\mathbf{H}}}\right) = \left(\left(\mathbf{T}_{\widetilde{\mathbf{H}}}, \le_{\widetilde{\mathbf{H}}}\right), \psi_{\widetilde{\mathbf{H}}}\right)$ of the oriented set $\mathcal{M}$.

$\Rrightarrow$ $\widetilde{\mathbf{H}}$**1.** Denote:

$$\mathbf{T}_{\widetilde{\mathbf{H}}} = \bigcup_{\mathbf{h}\in\mathscr{L}} \mathbf{T_h}. \tag{45}$$

$\Rrightarrow$ $\widetilde{\mathbf{H}}$**2.** For arbitrary $t_1, t_2 \in \mathbf{T}_{\widetilde{\mathbf{H}}}$ we will write $t_1 \le_{\widetilde{\mathbf{H}}} t_2$ if and only if a chronologization $\mathbf{h} \in \mathscr{L}$ exists such, that $t_1, t_2 \in \mathbf{T_h}$ and $t_1 \le_{\mathbf{h}} t_2$.

$\Rrightarrow$ $\widetilde{\mathbf{H}}$**3.** Let $t \in \mathbf{T}_{\widetilde{\mathbf{H}}}$. Then, according to formula (45), there exists a chronologization $\mathbf{h} \in \mathscr{L}$ such that $t \in \mathbf{T_h}$. Denote:

$$\psi_{\widetilde{\mathbf{H}}}(t) := \psi_{\mathbf{h}}(t)\,. \tag{46}$$

First of all we have to prove that the formula (46) determines the mapping $\psi_{\widetilde{\mathbf{H}}}(t) : \mathbf{T}_{\widetilde{\mathbf{H}}} \to 2^{\mathsf{N}_{\widetilde{\mathbf{H}}}}$ by a correct way, where

$$\mathsf{N}_{\widetilde{\mathbf{H}}} = \bigcup_{\mathbf{H}\in\mathscr{L}} \mathsf{N}_{\mathbf{H}}. \tag{47}$$

Suppose that $t \in \mathbf{T_h}$ and $t \in \mathbf{T}_{\mathbf{h}_1}$, where $\mathbf{h}, \mathbf{h}_1 \in \mathscr{L}$. To prove the correctness of definition of the mapping $\psi_{\widetilde{\mathbf{H}}}$ by the formula (46), it is necessary to verify the equality $\psi_{\mathbf{h}}(t) = \psi_{\mathbf{h}_1}(t)$. Since $\mathscr{L}$ is a chain of the ordered set $\left(\mathscr{H}, \le^{\mathscr{H}}\right)$ then $\mathscr{L}$ is a linearly ordered set under the relation $\le^{\mathscr{H}}$. Hence for the elements $\mathbf{h}, \mathbf{h}_1 \in \mathscr{L}$ at least one of the correlations $\mathbf{h} \le^{\mathscr{H}} \mathbf{h}_1$ or $\mathbf{h}_1 \le^{\mathscr{H}} \mathbf{h}$ must hold. But in the both cases, according to the item $\mathcal{H}o2$ of definition ($\mathcal{H}o$) of order $\le^{\mathscr{H}}$, we obtain the equality, $\psi_{\mathbf{h}}(t) = \psi_{\mathbf{h}_1}(t)$, which was to be proven.

**I.2.** Now the aim is to prove that the triple $\widetilde{\mathbf{H}} = \left(\mathbb{T}_{\widetilde{\mathbf{H}}}, \psi_{\widetilde{\mathbf{H}}}\right) = \left(\left(\mathbf{T}_{\widetilde{\mathbf{H}}}, \le_{\widetilde{\mathbf{H}}}\right), \psi_{\widetilde{\mathbf{H}}}\right)$, constructed above, is a partial 2-chronologization of the oriented set $\mathcal{M}$ relatively the subset $\mathsf{N}_{\widetilde{\mathbf{H}}} \subseteq \mathfrak{Bs}(\mathcal{M})$, determined by the formula (47).

**I.2.1.** Let us prove that $\mathbb{T}_{\widetilde{\mathbf{H}}} = \left(\mathbf{T}_{\widetilde{\mathbf{H}}}, \le_{\widetilde{\mathbf{H}}}\right)$ is a linearly ordered set.

**I.2.1.a)** Consider any element $t \in \mathbf{T}_{\widetilde{\mathbf{H}}}$. By the formula (45), there exist the chronologization $\mathbf{h} = (\mathbb{T}_{\mathbf{h}}, \psi_{\mathbf{h}}) = ((\mathbf{T_h}, \le_{\mathbf{h}}), \psi_{\mathbf{h}}) \in \mathscr{L}$ such, that $t \in \mathbf{T_h}$. So, since $\mathbb{T}_{\mathbf{h}} = (\mathbf{T_h}, \le_{\mathbf{h}})$ is a linearly ordered set, we have $t \le_{\mathbf{h}} t$, and, according to the item $\widetilde{\mathbf{H}}2$ (see above), we get $t \le_{\widetilde{\mathbf{H}}} t$. Hence, the reflexivity of the relation $\le_{\widetilde{\mathbf{H}}}$ has been proven.

**I.2.1.b)** Assume that for elements $t, \tau \in \mathbf{T}_{\widetilde{\mathbf{H}}}$ inequalities $t \le_{\widetilde{\mathbf{H}}} \tau$ and $\tau \le_{\widetilde{\mathbf{H}}} t$ hold. Then, by the item $\widetilde{\mathbf{H}}2$ (see above), the chronologizations $\mathbf{h}_1, \mathbf{h}_2 \in \mathscr{L}$ exist such, that $t, \tau \in \mathbf{T}_{\mathbf{h}_i}$ ($i \in \overline{1,2}$), $t \le_{\mathbf{h}_1} \tau$ and $\tau \le_{\mathbf{h}_2} t$. Since $\mathscr{L}$ is a chain of the ordered set $\left(\mathscr{H}, \le^{\mathscr{H}}\right)$ the element $\mathbf{h}_* \in \{\mathbf{h}_1, \mathbf{h}_2\}$ must exist such that $\mathbf{h}_1 \le^{\mathscr{H}} \mathbf{h}_*$ and $\mathbf{h}_2 \le^{\mathscr{H}} \mathbf{h}_*$. Indeed, to verify the existence of such element $\mathbf{h}_*$ it is sufficient to put:

$$\mathbf{h}_* := \begin{cases} \mathbf{h}_2, & \mathbf{h}_1 \le^{\mathscr{H}} \mathbf{h}_2 \\ \mathbf{h}_1, & \mathbf{h}_2 <^{\mathscr{H}} \mathbf{h}_1. \end{cases}$$

According to the item $\mathcal{H}o1$ of the definition ($\mathcal{H}o$) of the order relation $\le^{\mathscr{H}}$, the correlations $\mathbf{h}_1 \le^{\mathscr{H}} \mathbf{h}_*$ and $\mathbf{h}_2 \le^{\mathscr{H}} \mathbf{h}_*$ provide the correlations $\mathbb{T}_{\mathbf{h}_1} \sqsubseteq \mathbb{T}_{\mathbf{h}_*}$ and $\mathbb{T}_{\mathbf{h}_2} \sqsubseteq \mathbb{T}_{\mathbf{h}_*}$. Hence (according to Notation 5), we deliver the inclusions $\mathbf{T}_{\mathbf{h}_1} \subseteq \mathbf{T}_{\mathbf{h}_*}$

and $\mathbf{T}_{\mathbf{h}_2} \subseteq \mathbf{T}_{\mathbf{h}_*}$. So, taking into account that $t, \tau \in \mathbf{T}_{\mathbf{h}_i}$ $(i \in \overline{1,2})$, we obtain $t, \tau \in \mathbf{T}_{\mathbf{h}_*}$. Since $\mathbb{T}_{\mathbf{h}_1}, \mathbb{T}_{\mathbf{h}_2} \sqsubseteq \mathbb{T}_{\mathbf{h}_*}$, then the inequalities $t \le_{\mathbf{h}_1} \tau$ and $\tau \le_{\mathbf{h}_2} t$, lead to the inequalities $t \le_{\mathbf{h}_*} \tau$ and $\tau \le_{\mathbf{h}_*} t$ (see Notation 5). But from the last inequalities, taking into account the fact that $(\mathbf{T}_{\mathbf{h}_*}, \le_{\mathbf{h}_*})$ is a linearly ordered set, we obtain the equality $t = \tau$. Thus, the asymmetry of the relation $\le_{\widetilde{\mathbf{H}}}$ has been proven.

**I.2.1.c)** Let elements $t_1, t_2, t_3 \in \mathbf{T}_{\widetilde{\mathbf{H}}}$, be such that $t_1 \le_{\widetilde{\mathbf{H}}} t_2$ and $t_2 \le_{\widetilde{\mathbf{H}}} t_3$. Then, according to the item $\widetilde{\mathbf{H}}2$ (see above), there exist the chronologizations $\mathbf{h}_{12}, \mathbf{h}_{23} \in \mathscr{L}$ such, that $t_1, t_2 \in \mathbf{T}_{\mathbf{h}_{12}}$, $t_2, t_3 \in \mathbf{T}_{\mathbf{h}_{23}}$, $t_1 \le_{\mathbf{h}_{12}} t_2$ and $t_2 \le_{\mathbf{h}_{23}} t_3$. Since $\mathscr{L}$ is a chain of the ordered set $\left(\mathscr{H}, \le^{\mathscr{H}}\right)$ the element $\mathbf{h}_{13} \in \{\mathbf{h}_{12}, \mathbf{h}_{23}\}$ must exist such that $\mathbf{h}_{12} \le^{\mathscr{H}} \mathbf{h}_{13}$ and $\mathbf{h}_{23} \le^{\mathscr{H}} \mathbf{h}_{13}$. According to the item $\mathcal{H}o1$ of the definition $(\mathcal{H}o)$, the correlations $\mathbf{h}_{12} \le^{\mathscr{H}} \mathbf{h}_{13}$ and $\mathbf{h}_{23} \le^{\mathscr{H}} \mathbf{h}_{13}$ lead to the correlations $\mathbf{T}_{\mathbf{h}_{12}} \subseteq \mathbf{T}_{\mathbf{h}_{13}}$ and $\mathbf{T}_{\mathbf{h}_{23}} \subseteq \mathbf{T}_{\mathbf{h}_{13}}$. So, taking into account that $t_1, t_2 \in \mathbf{T}_{\mathbf{h}_{12}}$ and $t_2, t_3 \in \mathbf{T}_{\mathbf{h}_{23}}$, we get $t_1, t_2, t_3 \in \mathbf{T}_{\mathbf{h}_{13}}$. Since $\mathbb{T}_{\mathbf{h}_{12}}, \mathbb{T}_{\mathbf{h}_{23}} \sqsubseteq \mathbb{T}_{\mathbf{h}_{13}}$, then the inequalities $t_1 \le_{\mathbf{h}_{12}} t_2$ and $t_2 \le_{\mathbf{h}_{23}} t_3$ assure the inequalities $t_1 \le_{\mathbf{h}_{13}} t_2$ and $t_2 \le_{\mathbf{h}_{13}} t_3$ (in accordance with Notation 5). But, since $(\mathbf{T}_{\mathbf{h}_{13}}, \le_{\mathbf{h}_{13}})$ is a linearly ordered set, the last two inequalities ensure the inequality $t_1 \le_{\mathbf{h}_{13}} t_3$. Thus, we have $t_1, t_3 \in \mathbf{T}_{\mathbf{h}_{13}}$ and $t_1 \le_{\mathbf{h}_{13}} t_3$. Thence by the item $\widetilde{\mathbf{H}}2$ (see above), we deduce $t_1 \le_{\widetilde{\mathbf{H}}} t_3$. Hence, the transitivity of the relation $\le_{\widetilde{\mathbf{H}}}$ has been proven.

**I.2.1.d)** Consider any elements $t_1, t_2 \in \mathbf{T}_{\widetilde{\mathbf{H}}}$. From the formula (45) it follows the existence of chronologizations $\mathbf{h}_1, \mathbf{h}_2 \in \mathscr{L}$ such that $t_1 \in \mathbf{T}_{\mathbf{h}_1}$, $t_2 \in \mathbf{T}_{\mathbf{h}_2}$. Since $\mathscr{L}$ is a chain of the ordered set $\left(\mathscr{H}, \le^{\mathscr{H}}\right)$ the element $\mathbf{h}_* \in \{\mathbf{h}_1, \mathbf{h}_2\}$ must exist such that $\mathbf{h}_1 \le^{\mathscr{H}} \mathbf{h}_*$ and $\mathbf{h}_2 \le^{\mathscr{H}} \mathbf{h}_*$. According to the item $\mathcal{H}o1$ of the definition $(\mathcal{H}o)$, the correlations $\mathbf{h}_1 \le^{\mathscr{H}} \mathbf{h}_*$ and $\mathbf{h}_2 \le^{\mathscr{H}} \mathbf{h}_*$ lead to the correlations $\mathbf{T}_{\mathbf{h}_1} \subseteq \mathbf{T}_{\mathbf{h}_*}$ and $\mathbf{T}_{\mathbf{h}_2} \subseteq \mathbf{T}_{\mathbf{h}_*}$. That is why, taking into account that $t_i \in \mathbf{T}_{\mathbf{h}_i}$ $(i \in \overline{1,2})$, we get $t_1, t_2 \in \mathbf{T}_{\mathbf{h}_*}$. Thence, since $(\mathbf{T}_{\mathbf{h}_*}, \le_{\mathbf{h}_*})$ is a linearly ordered set, it follows that at least one of the inequalities $t_1 \le_{\mathbf{h}_*} t_2$ or $t_2 \le_{\mathbf{h}_*} t_1$ must be performed. But, in accordance with the item $\widetilde{\mathbf{H}}2$ (see above), the inequality $t_1 \le_{\mathbf{h}_*} t_2$ leads to the inequality $t_1 \le_{\widetilde{\mathbf{H}}} t_2$ as well as the inequality $t_2 \le_{\mathbf{h}_*} t_1$ ensures the inequality $t_2 \le_{\widetilde{\mathbf{H}}} t_1$. Thus for every $t_1, t_2 \in \mathbf{T}_{\widetilde{\mathbf{H}}}$ at least one of the correlations $t_1 \le_{\widetilde{\mathbf{H}}} t_2$ or $t_2 \le_{\widetilde{\mathbf{H}}} t_1$ must be true. Hence the ordering $\le_{\widetilde{\mathbf{H}}}$ is a linear, and so $\left(\mathbf{T}_{\widetilde{\mathbf{H}}}, \le_{\widetilde{\mathbf{H}}}\right)$ is a linearly ordered set, which was to be proven.

**I.2.2.** Now we are going to prove that the mapping $\psi_{\widetilde{\mathbf{H}}}(t) : \mathbf{T}_{\widetilde{\mathbf{H}}} \to 2^{\mathsf{N}_{\widetilde{\mathbf{H}}}}$ is a time on the oriented set $\mathcal{M}_{\restriction \mathsf{N}_{\widetilde{\mathbf{H}}}} = \left(\mathsf{N}_{\widetilde{\mathbf{H}}}, \leftarrow\right)$.

**I.2.2.a)** Let $x \in \mathsf{N}_{\widetilde{\mathbf{H}}} = \mathfrak{Bs}\left(\mathcal{M}_{\restriction \mathsf{N}_{\widetilde{\mathbf{H}}}}\right)$. Then by the formula (47), there exist the chronologization $\mathbf{h} \in \mathscr{L}$ such, that $x \in \mathsf{N}_{\mathbf{h}}$. Thence, according to the formula (44), it follows the existence of the element $t \in \mathbf{T}_{\mathbf{h}}$ such, that $x \in \psi_{\mathbf{h}}(t)$. Since $t \in \mathbf{T}_{\mathbf{h}}$ then, by the formula (45), we get $t \in \mathbf{T}_{\widetilde{\mathbf{H}}}$, as well, according to the formula (46), we obtain $\psi_{\widetilde{\mathbf{H}}}(t) = \psi_{\mathbf{h}}(t)$. Therefore we have $x \in \psi_{\widetilde{\mathbf{H}}}(t)$, where $t \in \mathbf{T}_{\widetilde{\mathbf{H}}}$. Thus, we have proven that for an arbitrary element $x \in \mathfrak{Bs}\left(\mathcal{M}_{\restriction \mathsf{N}_{\widetilde{\mathbf{H}}}}\right)$ the element $t \in \mathbf{T}_{\widetilde{\mathbf{H}}}$ exists such that $x \in \psi_{\widetilde{\mathbf{H}}}(t)$.

**I.2.2.b)** Let $x_1, x_2 \in \mathsf{N}_{\widetilde{\mathbf{H}}} = \mathfrak{Bs}\left(\mathcal{M}_{\restriction \mathsf{N}_{\widetilde{\mathbf{H}}}}\right)$, $x_2 \leftarrow x_1$ and $x_1 \ne x_2$. Then according to the formula (47), there exist the chronologizations $\mathbf{h}_1, \mathbf{h}_2 \in \mathscr{L}$ such, that $x_1 \in \mathsf{N}_{\mathbf{h}_1}$, $x_2 \in \mathsf{N}_{\mathbf{h}_2}$. Since $\mathscr{L}$ is a chain of the ordered set $\left(\mathscr{H}, \le^{\mathscr{H}}\right)$ the element $\mathbf{h}_* \in \{\mathbf{h}_1, \mathbf{h}_2\}$ must exist such that $\mathbf{h}_1 \le^{\mathscr{H}} \mathbf{h}_*$ and $\mathbf{h}_2 \le^{\mathscr{H}} \mathbf{h}_*$. According to the item $\mathcal{H}o1$ of the definition $(\mathcal{H}o)$, the correlations $\mathbf{h}_1 \le^{\mathscr{H}} \mathbf{h}_*$ and $\mathbf{h}_2 \le^{\mathscr{H}} \mathbf{h}_*$ lead to the correlation $\mathbf{T}_{\mathbf{h}_1}, \mathbf{T}_{\mathbf{h}_2} \subseteq \mathbf{T}_{\mathbf{h}_*}$. Consequently, applying the formulas (44) and (46) for $i \in \overline{1,2}$ we deduce:

$$\mathsf{N}_{\mathbf{h}_i} = \bigcup_{t \in \mathbf{T}_{\mathbf{h}_i}} \psi_{\mathbf{h_i}}(t) = \bigcup_{t \in \mathbf{T}_{\mathbf{h}_i}} \psi_{\widetilde{\mathbf{H}}}(t) \subseteq \bigcup_{t \in \mathbf{T}_{\mathbf{h}_*}} \psi_{\widetilde{\mathbf{H}}}(t) = \mathsf{N}_{\mathbf{h}_*}.$$

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Hence we have $x_1, x_2 \in \mathsf{N}_{\mathbf{h}_*}$, where $\mathbf{h}_* = ((\mathbf{T}_{\mathbf{h}_*}, \leq_{\mathbf{h}_*}), \psi_{\mathbf{h}_*}) \in \mathscr{L}$ is a partial 2-chronologization of the oriented set $\mathcal{M}$ relatively the subset $\mathsf{N}_{\mathbf{h}_*} \subseteq \mathfrak{Bs}(\mathcal{M})$, moreover $x_2 \leftarrow x_1$ and $x_1 \neq x_2$. And since $\psi_{\mathbf{h}_*}$ is a time on the oriented set $\mathcal{M}_{\restriction \mathsf{N}_{\mathbf{h}_*}} = (\mathsf{N}_{\mathbf{h}_*}, \leftarrow)$, then (by Definition 3) there exist elements $t_1, t_2 \in \mathbf{T}_{\mathbf{h}_*}$ such, that $x_1 \in \psi_{\mathbf{h}_*}(t_1)$, $x_2 \in \psi_{\mathbf{h}_*}(t_2)$ and $t_1 <_{\mathbf{h}_*} t_2$. Since $t_1, t_2 \in \mathbf{T}_{\mathbf{h}_*}$, where $\mathbf{h}_* \in \mathscr{L}$, then, by the formula (45), we have $t_1, t_2 \in \mathbf{T}_{\widetilde{\mathbf{H}}}$. Taking into account that $\mathbf{h}_* \in \mathscr{L}$ and $t_1, t_2 \in \mathbf{T}_{\mathbf{h}_*}$ and using the formula (46), we obtain, $\psi_{\mathbf{h}_*}(t_1) = \psi_{\widetilde{\mathbf{H}}}(t_1)$, $\psi_{\mathbf{h}_*}(t_2) = \psi_{\widetilde{\mathbf{H}}}(t_2)$. According to the item $\widetilde{\mathbf{H}}2$ (see above), the inequality $t_1 <_{\mathbf{h}_*} t_2$ stipulates the inequality $t_1 <_{\widetilde{\mathbf{H}}} t_2$. Hence we have, $t_1, t_2 \in \mathbf{T}_{\widetilde{\mathbf{H}}}$, $x_1 \in \psi_{\widetilde{\mathbf{H}}}(t_1)$, $x_2 \in \psi_{\widetilde{\mathbf{H}}}(t_2)$ and $t_1 <_{\widetilde{\mathbf{H}}} t_2$. Thus, we have proven that for arbitrary $x_1, x_2 \in \mathfrak{Bs}\left(\mathcal{M}_{\restriction \mathsf{N}_{\widetilde{\mathbf{H}}}}\right)$, satisfying $x_2 \leftarrow x_1$ and $x_1 \neq x_2$, there exist elements $t_1, t_2 \in \mathbf{T}_{\widetilde{\mathbf{H}}}$ such, that $x_1 \in \psi_{\widetilde{\mathbf{H}}}(t_1)$, $x_2 \in \psi_{\widetilde{\mathbf{H}}}(t_2)$ and $t_1 <_{\widetilde{\mathbf{H}}} t_2$.

Taking into account the results, obtained in the items I.2.2.a) and I.2.2.b), as well as Definition 3, we see that the mapping $\psi_{\widetilde{\mathbf{H}}}(t) : \mathbf{T}_{\widetilde{\mathbf{H}}} \to 2^{\mathsf{N}_{\widetilde{\mathbf{H}}}}$ is a time on the oriented set $\mathcal{M}_{\restriction \mathsf{N}_{\widetilde{\mathbf{H}}}}$, which was to be proven.

**I.2.3.** Now we prove that the time $\psi_{\widetilde{\mathbf{H}}}$ is one-point on the oriented set $\mathcal{M}_{\restriction \mathsf{N}_{\widetilde{\mathbf{H}}}}$.

**I.2.3.a)** From the formula (46) as well as the fact that $\psi_{\mathbf{h}}$ is an one-point time on the oriented set $\mathcal{M}_{\restriction \mathsf{N}_{\mathbf{h}}}$ (for each $\mathbf{h} \in \mathscr{L}$) it follows that for every $t \in \mathbf{T}_{\widetilde{\mathbf{H}}}$ the set $\psi_{\widetilde{\mathbf{H}}}(t)$ is a singleton.

**I.2.3.b)** Let $t_1, t_2 \in \mathbf{T}_{\widetilde{\mathbf{H}}}$, $x_1, x_2 \in \mathsf{N}_{\widetilde{\mathbf{H}}} = \mathfrak{Bs}\left(\mathcal{M}_{\restriction \mathsf{N}_{\widetilde{\mathbf{H}}}}\right)$, $t_1 \leq_{\widetilde{\mathbf{H}}} t_2$, $x_1 \in \psi_{\widetilde{\mathbf{H}}}(t_1)$, $x_2 \in \psi_{\widetilde{\mathbf{H}}}(t_2)$. The aim is to prove that in this case it is true that $x_2 \leftarrow x_1$. Since $t_1 \leq_{\widetilde{\mathbf{H}}} t_2$ then, by the item $\widetilde{\mathbf{H}}2$ (see above), the chronologization $\mathbf{h} \in \mathscr{L}$ exists such, that $t_1, t_2 \in \mathbf{T}_{\mathbf{h}}$ and $t_1 \leq_{\mathbf{h}} t_2$. Since $t_1, t_2 \in \mathbf{T}_{\mathbf{h}}$, then the formula (46) stipulates the equalities, $\psi_{\widetilde{\mathbf{H}}}(t_1) = \psi_{\mathbf{h}}(t_1)$, $\psi_{\widetilde{\mathbf{H}}}(t_2) = \psi_{\mathbf{h}}(t_2)$. Therefore we have:

$$t_1, t_2 \in \mathbf{T}_{\mathbf{h}}, \quad t_1 \leq_{\mathbf{h}} t_2, \quad x_1 \in \psi_{\mathbf{h}}(t_1), \quad x_2 \in \psi_{\mathbf{h}}(t_2). \tag{48}$$

Since $\mathbf{h}$ is a partial 2-chronologization of the oriented set $\mathcal{M}$ relatively the subset $\mathsf{N}_{\mathbf{h}} \subseteq \mathfrak{Bs}(\mathcal{M})$, then $\psi_{\mathbf{h}}$ is a one-point time on the oriented set $\mathcal{M}_{\restriction \mathsf{N}_{\mathbf{h}}}$. That is why, by Definition 4, the correlations (48) deliver the desired correlations $x_2 \leftarrow x_1$. Thus, for arbitrary $t_1, t_2 \in \mathbf{T}_{\widetilde{\mathbf{H}}}$, $x_1, x_2 \in \mathsf{N}_{\widetilde{\mathbf{H}}} = \mathfrak{Bs}\left(\mathcal{M}_{\restriction \mathsf{N}_{\widetilde{\mathbf{H}}}}\right)$ the conditions $t_1 \leq_{\widetilde{\mathbf{H}}} t_2$, $x_1 \in \psi_{\widetilde{\mathbf{H}}}(t_1)$, $x_2 \in \psi_{\widetilde{\mathbf{H}}}(t_2)$ stipulate the correlation $x_2 \leftarrow x_1$.

By Definition 4, results, established in the items I.2.3.a) and I.2.3.b), imply that $\psi_{\widetilde{\mathbf{H}}}$ is an one-point time on the oriented set $\mathcal{M}_{\restriction \mathsf{N}_{\widetilde{\mathbf{H}}}}$, which was to be proven.

**I.2.4.** The next aim is to prove that the time $\psi_{\widetilde{\mathbf{H}}}$ is 2-repeating. Let $x \in \mathsf{N}_{\widetilde{\mathbf{H}}}$. Then, by the formula (47), there exists the chronologization $\mathbf{h} = ((\mathbf{T}_{\mathbf{h}}, \leq_{\mathbf{h}}), \psi_{\mathbf{h}}) \in \mathscr{L}$ such, that $x \in \mathsf{N}_{\mathbf{h}}$. According to the item $1^0$ (see the definition of the the set $\mathscr{H}$ above), $\mathbf{h}$ is a partial 2-chronologization of the oriented set $\mathcal{M}$ relatively the subset $\mathsf{N}_{\mathbf{h}} \subseteq \mathfrak{Bs}(\mathcal{M})$. That is why the time $\psi_{\mathbf{h}}$ is 2-repeating and so by Definition 6, two elements $t_1, t_2 \in \mathbf{T}_{\mathbf{h}}$ $(t_1 \neq t_2)$ must exist such that $x \in \psi_{\mathbf{h}}(t_1)$ and $x \in \psi_{\mathbf{h}}(t_2)$. Since $\mathbf{h} \in \mathscr{L}$ and $t_1, t_2 \in \mathbf{T}_{\mathbf{h}}$, then, by the formula (45) we have $t_1, t_2 \in \mathbf{T}_{\widetilde{\mathbf{H}}}$, as well by the formula (46), we obtain $\psi_{\mathbf{h}}(t_1) = \psi_{\widetilde{\mathbf{H}}}(t_1)$, $\psi_{\mathbf{h}}(t_2) = \psi_{\widetilde{\mathbf{H}}}(t_2)$. Hence, we have proven that there exist the elements $t_1, t_2 \in \mathbf{T}_{\widetilde{\mathbf{H}}}$ such that $t_1 \neq t_2$ and $x \in \psi_{\widetilde{\mathbf{H}}}(t_1) \cap \psi_{\widetilde{\mathbf{H}}}(t_2)$. Consequently, by Definition 6, we ensure:

$$\mathsf{Rp}_x\left(\psi_{\widetilde{\mathbf{H}}}\right) = \mathbf{card}\left(\left\{t \in \mathbf{T}_{\widetilde{\mathbf{H}}} \,\middle|\, x \in \psi_{\widetilde{\mathbf{H}}}(t)\right\}\right) \geq 2. \tag{49}$$

Now we assume that there exist the elements $x \in \mathsf{N}_{\widetilde{\mathbf{H}}}$ and $t_1, t_2, t_3 \in \mathbf{T}_{\widetilde{\mathbf{H}}}$ such, that $t_i \neq t_j$ for $i \neq j$ $(i, j \in \overline{1,3})$ and $x \in \psi_{\widetilde{\mathbf{H}}}(t_1) \cap \psi_{\widetilde{\mathbf{H}}}(t_2) \cap \psi_{\widetilde{\mathbf{H}}}(t_3)$. In accordance with the formula (45), the correlation $t_1, t_2, t_3 \in \mathbf{T}_{\widetilde{\mathbf{H}}}$ provides the existence of chronologizations $\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3 \in \mathscr{L}$ such, that $t_i \in \mathbf{T}_{\mathbf{h}_i}$ $(\forall i \in \overline{1,3})$. Since $\mathscr{L}$ is a chain of the ordered set

$\left(\mathscr{H}, \leq^{\mathscr{H}}\right)$ the element $\mathbf{h}_0 \in \{\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3\}$ must exist such that $\mathbf{h}_i \leq^{\mathscr{H}} \mathbf{h}_0$ $(\forall i \in \overline{1,3})$. According to the item $\mathcal{H}o1$ of the definition $(\mathcal{H}o)$, the correlations $\mathbf{h}_i \leq^{\mathscr{H}} \mathbf{h}_0$ $(i \in \overline{1,3})$, lead to the correlations $\mathbf{T}_{\mathbf{h}_i} \subseteq \mathbf{T}_{\mathbf{h}_0}$ $(i \in \overline{1,3})$. Therefore, we have, $t_1, t_2, t_3 \in \mathbf{T}_{\mathbf{h}_0}$. Thence, by the formula (46), we ensure $\psi_{\widetilde{\mathbf{H}}}(t_i) = \psi_{\mathbf{h}_0}(t_i)$, $(i \in \overline{1,3})$. Thus, we get $x \in \psi_{\mathbf{h}_0}(t_1) \cap \psi_{\mathbf{h}_0}(t_2) \cap \psi_{\mathbf{h}_0}(t_3)$, where $t_1, t_2, t_3 \in \mathbf{T}_{\mathbf{h}_0}$ and $t_i \neq t_j$ for $i \neq j$ $(i, j \in \overline{1,3})$. Hence, by Definition 6, we have, $\mathsf{Rp}_x(\psi_{\mathbf{h}_0}) = \mathbf{card}\left(\left\{t \in \mathbf{T} \mid x \in \psi_{\mathbf{h}_0}(t)\right\}\right) \geq 3$. But the last inequality is in a contradiction to the fact that $\mathbf{h}_0 = \left(\left(\mathbf{T}_{\mathbf{h}_0}, \leq_{\mathbf{h}_0}\right), \psi_{\mathbf{h}_0}\right)$ a partial 2-chronologization of the oriented set $\mathcal{M}$ (ie to the fact that the time $\psi_{\mathbf{h}_0}$ is 2-repeating). Thus the assumption, made before, leads to a contradiction. Consequently, there do not exist the elements $x \in \mathsf{N}_{\widetilde{\mathbf{H}}}$ and $t_1, t_2, t_3 \in \mathbf{T}_{\widetilde{\mathbf{H}}}$ such, that $t_i \neq t_j$ for $i \neq j$ $(i, j \in \overline{1,3})$ and $x \in \psi_{\widetilde{\mathbf{H}}}(t_1) \cap \psi_{\widetilde{\mathbf{H}}}(t_2) \cap \psi_{\widetilde{\mathbf{H}}}(t_3)$. According to Definition 6, this means, that:

$$\mathsf{Rp}_x\left(\psi_{\widetilde{\mathbf{H}}}\right) = \mathbf{card}\left(\left\{t \in \mathbf{T}_{\widetilde{\mathbf{H}}} \mid x \in \psi_{\widetilde{\mathbf{H}}}(t)\right\}\right) \leq 2. \tag{50}$$

From the inequalities (49) and (50) it follows that the equality $\mathsf{Rp}_x\left(\psi_{\widetilde{\mathbf{H}}}\right) = 2$ holds for each $x \in \mathsf{N}_{\widetilde{\mathbf{H}}}$. That is why the time $\psi_{\widetilde{\mathbf{H}}}$ is 2-repeating, which was to be proven.

**I.2.5.** Let us prove that for every $\mathbf{h} \in \mathscr{L}$ and $x \in \mathsf{N}_{\mathbf{h}}$ the following equalities are performed:

$$\widehat{\psi}_{\mathbf{h}}^{-}(x) = \widehat{\psi}_{\widetilde{\mathbf{H}}}^{-}(x), \qquad \widehat{\psi}_{\mathbf{h}}^{+}(x) = \widehat{\psi}_{\widetilde{\mathbf{H}}}^{+}(x).$$

Indeed, let $\mathbf{h} \in \mathscr{L}$ and $x \in \mathsf{N}_{\mathbf{h}}$. Denote:

$$t_{-} := \widehat{\psi}_{\mathbf{h}}^{-}(x), \qquad t_{+} := \widehat{\psi}_{\mathbf{h}}^{+}(x).$$

Then, by Notation 4 (taking into account that the time $\psi_{\mathbf{h}} : \mathbf{T}_{\mathbf{h}} \to 2^{\mathsf{N}_{\mathbf{h}}}$ is an one-point and 2-repeating), we obtain:

$$t_{+}, t_{-} \in \mathbf{T}_{\mathbf{h}}, \qquad \psi_{\mathbf{h}}(t_{-}) = \psi_{\mathbf{h}}(t_{+}) = \{x\}, \qquad t_{-} <_{\mathbf{h}} t_{+}.$$

Thence, using the formulas (45) and (46), as well as item $\widetilde{\mathbf{H}}2$ (see above), we ensure:

$$t_{+}, t_{-} \in \mathbf{T}_{\widetilde{\mathbf{H}}}, \qquad \psi_{\widetilde{\mathbf{H}}}(t_{-}) = \psi_{\widetilde{\mathbf{H}}}(t_{+}) = \{x\}, \qquad t_{-} <_{\widetilde{\mathbf{H}}} t_{+}.$$

According to the result, established in the item I.2.4, the time $\psi_{\widetilde{\mathbf{H}}}$ is 2-repeating. Hence we have:

$$\begin{aligned}\widehat{\psi}_{\widetilde{\mathbf{H}}}^{+}(x) &= \max\left(\left\{t \in \mathbf{T} \mid x \in \psi_{\widetilde{\mathbf{H}}}(t)\right\}\right) = t_{+};\\ \widehat{\psi}_{\widetilde{\mathbf{H}}}^{-}(x) &= \min\left(\left\{t \in \mathbf{T} \mid x \in \psi_{\widetilde{\mathbf{H}}}(t)\right\}\right) = t_{-},\end{aligned}$$

which was to be proven.

**I.2.6.** The next aim is to prove that for arbitrary $x, y \in \mathsf{N}_{\widetilde{\mathbf{H}}} = \mathfrak{Bs}\left(\mathcal{M}_{\restriction \mathsf{N}_{\widetilde{\mathbf{H}}}}\right)$ the correlation $y \overset{+-}{\leftarrow} x$ leads to the inequality $\widehat{\psi}_{\widetilde{\mathbf{H}}}^{-}(x) <_{\widetilde{\mathbf{H}}} \widehat{\psi}_{\widetilde{\mathbf{H}}}^{-}(y)$. Let $x, y \in \mathsf{N}_{\widetilde{\mathbf{H}}}$ and $y \overset{+-}{\leftarrow} x$. In accordance with the equality (47), the correlation $x, y \in \mathsf{N}_{\widetilde{\mathbf{H}}}$ assures the existence of chronologizations $\mathbf{h}_x, \mathbf{h}_y \in \mathscr{L}$ such, that $x \in \mathsf{N}_{\mathbf{h}_x}$, $y \in \mathsf{N}_{\mathbf{h}_y}$. Since $\mathscr{L}$ is a chain of the ordered set $\left(\mathscr{H}, \leq^{\mathscr{H}}\right)$ the element $\mathbf{h}_{xy} \in \{\mathbf{h}_x, \mathbf{h}_y\} \subseteq \mathscr{L}$ must exist such that $\mathbf{h}_x \leq^{\mathscr{H}} \mathbf{h}_{xy}$, $\mathbf{h}_y \leq^{\mathscr{H}} \mathbf{h}_{xy}$. Then, from the item $\mathcal{H}o1$ of the definition $(\mathcal{H}o)$, we deduce the inclusions $\mathbf{T}_{\mathbf{h}_x} \subseteq \mathbf{T}_{\mathbf{h}_{xy}}$, $\mathbf{T}_{\mathbf{h}_y} \subseteq \mathbf{T}_{\mathbf{h}_{xy}}$. Hence, using the formula (44) as well as the item $\mathcal{H}o2$ of the definition $(\mathcal{H}o)$ we deliver:

$$\mathsf{N}_{\mathbf{h}_x} = \bigcup_{t \in \mathbf{T}_{\mathbf{h}_x}} \psi_{\mathbf{h}_x}(t) = \bigcup_{t \in \mathbf{T}_{\mathbf{h}_x}} \psi_{\mathbf{h}_{xy}}(t) \subseteq \bigcup_{t \in \mathbf{T}_{\mathbf{h}_{xy}}} \psi_{\mathbf{h}_{xy}}(t) = \mathsf{N}_{\mathbf{h}_{xy}}, \quad \mathsf{N}_{\mathbf{h}_y} \subseteq \mathsf{N}_{\mathbf{h}_{xy}}.$$

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Therefore, the chronologization $\mathbf{h}_{xy} = \left(\left(\mathbf{T}_{\mathbf{h}_{xy}}, \leq_{\mathbf{h}_{xy}}\right), \psi_{\mathbf{h}_{xy}}\right) \in \mathscr{L}$ satisfies the condition $x, y \in \mathsf{N}_{\mathbf{h}_{xy}}$, while the time $\psi_{\mathbf{h}_{x,y}} : \mathbf{T}_{\mathbf{h}_{xy}} \to 2^{\mathsf{N}_{\mathbf{h}_{xy}}}$ is an one-point, 2-repeating and satisfying the condition 3.1 of Definition 10, that is:

$$\widehat{\psi}^{-}_{\mathbf{h}_{xy}}(x) <_{\mathbf{h}_{xy}} \widehat{\psi}^{-}_{\mathbf{h}_{xy}}(y).$$

Thence, applying result, established in the item I.2.5, we obtain the inequality $\widehat{\psi}^{-}_{\widetilde{\mathbf{H}}}(x) <_{\mathbf{h}_{xy}} \widehat{\psi}^{-}_{\widetilde{\mathbf{H}}}(y)$, which together with the item $\widetilde{\mathbf{H}}2$ (see above), provides the desired inequality:

$$\widehat{\psi}^{-}_{\widetilde{\mathbf{H}}}(x) <_{\widetilde{\mathbf{H}}} \widehat{\psi}^{-}_{\widetilde{\mathbf{H}}}(y).$$

**I.2.7.** Now we prove that for arbitrary $x, y \in \mathsf{N}_{\widetilde{\mathbf{H}}} = \mathfrak{Bs}\left(\mathcal{M}_{\restriction \mathsf{N}_{\widetilde{\mathbf{H}}}}\right)$ the correlation $y \overset{-+}{\leftarrow} x$ leads to the inequality $\widehat{\psi}^{+}_{\widetilde{\mathbf{H}}}(x) <_{\widetilde{\mathbf{H}}} \widehat{\psi}^{+}_{\widetilde{\mathbf{H}}}(y)$. Let, $x, y \in \mathsf{N}_{\widetilde{\mathbf{H}}}$ and $y \overset{-+}{\leftarrow} x$. In accordance with the equality (47), the correlation $x, y \in \mathsf{N}_{\widetilde{\mathbf{H}}}$ ensures the existence of chronologizations $\mathbf{h}_x, \mathbf{h}_y \in \mathscr{L}$ such, that $x \in \mathsf{N}_{\mathbf{h}_x}$, $y \in \mathsf{N}_{\mathbf{h}_y}$. Since $\mathscr{L}$ is a chain of the ordered set $\left(\mathscr{H}, \leq^{\mathscr{H}}\right)$ the element $\mathbf{h}_{xy} \in \{\mathbf{h}_x, \mathbf{h}_y\} \subseteq \mathscr{L}$ must exist such that $\mathbf{h}_x \leq^{\mathscr{H}} \mathbf{h}_{xy}$, $\mathbf{h}_y \leq^{\mathscr{H}} \mathbf{h}_{xy}$. Then, from the item $\mathcal{H}o1$ of the definition ($\mathcal{H}o$), we get the inclusions $\mathbf{T}_{\mathbf{h}_x} \subseteq \mathbf{T}_{\mathbf{h}_{xy}}$, $\mathbf{T}_{\mathbf{h}_y} \subseteq \mathbf{T}_{\mathbf{h}_{xy}}$. Hence, using the formula (44) as well as the item $\mathcal{H}o2$ of the definition ($\mathcal{H}o$) we deliver:

$$\mathsf{N}_{\mathbf{h}_x} \subseteq \mathsf{N}_{\mathbf{h}_{xy}}, \qquad \mathsf{N}_{\mathbf{h}_y} \subseteq \mathsf{N}_{\mathbf{h}_{xy}}.$$

So, the chronologization $\mathbf{h}_{xy} = \left(\left(\mathbf{T}_{\mathbf{h}_{xy}}, \leq_{\mathbf{h}_{xy}}\right), \psi_{\mathbf{h}_{xy}}\right) \in \mathscr{L}$ satisfies the condition $x, y \in \mathsf{N}_{\mathbf{h}_{xy}}$, whereas the time $\psi_{\mathbf{h}_{x,y}} : \mathbf{T}_{\mathbf{h}_{xy}} \to 2^{\mathsf{N}_{\mathbf{h}_{xy}}}$ is an one-point, 2-repeating and satisfying the condition 3.2 of Definition 10, that is:

$$\widehat{\psi}^{+}_{\mathbf{h}_{xy}}(x) <_{\mathbf{h}_{xy}} \widehat{\psi}^{+}_{\mathbf{h}_{xy}}(y).$$

Thence, applying result, established in the item I.2.5, we obtain the inequality $\widehat{\psi}^{+}_{\widetilde{\mathbf{H}}}(x) <_{\mathbf{h}_{xy}} \widehat{\psi}^{+}_{\widetilde{\mathbf{H}}}(y)$, which together with the item $\widetilde{\mathbf{H}}2$ (see above), provides the desired inequality:

$$\widehat{\psi}^{+}_{\widetilde{\mathbf{H}}}(x) <_{\widetilde{\mathbf{H}}} \widehat{\psi}^{+}_{\widetilde{\mathbf{H}}}(y).$$

From the facts, established in the items I.2.1–I.2.7 it follows that the triple $\widetilde{\mathbf{H}} = \left(\mathbb{T}_{\widetilde{\mathbf{H}}}, \psi_{\widetilde{\mathbf{H}}}\right) = \left(\left(\mathbf{T}_{\widetilde{\mathbf{H}}}, \leq_{\widetilde{\mathbf{H}}}\right), \psi_{\widetilde{\mathbf{H}}}\right)$ is a partial 2-chronologization of the oriented set $\mathcal{M}$ relatively the subset $\mathsf{N}_{\widetilde{\mathbf{H}}}$.

**I.3.** Since for each chronologization $\mathbf{h} \in \mathscr{L}$ it is fulfilled the condition $\min^*(\mathcal{M}), \max^*(\mathcal{M}) \in \mathsf{N}_{\mathbf{h}}$, then, according to the formula (47), we obtain:

$$\min^*(\mathcal{M}), \max^*(\mathcal{M}) \in \mathsf{N}_{\widetilde{\mathbf{H}}}.$$

**I.4.** Taking into account the item $2^0$ of the definition of the chronologization set $\mathscr{H}$ as well as the including $\mathscr{L} \subseteq \mathscr{H}$, we conclude that for every chronologization $\mathbf{h} \in \mathscr{L}$ the inclusion $\mathbf{T}_{\mathbf{h}} \subseteq \mathcal{T}$ holds. That is why, based on the formula (45), we deduce:

$$\mathbf{T}_{\widetilde{\mathbf{H}}} = \bigcup_{\mathbf{h} \in \mathscr{L}} \mathbf{T}_{\mathbf{h}} \subseteq \mathcal{T}.$$

From the facts, established in the items **I.2–I.4** it follows that $\widetilde{\mathbf{H}} \in \mathscr{H}$.

**I.5.** Now the aim is to prove that the chronologization $\widetilde{\mathbf{H}}$ is an upper bound of the chain $\mathscr{L}$ relatively the ordered set $\left(\mathscr{H}, \leq^{\mathscr{H}}\right)$. Chose any $\mathbf{h} \in \mathscr{L}$. Let us prove that $\mathbf{h} \leq^{\mathscr{H}} \widetilde{\mathbf{H}}$.

**I.5.1.** Formula (45) provides the inclusion:

$$\mathbf{T}_{\mathbf{h}} \subseteq \bigcup_{\mathbf{H} \in \mathscr{L}} \mathbf{T}_{\mathbf{H}} = \mathbf{T}_{\widetilde{\mathbf{H}}}. \tag{51}$$

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Consider arbitrary $t_1, t_2 \in \mathbf{T_h}$. Suppose, that $t_1 \leq_{\mathbf{h}} t_2$. Then using the item $\widetilde{\mathbf{H}}2$ (see above), we get, $t_1 \leq_{\widetilde{\mathbf{H}}} t_2$. Conversely, suppose, that $t_1 \leq_{\widetilde{\mathbf{H}}} t_2$. If we assume that $t_2 <_{\mathbf{h}} t_1$, then, according to the item $\widetilde{\mathbf{H}}2$ (see above), we will obtain the inequality $t_2 <_{\widetilde{\mathbf{H}}} t_1$, which contradicts to the start assumption that $t_1 \leq_{\widetilde{\mathbf{H}}} t_2$. Hence, since $\mathbb{T}_{\mathbf{h}} = (\mathbf{T_h}, \leq_{\mathbf{h}})$ is a linearly ordered set, we ensure the inequality $t_1 \leq_{\mathbf{h}} t_2$. Thus we have:

$$\forall\, t_1,\, t_2 \in \mathbf{T_h}\ \left( (t_1 \leq_{\mathbf{h}} t_2) \Leftrightarrow \left(t_1 \leq_{\widetilde{\mathbf{H}}} t_2\right) \right). \tag{52}$$

In accordance with Notation 5, the correlations (51) and (52) lead to the correlation:

$$\mathbb{T}_{\mathbf{h}} \sqsubseteq \mathbb{T}_{\widetilde{\mathbf{H}}}.$$

**I.5.2.** From the formula (46) it follows that

$$\forall\, t \in \mathbf{T_h}\ \left(\psi_{\mathbf{h}}(t) = \psi_{\widetilde{\mathbf{H}}}(t)\right).$$

From the results, established in the items I.5.1 and I.5.2 it follows that $\mathbf{h} \leq^{\mathscr{H}} \widetilde{\mathbf{H}}$, moreover the last inequality is valid for each chronologization $\mathbf{h} \in \mathscr{L}$. Therefore $\widetilde{\mathbf{H}}$ is an upper bound of the chain $\mathscr{L}$ relatively the ordered set $\left(\mathscr{H}, \leq^{\mathscr{H}}\right)$, which was to be proven.

Taking into account the arbitrariness of choice of chain $\mathscr{L} \subseteq \mathscr{H}$, we have seen that in the ordered set $\left(\mathscr{H}, \leq^{\mathscr{H}}\right)$ every chain has an upper bound. Therefore, according to Zorn's lemma, this ordered set contains a maximal element.

**I.6.** Let chronologization $\mathbf{h}^* = (\mathbb{T}_{\mathbf{h}^*}, \psi_{\mathbf{h}^*}) = ((\mathbf{T}_{\mathbf{h}^*}, \leq_{\mathbf{h}^*}), \psi_{\mathbf{h}^*}) \in \mathscr{H}$ be a maximal element of the ordered set $\left(\mathscr{H}, \leq^{\mathscr{H}}\right)$. Let us prove that $\mathsf{N}_{\mathbf{h}^*} = \mathfrak{Bs}(\mathcal{M})$, where $\mathsf{N}_{\mathbf{h}^*} = \bigcup_{t \in \mathbf{T}_{\mathbf{h}^*}} \psi_{\mathbf{h}^*}(t)$ is the set, determined by the formula (44). It is evidently that $\mathsf{N}_{\mathbf{h}^*} \subseteq \mathfrak{Bs}(\mathcal{M})$. Assume, that $\mathsf{N}_{\mathbf{h}^*} \neq \mathfrak{Bs}(\mathcal{M})$. Then the element $x_0 \in \mathfrak{Bs}(\mathcal{M})$ exists such, that $x_0 \notin \mathsf{N}_{\mathbf{h}^*}$. We will estimate the cardinality of the set $\mathbf{T}_{\mathbf{h}^*}$. Since $\psi_{\mathbf{h}^*}$ is an one-point time, then, by Definition 4, the correlation $x \in \psi_{\mathbf{h}^*}(t)$ is equivalent to the correlation $\psi_{\mathbf{h}^*}(t) = \{x\}$ (for any $x \in \mathsf{N}_{\mathbf{h}^*}$ and $t \in \mathbf{T}_{\mathbf{h}^*}$). Hence, since the time $\psi_{\mathbf{h}^*}$ is 2-repeating, by Definition 6, for each $x \in \mathsf{N}_{\mathbf{h}^*}$ we obtain:

$$\begin{aligned}\mathbf{card}\left(\psi_{\mathbf{h}^*}^{[-1]}(\{x\})\right) &= \mathbf{card}\left(\left\{t \in \mathbf{T}_{\mathbf{h}^*} \,\middle|\, \psi_{\mathbf{h}^*}(t) = \{x\}\right\}\right) = \\ &= \mathbf{card}\left(\left\{t \in \mathbf{T}_{\mathbf{h}^*} \,\middle|\, x \in \psi_{\mathbf{h}^*}(t)\right\}\right) = \mathsf{Rp}_x(\psi_{\mathbf{h}^*}) = 2.\end{aligned}$$

Thence we deduce that for every $x \in \mathsf{N}_{\mathbf{h}^*}$ it is valid the equality:

$$\psi_{\mathbf{h}^*}^{[-1]}(\{x\}) = \left\{\widehat{\psi}_{\mathbf{h}^*}^{-}(x),\, \widehat{\psi}_{\mathbf{h}^*}^{+}(x)\right\}. \tag{53}$$

Since the time $\psi_{\mathbf{h}^*}$ is one-point, then for any $t \in \mathbf{T}_{\mathbf{h}^*}$ the set $\psi_{\mathbf{h}^*}(t)$, being singleton, is non-empty. Hence it is performed the equality, $\mathbf{T}_{\mathbf{h}^*} = \bigcup_{x \in \mathsf{N}_{\mathbf{h}^*}} \psi_{\mathbf{h}^*}^{[-1]}(\{x\})$. Thence, applying the equality (53), we obtain:

$$\begin{aligned}\mathbf{T}_{\mathbf{h}^*} &= \bigcup_{x \in \mathsf{N}_{\mathbf{h}^*}} \psi_{\mathbf{h}^*}^{[-1]}(\{x\}) = \bigcup_{x \in \mathsf{N}_{\mathbf{h}^*}} \left\{\widehat{\psi}_{\mathbf{h}^*}^{-}(x),\, \widehat{\psi}_{\mathbf{h}^*}^{+}(x)\right\} = \\ &= \left(\bigcup_{x \in \mathsf{N}_{\mathbf{h}^*}} \left\{\widehat{\psi}_{\mathbf{h}^*}^{-}(x)\right\}\right) \cup \left(\bigcup_{x \in \mathsf{N}_{\mathbf{h}^*}} \left\{\widehat{\psi}_{\mathbf{h}^*}^{+}(x)\right\}\right) = \\ &= \left\{\widehat{\psi}_{\mathbf{h}^*}^{-}(x) \,\middle|\, x \in \mathsf{N}_{\mathbf{h}^*}\right\} \cup \left\{\widehat{\psi}_{\mathbf{h}^*}^{+}(x) \,\middle|\, x \in \mathsf{N}_{\mathbf{h}^*}\right\}.\end{aligned} \tag{54}$$

Taking into account the fact that the set $\mathcal{T}$ is infinite and satisfying the condition, $\mathbf{card}(\mathcal{T}) > \mathbf{card}(\mathfrak{Bs}(\mathcal{M}))$, we deduce:

$$\begin{aligned}&\mathbf{card}\left(\left\{\widehat{\psi}_{\mathbf{h}^*}^{-}(x) \,\middle|\, x \in \mathsf{N}_{\mathbf{h}^*}\right\}\right) \leq \mathbf{card}(\mathsf{N}_{\mathbf{h}^*}) \leq \mathbf{card}(\mathfrak{Bs}(\mathcal{M})) < \mathbf{card}(\mathcal{T});\\ &\mathbf{card}\left(\left\{\widehat{\psi}_{\mathbf{h}^*}^{+}(x) \,\middle|\, x \in \mathsf{N}_{\mathbf{h}^*}\right\}\right) < \mathbf{card}(\mathcal{T}).\end{aligned}$$

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Thence, using the equality (54), we conclude:

$$\mathbf{card}\left(\mathbf{T}_{\mathbf{h}^*}\right) < \mathbf{card}\left(\mathcal{T}\right).$$

So, since the set $\mathcal{T}$ is infinite ($\mathbf{card}\left(\mathcal{T}\right) \geq \aleph_0$), there exist the elements $t_0, t_0' \in \mathcal{T}$ such, that $t_0, t_0' \notin \mathbf{T}_{\mathbf{h}^*}$ and $t_0 \neq t_0'$. According to condition of the lemma as well as additional assumption "Assumption *", the oriented set $\mathcal{M}$ is strictly evolutionary bounded and quasi-chain. And, since the chronologization $\mathbf{h}^*$ belongs to the chronologization set $\mathscr{H}$, then all conditions of Lemma 2 are fulfilled. Consequently, according to this lemma, there exists the partial 2-chronologization $\mathbf{h}_1^* = \left(\mathbb{T}_{\mathbf{h}_1^*}, \psi_{\mathbf{h}_1^*}\right) = \left(\left(\mathbf{T}_{\mathbf{h}_1^*}, \leq_{\mathbf{h}_1^*}\right), \psi_{\mathbf{h}_1^*}\right)$ of the oriented set $\mathcal{M}$ relatively the subset $\mathsf{N} \cup \{x_0\}$ such, that $\mathbb{T}_{\mathbf{h}^*} \sqsubseteq \mathbb{T}_{\mathbf{h}_1^*}$, $\mathbf{T}_{\mathbf{h}_1^*} = \mathbf{T}_{\mathbf{h}^*} \cup \{t_0, t_0'\}$ and $\forall t \in \mathbf{T}_{\mathbf{h}^*} \left(\psi_{\mathbf{h}_1^*}(t) = \psi_{\mathbf{h}^*}(t)\right)$. Then from the definition of the set $\mathscr{H}$ we conclude that $\mathbf{h}_1^* \in \mathscr{H}$, as well as the definition ($\mathcal{H}o$) of the ordering $\leq^{\mathscr{H}}$ results that $\mathbf{h}^* \leq^{\mathscr{H}} \mathbf{h}_1^*$. Further from the condition $\mathbf{T}_{\mathbf{h}_1^*} = \mathbf{T}_{\mathbf{h}^*} \cup \{t_0, t_0'\}$ it follows that $\mathbf{h}^* \neq \mathbf{h}_1^*$. Therefore, we deliver $\mathbf{h}^* <^{\mathscr{H}} \mathbf{h}_1^*$. Thus the assumption about the existence of element $x_0 \in \mathfrak{Bs}(\mathcal{M})$ satisfying $x_0 \notin \mathsf{N}_{\mathbf{h}^*}$ leads to the existence of chronologization $\mathbf{h}_1^* \in \mathscr{H}$ such, that $\mathbf{h}^* <^{\mathscr{H}} \mathbf{h}_1^*$, which contradicts to the fact that the chronologization $\mathbf{h}^* \in \mathscr{H}$ maximal element of the ordered set $\left(\mathscr{H}, \leq^{\mathscr{H}}\right)$. That is why, the assumption, made above is false, ie $\mathsf{N}_{\mathbf{h}^*} = \mathfrak{Bs}(\mathcal{M})$. Thence we conclude that the time $\psi_{\mathbf{h}^*}$ is an one-point, and 2-repeating on the oriented set $\mathcal{M}$, that is the oriented set $\mathcal{M}$ can be chronologized by means of some 2-repeating and one-point time.

Thus, under additional assumption "Assumption *" the lemma had been proven.

**II.** Now we consider any quasi-chain oriented set $\mathcal{M}$, which which is not necessarily strictly evolutionarily bounded (ie the the conditions of Assumption * may be not satisfied). Let, $x_*$ and $y_*$ be arbitrary elements (ie mathematical objects) such, that $x_*, y_* \notin \mathfrak{Bs}(\mathcal{M})$ and $x_* \neq y_*$. We construct the oriented set $\mathcal{M}_*$ by the following way.

We denote:

$$\mathfrak{Bs}\left(\mathcal{M}_*\right) := \mathfrak{Bs}(\mathcal{M}) \cup \{x_*, y_*\}$$

and for $x, y \in \mathfrak{Bs}\left(\mathcal{M}_*\right)$ we write $y \underset{\mathcal{M}_*}{\leftarrow} x$ if and only if at least one of the following conditions is fulfilled:

1. $x, y \in \mathfrak{Bs}(\mathcal{M})$ and $y \underset{\mathcal{M}}{\leftarrow} x$;

2. $x = x_*$;

3. $y = y_*$.

From the above conditions $1, 2, 3$ it follows that for each $\widetilde{x} \in \mathfrak{Bs}\left(\mathcal{M}_*\right)$ such, that $\widetilde{x} \neq x_*$ it is performed the condition $\widetilde{x} \overset{+}{\underset{\mathcal{M}_*}{\leftarrow}} x_*$, as well for each $\widetilde{y} \in \mathfrak{Bs}\left(\mathcal{M}_*\right)$ such, that $\widetilde{y} \neq y_*$ it is performed the condition $y_* \overset{+}{\underset{\mathcal{M}_*}{\leftarrow}} \widetilde{y}$. So, by Definition 8, we have:

$$\min{}^*\left(\mathcal{M}_*\right) = x_*, \qquad \max{}^*\left(\mathcal{M}_*\right) = y_*. \tag{55}$$

Therefore, according to Definition 9, the oriented set $\mathcal{M}_*$ is strictly evolutionary bounded. We are going to prove that it is a quasi-chain.

Let, $x, y \in \mathfrak{Bs}\left(\mathcal{M}_*\right)$. Since the oriented set $\mathcal{M}$ is quasi-chain, then in the case, where $x, y \in \mathfrak{Bs}(\mathcal{M})$ from the condition 1 we conclude that at least one of the conditions $y \underset{\mathcal{M}_*}{\leftarrow} x$ or $x \underset{\mathcal{M}_*}{\leftarrow} y$ is valid. In the cases, where $x \in \{x_*, y_*\}$ or $y \in \{x_*, y_*\}$ due to the conditions $2, 3$ we deduce that $y \underset{\mathcal{M}_*}{\leftarrow} x$ or $x \underset{\mathcal{M}_*}{\leftarrow} y$. Hence, condition (QL1) of Definition 5 is fulfilled for the oriented set $\mathcal{M}_*$.

Assume that $x_0, x_1, x_2, x_3 \in \mathfrak{Bs}\left(\mathcal{M}_*\right)$ and $x_3 \overset{+}{\underset{\mathcal{M}_*}{\leftarrow}} x_2 \underset{\mathcal{M}_*}{\leftarrow} x_1 \overset{+}{\underset{\mathcal{M}_*}{\leftarrow}} x_0$. Since the oriented set $\mathcal{M}$ is quasi-chain, then in the case, where $x_0, x_1, x_2, x_3 \in \mathfrak{Bs}(\mathcal{M})$, we have $x_3 \overset{+}{\underset{\mathcal{M}}{\leftarrow}} x_0$, and consequently, in accordance with the item 1, we get, $x_3 \overset{+}{\underset{\mathcal{M}_*}{\leftarrow}} x_0$. In the case $x_0 = x_*$ the equality $x_3 = x_*$ is impossible, because the correlation $x_3 = x_* \overset{+}{\underset{\mathcal{M}_*}{\leftarrow}} x_2$ is nonsensical according to the conditions $1, 2, 3$. Therefore we have, $x_3 \neq x_* = x_0$. Consequently, from the equalities (55) we conclude that $x_3 \overset{+}{\underset{\mathcal{M}_*}{\leftarrow}} x_* = x_0$. The cases $x_1 = x_*$, $x_3 = x_*$ are impossible since in these cases the correlations $x_* = x_1 \overset{+}{\underset{\mathcal{M}_*}{\leftarrow}} x_0$ or $x_* = x_3 \overset{+}{\underset{\mathcal{M}_*}{\leftarrow}} x_2$ are nonsensical in accordance with (55). Case $x_2 = x_*$ also is impossible, because in this case the conditions $1, 2, 3$ together with the correlation $x_2 \underset{\mathcal{M}_*}{\leftarrow} x_1$ (ie $x_* \underset{\mathcal{M}_*}{\leftarrow} x_1$) lead to the equality $x_1 = x_*$, which is impossible according to the case, explained before. In the case $x_3 = y_*$ the equality $x_0 = y_*$ is impossible, because the correlation $x_1 \overset{+}{\underset{\mathcal{M}_*}{\leftarrow}} x_0 = y_*$ is nonsensical according to the conditions $1, 2, 3$. Therefore we have got, $x_3 = y_* \neq x_0$. Consequently, from the equalities (55), we conclude that $x_3 = y_* \overset{+}{\underset{\mathcal{M}_*}{\leftarrow}} x_0$. The cases $x_2 = y_*$, $x_0 = y_*$ are impossible since in these cases the correlations $x_3 \overset{+}{\underset{\mathcal{M}_*}{\leftarrow}} x_2 = y_*$ or $x_1 \overset{+}{\underset{\mathcal{M}_*}{\leftarrow}} x_0 = y_*$ are nonsensical in accordance with (55). The case $x_1 = y_*$ also is impossible, because in this case the conditions $1, 2, 3$ together with the correlation $x_2 \underset{\mathcal{M}_*}{\leftarrow} x_1$ (ie $x_2 \underset{\mathcal{M}_*}{\leftarrow} y_*$) lead to the equality $x_2 = y_*$, which is impossible according to the case, explained before. Hence in all possible cases the condition $x_3 \overset{+}{\underset{\mathcal{M}_*}{\leftarrow}} x_2 \underset{\mathcal{M}_*}{\leftarrow} x_1 \overset{+}{\underset{\mathcal{M}_*}{\leftarrow}} x_0$ ensures the correlation $x_3 \overset{+}{\underset{\mathcal{M}_*}{\leftarrow}} x_0$. Consequently, the condition (QL2) of Definition 5 also is satisfied for the oriented set $\mathcal{M}_*$.

Thus, by Definition 5, the oriented set $\mathcal{M}_*$ is a quasi-chain, which was to be proven.

From the equalities (55) we see, that the quasi-chain oriented set $\mathcal{M}_*$ satisfies conditions of Assumption *. Hence, in accordance with result, proven in the item I, the oriented set $\mathcal{M}_*$ can be one-point chronologized by means of 2-repeating one-point time, that is there exist a linearly ordered set $\mathbb{T}_* = (\mathbf{T}_*, \leq)$ and an 2-repeating one-point time $\psi_* : \mathbf{T}_* \to 2^{\mathfrak{Bs}(\mathcal{M}_*)}$. Since the time $\psi_*$ is one-point, then for an arbitrary $t \in \mathbf{T}_*$ the element $x_t \in \mathfrak{Bs}\,(\mathcal{M}_*)$ exists such, that $\psi_*(t) = \{x_t\}$. Moreover, using definition of time (see Definition 3), we deliver:

$$\left\{x_t \middle| \, t \in \mathbf{T}_*\right\} = \bigcup_{t \in \mathbf{T}_*} \{x_t\} = \bigcup_{t \in \mathbf{T}_*} \psi_*(t) = \mathfrak{Bs}\,(\mathcal{M}_*)\,. \tag{56}$$

Denote:

$$\mathbf{T} := \left\{t \in \mathbf{T}_* \middle| \, x_t \in \mathfrak{Bs}(\mathcal{M})\right\}.$$

Since $\mathfrak{Bs}(\mathcal{M}) \neq \emptyset$ and $\mathfrak{Bs}(\mathcal{M}) \subseteq \mathfrak{Bs}\,(\mathcal{M}_*)$ then the equality (56) involves that $\mathbf{T} \neq \emptyset$. Denote:

$$\psi(t) := \psi_*(t) = \{x_t\} \qquad (t \in \mathbf{T})\,.$$

It is not hard to verify that the map $\psi : \mathbf{T} \to 2^{\mathfrak{Bs}(\mathcal{M})}$ is an 2-repeating one-point time on the oriented set $\mathcal{M}$. □

Now Theorem 3 follows from Lemma 1 and Lemma 3.

## 8 On images of linearly ordered sets

In this short section we deduce one interesting corollary from Theorem 3 in the theory of ordered sets. Namely it will be obtained the description of all oriented sets, which can be represented as images of linearly ordered sets. First of all we formulate the definition of image of linearly ordered set.

Let $\mathcal{M}$ be an oriented set and $\mathbf{U} : \mathfrak{Bs}(\mathcal{M}) \to \mathcal{X}$ be a mapping from $\mathfrak{Bs}(\mathcal{M})$ to $\mathcal{X}$. Then we can introduce the binary relation $\leftarrow_{(1)}$ on the set $M_1 = \mathbf{U}\,[\mathfrak{Bs}(\mathcal{M})] = \mathfrak{R}\,(\mathbf{U})$ by the following rule:

- ► For $\widetilde{x}, \widetilde{y} \in M_1$ we note $\widetilde{y} \leftarrow_{(1)} \widetilde{x}$ if and only if there exist $x, y \in \mathfrak{Bs}(\mathcal{M})$ such, that $\widetilde{x} = \mathbf{U}(x)$, $\widetilde{y} = \mathbf{U}(y)$ and $y \leftarrow x$.

It is not difficult to verify that the ordered pair $\mathcal{M}_1 = \left(M_1, \leftarrow_{(1)}\right)$ is an oriented set, moreover $\mathfrak{Bs}(\mathcal{M}_1) = M_1$ and $\underset{\mathcal{M}_1}{\leftarrow} = \leftarrow_{(1)}$.

**Definition 11.** *An oriented set $\mathcal{M}_1$ is referred to as **image** of the oriented set $\mathcal{M}$ under the mapping $\mathbf{U} : \mathfrak{Bs}(\mathcal{M}) \to \mathcal{X}$ if and only if:*

1. $\mathfrak{Bs}(\mathcal{M}_1) = \mathbf{U}[\mathfrak{Bs}(\mathcal{M})] = \mathfrak{R}(\mathbf{U})$.
2. *For $\widetilde{x}, \widetilde{y} \in \mathfrak{Bs}(\mathcal{M}_1)$ the correlation $\widetilde{y} \underset{\mathcal{M}_1}{\leftarrow} \widetilde{x}$ holds if and only if there exist $x, y \in \mathfrak{Bs}(\mathcal{M})$ such, that $\widetilde{x} = \mathbf{U}(x)$, $\widetilde{y} = \mathbf{U}(y)$ and $y \underset{\mathcal{M}}{\leftarrow} x$.*

It is apparently that for each mapping $\mathbf{U} : \mathfrak{Bs}(\mathcal{M}) \to \mathcal{X}$ there exists an unique image under the mapping $\mathbf{U}$. We will use the notation $\mathbf{U}[[\mathcal{M}]]$ for the image of the oriented set $\mathcal{M}$ under the mapping $\mathbf{U} : \mathfrak{Bs}(\mathcal{M}) \to \mathcal{X}$.

It is evidently that every linearly ordered set $\mathbb{T} = (\mathbf{T}, \leq)$ is an oriented set with:

$$\mathfrak{Bs}(\mathbb{T}) = \mathbf{T}, \qquad \underset{\mathbb{T}}{\leftarrow} = \leq .$$

Therefore, it is meaningful to consider the image of the linearly ordered set $\mathbb{T} = (\mathbf{T}, \leq)$ under some mapping of kind $\mathbf{U} : \mathbf{T} \to \mathcal{X}$. And the image of the linearly ordered set $\mathbb{T}$ is the oriented set $\mathbf{U}[[\mathbb{T}]]$. That is why the following problem naturally arises:

**Problem 2.** *Can an arbitrary oriented set be represented as the image $\mathbf{U}[[\mathbb{T}]]$ of some linearly ordered set $\mathbb{T}$? If it can not, describe all oriented sets that can be represented as an image of some linearly ordered set.*

The key for solution of Problem 2 gives the following Assertion.

**Assertion 10.** *An oriented set $\mathcal{M}$ can be represented as image of some linearly ordered set if and only if $\mathcal{M}$ can be one-point chronologized.*

*Proof.* Indeed, suppose that the ordered set $\mathcal{M}$ can be represented in the form $\mathcal{M} = \mathbf{U}[[\mathbb{T}]]$, where $\mathbb{T} = (\mathbf{T}, \leq)$ is a linearly ordered set. So, $\mathbf{U}$ is the mapping of kind $\mathbf{U} : \mathbf{T} \to \mathfrak{Bs}(\mathcal{M})$ with $\mathfrak{R}(\mathbf{U}) = \mathfrak{Bs}(\mathcal{M})$. Here we denote by $\geq$ the binary relation, inverse to $\leq$ (ie for $x, y \in \mathbf{T}$ the condition $y \geq x$ holds if and only if $x \leq y$). According to Duality Principle (see [14, page 14]), the ordered pair

$$\mathbb{T}_{\geq} = (\mathbf{T}, \geq) \tag{57}$$

is the linearly ordered set as well. It is not difficult to verify that the mapping:

$$\mathbf{T} \ni t \mapsto \psi(t) = \{\mathbf{U}(t)\} \subseteq \mathfrak{Bs}(\mathcal{M})$$

is an one-point time on $\mathcal{M}$ (relatively the linearly ordered set $\mathbb{T}_{\geq}$). Conversely, let $\mathbb{T} = (\mathbf{T}, \leq)$ be a linearly ordered set and $\psi : \mathbf{T} \to 2^{\mathfrak{Bs}(\mathcal{M})}$ be one-point time on the oriented set $\mathcal{M}$. Then, by Definition 4, for every time point $t \in \mathbf{T}$ the element $\boldsymbol{x}_{(t)} \in \mathfrak{Bs}(\mathcal{M})$ exists such, that $\psi(t) = \left\{\boldsymbol{x}_{(t)}\right\}$. Consider the mapping:

$$\mathbf{T} \ni t \mapsto \mathbf{U}(t) = \boldsymbol{x}_{(t)} \in \mathfrak{Bs}(\mathcal{M}).$$

It is easy to verify that for this mapping $\mathbf{U}$ it is performed the equality $\mathcal{M} = \mathbf{U}[[\mathbb{T}_{\geq}]]$, where the linearly ordered set $\mathbb{T}_{\geq}$ is determined by the formula (57). □

Assertion 10 together with Theorem 3 stipulate the following corollary.

**Corollary 1.** *An oriented set $\mathcal{M}$ can be represented as image of some linearly ordered set if and only if it is a quasi-chain.*

Note that Theorem 3 was announced in [15].

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