Article

A Mathematical Theory of Knowledge for Intelligent Agents

Zie Eya Ekolle[1*], Ryuji Kohno[1] and Hideki Ochiai[1]

[1] Department of Electrical and Computer Engineering, Yokohama National University, Yokohama, Japan.

[*] Corresponding author: zie-ekolle-cj@ynu.jp

Abstract: Knowledge is a property that measures the degree of awareness of an agent about a target in an environment. The goal in conventional intelligent and cognitive agent development is to build agents that can be trained to gain knowledge about a target. The definition and operations of this knowledge associated to the agent is not clear, whereas these are required for developing a reliable, scaleable and flexible a gent. In this paper, we take into account such requirements, and present a concise theoretical framework for the design of cognitive and rational intelligent agents, their properties and operations on targets. We present many illustrative examples and an experiment to show how the incorporation of different cognitive properties to an agent enables the agent to act rationally with improve generalization performance in an interactive environment.

Keywords: intelligent agents; cognitive agents; rational agents; epistemology; machine learning; information theory; information geometry; relativity theory; value systems; valuation modeling

1. Introduction

Intelligent agent design has been around for many years, with a major goal to build machines that can act as we (humans) think [1]. The purposes of such a goal are diverse, giving rise to many varieties of agent design methods and techniques. One important aspect in all agent design is the acquisition of a value about a target and the optimization of this value. Many of such values are defined in the literature [2],[3],[4] but are not exhaustive enough to capture what we (human) think. This also makes such values less useful in the design of a reliable, scaleable and flexible agent needed in today's fast growing intelligent agent industry.

To address this issue, we present a value quantification scheme for intelligent agent and use it to quantify knowledge, which we consider as a *justified true belief* [5],[6],[7] of an agent about a target. We start by introducing the cognitive nature of an agent, and define different cognitive properties required by an intelligent agent. We classify these cognitive properties into; *intelligence*, *action*, and *cognitive value*.

Many types of intelligence, actions, and cognitive values can be defined for an agent, but base on the title of this article, we focus more on the cognitive value property, specifically the knowledge property of an agent. More so, since the notion of knowledge is literally confused with that of *intelligence*, *belief*, and *information*, we mathematically distinguish it from and relate it to them as literally discussed in [8],[9],[10],[11]. Apart from this, other cognitive values such as *understanding*, *trust*, and *wisdom* are mathematically defined, distinguished and related to the knowledge value.

Logically, the knowledge quantity in most literature is considered as information [11],[12]. However, the definition of information itself is diverse [13],[14],[15][16]. If we focus on Shannon's definition of information [13], where information is considered as a type of uncertainty and defined using probabilistic logic, then such type of information theoretic knowledge is invariant to the environment in which it is generated, because it does not consider the environment influence on the events and observer during cognition.

Consider Shannon's information content and the expected information content (also called entropy):

Information content: $I = \log(\frac{1}{P(x)})$,

Entropy: $H = \sum_{i=1}^{n} P(x_i)\log(\frac{1}{P(x_i)})$,
where x is a single state event, x_i is a state in a multi-states event, and $P(x)$ is probability of a single state event.

In his logical definition of information [13], Shannon considers the information content as a value generated by an event which causes a *surprisal* to an observer of the event. The degree of this surprise is considered to increase when the event rarely occurs and vice versa. This implies that the initial state of the observer is not considered, because, if the observer already has prior knowledge about the event, no matter how long it takes for the event to occur, the observer may likely not be surprised about it. It is therefore clear that such definition is tied more to the event than to the observer.

In relation to this, Shannon's entropy, which is the expected information content, is actually a type of uncertainty measure of an event and not of the observer of the event. Together with its different variants such as, joint entropy, mutual information, cross entropy, Kullback–Leibler (KL) divergence, etc., they are all used to measure different quantities of an event rather than an observer of the event. More so, there is no link between this information and the epistemological definition of knowledge [5],[17]. In addition to this, the environment in which the event occurs or that in which the observer exists, is not considered in any entropy value generation.

Generally, the environment plays an important role in the justification of values such as belief, information, etc., and this type of justification is highly used in the epistemological definitions of knowledge [18],[19] in any rational agent.

Imagine a scenario where there are two human agents, Ekane and Aki, living in different environments defined by different cultural, religious, political, geographical, and social features. Assume Ekane's environment is defined culturally by language A, geographically by a hilly topography, religiously by monotheism, politically by democracy, and socially in support for free speech. On the other hand, Aki's environment is defined culturally by language B, geographically by a coastal topography, religiously by polytheism, politically by autocracy, and socially against free speech.

If both Ekane and Aki have high knowledge, understanding, and trust (consciously or unconsciously) in their respective environments, then any event that occurs in their environments will be perceived and interpreted based on how their environments define the event. Consider an event such as "occurrence of peace", how both agents perceive and interpret this event may be completely different, and this is simply based on the influence of their environments on their cognition and rationality about the event.

Ekane, living and trusting in a democratic system will likely have the belief that peace can only occur via democracy, while Aki who lives and trusts in an autocratic system will likely belief that peace can only occur via autocracy. Clearly, placing both agents together to work for peace without any consideration of their respective environments will instead lead to disagreements and conflicts, unless their belief systems and/or environments are revised to align with the required definition (environment) of peace.

Same phenomenon will arise on different events such as "occurrence of success on a project", "occurrence of a cyber attack in a network", etc., because all these events and how the agent perceives and interprets them are influenced by the environment of the event and agent. So, the environment forms an *invisible barrier* to knowledge generation and harmonization among agents, and taking into account its influence on the event and agent is important in justifying the information about the event and belief of an agent about the event in any rational process. Many researches in cognition such as the work of Lewin [20],[21], Audi [22],[23], and many others, consider the environment as a major factor in value generation of a cognitive and rational agent.

In this paper, we focus on the influence of the environment on an agent belief system and how such influence is used to generate knowledge in the agent. We derive the logical definition of knowledge from an epistemological viewpoint. This enables a concise and precise view about knowledge, its operations and properties. Furthermore, we relate this

 doi:10.20944/preprints202211.0450.v1

concept of knowledge to other cognitive values of an agent such as *understanding*, *trust*, and *wisdom*.

1.1. Related Work

In reality, a study of knowledge in relation to intelligent agent is challenging and exhaustive. This is due to the large amount of diversity, misunderstanding, and misconception related to the topic. To some, knowledge is a type of belief, to others, it is a type of information, and to some extent, it is considered to be both belief and information. In most cases, the entity which possesses knowledge is not clearly defined and how the *knowledge ecosystem* operates in an environment of entities is not mentioned. What is presented in most literature is a definition of knowledge using either logic without semantics or semantics without logic.

Considering the definition of knowledge in [5], it is clearly a literal description of knowledge, rooted from philosophical literature and with main focus on the meaning (semantics) of knowledge. More so, the article consider knowledge as a type of belief and much about its operation is not mentioned. A similar concept about knowledge is given in [7], where knowledge is considered to consist of *true belief* and can only exist if three conditions are satisfied; truth condition, belief condition, and evidence condition. The evidence condition is considered as a justification of the true belief. In many other theories [6] related to the semantic definition of knowledge, such as the ontological description of knowledge [24],[25], the operations on knowledge and the entity that generates it are not defined.

Related to the logical definition of knowledge, in [11], knowledge is defined using a probabilistic logic and it is considered at some point to be both belief and information. The article also focuses on defining the semantic content of information as the only property that can cause Shannon information [13] to yield knowledge. There have been many critics to this theory, e.g., in [26], due to its lack of coherence with the same information it tries to redefine. In [10], the logical definition of knowledge is not different to the definition of belief in Section 3.1 of this paper. In reality, such indifference between knowledge and belief is epistemologically misleading.

In [12], knowledge is defined as a mathematical function based on probability logic and information theory, but with no clear semantics and relation to epistemology.

In this research, we take into consideration both the semantic and logical definitions of knowledge, to enable a concise understanding of the subject matter.

1.2. Contributions

The contributions of this work are summarized below.

1. A detailed abstraction of cognitive properties and their interrelationships in a cognitive intelligent agent.
2. A concise mathematical definition of *belief, knowledge, ignorance, understanding, trust, wisdom*, and *attention* in relation to cognition and epistemology.
3. An operational mathematical logic for knowledge *acquisition, optimization, and transfer* that can be used in the design and development of any intelligent, cognitive and rational agent.
4. A mathematical quantification and description of the *semantic* and *mutual* difference between agents, and algorithms that can be used to reduce such differences.
5. A mathematical description of a *knowledge transformation factor* between agents and its application to solve the *triangle inequality* problem of *I-projection*.
6. A technique which we refer to as *cognimatics*, to analyse the dynamics of a cognitive process, and a knowledge graph to use as a tool in this analysis.
7. An approach to analyse and solve *multi-target* (or multi-domain) problems in cognition, including *multi-action* (or multi-service) problems.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 24 November 2022 doi:10.20944/preprints202211.0450.v1

1.3. Organization

This article is written simply, not actually to prove our understanding of the subject, but to ease its understanding and application by a large audience from different disciplines.

The rest of the article is organized as follows: Section 2 focuses on the mathematical description of entities and the properties that can be associated to them. Section 3 concerns the quantification of cognitive properties. Sections 4-6 focus on three main operations of an agent, in which, Section 4 is on the value acquisition process of an agent, Section 5 is on the cognitive value optimization process of an agent, and Section 6 is on the value transfer process between entities. In Section 7, we perform an experiment to prove the importance of value transfer, and conclude on the research in Section 8.

For easy comprehension of this paper, we recommend the readers to have basic understanding on epistemology, logic, machine learning, information theory, relativity, kinematics, physical energy, probability, and vector spaces.

2. Definition of Entities and Properties

2.1. Cognitive Property

For an agent to gain awareness about targets in its environment, the agent needs to have some properties that will define its abilities, such as: what type of value it can gain, what actions it take to acquire such value, and what strategy does it need to support such action. We call these sets of proprieties, the cognitive properties of an agent.

In accordance with the definitions in [27],[28],[29], we define cognitive property as:

Definition 1. Cognitive property *is a set of properties that define the values, actions, and intelligence of an agent.*

In this research, we group cognitive properties of an agent into three types: intelligence properties, action properties, and cognitive value properties, as shown in Figure 1.

Figure 1 shows a proposed cognitive property model for this research. Unlike other models [8],[9],[10] that depict the relationships between cognitive properties as a sequential chain of properties, this model describes cognitive properties as a parallel hierarchical chain of properties and processes. Starting with the intelligence, action, and cognitive value properties, each property has other attributes such as inverse, stability, and exactness. We shall provide mathematical descriptions of these attributes for the knowledge property. This model gives a detailed abstraction of cognitive properties in any cognitive intelligent agent such as a human agent.

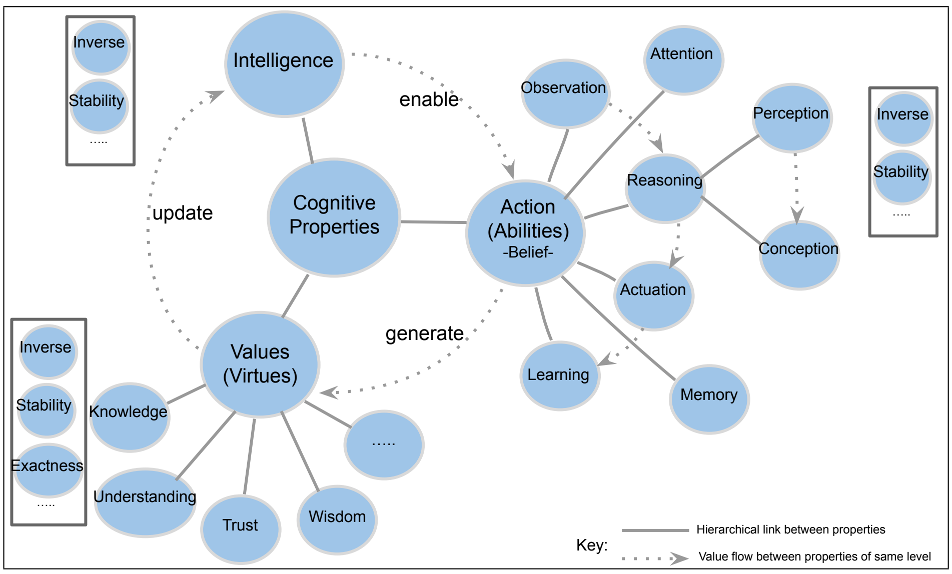

Figure 1. Cognitive property organization in a cognitive intelligent agent

 doi:10.20944/preprints202211.0450.v1

We focus on the value properties [30],[31],[32], specifically the knowledge value, and introduce its related properties and operations. With respect to this research, we define a cognitive value [32],[33],[31] as follows;

Definition 2. Cognitive value *is any property of an agent that depends on the action of the agent on a target.*

So the *action value*, also called *belief* in Section 3.1, is not a cognitive value but a quantity on which cognitive values are defined as shown in Figure 1 and discussed in Section 3.1.

Lastly, related to cognition, this paper is focus on conscious cognition [34] based on beliefs, e.g., knowledge. Other conscious cognition such as those based on emotion, e.g., happiness, are reserved for future publications. We consider the former as *hard consciousness* and the latter as *soft consciousness*. Unconscious cognition such as intuition [35] for cognitive agents will also be discussed in future publications.

2.1.1. Knowledge, Action, and Intelligence

Knowledge has been defined and studied extensively in the literature in relation to cognitive and intelligent agent design [8],[9],[10],[11]. Together with [36],[37],[38], we adopt a literal definition of knowledge link to cognition as follows:

Definition 3. Knowledge, K, *represents the level of comprehension an agent has or conclusion an agent can make about a target, with respect to the environment of the target.*

An agent may seek knowledge about a target for various reasons, but the acquisition of a complete knowledge about a target is a progressive process. The absence of knowledge in such case is also important. This has been less studied in agent design, but literally exists in [39] and is given the name ignorance [40], which we adopt as follows:

Definition 4. Ignorance, I, *is a property that represents the level of knowledge absence of an agent about a target.*

An agent may seek to reduce ignorance about a target for various reasons, but the reduction of the complete ignorance about a target is a progressive process.

Axiom 2.1. *The increase in knowledge of an agent about a target leads to a decrease in ignorance of the agent about same or related target, and vice versa.*

Hence, seeking knowledge implies reducing ignorance, and vice versa. The target in this case can be a task, a project, a discussion topic, an event, an agent, a property, etc.

Knowledge and ignorance are cognitive value properties whose operations are based on the action property.

Definition 5. Action, A, *is a property with the ability to receive, select, control, generate, store, optimize, and transfer cognitive values about a target.*

Much about actions exist in literature [41],[42],[43],[44]. We focus on knowledge based actions, which are actions carried on the knowledge value of an agent. We categorize actions into: receive (observation), select (attention), control (actuate), generate (reasoning), store (memorization), optimize (learning), and transfer (interoperability) operations.

These cognitive actions are supported by or based on the intelligence [45] of the agent, which we define as follows:

Definition 6. Intelligence, ϕ**,** *is a property with the ability to enable an action on a target.*

An intelligence can be; a strategy, logic, algorithm, etc., of the action. We will also focus more on intelligence that enables knowledge based actions.

Much literature exists about intelligence [46],[47],[48] and the different action operations: observation [49],[50],[51], attention [52],[53],[54],[55],[56], actuation [57], reasoning [58],[59],[60], memorization [61],[62],[63],[54],[56], learning [64],[65],[54],[56] and interoperability [66],[67],[68]. In this paper, we focus on the observation, reasoning, learning, and transfer action operations.

Axiom 2.2. *An agent is said to be intelligent about a target if it has the ability to take an action on the target, no matter how less likely the action may be.*

Axiom 2.3. *An agent is said to posses knowledge about a target if it has taken an action on the target, no matter how less likely the action may be.*

From the definitions above, we can logically deduce that,

Proposition 1. $((K \to A) \wedge (A \to \phi)) \vdash K \to \phi$

Proposition 2. *Given that* $(K \to A \to \phi)$,
If $\neg\phi \to \neg A \to \neg K \vdash \neg\phi \to (\neg A \wedge \neg K)$.
If $\neg K \to \neg A \to (\phi \vee \neg\phi) \vdash \neg K \to (\neg A \wedge (\phi \vee \neg\phi))$.,
where $\to$ *is logical dependency,* $y \to x$ *indicates that* y *has logical dependency on* x, $\vdash$ *is logical consequence,* $\wedge$ *indicate logical disjunction, and* $\neg$ *is logical negation.*

In Section 3, we shall transform these literal and logical constructs to mathematical definitions and relationships of knowledge, ignorance, action, and intelligence. These mathematical definitions and relationships of cognitive properties will be quantified, making it possible for various mathematical operations to be perform on them.

Apart from the generation of knowledge value, action and intelligence also generate other cognitive values such as *understanding*, *trust*, and *wisdom*, which we define below.

Definition 7. Understanding *represents the degree of convergence or divergence between the actions of agents about a target.*

Definition 8. Trust *represents the degree of mutual value between agents about a target.*

Definition 9. Wisdom *represents the complete value an agent can achieve about a target.*

The mathematical definition and features of these concepts are describe in Section 5.

2.1.2. Observation, Reasoning, and Actuation

An intelligent agent with a perceptual and conceptual abilities observes an environment and collects information about a target to perceive and conceive the target. Many approaches have been used to quantify information [13],[14],[69],[15],[16], but not entirely with respect to an agent. We adopt a definition that is based on the cognitive property of an agent, specifically the value property.

Definition 10. Information, X, *is any input value to an agent based on which the agent takes action about a target.*

As we shall discuss in Section 4, information can take any form, such as, beliefs, knowledge, ignorance, stability, etc., use as input during reasoning. So, all cognitive property values which are used as a source value to achieve a other values about a target are considered here as information.

To achieve a target, the agent collects information from its environment about the target using its senses [49],[51]. Through a reasoning process, it takes action on the target based on the collected information [70],[57]. During this process, the values of the cognitive properties may change to reflect the agent's current cognitive state about the target.

For an agent with an added ability to learn a value, these cognitive values can be optimized directly through a forward reasoning, or indirectly through a backward reasoning where the value is reinserted back to the intelligence that supports its generation [64],[65]. In this article, the word optimization will be used interchangeably with the word learning because we consider learning as a value optimization process.

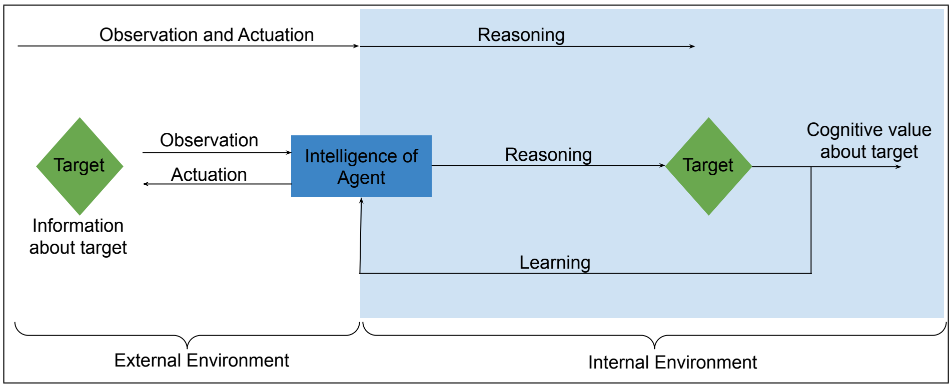

Figure 2. Cognitive action processes of an agent

For clarity, we present in Figure 2 an agent with four cognitive action operations: observation, reasoning, learning, and actuation.

In this research, we consider that during these operations, both the agent and target exist in an environment. The input and output cognitive values of an agent define its environment with respect to the target. Also, the environment can be external (real) and internal (abstract) environments with respect to the agent as shown in Figure 2. Examples of real environments include room space, city, etc., and abstract environment include perspectives, culture, context, consciousness, etc. Same description holds for a cyber-physical system. We assume that the internal environment is fully observable to the agent.

With respect to the agent and its environments, we define these four action operations.

Definition 11. Observation, O, *is the process of information identification and collection from an environment.*

Based on the agent, we can distinguish two types of observation: internal (abstract) and external (real) observation. During internal observation of an agent, the internal information can come from the actions of the agent (local value acquisition) or that of a remote agent (remote value acquisition).

Definition 12. Reasoning, R, *is the process of cognitive value generation about a target.*

It thus generates a value for all the actions of an agent and can be seen as the central process for value generation in an agent. An agent will therefore have a reasoning process for observation, attention, memory, actuation, and learning to generate their respective cognitive values. The output of a reasoning action is quantified and stored as *belief*.

We classify the reasoning actions related to observation into perception [71],[72] and conception [73],[74], and it can be executed for different purpose on the target such as for description, diagnosis, prediction, prescription, and cognition of a target [75],[76],[77]. Reasoning actions for actuation generate cognitive values related to the influence on both agent and target environment [57], and those for learning generate values related to cognitive value optimization about a target [58],[65].

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 24 November 2022 doi:10.20944/preprints202211.0450.v1

Also, based on its execution logic, reasoning can be inductive, deductive, abductive, etc., [78], and based on its input-output relationship, it can be classified as deterministic, non deterministic, stochastic, etc.

Since the reasoning action forms a large part of agent design, we use the terms reasoning and action interchangeably in this article, unless stated otherwise.

Definition 13. **Actuation, ζ,** *is the process through which an agent influences its environments.*

The purpose of the influence may consist of controlling, changing, etc., the target or agent environments. Actuation can be external (real) or internal (abstract) to an agent.

Definition 14. **Learning, γ,** *is an optimization process that involves cognitive value increases through revision and update.*

Agents can choose to learn from their self acquired value, remotely acquired values or both. More on learning will be discussed in Section 5.

Moreover, the agent can do any of these operations on a target by itself (unsupervised), assisted by another agent (supervised), by both itself and another agent (semi-supervised), or by another agent (remote). Most of these relationships with its target environment are well studied in the literature [65],[64] apart from the remote situation which we shall discuss in Section 6.

Also, related to the operations, we can classify agents based on the existence and non existence of any of the four operations, with reasoning being the basic operation for all intelligent agents. Any agent without reason is considered in this research as a non intelligent agent. Intelligent agents can thus be classify into different types, such as: non-observable non-actuatable and non-learning intelligent agent, observable non-actuatable and non-learning intelligent agent, etc.

2.2. Definitions of Entities

One cannot define the quantification and operations on knowledge without defining the entities that will generate and use it. An *entity* is generally considered as anything that exists. The main philosophical question related to existence is, how do we know if something exists? In response to this, we think anything that exists must possess a *value*. So, with respect to this research, we define an entity as follows:

Definition 15. *An **entity, ρ,** is anything that exists and possesses value.*

In this section, we mathematically define different types of entities, their logical relationships, and operations.

2.2.1. Description and Properties

We distinguish and define three types of entities: agent, target, and environment, together with their properties.

Definition 16. *An **agent, g,** is an entity that possesses cognitive property values.*

The different cognitive properties an agent can posses are presented in Figure 1. An agent can also be considered as a *cognitive actor* in an environment.

Definition 17. *A **target, t,** is an entity that an agent seeks, and defined by a set of input and output property values, and the relationship value between the two properties.*

The properties of the target represent the domains of the environment in which the target is found. The output domains *define the existence* of the target and are considered as

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 24 November 2022 doi:10.20944/preprints202211.0450.v1

the *existential properties* of the target. The input domains *influence the existence* of the target and are considered as the *evidential properties* of the target.

The process where an agent uses input properties of a target to act on the output properties of the target is called *Perception* [71],[72], and it is the main focus in conventional agent design. Whereas, the process where the agent seeks to reproduce the output properties of a target is called *Conceptualization* [72],[73]. In most conventional agent design, the output properties are given to the agent (with or without labels) during learning and testing, and this does not imply conceptualization.

To fully act on a target, an agent must not only identify the input and output properties of the target but also the relationship between them. Thus, the main goal of an agent during learning is to recreate the relationship between the input and output properties of a target. This is summarized in the axiom below.

Axiom 2.4. *The value an agent attributes to the input-output relationship of a target is a property of the agent and defines the relationship between the agent and the target.*

We define an environment entity, which acts as a *container* of agents and targets.

Definition 18. *An* ***environment, e,*** *is an entity which contains agents and targets and defines their relationships.*

Environments may contain other environments which we group into: *community, world, universe, multiverse*, and *infiniteverse*. A community is an environment with agents and targets, a world is a collection of communities, a universe is a collection of world, a multiverse is a collection of universes, and infiniteverse is an infinite hierarchy of multiverse. We consider all environments as *containers* as illustrated in Figure 3.

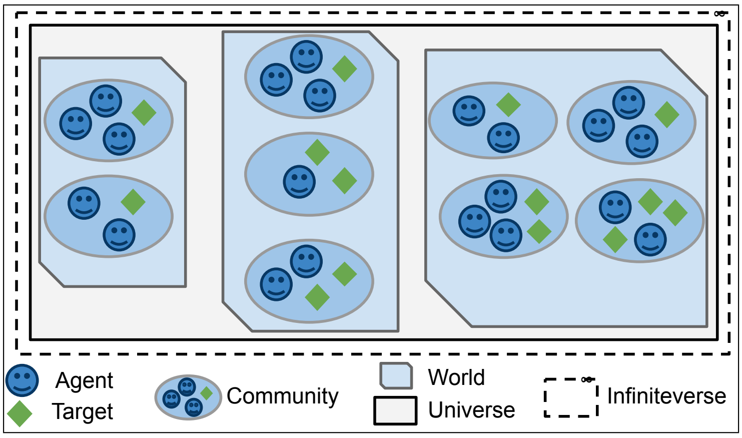

Figure 3. Entities and containers hierarchical relationship.

We logically define these structural properties as follows:

$e = \{g, t\}$, $Q = \{e, g, t\}$ and $\forall \rho \in Q, q = \{\rho_1, \rho_2, ...\rho_n\}$,

where e is an environment entity, g is an agent entity, t is a target entity, Q is the set of entity types, ρ is an entity, and q is a container or set of entities.

To support our operations on agent design, we define the following functional properties of the different entities.

Definition 19. **(Agent)** $\forall g \in e, g = (\Phi, A, C, f_g, \mathbb{P}_e)$,
where Φ *is intelligence property, A is action property, C is cognitive value property,* f_g *is relationship between properties, and* $\mathbb{P}_e$ *is inherited property from the environment.*

Generally, using the knowledge property, we can simply express the agent as follows:

$$G = \{g : \ g \text{ is an agent}\}, g = (\phi, A, K), \tag{2.1}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 24 November 2022 doi:10.20944/preprints202211.0450.v1

$$A(g)_t = f(y, X, \phi), \tag{2.2}$$
$$K(g)_t = f(A(g)_t, A(e)_t), \tag{2.3}$$

where g is an agent, t is target of the agent, y is a target output information, X is the set of target input information, $A(g)_t$ is action function (or belief) of the agent on the target, ϕ is the intelligence (or parameters) of the agent that enables the action, and $K(g)_t$ is knowledge generated by the agent based on its action and the environment action $A(e)_t$ on the target.

Definition 20. (Target) $\forall t \in e, t = (X, Y, f_t, \mathbb{P}_e)$,
where $f_t : X \Rightarrow Y$, *X is input property, Y is output property.*

Generally, we can simply express the target as an entity of the environment as follows:

$$T = (\mathcal{X}, \mathcal{Y}, f(\mathcal{X}, \mathcal{Y})),\ t = (X, Y, f(X, Y)), \tag{2.4}$$
$$X \in \mathcal{X}, Y \in \mathcal{Y}, \mathcal{X} \wedge \mathcal{Y} \in \mathcal{P},\ f(X, Y) \in \mathbb{R},\ t \in T$$

where t is a target in an environment, T is a set of targets, X is set of properties of the environment that act as input (or evidential) properties of the target, Y is set of properties of the environment that act as output (or existential) properties of the target, and $f(X, Y)$ is a property of the target that defines a logical relationship between X and Y.

Definition 21. (Environment) $\forall e \in q, e = (\mathbb{P}_t, \mathbb{P}_g, f_{tg}, \mathbb{P}_q)$,
where $\mathbb{P}_t$ *is property of target,* $\mathbb{P}_g$ *is property of agent,* f_{tg} *is relationship between properties, and* $\mathbb{P}_q$ *is inherited property from the environment container.*

Generally, we can simply express the environment as follows:

$$E = (\mathbb{P}, f(\mathbb{P})),\ e = (\mathcal{P}, f(\mathcal{P})), \tag{2.5}$$
$$\mathcal{P} : \mathbb{S} \rightarrow \mathbb{R}^n, f(\mathcal{P}) \in \mathbb{R}, e \in E, \mathcal{P} \in \mathbb{P}$$

where e is an environment, E is set of environments, $\mathcal{P}$ is a vector of all properties of an environment, $f(\mathcal{P})$ is a logical relationship define over all the properties of the environment, and $\mathbb{S}$ is the sample space of $\mathcal{P}$.

Figure 4 shows the interaction between agents and targets in different environments.

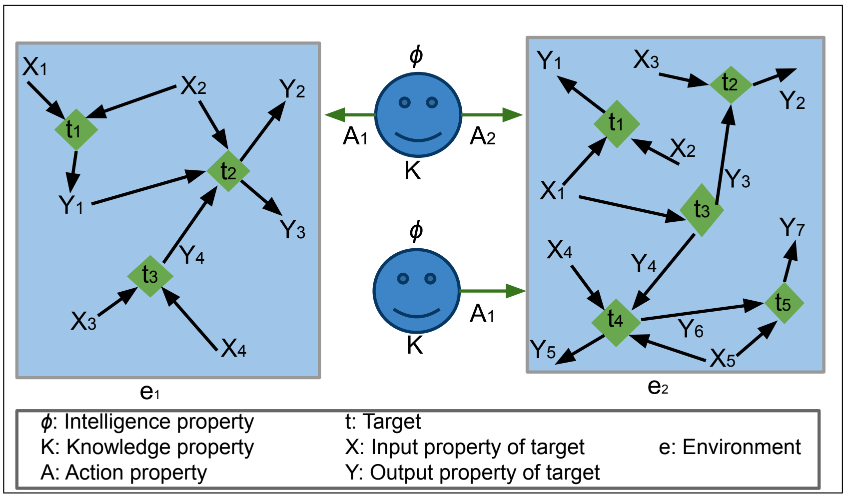

Figure 4. Two agents interacting with two environments. e_1 has one agent, e_2 has two agents.

After defining the entities and properties, we then define logical relations and operations on them. The next section answers questions about the equality, equivalence, superiority, inferiority, dependency, association, dissociation, and intersection of entities. For example, when are agents equal?

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 24 November 2022 doi:10.20944/preprints202211.0450.v1

2.2.2. Logical Relationships Between Entities

We define different logical relationships between entities, where ω is entity property, $(\rho_i * \rho_j)_\omega$ indicates the logical relationship $(*)$ between entities ρ_i and ρ_j over the property ω, $i \neq j$, and $i, j \in \mathbb{N}$. Also, iff means 'if and only if'.

i. Equivalence $(\equiv)$

Definition 22. *Two entities are equivalent over a set of properties if at least one of the properties (weak equivalence) or all (strong equivalence) have same value and structure.*

We distinguish two types of equivalence.
a) Agent equivalence: $\forall g_i, g_j \in q, (g_i \equiv g_j)_\omega$ iff $(\omega_i \equiv \omega_j) \wedge t_i \equiv t_j$.
b) Target equivalence: $\forall t_i, t_j \in q, (t_i \equiv t_j)_\omega$ iff $(X_i \equiv X_j \vee Y_i \equiv Y_j \vee f_i \equiv f_j)$.

Equivalent relationships must also be reflexive, symmetrical and transitive over the properties. Example of equivalent relationship between entities is an equality relationship.

ii. Equality (=)

Definition 23. *Two entities are equal over a set of properties if at least one of the properties (weak equality) or all (strong equality) have same value.*

We distinguish two types of equality.
a) Agent equality: $\forall g_i, g_j \in q, (g_i = g_j)_\omega$ iff $(\omega_{(1,i)} = \omega_{(1,j)} \vee \omega_{(2,i)} = \omega_{(2,j)} \vee \ldots \omega_{(n,i)} = \omega_{(n,j)}) \wedge t_i = t_j$.
b) Targets equality: $\forall t_i, t_j \in q, (t_i = t_j)_\omega$ iff $(X_{(1,i)} = X_{(1,j)} \vee X_{(2,i)} = X_{(2,j)} \ldots X_{(n,i)} = X_{(n,j)}) \vee (Y_{(1,i)} = Y_{(1,j)} \vee Y_{(2,i)} = Y_{(2,j)} \ldots Y_{(n,i)} = Y_{(n,j)} \vee (f_i = f_j)$.

iii. Superiority $(>)$

Definition 24. *One entity is superior over another entity on a set of properties if at least in one of the properties (weak equality) or all (strong equality), it has a greater value.*

We distinguish two types of superiority.
a) Agents superiority: $\forall g_i, g_j \in q, (g_i > g_j)_\omega$ iff $(\omega_{(1,i)} > \omega_{(1,j)} \vee \omega_{(2,i)} > \omega_{(2,j)} \vee \ldots \omega_{(n,i)} > \omega_{(n,j)}) \wedge t_i = t_j$.
b) Targets superiority: $\forall t_i, t_j \in q, (t_i > t_j)_\omega$ iff $(X_{(1,i)} > X_{(1,j)} \vee X_{(2,i)} > X_{(2,j)} \ldots X_{(n,i)} > X_{(n,j)}) \vee (Y_{(1,i)} > Y_{(1,j)} \vee Y_{(2,i)} > Y_{(2,j)} \ldots Y_{(n,i)} > Y_{(n,j)} \vee (f_i > f_j)$.

iv. Inferiority $(<)$

Definition 25. *One entity is inferior over another on a set of properties if at least in one the properties (weak equality) or all (strong equality), it has a lesser value.*

We distinguish two types of inferiority.
a) Agents inferiority: $\forall g_i, g_j \in q, (g_i < g_j)_\omega$ iff $(\omega_{(1,i)} < \omega_{(1,j)} \vee \omega_{(2,i)} < \omega_{(2,j)} \vee \ldots \omega_{(n,i)} < \omega_{(n,j)}) \wedge t_i = t_j$.
b) Targets inferiority: $\forall t_i, t_j \in (\rho, q), (t_i < t_j)_\omega$ iff $(X_{(1,i)} < X_{(1,j)} \vee X_{(2,i)} < X_{(2,j)} \ldots X_{(n,i)} < X_{(n,j)}) \vee (Y_{(1,i)} < Y_{(1,j)} \vee Y_{(2,i)} < Y_{(2,j)} \ldots Y_{(n,i)} < Y_{(n,j)} \vee (f_i < f_j)$.

v. Dependency $(\rightarrow)$

Definition 26. *An entity ρ_i depends on ρ_j over a set of properties if at least one (weak dependency) or all (strong dependency) properties of ρ_i are defined by those of ρ_j.*

We distinguish two types of dependency.
a) Agents dependency: $\forall g_i, g_j \in q, (g_i \rightarrow g_j)_\omega$ iff $(\omega_k)_i = f((\omega_1 \vee ...\omega_n)_j)$.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 24 November 2022 doi:10.20944/preprints202211.0450.v1

b) Target dependency: $\forall t_i, t_j \in q, (t_i \rightarrow t_j)_\omega$ iff $(X_k)_i = f((X_1 \vee \ldots X_n)_j) \vee (Y_k)_i = f((Y_1 \vee \ldots Y_n)_j) \vee (f_k)_i = f((f_1 \vee \ldots f_n)_j)$.

Proposition 3. *All entities depend on their properties.*

Proof: By definition, $\rho = f(\omega) \vdash \rho \rightarrow \omega$.
Dependency between entities can be bidirectional.

Definition 27. (Bidirectional dependency($\leftrightarrow$))
$\forall \rho_i, \rho_j \in q, (\rho_i \leftrightarrow \rho_j)_\omega$ iff $(\rho_i \rightarrow \rho_j)_\omega \wedge (\rho_j \rightarrow \rho_i)_\omega$.

vi. Conditional dependency($\rightarrow|$)

It is a type of dependency where the relationship between the properties of the two entities is conditioned on one of the entities or a third entity.

Definition 28. *An entity ρ_i is conditionally dependent on another entity ρ_j over a set of properties if (the occurrence of) at least one property (weak condition) or all properties (strong condition) of ρ_j are conditioned on those of ρ_i, or those of a third entity ρ_z.*

We distinguish two types of conditional dependency;
a) Agents conditional dependency: $\forall \rho_i, \rho_j \in q, (\rho_i \rightarrow| \rho_j)_\omega$ iff $(\omega_i)_i \rightarrow| (\omega_1 \vee ...\omega_n)_j$.
b) Targets conditional dependency: $\forall \rho_i, \rho_j \in q, (\rho_i \rightarrow| \rho_j)_\omega$ iff $\forall X, Y, f \in t, ((X_i)_i \rightarrow| (X_1 \vee \ldots X_n)_j) \vee ((Y_i)_i \rightarrow| (Y_1 \vee \ldots Y_n)_j) \vee (f_{i_i} \rightarrow| (f_1 \vee \ldots f_n)_j)$.
Conditional dependency can also be bidirectional

Definition 29. (Bidirectional CD($|\leftrightarrow|$))
$\forall \rho_i, \rho_j \in q, (\rho_i |\leftrightarrow| \rho_j)_\omega$ iff $(\rho_i \rightarrow| \rho_j)_\omega \wedge (\rho_j \rightarrow| \rho_i)_\omega$.

If $(\rho_i \rightarrow| \rho_j)_\omega = (\rho_j \rightarrow| \rho_i)_\omega$, we say the entities are *conditionally equivalent* over the property ω.

Proposition 4. $(\rho_i \rightarrow| \rho_j)_\omega = (\rho_j \rightarrow| \rho_i)_\omega \Rightarrow (\rho_i)_\omega = (\rho_j)_t$.

vii. Mutual dependency($\multimap$)

It is a type of dependency where the self relationships of the entities on their property are excluded from their conditional relationships.

Definition 30. *An entity ρ_i is mutually dependent on another entity ρ_j over a set of properties if (the occurrence of) at least one (weak condition) or all the self properties (strong condition) of ρ_i are excluded from its conditional relationship with those of ρ_j, or those of a third entity ρ_z.*
$(\rho_i \multimap \rho_j)_\omega = (\rho_i \rightarrow| \rho_j)_\omega \backslash (\rho_i)_\omega$.

Excluding their self-relationships imply that mutual dependency captures a bidirectional relationship between entities.

Proposition 5. *Two entities in a mutual dependency over a property have a birectional equality over the property.*
$(\rho_i \multimap \rho_j)_\omega = (\rho_j \multimap \rho_i)_\omega$ *(reciprocity).*

Similar to conditional dependency, mutual dependency can be established on the properties of targets and agents.

viii. Container ($\sqcup$, $\square$)

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 24 November 2022 doi:10.20944/preprints202211.0450.v1

Definition 31. *An entity q is a container to another entity ρ over a set of properties if for these properties, ρ is equal or inferior to q.* $(q \sqcup \rho)_\omega$ iff $\forall \omega, (\omega_\rho <= \omega_q)$.

Axiom 2.5. *Containers can be defined based on their entities and entities can be defined based on their containers.*

From Definition 31 and Axiom 2.5, we can deduce that,

Proposition 6. *All entities in a container depend on the container and all containers depend on their entities.*

Proof: $q \sqcup \rho \Rightarrow (\omega_\rho = f_\rho(\omega_q)) \wedge (\omega_q = f_q(\omega_\rho)) \vdash q \leftrightarrow \rho$.

We distinguish two categories of containers and entities dependency relation, that is, coupling and cohesion.

ix. Coupling ($\bowtie$)

It defines the dependency between containers.

Definition 32. *A container entity q_i is coupled with a container entity q_j* iff $(q_i \to q_j) \vee (q_i \leftarrow q_j) \vee (q_i \leftrightarrow q_j)$.

x. Cohesion ($\otimes$)

It defines the dependencies between entities in a container.

Definition 33. *An entity ρ_i is considered to be cohesive with an entity ρ_j,* iff $(\rho_i \to \rho_j) \vee (\rho_i \leftarrow \rho_j) \vee (\rho_i \leftrightarrow \rho_j)$.

Cohesive dependency can exist between entities in same container or different containers. We distinguish two types;
a) Local cohesion: $q \sqcup \rho_j, \rho_j$ and $(\rho_i \to \rho_j) \vee (\rho_i \leftarrow \rho_j) \vee (\rho_i \leftrightarrow \rho_j)$.
b) Remote cohesion: $(q_i \sqcup \rho_j) \wedge (q_j \sqcup \rho_j)$ and $(\rho_i \to \rho_j) \vee (\rho_i \leftarrow \rho_j) \vee (\rho_i \leftrightarrow \rho_j)$.

Also, based on the dependency and exchange of properties between containers and their host, we can distinguish three types of containers; open, close and isolated containers.

xi. Open container ($\sqcup$)

Definition 34. *A container q is open over a host h if values for some ω of q depend on h, i.e.,* $(q \to h)_\omega$*, and ρ or ω can be exchanged between q and h.*

xii. Close container ($\square$)

Definition 35. *A container q is closed over a host h if values for some ω of q depend on h, i.e.,* $(q \to h)_\omega$*, and ρ or ω cannot be exchanged between q and h.*

xiii. Isolated containers ($\square$)

Definition 36. *A container q is isolated over a host h if values for all ω of q is independent on h, i.e.,* $\neg(q \to h)_\omega$*, and ρ or ω cannot be exchanged between q and h.*

In addition, entities can be considered as a container because even if they may not contain other entities, they contain at least some properties, making them *property containers*.

xiv. Referencing ($\to||$)

 doi:10.20944/preprints202211.0450.v1

Definition 37. *A relationship where an entity uses another entity as its container or content in defining its property value.*

Axiom 2.6. *Any referencing to a referent equal to the referral on a value is insignificant (null) to the referral.*
If $(\rho_1 \perp \rho_2)_\omega \wedge (\rho_1 = \rho_2)_\omega \Rightarrow (\rho_1||\rho_2)_\omega = (\rho_1)_\omega$.

Axiom 2.7. *A referral will retrograde if it references a referent less significant than it on a value, and advances if otherwise.*
If $(\rho_1 \perp \rho_2)_\omega \wedge (\rho_1 > \rho_2)_\omega \Rightarrow (\rho_1||\rho_2)_\omega < (\rho_1)_\omega$ *(retrograde),*
if $(\rho_1 \perp \rho_2)_\omega \wedge (\rho_1 < \rho_2)_\omega \Rightarrow (\rho_1||\rho_2)_\omega > (\rho_1)_\omega$ *(advance).*

Referencing can be considered as a *relative* relationship or a type of dependency of an entity on another entity.

In general, an agent can relate to different entities for different purposes to achieve a value, to some (e.g., targets) it builds a conditional relation, while to others (e.g., environments) it builds a reference relation. The operation on the relationships of an entity is important, and requires a logical construct, which we define in the following axiom.

Axiom 2.8. *The operation on the relationships of an entity can be defined as a vector operation on the relationships.*

For example, if entities ρ_1 and ρ_2 are *non-referentially* (e.g., conditionally, mutually, jointly, etc.) related to ρ_3 independently, then any *referential* relationship we define between ρ_1 and ρ_2 on ρ_3 will be equal to the vector sum of their individual relationships on ρ_3.

Let $(\rho_1 \to \rho_3)_\omega||\rho_{e1} = \vec{r_1}$, $(\rho_2 \to \rho_3)_\omega||\rho_{e2} = \vec{r_2}$, and $(\rho_1 \,||\, \rho_2)_\omega = \vec{r_3}$.

Then, using an n-dimensional *Euclidean vector space* of ω, on a relationship $\vec{r}$ between entities, we propose that,

Proposition 7.

$$\vec{r_3} = \vec{r_1} + \vec{r_2} \text{ if } (\rho_{e1} = \rho_{e2})_{\omega,\rho_3} \text{ (collinear on } \rho_3), \tag{2.6}$$

$$\vec{r_3} \neq \vec{r_1} + \vec{r_2} \text{ if } (\rho_{e1} \neq \rho_{e2})_{\omega,\rho_3} \text{ (noncollinear on } \rho_3). \tag{2.7}$$

For $\vec{r_3} = \vec{r_1} + \vec{r_2}$ during non-collinearity, the environment of one entity needs to be transformed to the environment of the other, leading to a collinear situation. We provide such a transformation process in Section 4.4 for knowledge values.

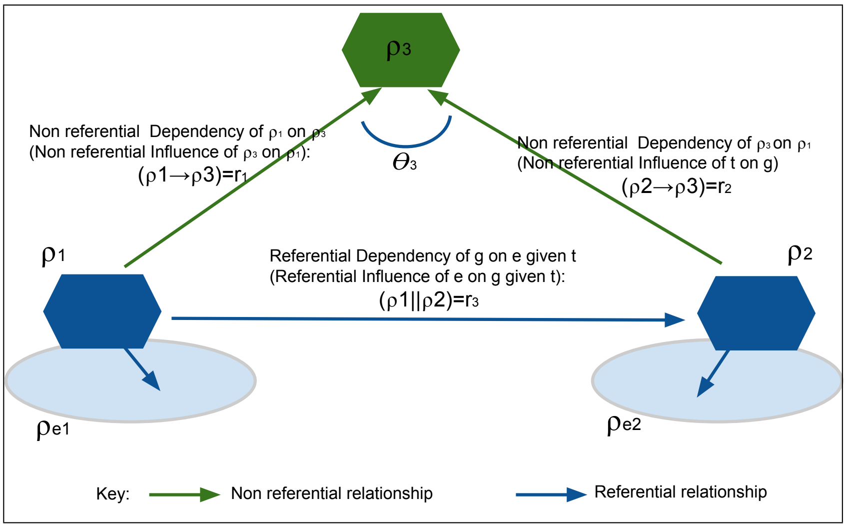

Figure 5. Interrelationship between the referential and non referential relationships of entities $\rho_1, \rho_2, \rho_{e1}$, and ρ_{e2}

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 24 November 2022 doi:10.20944/preprints202211.0450.v1

Furthermore, with collinearity, other vector algebra operations such as operations on vector magnitude can be evaluated easily using vector rules such as the cosine rule.

$$\vec{r_3}^2 = \vec{r_1}^2 + \vec{r_2}^2, \text{ if } \rho_1 \perp \rho_2, \tag{2.8}$$

$$\vec{r_3}^2 = \vec{r_1}^2 + \vec{r_2}^2 - 2\vec{r_1}\vec{r_2}cos(\theta_3), \text{ if } \rho_1 \not\perp \rho_2, \tag{2.9}$$

where $(\rho_{e1} = \rho_{e2})_{\omega,\rho_3}$.

In such case, the angle between the non-referential (i.e., absolute) values of two entities about a third entity represent their referential (i.e., relative) relationships about the entity.

Proposition 8. *The angle θ_3 between any two entities ρ_1 and ρ_2 on ρ_3 is a measure of their referential relationship.*

These relationships between non-referential and referential values of entities are represented in Figure 5.

For a relationship over space and time, we shall apply hyperbolic functions to the Euclidean vector in Section 4.4 to present important effects of time and space on cognition. The relationships can also be represented using *tensors* on the euclidean and non-euclidean (e.g., hyperbolic) space.

2.2.3. Logical operations between entities

We define three types of operations (association, dissociation, intersection) between containers and entities.

i. Operations Between Containers

a) Association(∪) of containers

This involves the union of the entities of many containers to form a new container.

Definition 38. $q_i \cup q_j = \{\forall\rho \in q_i \vee \rho \in q_j\}$.

b) Dissociation (-) of containers

This involves the separation of a container from another container to form a new container.

Definition 39. $q_i - q_j = \{\forall\rho_i \in q_i \vee \rho_j \notin q_j\}$.

c) Intersection (∩) of containers

This involves the intersection of many containers to form a new container based on their equivalent entities.

Definition 40. $q_i \cap q_j = \{\forall\rho \in q_i \wedge \rho \notin q_j\}$.

ii. Operations between entities

a) Association (∪) of entities

This involves the union of the properties of many entities to form a new entity.

Definition 41. $\rho_i \cup \rho_j = \{\forall\omega \in \rho_i \vee \omega \in \rho_j\}$.

b) Dissociation (-) of entities

This involves the separation of an entity from another to form a new entity.

Definition 42. $\rho_i - \rho_j = \{\forall\omega \in \rho_i \vee \omega \notin \rho_j\}$.

c) Intersection (∩) of entities

This involves the joining of many entities to form a new entity based on their equivalence properties.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 24 November 2022 doi:10.20944/preprints202211.0450.v1

Definition 43. $\rho_i \cap \rho_j = \{\forall \omega \in \rho_i \wedge \omega \in \rho_j\}$.

iii. Operations between properties

This involves operations between the values of the properties of entities, rather than the entities themselves. Due to the dependency possibility of the properties, we use probability logic operations [79] to define the operations between properties. Nevertheless, other logic, such as, functional, fuzzy, symbolic, etc., can also be used.

Concerning property and value, the main difference between them in this article is given in their definitions below;

Definition 44. *A **property** is a vessel (or variable) that holds a value which can be optimized within the property.*

Definition 45. *A **value** is the content of a property and defines the nature of the property.*

Values can be static, dynamic, discrete, continuous, etc.

In this section, we presented three types of entities and defined the properties, relationship and operations they can posses. In the next section, we will introduce the quantification of the knowledge property and different cognitive properties related to it. In Section 4-6, we shall discuss the different operations (acquisition, optimization, transfer) an agent can perform with the quantified properties.

3. Cognitive Property Quantification

As defined in Section 2.1, cognitive properties include the action, intelligence and cognitive values of an agent. We quantify the action and intelligence properties and, for the cognitive value property, we focus on the knowledge value.

3.1. Action Property

In an environment, an agent performs actions on one or more targets to acquire values. As discussed in Section 2.1.2, the reasoning action is the main action for cognitive value generation. The value generated by an agent irrespective of the target environment is considered as the *action value* and stored as *belief* in the agent. We quantify this value using epistemological perspective [7] as described below.

3.1.1. Action Quantification

Definition 46. ***Action value** of an agent on a target is the likelihood of that action on the target given an intelligence.*

$$A_\phi(g)_t = f(t, \phi) = L(\phi; t) \Rightarrow A_\phi(g)_t = P(t; \phi), \tag{3.1}$$

where $L(\phi; t)$ is likelihood of an intelligence ϕ on the target t, $P(t; \phi)$ is probability of t based on ϕ.

Each action value generation process about a target is considered as an *event* on which cognitive values depend. An agent can generates different types of *action values (belief)* about a target. Below is a summary of these action values.

3.1.2. Types of Action Values

i) Domain action and Specific action

These are action values that define an agent's belief about the relationship between the input and output features of a target.

Definition 47. ***Domain action** is an action that leads to the comprehension of a target.*

$$[A_\phi(g)_t]_d = L(\phi; Y \leftrightarrow X) \Rightarrow P(Y \cap X; \phi). \tag{3.2}$$

Definition 48. ***Specific or Causal action*** *is an action that leads to the conclusion about a target.*

$$[A_\phi(g)_t]_s = L(\phi; Y \rightarrow\mid X) \Rightarrow P(Y \mid X; \phi). \tag{3.3}$$

The conversion from domain action to specific action and vice versa is defined using the probability logic below.

From Domain to Specific action:

$$[A_\phi(g)_t]_s = \frac{[A_\phi(g)_t]_d}{P(X;\phi)}. \tag{3.4}$$

From Specific to Domain action:

$$[A_\phi(g)_t]_d = [A_\phi(g)_t]_s(P(X;\phi)). \tag{3.5}$$

From these expressions, we deduce the following:

Proposition 9. *It is required to exclude information about input space existence during domain to specific action conversion but such information is needed in the reverse process.*

This is intuitively true because during causality, such information will be less helpful and will lead to more noise.

ii) Abstract action and Real action

These are action values defined based on the agent's environment in which the actions are executed.

Definition 49. ***Abstract action*** *is an action performed on targets in the internal environment of an agent.*

Definition 50. ***Real action*** *is an action performed on targets in the external environment of an agent.*

An example of an abstract and a real action are respectively the *internal and external locus of control* [80] of an agent, which is also just a type of *actuation action*.

It is important to note that both the real and abstract actions can have domain and specific types. Also, real actions represent *practical actions* while abstract actions represent *theoretical actions*. Their conversion is described below.

Theoretical and practical action conversion:

$$[A_\phi(g)_t]_T = f_{P\rightarrow T}([A_\phi(g)_t]_P), \tag{3.6}$$

$$[A_\phi(g)_t]_P = f_{T\rightarrow P}([A_\phi(g)_t]_T), \tag{3.7}$$

where $f_{P\rightarrow T}()$ is a conversion from practical action to theoretical action, and $f_{T\rightarrow P}()$ is a conversion from theoretical action to practical action.

In a cognitive agent, the deviation from theoretical to practical action and vice versa of a single action is important.

Theoretical and practical action deviations:

$$D_{P\rightarrow T} = \frac{[A_\phi(g)_t]_P}{[A_\phi(g)_t]_T}, \tag{3.8}$$

$$D_{T\rightarrow P} = \frac{[A_\phi(g)_t]_T}{[A_\phi(g)_t]_P}, \tag{3.9}$$

$$D_{T\rightarrow P} = D_{T\rightarrow T}\ iff\ [A_\phi(g)_t]_P = [A_\phi(g)_t]_T, \tag{3.10}$$

where $D_{P\rightarrow T}$ is a deviation from practical action to theoretical action, and $D_{T\rightarrow P}$ is a deviation from theoretical action to practical action.

The deviation between theoretical and practical actions will lead to incoherence of actions on a target. This may not be desirable if the agents are required to collaborate on the target. To reduce such deviation, one action is optimized relative to the other. Such type of optimization process based on relative values forms an important part of this research.

3.1.3. Logical Operations on Action Values

The operations on the action values are the logical relationships that define two or more actions of agents on targets. As mentioned in Section 2.2.3, we use a probabilistic logic [79] for definition of properties relationships. We define the operations for two agents (g_i, g_j) with actions (A_i, A_j) on a set of targets, assuming $\phi_i \perp \phi_j$. Extension to multi-actions can simply be done using probability logic.

Definition 51. ***Self action*** *is action value on a target irrespective of other actions.*

$$A(g)_t = L(\phi;t) \Rightarrow A(g)_t = P(t;\phi). \tag{3.11}$$

Definition 52. ***Joint action*** *is the action value representing the joint relationship between agents or targets .*

$$A(g_1, g_2)_t = P(t;\phi_1, \phi_2) = P(t;\phi_1)P(t;\phi_2), \phi_i \perp \phi_j, \tag{3.12}$$

$$A(g)_{t_1,t_2} = P(t_1, t_2;\phi) = P(t_1;\phi)P(t_2;\phi)r, \neg(t_i \perp t_j), \tag{3.13}$$

where, $r = P(t_1, t_2;\phi)/[P(t_1;\phi)P(t_2;\phi)]$.

Definition 53. ***Mutual action*** *is the action value representing the mutual relationships between agents or targets.*

$$A(g_1;g_2)_t = \frac{P(t;\phi_1, \phi_2)}{P(t;\phi_1)P(t;\phi_2)} = 1, \phi_i \perp \phi_j, \tag{3.14}$$

$$A(g)_{t_1;t_2} = \frac{P(t_1, t_2;\phi)}{P(t_1;\phi)P(t_2;\phi)}, \neg(t_i \perp t_j). \tag{3.15}$$

Definition 54. ***Conditional action*** *is the action of an agent on a target conditioned on another target or action.*

$$A(g_1|g_2)_t = \frac{P(t;\phi_1, \phi_2)}{P(t;\phi_2)} = P(t;\phi_1), \phi_i \perp \phi_j, \tag{3.16}$$

$$A(g)_{t_1|t_2} = \frac{P(t_1, t_2;\phi)}{P(t_2;\phi)}, \neg(t_i \perp t_j). \tag{3.17}$$

Definition 55. ***Relative action*** *is the action of an agent on a target referenced to another target or action*

$$A(g_1||g_2)_t = \frac{P(t;\phi_1)}{P(t;\phi_2)}, \tag{3.18}$$

$$A(g)_{t_1||t_2} = \frac{P(t_1;\phi)}{P(t_2;\phi)}. \tag{3.19}$$

3.2. Intelligence Property (Φ)

As defined in Section 2.1.1, the intelligence is the *enabler* of an action. Based on Proposition 2, we express the intelligence property with respect to knowledge.

3.2.1. Intelligence Quantification

Definition 56. ***Intelligence value,*** ϕ***,*** *is a function of the knowledge value generated by an action value*

$$\Phi = f_{k\to\phi}(K), \tag{3.20}$$

where $f_{k\to\phi}()$ *is the conversion function from knowledge value to intelligence value.*

Likewise, knowledge can be defined with respect to intelligence using a knowledge to intelligence conversion function.

$$K = f_{\phi\to k}(\phi). \tag{3.21}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 24 November 2022 doi:10.20944/preprints202211.0450.v1

Proposition 10. *The conversion of an Intelligence value to a Knowledge value is only possible through an action value and the convention of a Knowledge value to an Intelligence value is only possible through a reverse action value.*

$$f_{k\to\phi} : (f_{a\to k} : A \to K) \to \Phi, \tag{3.22}$$

$$f_{\phi\to k} : (f_{\phi\to a} : \Phi \to A) \to K. \tag{3.23}$$

The proof of Proposition 10 is given in Appendix A.1.

In this article, we focus on the definition of knowledge with respect to action and not intelligence and assume an equality relationship between knowledge and intelligence.

Just like any cognitive property, intelligence can be abstract or real, based on the environment in which it operates. Most importantly, intelligence can also be intermediary, where it supports the convention of one property to another. More on intelligence will be discussed in a future publication.

3.3. Cognitive Value Property

Cognitive values as defined in Section 2.1 are values that are generated by agents based on the actions of the agents on a target. One of such values is the *knowledge value* of an agent, whose inverse is the *ignorance value*. Other cognitive values mentioned in this article include *understanding*, *trust*, and *wisdom*.

In this research, we shall quantify the *knowledge* value, its properties and operations in an agent. In Section 4, we shall quantify *understanding*, *trust*, and *wisdom* and reserve their operations for future publications.

Knowledge as defined in Section 2.1.1 is a cognitive value property that an agent seeks from a target. Here, we quantify this knowledge from an epistemological viewpoint [5],[7].

3.3.1. Knowledge Quantification

Epistemologically, knowledge is consider as *a justify true belief* [5],[6],[7]. From this universally accepted axiom, we then build a mathematical definition of knowledge.

In epistemology, the concept of justification, truth, and belief are crucial concepts for any rational agent with cognitive and intelligent abilities. As defined in this article, belief is the output of the reasoning action. To address the issue of justification and the truthfulness of belief, the question we may ask is, why do agents hold beliefs about a target, and if such beliefs have met a standard that renders them true and rational for the agent to hold?

As discussed in Section 1 of this paper, the aim for an agent to hold a belief about a target is to guide the agent in achieving a cognitive value about the target. Thus, belief is considered as the base value on which cognitive values are generated.

The truthfulness of a belief in this research is the *truth value* [81] of the belief, as oppose to the *false value*. The truthfulness or falsehood of a belief can depend only on the agent that generates it or another agent. These are considered in the literature as *absolute* and *relative truth*[81]. We mathematically represent these two types of truth as follows:

$$\text{Absolute true belief} = A(g)_t, \tag{3.24}$$

$$\text{Relative true belief} = \frac{A(g)_t}{A(e)_t} = A(g||e)_t, \tag{3.25}$$

where $A(g)$ is the absolute belief of an agent g on a target t, and $A(e)$ is the absolute belief of an agent e which is referenced by g as g generates beliefs on t. In the definition of knowledge, e is considered to have more influence or accurate definition on t than g. With respect to the definition of knowledge, we consider e as the environment of t.

Justification, also called epistemic justification [18], is a controversial concept in epistemology [17] and is considered to determine the rational ability of an agent about a target. Normally, a *rational agent* is an agent that possesses and seeks to optimize a justified true belief (i.e., knowledge) about a target.

Conventionally, the relative truth value of belief is considered as a justified true belief [81]. This is called the *fundamentalism* view of justification [19]. In this view, the belief of an

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 24 November 2022 doi:10.20944/preprints202211.0450.v1

agent about a target is justified by referencing it to the belief of another agent considered to possess a more rational belief value about the target. Such a rational chain can be endless, unless it ends with an absolute true belief value. Another problem with this view is that, what if the referenced belief is wrong?

Our view of justification in this paper is that of *externalism* [82][83], without undermining *internalism* [83], because we think both views are required for a complete justification. In this context, we consider justification not only as a relative truth value but also as a process of standardizing a truth value. This is because, not all beliefs that are true maybe rational and not all belief that are rational maybe true.

This external layer of justification is a logical process rather than a logical value, independent on the belief generation process of the agent, but which is a requirement to the agent in order to justify its belief value. This accords with the suggestion of the Greek philosopher (Socrates) through Theatetus, that knowledge is true judgment plus a logos - an account or argument [18][84].

Defining such an external justification as a standardization process is important in the mathematical definition of knowledge. Mathematically, standardization is a scaling operation. The likelihood function we used in Section 46 to define the belief value is considered to generate an unstandardized value, and even the transformed version using the *probabilistic constraint* ($\sum_{i=1}^{i=n} P(t_i;\theta) = 1$) will not guarantee a sound logic for knowledge quantification, because although the probabilistic logic is a good measure for the uncertainties of random quantities, it is not suitable as a scaling function for this justification.

For this purpose, we use the *logarithmic scale* as a standardization logic for belief value defined either using the likelihood function or the probabilistic logic. This is because of the operational simplicity, computational advantage, and application diversity of the logarithmic scale. Nevertheless, other sound deterministic scaling logics can still be used.

We express the knowledge value of an agent about a target by justifying the expressions for the relative true belief and absolute true belief of an agent.

Knowledge based on relative true belief:

Definition 57. *Relative knowledge is the relative likelihood of the true action of an agent to that of its environment.*

$$K_a((g \to t)||e) = \log\left(\frac{A(g)_t}{A(e)_t}\right) \equiv J[A(g||e)_t]. \tag{3.26}$$

Knowledge based on absolute true belief:

Definition 58. *Absolute knowledge is the log likelihood of the true action of an agent irrespective of the environment.*

$$K_a(g \to t) = \log(A(g)_t) \equiv J[A(g)_t], \tag{3.27}$$

where g is agent, e is environment of the agent influencing the target, $A(g)_t$ is true dependency action of g on t, $A(e)_t$ is true influence action of e on t, $K_a((g \to t)||e) = K_a(g||e)_t$ is knowledge of g on t referenced to e, $K_a(g \to t)$ or $K_a(g)_t$ are identical and represent absolute knowledge of g on t, and $J[\cdot]$ is justification function.

It should be noted that, false action of environment $\bar{A}(e)_t$ can also be used for knowledge definition but this should be the same for ignorance.

The use of the environment as a reference point in a cognitive value generation of an agent is similar to that of the *field theory* of psychology [20],[21] by Lewin, where he presented the behavior of an individual (i.e., agent) as a function of the ability of the individual and his environment.

One important aspect here is the case where the environment of the agent has little or no influence on the target. In such case, the agent needs to consider generating knowledge using the influences of remote environments. While doing so, it needs to take into account the divergence between its environment influence on the target to that of the remote environments. This divergence is considered as a *semantic difference* of agents on targets, and will lead to an *environment divergence problem* of the agents during knowledge acquisition, optimization, and transfer. We discuss this in Section 4.4.

Furthermore, considering the fact that an agent can receive influence from and exert influence to another agent about a target, we classify the justification (or rationality) process of an agent in this research into *exopistemic* and *endopistemic* justification (or rationality).

Definition 59. ***Endopistemic*** *justification is that which is based on the dependency of an agent on other entities such that values flow from the other entities to the agent.*

Definition 60. ***Exopistemic*** *justification is that which is based on the dependency of other entities to an agent such that values flow from the agent to the other entities.*

This implies, the endopistemic process is a process through which value enters or is pulled into the agent, while exopistemic process is a process through which value leave or is pushed out of the agent. If values are consider as a type of energy, then these process can be seen as a type of *energy input and output* processes. If we consider the values as a type of influences, then the processes can be seen as *push and pull* processes, similar in operation to the forces defined by Sir Isaac Newton in his *Principia* [85], but whose evaluation will require a *Lagrangian* or *Hamiltonian* approach. The mechanics of such dynamics is considered in this research as *cognimatics*.

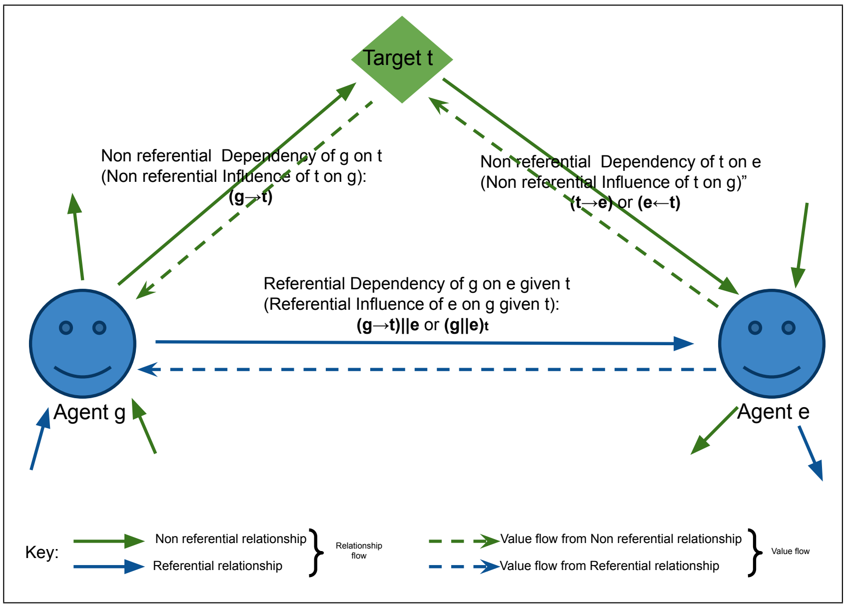

Figure 6. Dependency and value flow diagrams. (a) Endopistemic and Exopistemic processes of g and e on t. (b) Cost dynamics of g and e on t

For the fact that endopistemic process entails influence from the environment, we consider this process as an *environment driven cognition*. During endopistemic learning, the agent seeks to reach the environment. On the other hand, in an exopistemic process, since the agent influences the environment, we considered this process as an *agent-driven cognition*. During exopistemic learning, the agent seeks to lead the environment.

Examples of an endopistemic and exopistemic cognition are observation and actuation, respectively. The ability to switch between endopistemic and exopistemic processes during cognition is important to an agent, but such operation is not considered in conventional agent design. We shall discuss more about this in future publications.

The endopstemic and exopistemic processes for referential and non-referential relationships are presented in Figure 6.

During a referential (i.e., relative) endopistemic process of any agent, the agent depends on other entities and the values the agent generates are based on how the agent

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 24 November 2022 doi:10.20944/preprints202211.0450.v1

perceives such dependencies. Whereas, during a referential exopistemic process of same agent, the agent influences other agents and the values it generates represent how the agent perceives its influence on other entities.

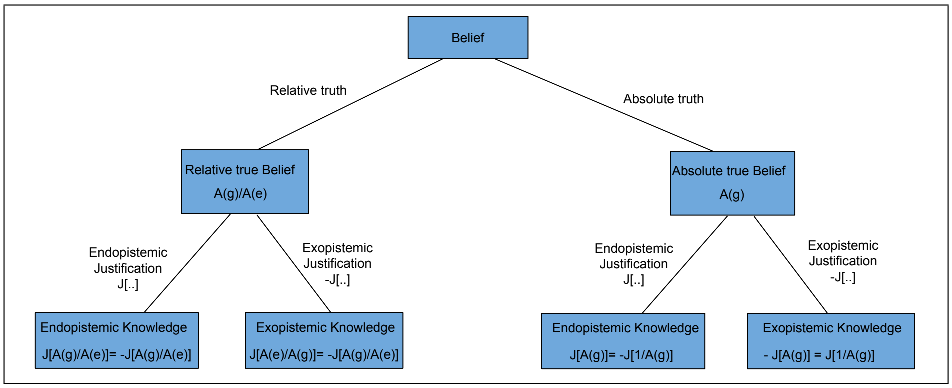

Figure 7. Structure semantic of Knowledge from Belief and Truth

Also, in both endopistemic and exopistemic cognition, the referential value may either be fixed or varied. This is analogous to the *initial and non-initial reference frames* in relativity [86] where initial reference frame implies fixed referential value and non-initial reference frame implies variable referential value.

If we consider that an agent's environment can be distinguished into internal and external environments, then the endopistemic and exopistemic processes of an agent can take place between these environments, where values of the agent about a target can flow from the external environment to the internal environment and vice versa.

Examples of such environments are the theoretical (internal) and practical (external) environments of an agent. The values an agent generates in a theoretical environment are transferred to the practical environment and vice versa. Such a value exchange process will enable the agent to have a more balanced and reliable rational actions. This process is also discussed in the literature [22],[23] by Audi, where he proposed a practical and theoretical reasoning structure for a rational agent.

In general, as defined in this our research, an endopistemic process of an agent about a target is the *inverse* of its exopistemic process about same target and vice versa. These processes can be interactively carried out by an agent during cognition. Such an interactive combination of both endopistemic and exopistemic values of an agent leads to an important value that can be used to balance the cognitive processes of an agent. We consider this value as a *Cognitive balancing factor* (CBF) of an agent during cognition. Other types of resultant values will be discussed in Section 4. For a knowledge based on true relative belief (i.e., relative knowledge), the CBF of knowledge for an agent during cognition between any two environments is defined as:

Definition 61.
Endopistemic Knowledge-Exopistemic Knowledge = CBF,

$$\log(\frac{A(g)_t}{A(e)_t}) - \log(\frac{A'(e)_t}{A'(g)_t}) = \text{CBF}, \tag{3.28}$$

where $A'(e)$ is dependency of e on t as perceived by g, $A'(g)$ is influence of g on t as perceived by g, $A(g)$ is real dependency of g on t, and $A(e)$ is real dependency of e on t.

For a perfect balance of g on t in the different environments (in this case, the abstract and real environments),

$$\text{CBF=0} \Leftrightarrow A'(e) = A(g) \; and \; A'(g) = A(e). \tag{3.29}$$

Being an inverse process to one another, and using a logarithmic justification, an endopistemic process for knowledge value is the *additive inverse* of its exopistemic process

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 24 November 2022 doi:10.20944/preprints202211.0450.v1

and vice versa. The value generated by such processes will depend on the ability and CBF value of the agent.

The structure semantics of knowledge from the definition of belief and the truth property of belief, are presented in Figure 7. Using the structure semantics in Figure 7, we can relate the absolute properties of knowledge and action values to the relative properties.

For example, if $g \to t$ and $t \to e$ non-referentially, and $g \to e$ referentially, as shown in Figure 6, then the knowledge value about t that flows from e to g is given using vector addition as,

$$K((g \to t)||e) = K(g \to t) + K(t \to e), \tag{3.30}$$

$$K((g \to t)||e) = \log A(g) + (-\log A(e)) = \log \frac{A(g)}{A(e)}, \tag{3.31}$$

where $g \to t$ generates an absolute endopistemic knowledge value $(\log A(g))$ of g from t, $e \to t$ generates an absolute exopistemic knowledge value $(-\log A(e))$ of e to t, $(g \to t)||e$ generates a relative endopistemic knowledge value $(\log(A(g)/A(e)))$ of g from e about t.

Similarly, without a *logarithmic justification*, the endopistemic and exopistemic values for referential and non-referential relationships are expressed as *beliefs*.

$$A((g \to t)||e) = A(g \to t)A(t \to e), \tag{3.32}$$

$$A((g \to t)||e) = A(g).A^{-1}(e), \tag{3.33}$$

$$\text{Considering, } A^{-1}(e) = \frac{1}{A(e)}, \tag{3.34}$$

$$\Rightarrow A((g \to t)||e) = A(g).\frac{1}{A(e)} = \frac{A(g)}{A(e)}, \tag{3.35}$$

where $A(g)$ is endopistemic action of g, $(A^{-1}(e))$ is exopistemic action of e, i.e., reciprocal (or multiplicative inverse) of the endopistemic action of e, $A(g)/A(e)$ is *dot product* of endopistemic and exopistemic actions of g and e, respectively.

In reality, during a *relative endopistemic* cognition on a target, such as learning to observe a target with respect to a referenced observer of the target, e.g., a teacher, the learning (dependent) agent tries to acquire the *complete relative endopistemic value* of the target as defined by the referenced (influencing) agents. This entails *maximizing its relative dependency* on t as referenced from its influencers.

The maximum value of an agent during cognition is discussed with more details in Section 4, where the variation of the relative endopistemic value of an agent is achieved by varying its absolute endopistemic value using a corresponding absolute exopistemic value. These two values are considered as the *cost or work* of the agent on the target, and the maximum value is considered as the *complete value* of the agent.

This absolute exopistemic value of an agent on a target turns to act as a counter-action on its absolute endopistemic value and the absolute exopistemic value of its influencer (i.e., environment) on the target. This implies that cognition is analogous to a *battle of influences* between an agent and the environment, about a target. The number of entities under such influences constitute an *exopistemic sphere* (i.e., sphere of influence) and the number of influencers on which an entity dependence constitute its *endopistemic sphere* (i.e., sphere of dependency). An entity can have both influences and dependencies as shown in Figure 6.

Using the cost value, the knowledge value about t that flow from e to g can also be expressed as,

$$K((g \to t)||e) = K(e \leftarrow t) - K(g \leftarrow t), \tag{3.36}$$

$$K((g \to t)||e) = (-\log A(e)) - (-\log A(g)), \tag{3.37}$$

$$K((g \to t)||e) = \log \frac{A(g)}{A(e)}, \tag{3.38}$$

where $K(e \leftarrow t)$ is exopistemic knowledge of e about t, $K(g \leftarrow t)$ is counteractive (i.e., counter-intuitive) or exopistemic knowledge of g about t, and $K((g \to t)||e)$ is resultant knowledge from the counteractive interaction process.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 24 November 2022 doi:10.20944/preprints202211.0450.v1

Similarly, the action value of g about t relative to e can also be expressed as,

$$A((g \to t)||e) = A(e \leftarrow t)(A(g \leftarrow t))^{-1}, \tag{3.39}$$

$$A((g \to t)||e) = (1/A(e)) \div (1/A(g)) = \frac{A(g)}{A(e)}, \tag{3.40}$$

where $A(e \leftarrow t)$ is exopistemic action of e about t, $A(g \leftarrow t)$ is counteractive (i.e., counter-intuitive) or exopistemic action of g about t, and $A((g \to t)||e)$ is resultant action from the counteractive interaction of exopistemic processes on t.

This logic of relating absolute values to relative values can be elaborated for multiple interactive targets and agents. Focusing on a *logarithmic justification relationships*, we present in Figure 8, multiple interactive targets and agents.

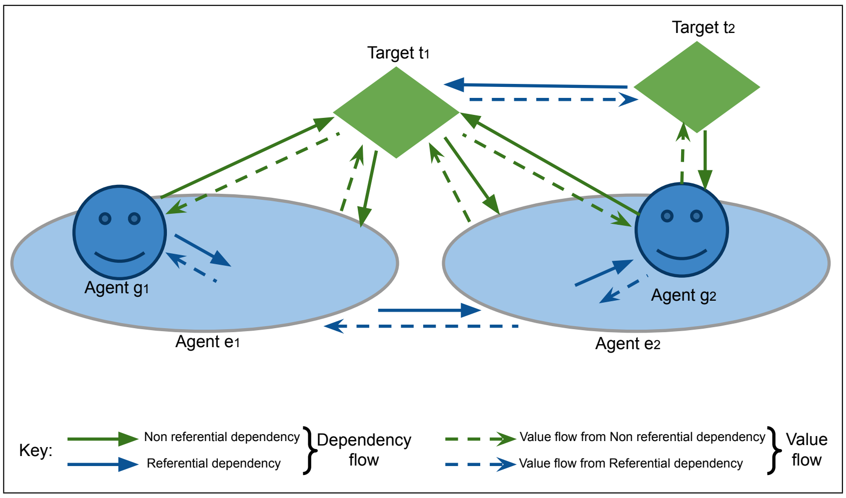

Figure 8. Dependency and value flow diagrams of endopistemic and exopistemic processes of $g_1, g_2, e_1, e_1, t_1, t_2, t_3$.

We can identify the dependencies and express the value flow between entities in Figure 8 as shown in Table 1.

Analysis of such cognitive relationships and value flows between entities is one of the basis of this mathematical theory, a process called *cognimatics*; the dynamics of cognition. In Section 4.4, we shall extend it to solve the environmental divergence problem of knowledge acquisition, which may arise in situations such as finding the relationship and valuation between g_1 and e_2 or g_1 and g_2 on t_1.

Table 1. Entity dependencies and values

Symbols	Dependency types	Action values	Knowledge values
$g_1 \to t_1$	non-referential	$A(g_1)_{t_1}$	$K(g_1)_{t_1} = \log A(g_1)_{t_1}$
$g_1 \to e_1$	referential	$A(g_1\|\|e_1)_{t_1}$	$K(g_1\|\|e_1)_{t_1} = \log \frac{A(g_1)_{t_1}}{A(e_1)_{t_1}}$
$g_2 \to t_1$	non-referential	$A(g_2)_{t_1}$	$K(g_2)_{t_1} = \log A(g_2)_{t_1}$
$g_2 \to t_2$	non-referential	$A(g_2)_{t_2}$	$K(g_2)_{t_2} = \log A(g_2)_{t_2}$
$g_2 \to e_2$	referential	$A(g_2\|\|e_2)_{t_2}$	$K(g_2\|\|e_2)_{t_2} = \log \frac{A(g_2)_{t_2}}{A(e_2)_{t_2}}$
$e_1 \to t_1$	non-referential	$A(e_1)_{t_1}$	$K(e_1)_{t_1} = -\log A(e_1)_{t_1}$
$e_2 \to t_1$	non-referential	$A(e_2)_{t_2}$	$K(e_2)_{t_1} = -\log A(e_2)_{t_1}$
$e_1 \to e_2$	referential	$A(e_1\|\|e_2)_{t_1}$	$K(e_1\|\|e_2)_{t_1} = -\log \frac{A(e_1)_{t_1}}{A(e_2)_{t_1}}$
$t_2 \to t_1$	referential	$A(g_2)_{t_2\|\|t_1}$	$K(g_2)_{t_2\|\|t_1} = \log \frac{A(g_2)_{t_2}}{A(g_2)_{t_1}}$
$t_3 \to t_1$	non-referential	$A(g_1)_{t_3\|t_1}$	$K(g_1)_{t_3\|t_1} = \log A(g_1)_{t_3\|t_1}$

It should be noted that, other non-referential dependencies can be used such as mutual (;), joint (,), etc., apart from conditional dependency (|).

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 24 November 2022 doi:10.20944/preprints202211.0450.v1

The following axiom defines the basic rules in analysing any cognitive entity dependency relationships.

Axiom 3.1.

1. *For any relationship, and at any time instance, all entities are either of two types: agent or target.*
2. *The target is the center (purpose) of all cognitive actions and value generation of an agent.*
3. *All value generated during cognition flow from the influencer entity to the dependent entity, contrary to the flow of dependency.*
4. *All relationships between same entity type are dependent relationships: self, conditional, mutual joint, referencing, etc.*
5. *The relationship between targets is defined by agent (action) and the relationship between agents is defined by target (state).*
6. *The relationship between agent and its environment is referential but between agent and its target is non-referential.*

It should be noted that an entity can transition between entity types over space and time, and one entity can have different types in different relationships. Such complexities will be avoided in this article and reserved for a future publication. Also, relationships can be *non-referential* (self, conditional, joint, etc.) or *referential*, *base* (self, conditional, mutual, joint, and referencing) or *composite* (conditional mutual, etc.).

The acquisition, optimization, and transfer of endopistemic and exopistemic values are important processes to an agent because all actions possesses endopistemic and exopistemic properties consciously or unconsciously. For example, the observation action of a human agent is consciously endopistemic in the external environment but unconsciously exopistemic in the internal environment. We shall focus on the endopistemic process and value in a relativistic setting.

As discussed in Section 2.1.1, cognitive values also have their respective inverse properties. The inverse of knowledge was considered as ignorance. With respect to the epistemological definition of knowledge, we can define ignorance as a *justified true disbelief*. Disbelief is a type of *irrational action*, and defined in the Cambridge dictionary as the inability or refusal to accept that something is true. Hence, ignorance can also be considered as a *justify false belief* of an agent.

Thus, similar to knowledge, ignorance can have a relative and absolute dimension based on the truth property. We express these in the following statements below.

Ignorance based on relative false belief:

Definition 62. *Relative ignorance is relative likelihood of the false action of an agent to true action of its environment.*

$$I_a((g \to t)||e) = \log\left(\frac{\bar{A}(g)_t}{A(e)_t}\right) \equiv J[\bar{A}(g||e)_t]. \tag{3.41}$$

Ignorance base on absolute false belief:

Definition 63. *Absolute ignorance is the log likelihood of the false action of an agent irrespective of the environment.*

$$I_a(g \to t) = \log(\bar{A}(g)_t) \equiv J[\bar{A}(g)_t], \tag{3.42}$$

where g is agent, e is environment of the agent influencing the target, $\bar{A}(g)_t$ is false action of g on t, $A(e)_t$ is true influence action of e on t, $I_a((g \to t)||e) = I_a(g||e)_t$ is ignorance of g on t referenced to e, $I_a(g \to t) = I_a(g)_t$ is absolute ignorance of g on t, and $J[\cdot]$ is justification function.

It should be noted that, false action of environment $\bar{A}(e)_t$ can also be used for ignorance definition but this should be same for the corresponding knowledge.

 doi:10.20944/preprints202211.0450.v1

Similar to knowledge, the ignorance value of an agent possesses an endopistemic and exopistemic property as presented in the structure semantic shown in Figure 9.

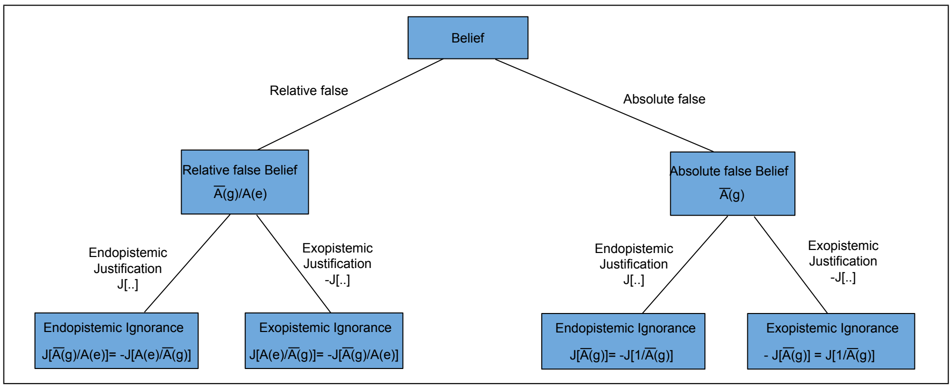

Figure 9. Structure semantic of Ignorance from Belief and Falsehood

Also, the ignorance value can be considered as a measure of the amount of uncertainty (or impurity) in cognition.

Moreover, considering the relative property of knowledge and ignorance, using the first two terms of a *Taylor expansion*, we can prove that the relative change between two actions is a linear approximation of the logarithm of their ratio.

The first order expansion of $\log(x)$ around $x = 1$ is,

$$\log(x) \approx \log(1) + \frac{d}{dx}\log(x)_{x=1}(x-1) = (x-1), \tag{3.43}$$

Hence, for $x \approx 1$, if $x = \frac{A(g)}{A(e)}$, then,

$$\log(\frac{A(g)}{A(e)}) = \frac{A(g)}{A(e)} - 1, \tag{3.44}$$

Also, for all $x > 0$, it can also be proven that,

$$\log(x) \leq x - 1, \tag{3.45}$$

This implies, our relativistic definitions of knowledge and ignorance are relative change of belief (action) values.

3.3.2. Cognitropy: Expected Cognitive Property Value

The above mathematical definitions of knowledge and ignorance focused on a single target. In case of a set of targets, related or not related to each other, we can actually estimate the average knowledge or ignorance over all the target set. We call such expected property in this research as *cognitropy*, defined as follows:

Definition 64. ***Cognitropy*** *is the expectation of any cognitive property value of an agent over a set of targets.*

It is actually a *summarization* of a cognitive property value over targets or actions states. In relation to knowledge and ignorance, the cognitropy is given as follows:

Definition 65. *The* ***Cognitropy of knowledge or ignorance*** *is the expected knowledge or ignorance of an agent about a target over its environment influences on the target.*

$$Z_a(g)_t = \sum_{i,j} A_j(e)_{t_i} V_{a_j}(g)_{t_i} \text{ (discrete states)}, \tag{3.46}$$

$$Z_a(g)_t = \int_{i,j} A_j(e)_{t_i} V_{a_j}(g)_{t_i} \text{ (continuous states)}, \tag{3.47}$$

 doi:10.20944/preprints202211.0450.v1

where V is K or I (values for a single independent action and target states), $A_j(e)$ is action of environment e, i is independent target states, j is independent action states, and the expressions $Z_a(g)_t \equiv Z_V(g)_t$.

In a similar way, the cognitropy value for both knowledge and ignorance can be endopistemic and exopistemic.

Endopistemic cognitropy of knowledge or ignorance:

$$Z_a(g)_t = \sum_{i,j} A_j(e)_{t_i} V_{a_j}(g)_{t_i}. \tag{3.48}$$

Exopistemic cognitropy of knowledge or ignorance:

$$-Z_a(g)_t = -\sum_{i,j} A_j(e)_{t_i} V_{a_j}(g)_{t_i}, \tag{3.49}$$

where V can be the relative or absolute value of K or I. Thus, endopistemic and exospisemic cognitropies can also be relative or absolute.

For example, using Figure 6, the relative endopistemic cognitropy of g from e is given as,

$$Z(g||e)_t = Z((g \to e)_t) = Z(g \to t) + Z(t \to e), \tag{3.50}$$

$$Z(g||e)_t = \sum_{i=1}^{n} A(e) \log A(g) + (-\sum_{i=1}^{n} A(e) \log A(e)), \tag{3.51}$$

$$Z(g||e)_t = \sum_{i=1}^{n} A(e) \log \frac{A(g)}{A(e)}, \tag{3.52}$$

where $g \to t$ is an absolute endopistemic cognitropy $\sum_{i=1}^{n} A(e) \log A(g)$ of g from t, $e \to t$ is an absolute exopistemic cognitropy value $(-\sum_{i=1}^{n} A(e) \log A(e))$ of e to t, and $(g \to e)_t$ is a relative endopistemic cognitropy $\sum_{i=1}^{n} A(e) \log \frac{A(g)}{A(e)}$ of g from e about t.

Concerning the action value, we consider the expected action of an agent g on a target t over an environment e as the action cognitropy and its definition is given by:

Definition 66. *The **Cognitropy of action** is the expected action of an agent on a target.*

$$A_Z(g)_t = \sum_{i=1}^{n} t_i A(g)_{t_i}, \tag{3.53}$$

where t_i is possible states of the target, and $A(g)_{t_i}$ is action value of the agent on the target outcomes.

Furthermore, the endopistemic and exopistemic values of the Action cognitropy are given as follows:

$$\text{Endopistemic cognitropy of action} = A_Z(g)_t, \tag{3.54}$$

$$\text{Exopistemic cognitropy of action} = \frac{1}{A_Z(g)_t}. \tag{3.55}$$

Apart from the *expected values* of agents on targets, i.e., cognitropy, the *resultant values* of agents on targets are also important quantities that we use in this research. They enable the resultant effect of a set of values from agents on a targets in same or different environments.

For example, the resultant action value of two referential related environments is defined as the average of their values.

$$A_r(g)_t = \frac{A(g)_t + A(e)_t}{2} \; (single\ action\ pair)\,, \tag{3.56}$$

$$A_r(g)_t = \frac{\sum_{i=1}^{n} A(g)_t + \sum_{j=1}^{m} A(e)_t}{n+m} (multiple\ action)\,, \tag{3.57}$$

where $(g||e)_t$ and if $A_r(g)_t = A(e)_t$, then $A(g)_t = A(e)_t$

More about resultant values and their operations are described in Section 4.5.1.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 24 November 2022 doi:10.20944/preprints202211.0450.v1

Same logic of cognitropy and resultant values can be used for multiple relationships to generate a complex value analysis of the cognitive processes of an agent.

Moreover, while the environment is the container of the agent and target, the target and agent can be ubiquitous over different environments which have different action values on the target. In this paper, we focus more on *ubiquitous targets and non-ubiquitous agents*.

Furthermore, concerning the unit for cognitive values, we introduce the *binary cognitive value (bcv) or binary cognition (bic)* if the logarithm is in base 2, the *zie-ochiai-kohno (zok)* if in base 10, and the *natural cognitive value (ncv) or natural cognition (nac)* if in the natural base. The action value is unit-less under this scheme.

3.3.3. Dissimilarities of Cognitropy from Other Quantities

It should be noted that the *cognitropy value of knowledge or ignorance* of an agent depends on the intelligence and belief of the agent, the value of its environment, its number of targets and actions together with their states. This makes cognitropy different from *entropy* [13] and *Kullback–Leibler* (KL) divergence [87].

It is worth noting that, the environment for a single and multiple states target in this research are considered respectively as *surprise* and *entropy* by Claude Shannon in his theory of communication [13]. Linking his research to ours, an environment with high surprise about a target will require more value from an agent. In other words, an agent in a highly surprising situation will require more value to reach completeness about the situation. Also, the degree of *surprise* and *entropy* of an agent about a target is a measure of the influence of the target on the agent. So, Shannon *entropy* and *surprise* are properties of only the target and not the agent, but *cognitropy* is a property of both agent and target.

Unlike entropy which quantifies the uncertainty of a target, cognitropy quantifies the uncertainty in an agent about a target. So, while entropy measures information about a target, cognitropy measure the value of that information in an agent. The relationship between entropy and cognitropy produces an interesting phenomenon which we describe in Section 4.3 as *the law of conservation of value*.

Another important difference is that cognitropy can be *negative or positive* depending on the direction of value flow, but entropy can only be *positive*.

Also, our expression for relative exopistemic knowledge should not be confused with *Kullback–Leibler* (KL) divergence because even if they are mathematically identical, their interpretation and significance are different. The main difference being that KL divergence is a measure of the divergence between two distributions, while relative exopistemic knowledge is a measure of the expectation of a justified relative true belief of an agent over the environment of its target. As we shall see in Section 4, this environment can be mixed, isolated, or non-isolated.

Similar to knowledge and ignorance, the cognitropy of knowledge and ignorance can be relative, absolute, endopistemic, and exopistemic. As previously mentioned, we shall focus more on the relative nature of knowledge.

Similar to the action property, the cognitive value property such as knowledge can be classified into different types. We shall focus on relative knowledge because of its double justification: fundamentalism and logical justification.

3.3.4. Types of Knowledge

i. Domain and Specific Knowledge

Definition 67. ***Domain knowledge*** *is knowledge acquired through domain actions.*

$$[V_a(g)_t]_d = J\left[\frac{[A(g)_t]_d}{[A(e)_t]_d}\right] = J\left[\frac{L(\phi_g; Y \cap X)}{L(\phi_e; Y \cap X)}\right], \tag{3.58}$$

$$[Z_a(g)_t]_d = \sum_{i=1}^{n} [A(e)_t]_d [V_a(g)_t]_d. \tag{3.59}$$

Definition 68. ***Specific knowledge*** *is knowledge acquired through specific actions.*

$$[V_a(g)_t]_s = J\left[\frac{[A(g)_t]_s}{[A(e)_t]_s}\right] = J\left[\frac{L(\phi_g; Y|X)}{((\phi_e; Y|X)}\right], \tag{3.60}$$

$$[Z_a(g)_t]_s = \sum_{i=1}^{n} [A(e)_t]_s [V_a(g)_t]_s. \tag{3.61}$$

The conversion from domain to specific knowledge and vice versa is possible and can be defined as describe below.

From Domain to Specific knowledge:

$$[V_a(g)_t]_s = [V_a(g)_t]_d - J[\frac{P(X;\phi_g)}{P(X;\phi_e)}], \tag{3.62}$$

$$[Z_a(g)_t]_s = [Z_a(g)_t]_d - \sum_{i=1}^{n} P(X;\phi_e) J[\frac{P(X;\phi_g)}{P(X;\phi_e)}]. \tag{3.63}$$

From Specific to Domain knowledge:

$$[V_a(g)_t]_d = [V_a(g)_t]_s + J[\frac{P(X;\phi_g)}{P(X;\phi_e)}], \tag{3.64}$$

$$[Z_a(g)_t]_d = [Z_a(g)_t]_s + \sum_{i=1}^{n} P(X;\phi_e) J[\frac{P(X;\phi_g)}{P(X;\phi_e)}]. \tag{3.65}$$

From these expressions, we can deduce the following.

Proposition 11. *Excluding the knowledge about the input space existence during domain to specific action conversion is required, but such knowledge is needed in the reverse process.*

Such knowledge will be less helpful during causality.

ii. Abstract and Real Knowledge

Definition 69. ***Abstract knowledge*** *is knowledge acquired through abstract action.*

$$[V_a(g)_t]_\mu = J[(A(g_\mu||e_\mu)], \tag{3.66}$$

$$[Z_a(g)_t]_\mu = \sum_{i=1}^{i=n} (A(e_\mu) J[(A(g_\mu||e_\mu)]. \tag{3.67}$$

Definition 70. ***Real knowledge*** *is knowledge acquired through real action.*

$$[V_a(g)_t]_v = J[(A(g_v||e_v)], \tag{3.68}$$

$$[Z_a(g)_t]_v = \sum_{i=1}^{n} (A(e_v) J[(A(g_v||e_v)]. \tag{3.69}$$

Both the real and abstract knowledge can have domain and specific types. Furthermore, abstract knowledge represents theoretical knowledge while real knowledge represents practical knowledge. Their conversion process is expressed below.

Theoretical and practical knowledge conversion:

$$[Z_a(g)_t]_T = f_{P\to T}([Z_a(g)_t]_T), \tag{3.70}$$

$$[Z_a(g)_t]_P = f_{T\to P}([Z_a(g)_t]_P), \tag{3.71}$$

where $f_{P\to T}()$ is a conversion function from practical to theoretical knowledge, and $f_{T\to P}()$ is a conversion function from theoretical to practical knowledge.

The deviation between theory and practical is also an important quantity considered as a *knowledge gap*.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 24 November 2022 doi:10.20944/preprints202211.0450.v1

Theoretical and practical knowledge gap:

$$D_{P\to T} = [Z_a(g)_t]_P - [Z_a(g)_t]_T, \quad (3.72)$$
$$D_{T\to P} = [Z_a(g)_t]_T - [Z_a(g)_t]_P, \quad (3.73)$$
$$D_{P\to T} = D_{T\to P} \; iff \; [Z_a(g)_t]_P = [Z_a(g)_t]_T. \quad (3.74)$$

where $D_{P\to T}$ is practical to theoretical knowledge deviation, and $D_{T\to P}$ is theoretical to practical knowledge deviation.

The deviation between theoretical and practical actions will lead to incoherence of knowledge about a target. This may not be desirable if the agents are required to collaborate on the target. To reduce such deviation, the deviated knowledge is optimized, through learning, relative to the other. Such type of learning of a relative knowledge will entail both agent and environment action. As we shall discuss in Section 5, such type of environment related learning will require a *semantic learning* approach.

The aspect of theoretical and practical knowledge can be related to the works of *Audi* [22],[23], who distinguished practical reasoning from theoretical reasoning and provided a philosophical structure of reasoning in this context.

Apart from knowledge acquired through actions (abstract, real, domain or specific), there is another classification of knowledge based on the ability of an agent to do action (i.e., actuate) with the knowledge acquired. In the literature [88], these are called, *procedural and declarative knowledge*.

iii. Procedural and Declarative Knowledge

Definition 71. ***Procedural knowledge*** *is knowledge acquired to do real action.*

$$[V_a(g_v)_t] = \alpha_1[V_a(g)_t]_\mu + \alpha_2[V_a(g)_t]_v, \quad (3.75)$$
$$[Z_a(g_v)_t] = \alpha_1[Z_a(g)_t]_\mu + \alpha_2[Z_a(g)_t]_v. \quad (3.76)$$

Definition 72. ***Declarative knowledge*** *is knowledge acquired to do abstract action.*

$$[V_a(g_\mu)_t] = \alpha_1[V_a(g)_t]_\mu + \alpha_2[V_a(g)_t]_v, \quad (3.77)$$
$$[Z_a(g_\mu)_t] = \alpha_1[Z_a(g)_t]_\mu + \alpha_2[Z_a(g)_t]_v, \quad (3.78)$$

where $\alpha_1, \alpha_2 \in [0, 1]$

.

We consider that both the abstract and real knowledge are needed by an agent to take any action. Also, α and β are the proportion of the abstract and real environment involve in generating the procedural and declarative values. This expression is generalized in Theorem 6 as the *law of total value* for action in multiple environments.

Based on Proposition 10, for knowledge to enable an action, it must be converted to intelligence through a reverse action because only intelligence can directly enable an action. Thus, the procedural and declarative knowledge represent the intelligence of an agent to take practical and theoretical actions, respective. They do not represent the knowledge property as conventionally presented [88].

3.3.5. Knowledge Structures

An agent can generate knowledge over many targets for same or different actions. We present three knowledge structures to describe this phenomenon: *knowledge matrix*, *knowledge block*, and *knowledge area*.

i. Knowledge Matrix (KM)

Definition 73. *A* ***Knowledge matrix, KM,*** *is a set of all knowledge values for all actions and targets of an agent.*

$$\mathrm{KM(g)}_{a_i,t_j} = \left\{ Z_{a_i}(g)_{t_j}, \forall a, t \in g \right\}. \quad (3.79)$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 24 November 2022 doi:10.20944/preprints202211.0450.v1

ii. Knowledge Block (KB)

Definition 74. *A **Knowledge block, KB,** is a set of knowledge values for an action on different targets or different actions on a given target.*

$$\mathrm{KB}(g)_{a,t_j} = \left\{ Z_a(g)_{t_j}, \forall t \in g \right\}, \tag{3.80}$$

$$\mathrm{KB}(g)_{a_i,t} = \{ Z_{a_i}(g)_t, \forall a \in g \}. \tag{3.81}$$

This represents a row or column in the knowledge matrix.

iii. Knowledge Area (KA)

Definition 75. *A **Knowledge area, KA,** is a set of targets and actions on which the agent have high knowledge value beyond a certain knowledge limit.*

$$\mathrm{KA}(g)_{a_i,t_j} = \left\{ Z_{a_i}(g)_{t_j}, \forall a, t \in g \wedge Z = c \right\}, \tag{3.82}$$

where c is the limit value.

The knowledge limit, can be the average or maximum knowledge about a target or by an action in an environment. It can also be defined otherwise.

KM, KB, and KA are important features for knowledge *structurization* and *representation* during *memorization*. We shall discuss this in a future publication. More so, other values such as the action value i.e., beliefs, the ignorance value, the understanding value, the trust value, the wisdom value, the attention value, the exactness value, the stability value, etc., of an agent can be structured and represented in the same way, that is, as a matrix, a block, and an area over a set of targets.

3.3.6. Logical Operations on Knowledge

Similar to operations on the action value in Section 3.1.3, we introduce in this section different operations on knowledge, which can also be applied to ignorance. We focus on the *relative endopistemic knowledge value* based on relative truth, but same approach can be applied to exopistmeic value. We define the operations for *action by multi-agents on multi-targets with discrete states*, assuming $\phi_i \perp \phi_j$. Extension to continuous state can be done using integral calculus.

Definition 76. ***Self knowledge** is the knowledge value of an agent about a target based on a self action.*

$$Z_a(g)_t = \sum_{i=1}^{n} A(e)_{t_i} V_a(g)_{t_i}, \tag{3.83}$$

$$V_a(g)_{t_i} = J\left[\frac{A(g)_{t_i}}{A(e)_{t_i}}\right] = J[A(g||e)_{t_i}] = J[a(g)_{t_i}], \tag{3.84}$$

where i is state of the target.

Definition 77. ***Joint knowledge** is the knowledge value of agents about targets based on their joint actions.*

1) Multiple agents with same target and action

$$Z_a(g_1, g_2)_t = \sum_{i=1}^{n} A(e_1, e_2)_{t_i} V_a(g_1, g_2)_{t_i}, \phi_1 \perp \phi_2, \tag{3.85}$$

$$V_a(g_1, g_2)_{t_i} = J[a(\phi_1, \phi_2)], \phi_1 \perp \phi_2, \tag{3.86}$$

$$J[a(\phi_1, \phi_2)] = J\left[\frac{A(g_1, g_2)_{t_i}}{A(e_1, e_2)_{t_i}}\right] = V_a(g_1)_{t_i} + V_a(g_2)_{t_i}. \tag{3.87}$$

 doi:10.20944/preprints202211.0450.v1

Proposition 12. *The joint knowledge value of independent collaborative agents to an action on a target is the sum of their individual knowledge values.*

2) Multiple targets by same agent and action

$$Z_a(g)_{t_1,t_2} = \sum_{t_1,t_2} A(e)_{t_1,t_2} V_a(g)_{t_1,t_2}, \neg(t_1 \perp t_2), \tag{3.88}$$

$$V_a(g)_{t_1,t_2} = J\left[\frac{A(g)_{t_1,t_2}}{A(e)_{t_1,t_2}}\right], \neg(t_1 \perp t_2), \tag{3.89}$$

$$J[\frac{A(g)_{t_1,t_2}}{A(e)_{t_1,t_2}}] = V_a(g)_{t_1} + V_a(g)_{t_2} + V_a(g)_{t_1;t_2}. \tag{3.90}$$

3) Multiple actions of same agent and target

$$Z_{a_1,a_2}(g)_t = \sum_{i=1}^{n} A_{12}(e)_{t_i} V_{a_1,a_2}(g)_{t_i}, \neg(a_1 \perp a_2), \tag{3.91}$$

$$V_{a_1,a_2}(g)_{t_i} = J[a_1(\phi), a_2(\phi)], \neg(a_1 \perp a_2). \tag{3.92}$$

$$J[a_1(\phi), a_2(\phi)] = V_{a_1}(g)_{t_i} + V_{a_2}(g)_{t_i} + V_{a_1;a_2}(g)_{t_i}. \tag{3.93}$$

4) Multiple agents with multiple actions and targets

$$Z_{a_1,a_2}(g_1, g_2)_{t_1,t_2} = \sum_{t_1,t_2} A(e_1, e_2)_{t_1,t_2} V_a(g_1, g_2)_{t_1,t_2}, \tag{3.94}$$

$$\begin{aligned} V_{a_1 a_2}(g_1, g_2)_{t_1,t_2} &= J[a_1(\phi_1, \phi_2)_{t_1,t_2}, a_2(\phi_1, \phi_2)_{t_1 t_2}] \\ &= V_{a_1}(g_1, g_2)_{t_1,t_2} + V_{a_2}(g_1, g_2)_{t_1,t_2} + V_{a_1;a_2}(g_1, g_2)_{t_1,t_2}, \end{aligned} \tag{3.95}$$

where $A_{12}(e)$ is joint action of state 1 and 2 of agent e, $A(e_1, e_2)$ is action enable by intelligence of agent e_1 and e_2.

Definition 78. ***Mutual knowledge*** *is the knowledge value of agents about targets based on their mutual actions.*

1) Multiple agents with same target and action

$$Z_a(g_1; g_2)_{t_i} = \sum_{i=1}^{n} A(e_1, e_2)_{t_i} V_a(g_1; g_2)_{t_i}, \phi_1 \perp \phi_2, \tag{3.96}$$

$$V_a(g_1; g_2)_{t_i} = J[a(\phi_1; \phi_2)_{t_i}], \phi_1 \perp \phi_2, \tag{3.97}$$

$$J[a(\phi_1; \phi_2)_{t_i}] = J[\frac{a(g_1, g_2)}{a(g_1)a(g_2)}] = J[\frac{a(g_1)a(g_2)}{a(g_1)a(g_2)}] = 0. \tag{3.98}$$

Proposition 13. *Independent collaborative agents to an action on a target have no mutual knowledge on the target.*

2) Multiple targets by same agent and action

$$Z_a(g)_{t_1;t_2} = \sum_{t_1,t_2} A(e)_{t_1,t_2} V_a(g)_{t_1;t_2}, \neg(t_1 \perp t_2), \tag{3.99}$$

$$V_a(g)_{t_1;t_2} = J[\frac{a(g)_{t_1,t_2}}{a(g)_{t_1} a(g)_{t_2}}], \neg(t_1 \perp t_2), \tag{3.100}$$

$$J[\frac{a(g)_{t_1,t_2}}{a(g)_{t_1} a(g)_{t_2}}] = V_a(g)_{t_1,t_2} - V_a(g)_{t_1} - V_a(g)_{t_2}. \tag{3.101}$$

3) Multiple actions of same agent and target

$$Z_{a_1;a_2}(g)_t = \sum_{i=1}^{n} A_{12}(e)_{t_i} V_{a_1;a_2}(g)_{t_i}, \neg(a_1 \perp a_2), \tag{3.102}$$

$$V_{a_1;a_2}(g)_t = J[\frac{a_{12}(g)_{t_i}}{a_1(g)_t a_2(g)_{t_i}}], \neg(a_1 \perp a_2), \tag{3.103}$$

$$J[\frac{a_{12}(g)_{t_i}}{a_1(g)_{t_i} a_2(g)_{t_i}}] = V_{a_1,a_2}(g)_{t_i} - V_{a_1}(g)_{t_i} - V_{a_2}(g)_{t_i}. \tag{3.104}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 24 November 2022 doi:10.20944/preprints202211.0450.v1

4) Multiple agents with multiple actions and targets

$$Z_{a_1;a_1}(g_1;g_2)_{t_1;t_2} = \sum_{t_1,t_2} A(e_1,g_2)_{t_1;t_2} f(a_1,a_2,t_1,t_2), \tag{3.105}$$

$$f(a_1,a_2,t_1,t_2) = V_{a_1;a_2}(g_1;g_2)_{t_1;t_2} = V_{a_1,a_2}(g_1;g_2)_{t_1;t_2}, \\ - V_{a_1}(g_1;g_2)_{t_1;t_2} - V_{a_2}(g_1;g_2)_{t_1;t_2}. \tag{3.106}$$

Definition 79. ***Conditional knowledge*** *is knowledge of agents about targets based on their conditional actions.* 1026 1027

1) Multiple agents with same target and action

$$Z_a(g_1|g_2)_t = \sum_{i=1}^{n} A(e_1,e_2)_{t_i} V_a(g_1|g_2)_{t_i}, \phi_1 \perp \phi_2, \tag{3.107}$$

$$V_a(g_1|g_2)_{t_i} = J[a(\phi_1|\phi_2)_{t_i}] = J[a(\phi_1)_{t_i}] = V_a(g_1)_{t_i}. \tag{3.108}$$

Proposition 14. *The conditional knowledge value of independent collaborative agents on a target is the knowledge value of the conditioned agent.* 1028 1029

2) Multiple targets by same agent and action

$$Z_a(g)_{t_1|t_2} = \sum_{t_1,t_2} A(e)_{t_1,t_2} V_a(g)_{t_1|t_2}, \neg(t_1 \perp t_2), \tag{3.109}$$

$$V_a(g)_{t_1|t_2} = J\left[\frac{a(g)_{t_1,t_2}}{a(g)_{t_2}}\right] = V_a(g)_{t_1,t_2} - V_a(g)_{t_2}, \tag{3.110}$$

$$J\left[\frac{a(g)_{t_1,t_2}}{a(g)_{t_2}}\right] = V_a(g)_{t_1} + V_a(g)_{t_1;t_2}, \neg(t_1 \perp t_2). \tag{3.111}$$

3) Multiple actions of same agent and target

$$Z_{a_1|a_2}(g)_t = \sum_{i=1}^{i=n} A_{12}(e)_{t_i} V_{a_1|a_2}(g)_{t_i}, \neg(a_1 \perp a_2), \tag{3.112}$$

$$V_{a_1|a_2}(g)_{t_i} = J\left[\frac{a_{12}(g)_{t_i}}{a_2(g)_{t_i}}\right] = V_{a_1,a_2}(g)_{t_i} - V_{a_2}(g)_{t_i}, \tag{3.113}$$

$$J\left[\frac{a_{12}(g)_{t_i}}{a_2(g)_{t_i}}\right] = V_{a_1}(g)_t + V_{a_1;a_2}(g)_{t_i}, \neg(a_1 \perp a_2). \tag{3.114}$$

4) Multiple agents with multiple actions and targets

$$Z_{a_1|a_2}(g_1|g_2)_{t_1|t_2} = \sum_{t_1,t_2} A(e_1,e_2)_{t_1|t_2} f(a_1,a_2,t_1,t_2), \tag{3.115}$$

$$f(a_1,a_2,t_1,t_2) = V_{a_1|a_2}(g_1;g_2)_{t_1|t_2}, \tag{3.116}$$

$$f(a_1,a_2,t_1,t_2) = V_{a_1,a_2}(g_1|g_2)_{t_1|t_2} - V_{a_2}(g_1|g_2)_{t_1|t_2}. \tag{3.117}$$

3.3.7. Properties of the Knowledge Value 1030

As we mentioned in Section 2, cognitive value also have properties that define their nature. These properties can also be considered as separate cognitive values. One of such property is the inverse property of a cognitive value, which for the knowledge property, we consider it as the ignorance of an agent. Other properties of a cognitive value introduce in this article include: *exactness, stability, energy, relativistic, and diffusion*. 1031 1032 1033 1034 1035

i. Exactness property 1036

Definition 80. *The* ***exactness*** *of a cognitive property is the precision of its value about a target.*

$$E_v(g) = \frac{V_a(g)}{V_a(e)}, \tag{3.118}$$

$$E_{Z_v}(g) = \frac{Z_a(g)}{Z_a(e)}. \tag{3.119}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 24 November 2022 doi:10.20944/preprints202211.0450.v1

This is the exactness property of the agent's knowledge over that of the environment. In Section 4.1, the environment value is expressed as a sum of the knowledge and ignorance value. This transformed the exactness value as follows:

$$E_v(g)_c = \frac{V_a(g)}{V_a(e)} = \frac{V_a(g)}{V_a(g) + \bar{V}_a(g) - V\bar{V}_a(g)}, \tag{3.120}$$

$$E_{Z_v}(g)_c = \frac{Z_a(g)}{Z_a(e)} = \frac{Z_a(g)}{Z_a(g) + \bar{Z}_a(g) - Z\bar{Z}_a(g)}. \tag{3.121}$$

Furthermore, the exactness property of the agent's knowledge or ignorance can also be expressed over the cost value of the agent in an environment as follows.

$$E_v(g)_{cr} = \frac{V_a(g)}{V_{(cr)}a(g)}, \tag{3.122}$$

$$E_{Z_v}(g)_{cr} = \frac{Z_a(g)}{Z_{(cr)}a(g)}. \tag{3.123}$$

More of the cost value is discussed in Section 4.2. The exactness of knowledge over the cost value will be used in Section 4.4 to estimate the *knowledge transformation factor* (KTF) during the knowledge transformation process between different environments.

An important aspect of the exactness value is the fact that it can be expressed as a gradient of a line.

$$Y = mX + C, \tag{3.124}$$

where Y is a knowledge value of the agent, X is value of the environment or the cost of the agent, m is the exactness value on environment or cost, and C is a constant of exactness in cognition of an agent g about a target t given intelligence ϕ.

$$V_a(g) = E_v(g)_c + C_v, \;\; Z_a(g) = E_{Z_v}(g) + C_z, \tag{3.125}$$

$$V_a(g) = E_v(g)_{cr} + C_v, \;\; Z_a(g) = E_{Z_v}(g)_{cr} + C_z. \tag{3.126}$$

Thus, Equation (3.124) gives a linear relationship between knowledge of an agent and its cost or knowledge of a referenced agents. This can be used as a cognitive tool for a knowledge value acquisition and optimization over a single state or multi-state target.

Furthermore, on a single state target, the exactness value can be considered as log-log ratio, hence, the action values can be extracted at any moment in a cognitive process or projected for a future value defined by the *exactness linear equation*.

$$E_v(g) = \frac{V_g}{V_e} = \log_{\frac{1}{p}} \frac{q}{p} \Rightarrow \frac{q}{p} = (\frac{1}{p})^{E_v(g)}, \tag{3.127}$$

$$E_v(g) = \frac{V_g}{V_cr} = \log_{\frac{1}{q}} \frac{q}{p} \Rightarrow \frac{q}{p} = (\frac{1}{q})^{E_v(g)}, \tag{3.128}$$

where $V_g = \log \frac{q}{p}$, $V_e = \log \frac{1}{p}$, and $V_{cr} = \log \frac{1}{q}$.

We shall discuss more about exactness of knowledge and other cognitive properties in a future publication.

ii. Stability property

Definition 81. *The* ***stability*** *of a cognitive property is the rate of change of its value with respect to space or time.*

Stability can be expressed with respect to time such as the learning time (defined in Section 5.2), or with respect to space such as the action space, intelligence space, target space, information space, and cognitive value space.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 24 November 2022 doi:10.20944/preprints202211.0450.v1

1) Stability with respect to action:

$$S_V(g)_{a(e)} = \frac{\partial V_a(g)}{\partial A(e)}, \tag{3.129}$$

$$\Rightarrow S_K(g)_{a(e)} = \frac{1}{A(e)}, \;\; S_I(g)_{a(e)} = -\frac{1}{A(e)}, \tag{3.130}$$

$$S_{Z_v}(g)_{a(g)} = \frac{\partial Z_a(g)}{\partial A(g)}, \tag{3.131}$$

$$S_{Z_K}(g)_{a(g)} = \frac{\partial Z_K(g)}{\partial A(g)}, \;\; S_{Z_I}(g)_{a(g)} = \frac{\partial Z_I(g)}{\partial A(g)}, \tag{3.132}$$

where $V = \{K, I\}$, $K_a(g) = \log \frac{A(g)}{A(e)}$, $I_a(g) = \log \frac{A(e)}{A(g)}$.

2) Stability with respect to information:

$$S_V(g)_X = \frac{\partial V_a(g)}{\partial X} = \frac{\partial \log(A(g))}{\partial X} - \frac{\partial \log(A(e))}{\partial X}, \tag{3.133}$$

$$S_{Z_v}(g)_X = \frac{\partial Z_a(g)}{\partial X}, \tag{3.134}$$

$$\frac{\partial Z_a(g)}{\partial X} = \frac{\partial \sum A(e) \log(A(g))}{\partial X} - \frac{\partial \sum A(e) \log(A(e))}{\partial X}, \tag{3.135}$$

where $A(.)_t = f(X, \phi)$, $V_a(g)$, and $Z_a(g) = f(A(g), A(e))$.

The stability with respect to information defines the *variance* of the agent. The information change can be; the change in input domain or input instances about the target.

For the stability with respect to the target, the change in target can be the change in the target instance or domain. This is considered as the *variance over the output space* of the target. We shall handle this case in a future publication. Also, the case with the stability of one cognitive value property over another type of cognitive value property is treated in a future publication.

We shall discuss more on the wave property of cognition and its operations such as modulation in future articles.

In Section 5, we shall present and apply the stability of cognitive values with respect to intelligence in a gradient based value optimization. In Section 4, we shall discuss the *relativistic and energy properties* of cognitive value during cognition and apply it through the research. The *energy* property in this case can also be considered as an *economic value* to and of the agent.

Concerning the *diffusion property* of cognitive value, we shall present it in Section 6, since it is associated with value transfer. However, the description of the transfer of this value, packaged in wave, through cognitive vibrations is a property of the transfer agent and not the value, this shall be presented in a future publication.

4. Value Acquisition

Value acquisition is the process through which an agent generates value about a target. Once generated, this value can be optimized and transferred as explained in Section 5 and Section 6, respectively. The value generation process may involve many actions, such as observation and reasoning for an observable intelligent agent (most conventional intelligent and cognitive agents). Furthermore, the amount of value acquired can be influence by other factors such as the environment.

In this section, we introduce environmental factors that can affect the acquisition of relative knowledge defined in the previous section. All the types of relative knowledge defined and quantified in Section 3 can be affected by the following environmental factors; uncertainty of the environment and the divergence of environments.

To understand the influence of the environment on value acquisition, we first present the process of *complete knowledge* acquisition and the *cost of knowledge acquisition*, then define a *law of conservation of value* in cognition.

These preliminary concepts are important in this research as they lead to a new discovery in agent development where the knowledge value quantity of an agent is proven to be equivalent in nature to the energy quantity in conventional physical systems. This theoretical findings can lead to a new direction in intelligent agent development, and the merging of cognitive science with energy science. As we shall show, a cognitive process is just a type of energy process.

4.1. The Complete Knowledge Phenomenon

As discussed in Section 3.3, the maximum value an agent can acquire about a target is considered as its *complete value*. Such completeness can also be viewed from the influencer's perspective i.e., based on the variation of its values. In this section, we present more about the complete endopistemic and exopisemic values of an agent, which we also considered to be the *wisdom values* of an agent about a target.

4.1.1. Endopistemic Completeness

During an endopistemic cognition, the key issue is how the agent needs to vary its endopistemic value to get closer to or away from the value of its influencer (environment) on the target. In an endopistemic process, since the agent seeks to depend rather than influence, the agent can only vary its own values; its absolute or relative endopistemic values.

To reach the value of its influencer on the target, the agent is required to maximize its absolute and relative values. Both values are considered as the *dynamic values* of an agent in an endopistemic process. This can also be seen as two types of *dynamic energy or forces* of an agent that keeps it in *cognitive motion* (i.e., optimization) as discussed in Section 5.

But since the absolute value is used to obtain the relative value as shown in Equations (3.31) and (3.35), the agent can focus just on the maximization of its absolute value on the target. This absolute value represents the amount of *work or cost* required by the agent to vary its relative value toward or away from the environment value of the target.

From the expression of absolute values in Equations (3.31) and (3.35), the variation of absolute value is expressed as:

$$\max[K(g \to t)] = \max[\log(A(g))], \tag{4.1}$$

but, $\log(A(g)) = -\log(\frac{1}{A(g)}) = -K(t \to g)$,

$$\Rightarrow \max[K(g \to t)] = \min[K(t \to g)]. \tag{4.2}$$

Also, using cognitropy of knowledge,

$$\max[Z(g \to t)] = \max[\sum_{i=1}^{i=n} A(e)_{t_i} \log(A(g)_{t_i})], \tag{4.3}$$

but, $\sum_{i=1}^{i=n} A(e)_{t_i} \log(A(g)) = -\sum_{i=1}^{i=n} A(e)_{t_i} \log(\frac{1}{A(g)}))$,

$$\Rightarrow \max[Z(g \to t)] = \min[Z(t \to g)]. \tag{4.4}$$

This implies, the maximization of the endopistemic value is the minimization of its endopistemic equivalence. Hence, we have the following proposition.

Proposition 15. *Maximizing a dependency of an agent about a target implies minimizing its influence on the target and vice versa.*

Also, since the influencer referenced by a dependent agent defines the value seeks by the agent on the target, then,

Axiom 4.1. *The maximum dependency value an agent can reach about a target in an endopistemic process must be equal in magnitude to the value of its influence(s) on the target.*

We call this maximum value the *complete endopistemic value* of the agent about the target.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 24 November 2022 doi:10.20944/preprints202211.0450.v1

Using the absolute value as a cost value for dependency measure, the state of complete value can be expressed as,

$$\omega_c(g \to t) = \omega(e \to t), \tag{4.5}$$

where ω_c is the complete value of a property value.

This implies that during an endopistemic process, the absolute value of the dependent agent is maximum when it is equal to the absolute value of the influencer.

Using Equation (3.35), at complete action value,

$$A(g \to t) = A(e)_t \;\Rightarrow A((g \to t)||e) = 1\,. \tag{4.6}$$

Using Equation (3.31), at complete knowledge,

$$K_c(g \to t) = \log(A(e)_t) \;\Rightarrow K((g \to t)||e) = 0\,, \tag{4.7}$$

$$Z_c(g \to t) = \sum_{i=1}^{n} \log(A(e)_{t_i}) \;\Rightarrow Z((g \to t)||e) = 0. \tag{4.8}$$

This implies that at maximum absolute value, the relative action is 1 and the relative knowledge is zero.

In reality, other variants of completeness can also be defined for an agent, apart from that based on its absolute value. For example, consider a complete value based on the dynamism of the relative value, i.e., the relative value being used as the cost for dependency measure, then,

$$\omega_c((g \to t)||e) = \omega(e \to t)\,, \tag{4.9}$$

where ω_c is the complete value of a property value.

Then, at complete relative action value,

$$A((g \to t)||e) = A(e)_t \;\Rightarrow A(g \to t) = A^2(e)_t\,. \tag{4.10}$$

Then, at complete relative knowledge,

$$K_c((g \to t)||e) = \log A(e)_t \Rightarrow K(g \to t) = \log A^2(e)_t, \tag{4.11}$$

$$Z_c((g \to t)||e) = \sum_{i=1}^{n} \log A(e)_{t_i} \Rightarrow Z(g \to t) = 2\sum_{i=1}^{n} \log A(e)_{t_i}. \tag{4.12}$$

So, based on the definition of the complete value of an agent with respect to the absolute value of the influencer, different relative and absolute values of the agent arise at the state of completeness. Conventionally, most agents are designed and trained with a *complete value model* based on a cost defined using the absolute values, where the relative value is expected to be zero. We shall focus on this model in this research.

4.1.2. Exopistemic Completeness

Apart from the dependency evaluation during an endopistemic process, the influence on agents during such process can be evaluated using a similar manner based on the variation of the influencer's action. In this case, the total influence and the work to sustain such influence are much more important.

The total influence on a set of agent about a target is the maximum they can reach about the target, and this can be proven to be equal to the sum of their respective dependencies on the influencer as proposed below.

Theorem 1. *During cognition on a target, the influence of the environment about the target is equal to the logical sum of its dependent values about the target.*

For a single action with only knowledge and ignorance:

$$V_a(g)_t + \bar{V}_a(g)_t - V\bar{V}_a(g)_t = V_a(e)_t (= \bar{V}_a(e)_t), \tag{4.13}$$

$$Z_a(g)_t + \bar{Z}_a(g)_t - Z\bar{Z}_a(g)_t = Z_a(e)_t (= \bar{Z}_a(e)_t), \tag{4.14}$$

where $V\bar{V}_a(g)_t$ and $Z\bar{Z}_a(g)_t$ are conjunction values,
$V\bar{V}_a(g)_t = J[\frac{A(g)_t \bar{A}(g)_t}{A(e)_t}]$, $Z\bar{Z}_a(g)_t = \sum_i A(e)_{t_i} V\bar{V}_a(g)_{t_i}$.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 24 November 2022 doi:10.20944/preprints202211.0450.v1

For knowledge or ignorance of multiple agents or actions:

$$V_1(g)_t + V_2(g)_t - V_1V_2(g)_t = V(e)_t, \tag{4.15}$$

$$Z_1(g)_t + Z_2(g)_t - Z_1Z_2(g)_t = Z(e)_t, \tag{4.16}$$

where $V_1V_2(g)_t$ and $Z_1Z_2(g)_t$ are conjunction values,
$V_1V_2(g)_t = J[\frac{A_1(g)_t A_2(g)_t}{A(e)_t}]$, $Z_1Z_2(g)_t = \sum_i A(e)_{t_i} V_1V_2(g)_{t_i}$.

We consider this theorem as *the law of complete cost*. We provide a generalization of this law in Section 4.5.1.

Furthermore, for a single action, the complete knowledge and ignorance are the same. This implies that,

Proposition 16. *An agent cannot distinguish between a value and its inverse of an environment on a target because the limits of the values are the same.*

$$V_{(c)a}(g)_t = \bar{V}_{(c)a}(g)_t, \tag{4.17}$$

$$Z_{(c)a}(g)_t = \bar{Z}_{(c)a}(g)_t. \tag{4.18}$$

Hence, the agent cannot distinguish between the complete knowledge and ignorance of an environment. Together with Proposition 6, this leads us to the following assumption;

Axiom 4.2. *An agent cannot act outside its environment except it has an ability that permits it to do so or it is elevated to that level through a value operation.*

Using the endopistemic relative knowledge value, the cost, $C_a(g)_t$, required by the influencer to maintain its influence on the dependent entities over the target is given below.

$$C_a(g)_t = \log A(g), \text{ for } V_a(g)_t = \log \frac{A(g)}{A(e)}, \tag{4.19}$$

$$C_a(g)_t = \sum_{i=1}^{n} A(e)_{t_i} \log A(g)_{t_i}, Z(g)_t = \sum_{i=1}^{n} A(e)_{t_i} \log \frac{A(g)_{t_i}}{A(e)_{t_i}}, \tag{4.20}$$

where $V_a(g)_t = \log \frac{1}{A(e)} + \log A(g)$, and
$Z(g)_t = \sum_{i=1}^{i=n} A(e)_{t_i} \log \frac{1}{A(e)_{t_i}} + \sum_{i=1}^{n} A(e)_{t_i} \log A(g)_{t_i}$.

Maximizing this cost value with respect to the influencer's action for multi-state target will maximize the total influence, thereby increasing the complete value of the dependent agents. This implies, dependent agents will need to do more work to reach complete value. Minimizing the cost from the influencer will produce an opposite effect. But for single state target, maximization or minimization of the cost with respect to the influencer is not possible and the relative knowledge can only be influenced directly by the influencer's action.

$$\max[\log(A(g))]_e = 0, \tag{4.21}$$

$$\max[\sum_{i=1}^{n} A(e)_{t_i} \log A(g)_{t_i}]_e = [\sum_{i=1}^{n} \log A(g)_{t_i}]_e. \tag{4.22}$$

In this situation, the minimum influence on the dependent entities during endopistemic process occurs when the *total value* is equal to the cost value, or when the influencer action equals to the dependency actions, i.e., influence value equals dependency value.

$$\min[\log(A(g))]_e \Leftrightarrow A(g) = A(e), \tag{4.23}$$

$$\min[\sum_{i=1}^{n} A(e)_{t_i} \log A(g)_{t_i}]_e \Leftrightarrow A(g) = A(e). \tag{4.24}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 24 November 2022 doi:10.20944/preprints202211.0450.v1

Same process can be interpreted using an *exopistemic value*, i.e., from the influencer's point of view of the dependent agents, where the exopistemic cost, $C_a(g)_t$, for the relative knowledge is given as follows:

$$C_a(g)_t = -\log A(g),\ \text{for}\ \ V_a(g)_t, \tag{4.25}$$

$$C_a(g)_t = -\sum_{i=1}^{i=n} A(e)_{t_i} \log A(g)_{t_i},\ \text{for}\ Z(g)_t, \tag{4.26}$$

where $V_a(g)_t = \log \frac{1}{A(g)} + \log A(e)$, and
$Z(g)_t = \sum_{i=1}^{i=n} A(e)_{t_i} \log \frac{1}{A(g)_{t_i}} + \sum_{i=1}^{i=n} A(e)_{t_i} \log Ae)_{t_i}$.

The values $\log(A(e))$ and $\sum_{i=1}^{i=n} A(e)_{t_i} \log A(e)_{t_i}$ are the *total value* of dependencies under the force of the influencer.

Depending on the type of action taken by an agent on a target, the ability of the agent to reach complete value about the target is described as its *reachability*. The reachability of the observation action is considered as *observability*, while that of an actuation action such as control is considered as *actuatability*. Others include, *learnability*, *controllability*, etc. More about *reachability* measure and analysis of intelligent agents on a target will be presented in a future article.

Both endopistemic and exopistemic complete values are important in cognition, and together with their cost values, they can be used in building an effective *cognitive balancing* process of an agent. More on the cost value is discussed in the next section.

4.2. The Cost of Value Acquisition

In Section 4.1, we presented the complete value of an agent about a target in both endopistemic and exopistemic cognition. The cost required by the agent to acquire values was also considered. In this section, we discuss more about the cost values associated to these two processes.

4.2.1. Endopistemic Cost

The cost value of the dependent agent for a relative endopistemic knowledge value can be expressed in terms of the absolute values of the influencing and dependent agents about the target as follows:

$$C_a(g)_t = V_a(e)_t + (-V_a(g)_t), \tag{4.27}$$

$$C_a(g)_t = Z_a(e)_t + (-Z_a(g)_t). \tag{4.28}$$

Using Theorem 1 *(law of complete cost)*, we can express the cost in terms of the ignorance and conjunction values.

$$C_a(g)_t = \bar{Z}_a(g)_t - Z\bar{Z}_a(g)_t. \tag{4.29}$$

This also implies the cost of ignorance can be expressed as a relationship between knowledge and the conjunction value.

$$\bar{C}_a(g)_t = Z_a(e)_t + (-\bar{Z}_a(g)_t), \tag{4.30}$$

$$\Rightarrow \bar{C}_a(g)_t = Z_a(g)_t - Z\bar{Z}_a(g)_t. \tag{4.31}$$

The agent varies this cost in order to vary its knowledge value and to reach complete value about the target. The cost required to reach such complete value is considered as the *complete cost value*.

In Section 4.1, we discussed that at complete value, the relative endopistemic value is zero. Using Equation (4.27) or (4.28) above, it implies that at complete value, the cost is equal to the influencer value.

$$C_a(g)_t = V_a(e)_t,\ \text{if}\ (-V_a(g)_t) = 0, \tag{4.32}$$

$$C_a(g)_t = Z_a(e)_t,\ \text{if}\ (-Z_a(g)_t) = 0. \tag{4.33}$$

This is also similar for the cost of ignorance.

$$\bar{C}_a(g)_t = Z_a(e)_t,\ \text{if}\ (-\bar{Z}_a(g)_t) = 0. \tag{4.34}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 24 November 2022 doi:10.20944/preprints202211.0450.v1

Thus, the complete cost of an agent is the same for both ignorance and knowledge acquisition. This also confirms Proposition 16. In fact, the complete cost of an agent can be considered to the agent as the *semantic value* [89] of its *referential influencer* (i.e., its environment) on a target.

Also, given that the knowledge influence is zero, then the cost of an agent will be equal to its relative value. This implies that, in a relative cognition, the agent acts as if it is in a non-referential cognition. We consider an environment with a zero knowledge influence as a *vacuum*.

Definition 82. *A **vacuum** is an environment with an optimum value about a target.*

$$C_a(g)_t = (-V_a(g)_t), \text{ if } V_a(e)_t = 0 \Rightarrow A(e) = 1, \quad (4.35)$$
$$C_a(g)_t = (-Z_a(g)_t), \text{ if } Z_a(e)_t = 0 \Rightarrow A(e) = 1. \quad (4.36)$$

Every action and knowledge acquired by the agent in such environment depends only on the agent and not the environment, because the environment has the optimum action and knowledge value already available for the agent to acquire.

This is also true for the cost of ignorance in a knowledge based environment, where less uncertainty is desired.

$$\bar{C}_a(g)_t = (-\bar{Z}_a(g)_t), \text{ if } Z_a(e)_t = 0. \Rightarrow A(e) = 1. \quad (4.37)$$

But in an ignorance based environment, where the ignorance value of the environment is referenced, and the optimum influence is when there is maximum uncertainty, then such a vacuum environment only occurs when $A(e) = 0$.

$$C_a(g)_t = (-V_a(g)_t), \text{ if } V_a(e)_t = \infty \Rightarrow A(e) = 0, \quad (4.38)$$
$$C_a(g)_t = (-Z_a(g)_t), \text{ if } Z_a(e)_t = 0 \Rightarrow A(e) = 0. \quad (4.39)$$

If we assume $0 \log \infty = 0$, $\log \infty = \infty$, $x/0 = \infty$. where ∞ and 0 are optimum environment values for a single state and multi-state target, respectively. More on this is discussed in Section 4.5.

Being the *cost of wisdom* on a target, the complete cost can also be used to represent the wisdom of the agent. In this context, the more targets in an environment an agent gains wisdom on, the more wisdom the agent have about the entire environment. In Section 4, this environment is distinguished into two types; *isolated and non-isolated environments*.

Proposition 17. *The complete cost an agent can acquire about a non-isolated environment is the sum of the complete cost of the domain action of all targets of the environment.*

$$C_{(c)a}(g)_e = \sum_{i=1}^{n} C_{(c)a}(g)_{t_i}, e = \{t_1, t_2, ..t_n\}. \quad (4.40)$$

Proposition 18. *The complete cost about a non-isolated environment is the sum of the complete cost of the domain action of all targets and dependencies of the environment.*

$$C_{(c)a}(g)_e = \sum_{i=1}^{n} C_{(c)a}(g)_{t_i} + \sum_{j=1}^{m} Z_{(c)a}(g)_{d_j}, \quad (4.41)$$

$e = \{t_1, t_2, .., t_n, d_1, d_2, .., d_m\}$, t is target, d is dependency.

4.2.2. Exopistemic Cost

This is the cost an influencer varies and maintains its influence on the dependent agents. As explained in Section 4.1, using the endopistemic value, we can express it as:

$$C_a(e)_t = V_a(e)_t + (-V_a(g)_t), \quad (4.42)$$
$$C_a(e)_t = Z_a(e)_t + (-Z_a(g)_t). \quad (4.43)$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 24 November 2022 doi:10.20944/preprints202211.0450.v1

Apart from the endopistemic value, we can also represent this cost directly using the exopistemic value as follows:

$$C_a(e)_t = (-V_a(e)_t) + V_a(g)_t, \tag{4.44}$$

$$C_a(e)_t = (-Z_a(e)_t) + Z_a(g)_t. \tag{4.45}$$

At complete influence, i.e., complete exopistemic value, the cost of influence is equal to the total dependency value.

$$C_a(e)_t = Z_a(e)_t, \text{ if } (-Z_a(g)_t) = 0 \text{ (endopistemic)}, \tag{4.46}$$

$$C_a(e)_t = (-Z_a(e)_t), \text{ if } Z_a(g)_t = 0 \text{ (exopitsemic)}, \tag{4.47}$$

where, $Z_a(g)_t$ is influence value as discussed in Section 4.1.

In general, regarding endopistemic and exopistemic cost, since cognitive value acquisition depends on both action (A) and intelligence (ϕ) abilities of an agent, this implies that the higher the cost value, the more cognitive ability and intelligence is require by an agent. In fact, given an input information (X), the cost defines the cognition "engine" requirement.

$$C_a(Z_c(X), Z_c(Y)) = Z_c(Y) + (-Z_a(Z_c(Y), C_a)), \tag{4.48}$$

$$C_a(Z_c(X), Z_c(Y)) = (-Z_c(Y)) + Z_a(Z_c(Y), C_a), \tag{4.49}$$

where $Z_c(X)$ and $Z_c(Y)$ are input and output uncertainties.

4.3. The Law of Conservation of Value

For a fixed environment value on a target, the cognitive value and cost value are antagonistic. In an open environment, the environment values about targets can change due to external (or remote) influences, which will change the complete value limits on the targets but not the antagonistic nature of agent's value and cost about the target. This implies, the value and cost of an agent about a target are conserved with respect to that of the environment, as generalized below:

Theorem 2. *In any environment, the value and cost of an agent about a target is conserved over time and space with respect to the value of the environment on the target.*

We consider this theorem as the *law of conservation of value*, represented by the following equation of variation.

$$\Delta_e Z_a(e)_t = \Delta_{g,e} Z_a(g)_t + \Delta_{g,e} C_a(g)_t \text{ (knowledge)}, \tag{4.50}$$

$$\Delta_e Z_a(e)_t = \Delta_{g,e} \bar{Z}_a(g)_t + \Delta_{g,e} \bar{C}_a(g)_t \text{ (ignorance)}. \tag{4.51}$$

In other words, we can say that the *value about a target is neither created nor destroyed but being transferred relatively from one environment and agent to another*. This is true for both exopistemic and endopistemic cognition as their difference is just a *sign change* in front of their values.

This implies that cognitive values have an energy-like nature according to this research. It is also easy to see that they can be used to represent economic related concepts such as profit(or loss), capital optimization, and diminishing returns for an economic valuation to and of an entity.

Furthermore, if an environment has the potential to define the value on a target, then this environment possesses cognitive abilities and can be considered as an agent according to Definition 16. We can firmly assume that;

Axiom 4.3. *The value on a target is defined by an agent.*

Similarly, all agents contained in an environment are under its influence (referential) and will definitely end up in a complete environment influence if they do not take any action on the environment to overcome its influence on them. This is because unlike targets, agents are equipped with such ability to influence their environments (e.g abstract and real environments) through action. Well, the issue is not only to take action but to take the right action that will generate high values. We summarize this in the following statement.

Theorem 3. *An actionless agent have a natural tendency to move toward the limit of its environment influence, unless it takes a value optimization action on the environment.*

$$V(g)_t = \log \frac{A(g)_t}{A(e)_t},\ Z(g)_t = \sum_{i=1}^{i=n} A(e)_{t_i} \log \frac{A(g)_{t_i}}{A(e)_{t_i}}, \tag{4.52}$$

$$\text{if } A(g)_t = \text{const}, A(e)_t = \neg\,\text{const},$$

$$\Rightarrow V(g)_t, Z(g)_t = \neg\,\text{const}. \tag{4.53}$$

For example, if an agent does not seek to influence (e.g through exopistemic learning) its environment about a target, its environment will influence (e.g via exopistemic learning) the agent about the target because cognition is a battle of influences. So, an agent which does not learn, consequently, unlearns. Such agent will end-up with an infinitely small knowledge and a complete ignorance with respect to its environment about the target.

In a situation with *ubiquitous target and non-ubiquitous agent*, if the environment of the agent is less reliable about the target than a remote environment, then the agent will need to take into account the deviation between its environment and that of the remote environment in order to acquire a reliable complete value about the target. For *ubiquitous agents*, as we shall show in a future publication, this may not be necessary as it can possess cognitive instances in multiple environments.

In the next sections, we present different ways the environment influences cognition and define the complete value require by an agent in each case.

4.4. Environment Divergence

Cognitive properties can be used as a valuation measure of the relationship between entities. Based on Proposition 7, such valuation of a relationship between entities can be expressed as a vector algebra. For example, the knowledge value between an agent and environment on the target can be expressed using vector algebra as follows:

$$Z_v(g \to t) = Z_v(e \to t) + (-Z_v(g \to t)||e), \tag{4.54}$$

where $Z_v((g \to t)||e) = Z_v(g)_t$, $Z_v(e \to t) = Z_v(e)_t$ and $e \sqcup g, t$.

This is in a situation where g and t are contained in e, implying g and e are collinear on t and Z_v. Actually, t may not be perfectly defined by e but by a remote environment e_r different from e, making g and e_r non-collinear except $e_r = e$. Thus, g's action on t may not be reliable if t is perfectly defined by or more contained in e_r than in e.

To resolve such unreliability in cognition, the agent needs to take into account the divergence value between the two environments by referencing the remote environment using its local environment. To estimate this value, we make use of the non-collinear vector expressions in Proposition 7.

Using the rules in Axiom 3.1 together with conventional vector logic (Euclidean) and related propositions in Section 3.3.1, we can construct a *cognitive entity relationship diagram* for any cognitive process, and evaluate any require value for the process and its system.

For example, a cognitive system with two targets (T_1, T_2) and four agents (g_1, g_2, e_1, e_2), with each agent carrying cognitive action on the targets and all the entities are collinear, i.e., exist in same environment, can be designed using a cognitive entity relationship diagram as shown in Figure 10.a.

This diagram simplifies the evaluation of cognitive processes and systems using the knowledge value. For example, the valuation for Figure 10.a is done simply by adding or subtracting the knowledge value of their relationships about the concerned target using vector logic. But the case of multiple environments as shown in Figure 10.b requires the consideration of their divergence on a target, which depends on their inter-relationship with respect to the target.

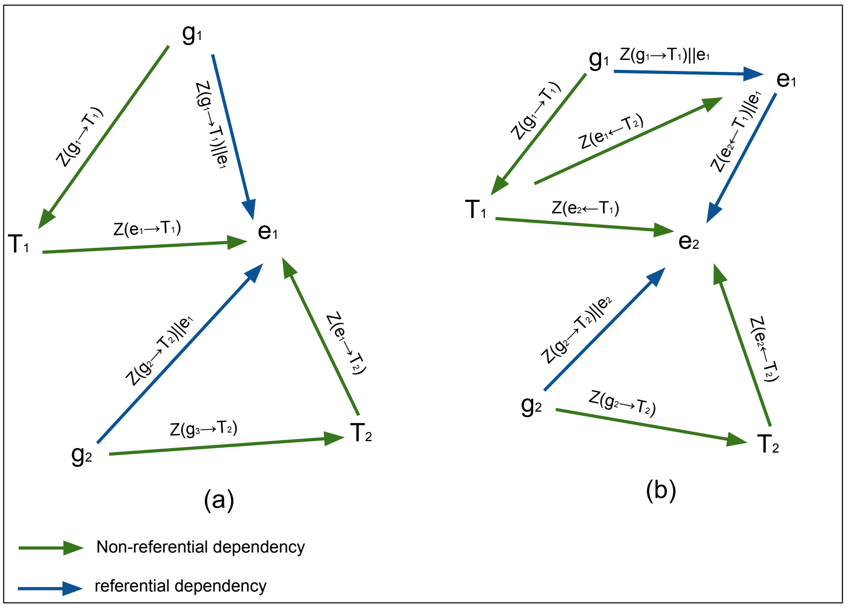

Figure 10. Cognitive entity relationships based on cognitropy valuations.

Using the knowledge valuation logic on a cognitive entity relationship diagram, we can analyze and solve the *environment divergence problem* in cognition.

Consider the relationships between g_1, e_1 and e_2 on T_1 in Figure 10.b. If e_2 has the true value about T_1 and the objective of g_1 is to achieve this true value from e_2 through e_1, then g_1 needs to acquire not only the values defines by e_1 but also that defined by the relationship between e_1 and e_2. In Figure 11.a, these relations are represented using vector.

As mentioned in Proposition 7, the angle between two entities defined their referential dependency. In this case, let us assume θ in Figure 11.a represents the measure of the referential dependency of e_2 on e_1 about T_1. Using collinearity logic, the amount of referential value (knowledge) left for g_1 to acquire the complete value of T_1 as defined by e_2 will be,

$$Z_v(g_1 \rightarrow T_1)||e_2 = Z_v(g_1 \mid\mid e_2)_{T_1} = b_{(g_1,e_1)} + b_{(e_1,e_2)}. \tag{4.55}$$

From the KTG in Figure 11.b, the cost of g_1 in e_2 is,

$$Z_v(g_1 \rightarrow T_1) = Z_{(cr)}(g_1 \mid\mid e_2)_{T_1} = a_{(e_1,e_2)} - b_{(g_1,e_1)}. \tag{4.56}$$

This cost can also be gotten directly from Equation (4.57). Using a relative endpoistemic knowledge,

$$Z_v(g_1 \mid\mid e_2)_{T_1} = b_{(g_1,e_2)} = b_{(g_1,e_1)} + b_{(e_1,e_2)}, \tag{4.57}$$

$$(a_{(e_2)} - a_{(g_1,e_2)}) = b_{(g_1,e_1)} + (a_{(e_2)} - a_{(e_1,e_2)}), \tag{4.58}$$

$$a_{(g_1,e_2)} = -b_{(g_1,e_1)} + a_{(e_1,e_2)}, \tag{4.59}$$

$a_{(\rho_1,\rho_2)} = \sum_i A(\rho_2)_{t_i} \log \frac{1}{A(\rho_1)_{t_i}}$,

$b_{(\rho_1,\rho_2)} = \sum_i A(\rho_2)_{t_i} \log \frac{A(\rho_1)_{t_i}}{A(\rho_2)_{t_i}}$,

where $a_{(gi,ei)}$ is cost of agent g_i in environment e_i about T_1, $a_{(ei,ej)}$ is cost of the sub environment e_i in super environment e_j about T_1, $b_{(gi,ei)}$ is knowledge to be achieved by agent g_i in environment e_i about T_1, and $b_{(ei,ej)}$ is knowledge to be achieved by sub environment e_i in super environment e_j about T_1.

It is easy to realize that such an algebraic vector operation assumed the agent g_1 is collinear with both e_1 and e_2, and can acquire knowledge about T_1 from each of them independently. But in our description, e_1 depends on e_2 about T_1, and g_1 depends on e_1 about T_1. Hence, they are not all in same environment, i.e., not-collinear, but in different *hierarchical and parallel* environments. In such case, g_1 cannot gain knowledge about T_1 from e_2 without the aide of e_1.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 24 November 2022 doi:10.20944/preprints202211.0450.v1

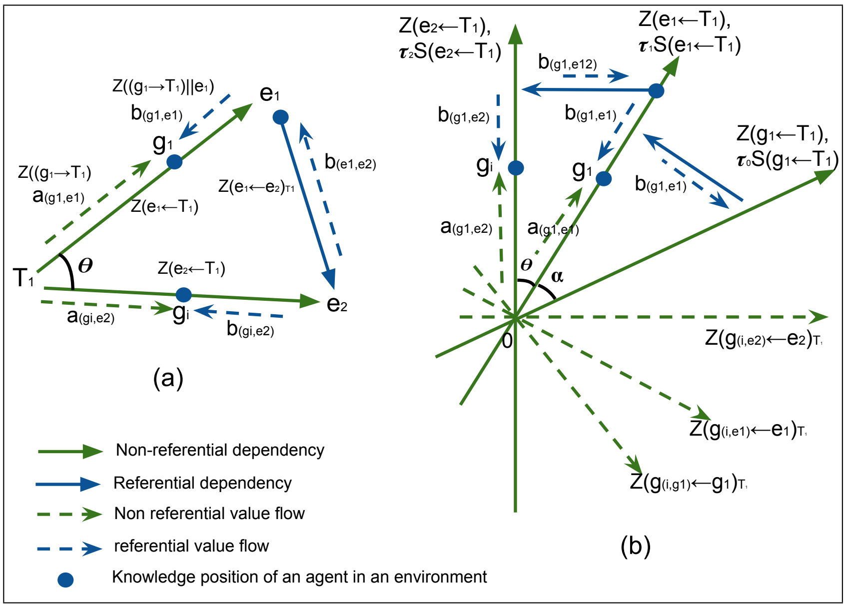

Figure 11. (a) Vector representation of a cognitive entity relationship of T_1, g_1, e_1 and e_2 (b) Knowledge transformation graph of g_1, e_1 and e_2 about T_1

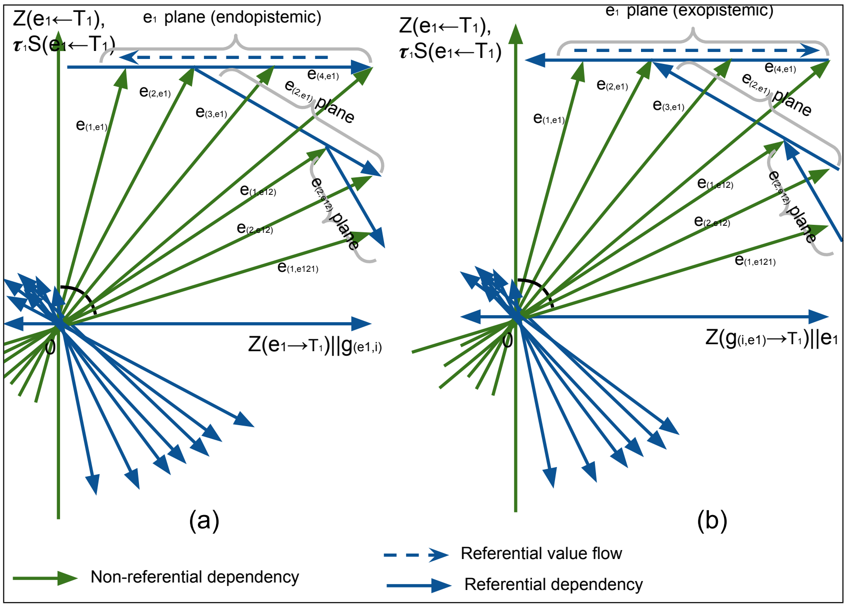

Figure 12. KTG of endopistemic and exopistemic cognition of an agent relative to other agents (a) endopisemic cognition (b) exopistemic cognition

The result of such a collinear algebraic operation if applied to a non-collinear relationship may result in a complete value on the target different from that defined by e_2. So, it is most likely required as in the field of relativity [86], to transform values from one environment to another in order to account for such value inconsistencies. This is also true in *Information geometry* [90],[91], where the *generalized Pythagoras* [92] is use in *I-projection* to capture the *triangle inequality* property for divergences.

However, using a vector logic and hyperbolic trigonometry in Figure 11.b, we seek to achieve this using a relativistic approach. Also, since the agents work toward a well defined complete cost line and have no awareness about the curve nature of the line, we avoid using curves lines for the knowledge path as done in information geometry.

We refer to the graph in Figure 11.b, as a *Knowledge transformation graph* (KTG). In general, the KTG is a type of *Value transformation graph* (VTG), that represents the transformation of knowledge in one environment to another environment. As represented in

 doi:10.20944/preprints202211.0450.v1

Figure 12, in both endopistemic an exopistemic cognition, each environment is a plane that contains other environments, which may contain other environments, forming a *hierarchical* environment layers. The vertical line in each environment represents the cost or complete value of the main agent (i.e., agent referencing others or that is referenced by others) e.g., observer and actuator agents. The horizontal line represents the referential value of each environment with the main environment. The slanted lines are the *transformed cost or completed values* of the main agent about its related agents.

An example of a KTG is given in Figure 13, which represents cognition on a binary state target. From Figures 13.a and 13.b, the KTG for both exopistemic and endopistemic cognition on a binary target is a two inverted cone shape structure. Different shapes will be generated for different types of targets and cognition process. We shall discuss more on such dynamics in a future publication on cognimatics.

Regarding Figure 11.b, for g_1 to acquire knowledge about e_2 through e_1, thereby solving the environment divergence problem it faces about the target T_1 in e_2, it is required to transform its values in e_1 to that of the remote environment e_2 based on a transformation operation. We develop two of such operations in this research; based on mathematical *factorization* and *optimization* techniques.

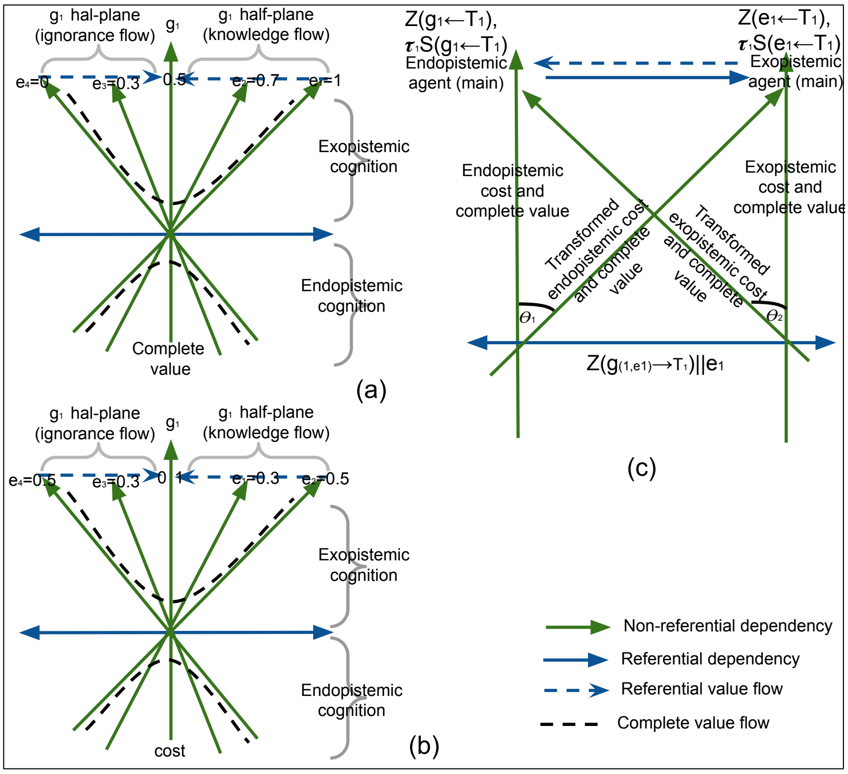

Figure 13. KTG cognition on a binary agent (a) over a complete value (b) over a cost value (c) exopistemic and endopistemic relationship of main agents

4.4.1. Factorization Method of Divergence Reduction

In the factorization approach, the agent needs to transform its absolute (i.e., cost) or relative knowledge in e_1 using a *transformation factor*, to a corresponding value in e_2. This is to avoid irregularities (or distortion) of these values in e_2 and preserved them as required in the new environment e_2, for learning and other purposes on T_1 in e_2.

Using *Knowledge transformation graph* (KTG) as shown in Figure 11.b, we developed a *Knowledge transformation factor* (KTF), to capture the transformation of an agent's cognition from one environment to another. This was done by applying hyperbolic trigonometric functions on the vector quantities in Figure 11.b.

In the Figure 11.b, a plane on the graph represents cognition in an environment. The vertical axis for each environment represents their cost value on the target with repsect to their main environment and a *complete cost* defined over the target for all sub environment under them. The horizontal axis for each environment represents their referential relationship with their sub environments and the amount of knowledge any agent from a different environment will need to shift to the environment. The values for the y and x axis of each environment or agent in e_2 and e_1 of Figure 11.b for an endopistemic cognition are given as,

Environment e_2 (absolute environment),

$$y = Z_c(e_2)_{T_1} = \sum_i A(e_2)_{t_i} \log \frac{1}{A(e_2)_{t_i}}, \tag{4.60}$$

$$x = Z_v(e_2)_{T_1} = 0. \tag{4.61}$$

Environment e_1 (contains in e_2),

$$y = Z_c(e_1)_{T_1} = \sum_i A(e_1)_{t_i} \log \frac{1}{A(e_1)_{t_i}}, \tag{4.62}$$

$$x = Z_v(e_1)_{T_1} = \sum_i A(e_2)_{t_i} \log \frac{A(e_1)_{t_i}}{A(e_2)_{t_i}}. \tag{4.63}$$

Agent g_1 (contains in e_1),

$$y = Z_c(g_1)_{T_1} = \sum_i A(g_1)_{t_i} \log \frac{1}{A(g_1)_{t_i}}, \tag{4.64}$$

$$x = Z_v(g_1)_{T_1} = \sum_i A(e_1)_{t_i} \log \frac{A(g_1)_{t_i}}{A(e_1)_{t_i}}. \tag{4.65}$$

Where e_2 has the absolute true action values on the target, also considered as the originator of the target values, hence needs no knowledge and transformation, except agents in the environment are to be *downgraded* to environments with lower values on the target. In some scenario, e_2 may not possess the true complete knowledge about the target but depends on another environments. This can lead to an endless chain of interconnected environments and agents on a target.

For an agent g_1 in Figure 11, the cost, a, and knowledge, b, that it generates in its environment e_1, are given as,

$$a_{(g_1,e_1)} = Z_{cr}(g_1||e_1)_{T_1} = \sum_i A(e_1)_{t_i} \log \frac{1}{A(g_1)_{t_i}}, \tag{4.66}$$

$$b_{(g_1,e_1)} = Z_v(g_1||e_1)_{T_1} = \sum_i A(e_1)_{t_i} \log \frac{A(g_1)_{t_i}}{A(e_1)_{t_i}}. \tag{4.67}$$

The objective is to have the knowledge and cost of g_1 in e_2 through e_1. By applying collinearity logic as given in Equation (4.57) for knowledge value, we have,

$$Z_v(g_1 \mid\mid e_2)_{T_1} = b_{(g_1,e_1)} + b_{(e_1,e_2)}, \tag{4.68}$$

$$b_{(g_1,e_1)} = \sum_i A(e_1)_{t_i} \log \frac{A(g_1)_{t_i}}{A(e_1)_{t_i}}, \tag{4.69}$$

$$b_{(e_1,e_2)} = \sum_i A(e_2)_{t_i} \log \frac{A(e_1)_{t_i}}{A(e_2)_{t_i}}. \tag{4.70}$$

Similarly, for the cost given in Equation (4.56), we have,

$$Z_{(cr)}(g_1 \mid\mid e_2)_{T_1} = a_{(e_1,e_2)} - b_{(g_1,e_1)}, \tag{4.71}$$

$$a_{(e_1,e_2)} = \sum_i A(e_2)_{t_i} \log \frac{1}{A(e_1)_{t_i}}, \tag{4.72}$$

$$b_{(g_1,e_1)} = \sum_i A(e_1)_{t_i} \log \frac{A(g_1)_{t_i}}{A(e_1)_{t_i}}. \tag{4.73}$$

It is mathematically proven that Equation (4.68), which represents the vector sum of the knowledge value of g_1 in e_2, does not satisfy the triangle inequality because of their

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 24 November 2022 doi:10.20944/preprints202211.0450.v1

divergence property [92]. So, adding the knowledge value of g_1 in e_1 and the knowledge value of e_1 in e_2 will not capture the entire knowledge requires by g_1 in e_2.

$$Z_v(g_1 \mid\mid e_2)_{T_1} = b_{(g_1,e_2)} \neq b_{(g_1,e_1)} + b_{(e_1,e_2)}. \tag{4.74}$$

If e_2 is convex, i.e., with a convex absolute cognitropy, then $b_{(g_1,e_2)} \geq 0$, which implies an exopistemic cognition.

$$b_{(g_1,e_2)} \geq b_{(g_1,e_1)} + b_{(e_1,e_2)}. \tag{4.75}$$

If e_2 is concave, i.e., with a concave complete cognitropy, then $b_{(g_1,e_2)} \leq 0$, which implies an endopistemic cognition.

$$b_{(g_1,e_2)} \leq b_{(g_1,e_1)} + b_{(e_1,e_2)}. \tag{4.76}$$

Also, Equation (4.71), representing a vector operation on the cost of g_1 in e_2 does not satisfy equality as proven below.

$$Z_{(cr)}(g_1 \mid\mid e_2)_{T_1} = a_{(g_1,e_2)} = a_{(e_1,e_2)} - b_{(g_1,e_1)}. \tag{4.77}$$

Applying *conservation of value law* (Theorem 2) to $b_{(g1,e1)}$

$$b_{(g_1,e_1)} = a_{(e_1)} - a_{(g_1,e_1)}, \tag{4.78}$$

$$\Rightarrow a_{(g_1,e_2)} = a_{(e_1,e_2)} - a_{(e_1)} + a_{(g_1,e_1)}. \tag{4.79}$$

For this to be equal,

$$a_{(e_1,e_2)} = a_{(e_1)} \Rightarrow A(e_1) = A(e_2), \tag{4.80}$$

$$\text{or } \; a_{(g_1,e_1)} = a_{(e_1)} \Rightarrow A(g_1) = A(e_1). \tag{4.81}$$

Hence, without any of such constrain,

$$Z_{(cr)}(g_1 \mid\mid e_2)_{T_1} = a_{(g_1,e_2)} \neq a_{(e_1,e_2)} - b_{(g_1,e_1)}. \tag{4.82}$$

So, combining the cost value of e_1 in e_2 and the knowledge value of g_1 in e_1 will not capture the entire cost of g_1 in e_2.

Such irregularities of the resultant cost and knowledge values occurred because of the consideration of a collinear vector addition in a non-collinear transformation process. To provide an adjustment for these irregularities, we propose a KTF using the KTG in Figure 11.b as described below.

Proposition 19. *In any entity dependency network, if $g_i \rightarrow g_j$, then cost transformation from g_i to g_j is given by,*

$$u = w(\cosh\theta) \;\; and \;\; \tanh\theta = r/u. \tag{4.83}$$

Since, $\tau_r = \tau_u \Rightarrow \tanh\theta = S_r/S_u$.
Also, $\cosh\theta = 1/\mathrm{sech}\theta$ and $\mathrm{sech}\theta = \sqrt{1 - \mathrm{tansh}^2\theta}$,

$$\Rightarrow u = \frac{w}{\sqrt{1-(\frac{S_r}{S_u})^2}}, \;\; \cosh\theta = \frac{1}{\sqrt{1-(\frac{S_r}{S_u})^2}} = \eta_K, \tag{4.84}$$

where $u = a_{(g_i,g_j)}$ is actual cost of g_i in g_j, $w = a_{(g_i)}$ is transformed cost of g_i in g_j and acting as a complete cost on g_j, $r = b_{(g_i,g_j)}$ is relative knowledge to be achieved by g_i in g_j, S_r and S_u are stability for the values r and u, respectively, τ_r and τ_u are the action times for the value r and u, respectively.

 doi:10.20944/preprints202211.0450.v1

A mathematical proof of this transform is in Appendix A.2. We can also express the KTF, into cost value by applying the *value conservation law* (Theorem 2) on r as follows:

$$\eta_K = \frac{1}{\sqrt{1-(\frac{S_u - S_{rc}}{S_u})^2}} = \frac{1}{\sqrt{1-(1-\frac{S_{rc}}{S_u})^2}}, \tag{4.85}$$

where $r = u - r_c$, r_c = $a_{(g_j)}$ is complete cost of g_i in g_j

This gives the valuation of the independent cost w of one agent to a cost in a directly dependent agent, such as g_i to g_j. For example, if an independent g_1 is to be trained in e_1, using its actual cost, then the transformed cost it incurs from its relationship with e_1 is the cost it will need to be trained as e_1 in e_2 without any distortion. This is important in *multitask learning*, where different related data sources are needed to train an agent. More on this in a future publication.

The value $\frac{S_r}{S_u} = \beta_K$ is actually the *exactness value* of knowledge and can be considered as a performance parameters of the agent during cognition, such as during learning where the agent is expected to transform its u to the complete value of the environment which can be defined using w. But in some cases, β_K can also be considered as a design parameter, which is set for each specific cognition. Also, since β_K is strongly related to the angle of divergence θ, as $\theta = tanh^{-1}\beta_K \Rightarrow \theta = f(\beta_K)$, β_K can be used to define θ in some applications.

Using this basic transform, we can build a transform of the cost of an endopistemic and exopistemic agent over multiple inter-ralated environments, such as the transform of g_1 to e_2 through e_1 as defined in Equation (4.71). Such transformation is defined in the proposition below;

Proposition 20. *In any entity dependency or influence network, if $g_i \to g_j \to g_z$, then,*

$$tanh(\theta+\alpha) = \frac{r_{zi}}{u_{zi}} \Rightarrow \frac{tanh(\theta)+tanh(\alpha)}{1+tanh(\theta)tanh(\alpha)} = \frac{r_{zi}}{u_{zi}}. \tag{4.86}$$

Applying Proposition 19,

$$tanh(\theta) = \frac{r_{ji}}{u_{ji}},\ tanh(\alpha) = \frac{r_{zj}}{u_{zj}} \Rightarrow \frac{\frac{r_{ji}}{u_{ji}} + \frac{r_{zj}}{u_{zj}}}{1+\frac{r_{ji}}{u_{ji}}\frac{r_{zj}}{u_{zj}}} = \frac{r_{zi}}{u_{zi}}. \tag{4.87}$$

If g_j and g_z are forced to have same valuation in their referenced entity, that is, $u_{ji} = u_{zj} = u_{zi}$, then,

$$\frac{\frac{r_{ji}}{u} + \frac{r_{zj}}{u}}{1+\frac{r_{ji}}{u}\frac{r_{zj}}{u}} = \frac{r_{zi}}{u} \Rightarrow \frac{r_{ji}+r_{zj}}{1+r_{ji}r_{zj}} = r_{zi}. \tag{4.88}$$

Since, $\tau_{rji} = \tau_{uij}$, $\tau_{rzj} = \tau_{uzj}$, and $\tau_{rzi} = \tau_{uzi}$,

$$\frac{\frac{r_{ji}}{u} + \frac{r_{zj}}{u}}{1+\frac{r_{ji}}{u}\frac{r_{zj}}{u}} = \frac{r_{zi}}{u} \Rightarrow \frac{S_{rji}+S_{rzj}}{1+S_{rji}S_{rzj}} = S_{rzi}, \tag{4.89}$$

where $u_{x,y} = a_{(g_x,g_y)}$ is cost of g_x in g_y, $r_{x,y} = b_{(g_x,g_y)}$ is knowledge to be achieve by g_x in g_y, S_r and S_u are stability for the values r and v, respectively, τ_r and τ_u are the action times for the value r and v, respectively.

This transform can be used to find the resultant relative knowledge transform of g_1 in e_2 through e_1. The resultant relative knowledge of e_2 can then be used to calculate the transformed cost of g_1 in e_2 using Proposition 19 by replacing r with r_{zi} and u with u_{zi}.

Indeed, the transformed cost in such case can also be obtained by applying a double operation of η_K on the actual cost value of g_1 in e_1, replacing the actual cost in the next operation with the transformed value w of the previous operation.

Table 2 shows results of this transformation when applied to resolve the *triangle inequality* problem of a random sets of three actions representing actions of agent g_1, environments e_1 and e_2 on a binary state target, while respecting the probabilistic constrain, $\sum_{i=1}^{i=n} A(g)_{t_i} = 1$, on each action set.

 doi:10.20944/preprints202211.0450.v1

Table 2. Result of KTF application to resolve the triangle inequality problem in I-projection of actions on binary state target

g_1	e_1	e_2	b_1	b_2	b_3	b_4
$0.4, 0.6$	$0.7, 0.3$	$0.1, 0.9$	-0.57	-0.51	0.04 (45 tf)	0.08 ($\beta_k = 0.96$)
$0.5, 0.5$	$0.2, 0.8$	$0.4, 0.6$	-0.28	-0.07	0.00 (2 tf)	0.01 ($\beta_k = 0.48$)
$0.1, 0.9$	$0.7, 0.3$	$0.1, 0.9$	-0.84	-0.56	0.30(2 tf)	0.26 ($\beta_k = 0.88$)
$0.3, 0.7$	$0.3, 0.7$	$0.6, 0.4$	0	0	0(1 tf)	0 ($\beta_k = 0\ to\ 1$)
$0.2, 0.8$	$0.3, 0.7$	$0.3, 0.7$	0	0	0(1 tf)	0 ($\beta_k = 0\ to\ 1$)
$0.2, 0.8$	$0.2, 0.8$	$0.2, 0.8$	0	0	0(1 tf)	0 ($\beta_k = 0\ to\ 1$)

Note that tf denotes transform. $b = \Delta b_{g_1,e_2} = \frac{b_{g_1,e_2} - (b_{g_1,e_1} + b_{e_1,e_2})}{(b_{g_1,e_1} + b_{e_1,e_2})}$.

b is the change in value between $(b_{g_1,e_1} + b_{e_1,e_2})$ and b_{g_1,e_2}, and the objective is for this change to be zero, implying $(b_{g_1,e_1} + b_{e_1,e_2}) = b_{g_1,e_2}$. The value b_1 is a change resulting from direct application of distributions. b_2 is a change resulting from the application of KTF on the cost values using w for a_{e_1} and a_{e_2} in b_{g_1,e_1} and b_{e_1,e_2}, respectively. b_3 is a change resulting from multiple KTF transforms that generate lowest change. b_3 is a change resulting from fixed values that generate zero change.

From the table, it can be seen that the application of KTF to a set of three distribution in an I-projection will force the distribution to respect the triangle inequality law of distance measure. But since such inequality can still be respected if g_1 completes e_1, then resolving triangular inequality does not solve the non-collinearity problem because g_1 has not access e_2. A simple solution is to assume the cost and knowledge of e_1 and e_2 are given to g_1 during its cognition with respect to e_1, making it not to spend resources to generating such values anymore but to rather use them directly.

We propose another solution which involves the sequential transformation of the values of g_1 in e_1 to that of e_2 using values obtained in e_1. The goal is to attain completeness with e_1 and use the perceived value as the new cost value for achieving e_2. This process of sequentially bridging the gap between two referentially related environments of an agent is considered in this article as *sequential semantic learning*. An algorithm for such learning process is provided in Section 5.

In the case of multiple transforms, as in b_3 above, the optimal transform can be reached through a combination of exopistemic and endopistemic cognitions, i.e., a process and its inverse, as shown in Figure 13.c. Furthermore, if the environment has structure such that the dependency order are defined by the complete cost, e.g., $g_1 \to e_1 \to e_2 \Rightarrow Z_c(g_1) < Z_c(e_1) < Z_c(e_2)$, then an agent is required to search the environment space of the target by moving between high and low environment values, up to when it reaches the optimal divergence with the destination environment. Such search may include finding environments with maximum or minimum complete cost about the target, using different techniques such those related to the *principle of maximum entropy* [93]. By forcing the three distributions to respect the triangle inequality law, a condition that can only be satisfied if $e_1 = e_2$ or $g_1 = e_1$, the KTF actually enables the reduction of the divergence between any two environments, and together with a *sequential semantic learning*, the divergence of any two distribution can be reduced to zero given another distribution.

4.4.2. Important Facts and Significance of the Transform

The results from Proposition 19 and 20 are important and revolutionary for cognition and agent design. The following three important facts can be extracted from them.

1) The knowledge and cost of an agent will contract as it is transposed from a higher to a referential lower complete cost.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 24 November 2022 doi:10.20944/preprints202211.0450.v1

2) The transformed knowledge and cost expression in Proposition 19 can be expressed in both space (information, beliefs, intelligence, etc.) and time stability values.

$$\tau_u S_u = \tau_w S_w \eta_K \ \text{(time distortion)} \tag{4.90}$$

$$X_u S_u = X_w S_w \eta_K \ \text{(input information distortion)} \tag{4.91}$$

$$A_u S_u = A_w S_w \eta_K \ \text{(belief distortion)} \tag{4.92}$$

$$\phi_u S_u = \phi_w S_w \eta_K \ \text{(Intelligence distortion)} \tag{4.93}$$

This is important as we generalize in a theorem below.

Theorem 4. *The time and space properties of an agent during cognition in one environment are distorted as they are transposed in another environment of a different cost value.*

Such distortion can lead to different observation and actuation valuation on a target in different environments by an agent. Also, the distortion will be high when the relative difference between the action values environments is high. For example, the information an agent gets about a target from the external environment can be distorted as it enters the internal environment. Such distrotion of information will definitely affect the belief and knowledge of the agent on the target, which will lead to misconception and misinformation.

3) From Theorem 4, we can conclude that cognitive properties can dilate or contract during a cognitive distortion process and be re-adjusted through a reverse process to cancel such distortion.

The significance of the transformation models defined in Proposition 19 and 20, is that it can be used to explain cognitive phenomenon such as forgetfulness, memory loss, recall, etc., in an agent as a value contraction or dilation phenomenon caused by environment divergences over a target.

For example, if our mind (i.e., an abstract environment) is busy with many activities relative to our external (i.e., real) environment, we will experience (i.e observe) time in the external environment to be surprisingly fast; hours will passed without us awaring of it. But if our mind has less activity relative to the external environment, then real time seems to slow down and we will think that real time does not move. This is because while our observation stability values, S_u and S_r, about a target are constant, any increase in our perceptual value, w, about the target will entails a distortion of our real and perceptual time. From Proposition 19, we have that,

$$w = u\sqrt{1 - \frac{S_r}{S_u}} \Rightarrow \Delta t_w S_w = \Delta t_u S_u \sqrt{1 - \frac{S_r}{S_u}} \tag{4.94}$$

So, if w or S_w increases, while S_u and S_r are constant, then the only change required in u to accommodate such change from a related environment is a change in the time Δt_u of u. This is what create a different impression about time between the two referentially related environment, and this can also be use to explain other similar cognitive contraction and dilation phenomena such as sight awareness distortion when we leave a dark room to a light room and vice versa.

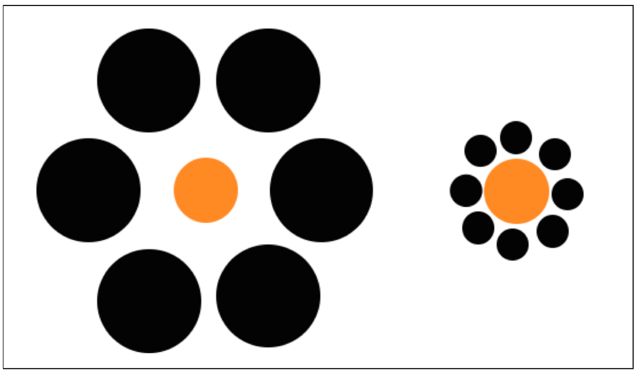

Figure 14. cognitive distortion in visual perception

A solution to these cognitive phenomenon is for the agents to change the cost or relative knowledge between the environments in order to balance cognitive awareness without changing time. The process of varying such values entails semantic learning, which is an optimization process.

Another interesting cognitive distortion practical application is the *visual perception* exercise in Figure 14.

This exercise is conventionally called the *Ebbinghaus illusion*. Looking at the orange circles in the middle, one will appear bigger than the other, whereas there are the same. A common explanation to this distortion is that our minds perceive the size of each middle circle relative to its surrounding black circles, and since the left surrounding circles are larger, we perceive that their inner circle is smaller than the right one.

Using the relativistic knowledge values we have described in this article, we can explain and resolve this phenomenon. Our minds attribute more attention toward the larger surrounding circle in the left and use them as a reference to their inner circle, then we turn our attention directly to the inner large circle in the right. The larger the circle, the more *attention action value* is attributed to it. Our high attention on the larger right inner circle causes us to perceive it as the environment of its surrounding circles, unlike the left inner and surrounding circles. Consider the left surrounding circle to represent the environment e_1 of the inner circle g_1, and the right surrounding circle to be environment e_2 with inner circle g_2, then the perceptual value and attention dependency flow is given by:

$$A(e_1) \leftarrow A(g_1), A(g_2) \leftarrow A(e_2) \;\; \text{(dependency)}, \tag{4.95}$$

$$A(e_1) \rightarrow A(g_1), A(g_2) \rightarrow A(e_2) \;\; \text{(value)}. \tag{4.96}$$

The relative knowledge value for each figure will be,

$$Z(g_1||e_1)_t = \sum A(e_1)_t \log \frac{A(g_1)_t}{A(e_1)_t}, \tag{4.97}$$

$$Z(g_2||g_1)_t = \sum A(g_1)_t \log \frac{A(g_2)_t}{A(g_1)_t}. \tag{4.98}$$

Since the perceptions of the inner circles are to be compared for equality, we can thus apply *cognitive balancing factor* (CBF) on the cognitive process.

$$\text{CBF} = Z(g_1||e_1)_t - Z(g_2||g_1)_t. \tag{4.99}$$

If our attention actions produce a CBF$\neq$0, then our perception will be distorted i.e., imbalance. Since we have different attention actions value on the figure, we therefore should experience distortion because our CBF$\neq$0. For our CBF=0, we need to distribute the attentions in order to cancel the CBF effect on our perception. This may require a sequential or parallel adjustment of one perception value to match the other.

Generating multiple attention actions on different targets simultaneously is practically difficult due to a limitation of the intelligence for that purpose. In this case, we can distribute our attention actions over the limited intelligence to achieve maximum perception. To effectively distribute our attention and easily resolve such distortion, we need to first describe a value flow chain between the different objects, and then learn these values to reduce their divergence. A sample perceptual value and dependency flow for this cognition is given below.

$$A(g_1) \leftarrow A(e_1) \leftarrow A(e_2) \leftarrow A(g_2) \;\; \text{(dependency)}, \tag{4.100}$$

$$A(g_1) \rightarrow A(e_1) \rightarrow A(e_2) \rightarrow A(g_2) \;\; \text{(value)}. \tag{4.101}$$

The goal in this flow is to reduce the cognitive divergence between g_1 and g_2 to zero, i.e., to acquire complete value. Well, other strategies can also be used. Expressing the combined relative knowledge value based on that between g_1 and g_2.

$$Z(g_1||g_2)_t = Z(g_1||e_1)_t + Z(e_1||e_2)_t + Z(e_2||g_2)_t. \tag{4.102}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 24 November 2022 doi:10.20944/preprints202211.0450.v1

Even if we assume we have an intelligence that can enable us to simultaneously generate the attention actions on the two inner circles and optimize them with respect to their surrounding circles easily, the divergence between the surrounding circles $Z(e_1||e_2)_t$ will remain as a nuance that we need to reduce semantically in order to actually get the complete value, $Z(g_1||g_2)_t = 0$. Both the KTF and other divergence reduction techniques, presented in the next section, can be used to achieve such complete value using semantic abilities.

4.4.3. Optimization Method of Divergence Reduction

If g_1 is to learn both environment at same time, i.e., parallelly, then g_1 will require to have access to the knowledge or cost of e_1 on e_2 about T_1 since it does not have the ability to generate such remote values about e_2 without the aide of e_1. For g_1 to generate values between two referentially related environments, it requires an intelligence to enable it take such actions, because e_1 and g_1 are separate entities with different abilities. This process of parallelly generating cost or knowledge about a target from referentially related environments is considered here as *parallel semantic learning*.

The value defining the relationship between the two environments is considered as "fictitious" to the agent in a parallel process because the agent have not yet completed e_1 but will need to also capture e_2. Defining the action value of this "fictitious" environment is important to the agent as it will be based on such value the agent will use to generate cost and knowledge about the target with respect to e_2.

Many techniques can be used to define such "fictitious" action value. One approach is for g_1 to estimate the action of the maximum relative knowledge between e_1 and e_2, then based on this value, it then trains two different actions simultaneously using two different intelligence. For a concave e_2,

$$A(e_1') = \underset{A(e_2)}{\text{argmax}}[Z(e_1||e_2] = \underset{A(e_2)}{\text{argmax}}[b_{e_1,e_2}], \tag{4.103}$$

$$b_{g_1,e_2} = b_{g_1,e_1'} + b_{e_1',e_2}, \tag{4.104}$$

where e_1' is action of g_1 approximating the closest action of e_1 to e_2. Thus, e_1 is used by g_1 only to estimate e_1'.

This technique is applied conventionally in approximating the triangle inequality problem in I-projection [90],[91],[92]. Another solution is to use the resultant action value of the actions of e_1 and e_2 to define a fixed "fictitious" action value. Then carry a second action on this value rather than on e_2.

$$A(e_1') = \frac{A(e_1) + A(e_2)}{2}, \tag{4.105}$$

$$b_{g_1,e_2} = b_{g_1,e_1} + b_{g_2,e_1'}, \tag{4.106}$$

where e_1' is action value of g_1 estimating the the environment action defined by e_1 and e_2, g_2 is a second action of agent g_1 on its estimated environment action e_1'.

Since, the agent can do optimization on both its local environment and the "fictitious" environment simultaneously, we consider this process as *parallel semantic learning*.

The abilities and values of an agent on the "fictitious" environment are different from those it has on its local environment e_1. These abilities and values are considered in this research as *semantic* abilities and values, notably; *Semantic intelligence*, *action* and *cognitive value*. We distinguished them from conventional abilities and values as follows:

Definition 83. ***Computational abilities and values*** *are those required by an agent in same (local) environment as the target.*

Definition 84. ***Semantic abilities and values*** *are those required by agent in different (remote) environment as the target.*

 doi:10.20944/preprints202211.0450.v1

From a relativistic point of view, all conventional intelligent and action (e.g., prediction, learning, etc.) models are computational (i.e., collinear) models. We think there is needed to integrate semantic (i.e., non-collinear) logic in agent development to increase flexibility of cognition between environments. Another type of intelligence required for an agent include, *mutual intelligence*, to enable mutual actions and generate mutual values between multiple actions or targets. Conventionally, *conditional intelligence* to enable conditional action is widely used. We propose semantic learning techniques in Section 5.

4.4.4. Understanding and Trust

Any two agents whose cognitive environments diverge in such a way that they cannot attain their respective complete semantic values during training to capture each other's environment, will result to an *irreconcilable cognitive difference on semantic values*. Such can lead to misunderstanding between agents over same or different environments about a target. Misunderstanding between agents is actually the absence of understanding as we define below.

Definition 85. ***Understanding*** *is the process of value acquisition with respect to other agents about a target.*

Understanding of g_1 on g_2:

$$V_a(g_1||g_2)_t = \log \frac{A(g_1)_t}{A(g_2)_t} \,. \tag{4.107}$$

Self understanding of g_2:

$$V_a(g_2)_t = \log \frac{1}{A(g_2)_t}. \tag{4.108}$$

Expected understanding (cognitropy of understanding):

$$Z_a(g_1||g_2)_t = \sum_i A(g_2)_{t_i} V_{a_j}(g)_{t_i}. \tag{4.109}$$

Hence, based on Definition 2 and 85, we can consider knowledge as a type of understanding with respect to an *environment* which is considered to possess a *higher value* on the target than the agent. Such a value constrained is not required for understanding. Epistemologically, the difference between knowledge and understanding is still a topic of debate [94],[95], and our definitions should be considered as a proposal to such debate. The concept of understanding is very important because its absence is a major source of conflict and disunity between cognitive agents, such as humans.

Based on our definition, we can distinguish two types of understanding between agents; *understanding of purpose* and *understanding of meaning*. The former is related to the divergence of the actions of the agents on the states of the target, as we have been doing. But the latter is related to the divergence of the actions on the "fictitious" or semantic environment which defines the relationship between their individual environments. For a concave g_1 and g_2, the understanding of meaning is given as,

$$A(g_1') = \operatorname*{argmax}_{A(e_2)}[Z(e_1||e_2] = \operatorname*{argmax}_{A(e_2)}[b_{e_1,e_2}], \tag{4.110}$$

$$A(g_2') = \operatorname*{argmax}_{A(e_1)}[Z(e_2||e_1] = \operatorname*{argmax}_{A(e_1)}[b_{e_2,e_1}], \tag{4.111}$$

$$Z_a(g_1'||g_2')_t = \sum_i A(g_2')_{t_i} V_a(g)_{t_i}, \tag{4.112}$$

where g_1' is action of g_1 estimating the closest action of e_1 to e_2, g_2' is action of g_2 estimating the closest action of e_2 to e_1, and $Z_a(g_1'||g_2')_t$ is the understanding of meaning, i.e., semantic understanding of $g'1$ on g_2' about the target.

We can also use the resultant value of the environment;

$$A(e') = \frac{A(e_1) + A(e_2)}{2}, \tag{4.113}$$

$$A(g_1') = \underset{A(e)}{\operatorname{argmax}}[Z(g_1||e] = \underset{A(e_2)}{\operatorname{argmax}}[b_{g_1,e_2}], \tag{4.114}$$

$$A(g_2') = \underset{A(e)}{\operatorname{argmax}}[Z(g_2||e] = \underset{A(e_1)}{\operatorname{argmax}}[b_{g_2,e_1}], \tag{4.115}$$

$$Z_a(g_1'||g_2')_t = \sum_i A(g_2')_{t_i} V_a(g)_{t_i}. \tag{4.116}$$

It is important for an agent to achieve a complete understanding of purpose and meaning, but this entails learning.

Similar to semantic intelligence, the inability to train mutual intelligence between targets and actions of agents will lead to *irreconcilable cognitive difference on mutual values*, which will lead to *mistrust*. Mistrust between agent is the absence of trust which we mathematically define as follows.

Definition 86. ***Trust*** *is a value defined by the mutual actions of agents on a target.*

Trust of g_1 on g_2:

$$A(g_1')_t = A(g_1|g_2;\phi_1)_t, \tag{4.117}$$

$$V_a(g_1;g_2)_t = \log \frac{A(g_1')_t}{A(g_1)_t}, \tag{4.118}$$

$$Z_a(g_1;g_2)_t = \sum_i A_{12}(g)_{t_i} V_{a_j}(g)_{t_i}, \tag{4.119}$$

Trust between g_1 and g_2:

$$T(g_1,g_2) = Z_a(g_1;g_2)_t - Z_a(g_2;g_1)_t, \tag{4.120}$$

where $A(g_1')_t = A(g_1|g_2;\phi_1)_t$ is the action of g_1 conditioned on the action of g_2, $Z_a(g_1;g_2)_t$ is the expected value of $V_a(g_1;g_2)_t$ over the joint action of g_1 and g_2 on t, $V_a(g_1;g_2)_t$ is the value of the conditional action of g_1 given g_2 about t reference to it self action about t.

Well, $V_a(g_1;g_2)_t$ can still be defined using the *perceptual value* of g_1 about the conditional action of g_2 given g_1 reference to the self action of g_2, that is,

$$V_a(g_1;g_2)_t = \log \frac{A(g_2|g_1;\phi_1)_t}{A(g_2)_t}. \tag{4.121}$$

Trust can actually be considered as a *conditional understanding* between agents about a target. Also, the trust value between any two agents, $T(g_1,g_2)$, depends highly on the intelligence of the agents.

$$\text{if } \phi_1 = \phi_2, \ \text{ then } \ Z_a(g_1;g_2)_t = Z_a(g_2;g_1)_t, \tag{4.122}$$

$$\text{if } \phi_1 \neq \phi_2, \ \text{ then } \ Z_a(g_1;g_2)_t \neq Z_a(g_2;g_1)_t. \tag{4.123}$$

For a trust value between two unequal intelligence to be rendered equal, i.e., for $T(g_1,g_2) = 0$ when $\phi_1 \neq \phi_2$, both or a single agent is required to *learn* its understanding of the other agent.

$$\text{for } \ Z_a(g_1;g_2)_t = Z_a(g_2;g_1)_t \ \text{ if } \phi_1 \neq \phi_2,$$

$$\underset{\omega_1}{\operatorname{argopt}}[Z_a(g_1;g_2)_t] = \underset{\omega_2}{\operatorname{argopt}}[Z_a(g_2;g_1)_t], \tag{4.124}$$

where $\operatorname{argopt}_\omega[...]$ is optimization operation over ω, and ω can be intelligence (ϕ), action (A) or cognitive values (K, I, etc.). Such optimization can be a minimization, $\operatorname{argmin}_x[...]$, or maximization, $\operatorname{argmax}_x[...]$, i.e., $\operatorname{argopt}_x[...] = \{\operatorname{argmin}_x[...], \operatorname{argmin}_x[...]\}$ as we shall discuss in Section 5.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 24 November 2022 doi:10.20944/preprints202211.0450.v1

In general, both *understanding* and *trust* can manifest in the real and abstract environments. In the real environment, for example, they can be transformed to a *social differences* between agents, based on their social actions on same target.

We conclude this section by mentioning the similarities of cognitropy transformation expression to that defined in *Lorentz transformation* [86], *Einstein's special theory of relativity* [96], and *Minkowski vector space* [97]. These transformations are similar to our cognitive transformation only when the stability values of the environments are equal and invariant.

Well, as we have shown, the environment stability that defines the stability of an agent on a target may differ from one environment to another. Thus, in a mixed environment with continuous or discrete value, the complete value and stability is not uniform, hence affecting an agent's value acquisition process as we shall present in the next section. In general, the cognitropy transformation process is analogous to conventional currency conversion process.

4.5. *Environment Uncertainty Variation.*

The environment as an agent, takes actions on a target. This action defines its degree of uncertainty on the target, which influence the cognitive value acquisition of any agent on that target. Also, this uncertainty can be fixed or dynamic, and acts as a force (i.e., influence) on the target, which the agent needs to overcome in order to acquire value about the target. The uncertainty therefore act as an *unconscious conditioner* [98] of an agent's value acquisition process about a target.

We can describe the influence of environment uncertainty on the knowledge value of an agent for single state targets and continuous action value in a endopistemic process as shown in Figure 15(b) as follows:

$$\lim_{A(e)\to 1} V_a(g)_t = \log A(g)_t, \tag{4.125}$$

$$\lim_{A(e)\to 0} V_a(g)_t = \infty, \tag{4.126}$$

$$\lim_{(A(e),A(g))\to(0,0)} V_a(g)_t = \infty, \tag{4.127}$$

$$\lim_{(A(e),A(g))\to(0,1)} V_a(g)_t = \infty, \tag{4.128}$$

$$\lim_{(A(e),A(g))\to(1,0)} V_a(g)_t = -\infty, \tag{4.129}$$

$$\lim_{(A(e),A(g))\to(1,1)} V_a(g)_t = 0. \tag{4.130}$$

Where result of $0/0$ is based on the following axiom.

Axiom 4.4. *If by definition $a \to b$, then $\lim_{(a,b)\to(0,0)} a/b = \infty$, and if $b \to a$, then $\lim_{(a,b)\to(0,0)} a/b = 0$, where $a, b \in R^+$.*

Literally, the expression $0/0$ is mathematically undefined but is defined in the context of this research. Since both the numerator and denominator have been attributed logical meaning, and the denominator defines the numerator, hence, based on Axiom 4.4, we can conveniently say $0/0 = \infty$. In same logic, if the numerator defines the denominator, then $0/0 = 0$. These can be considered as part of an assumed *projective space*.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 24 November 2022 doi:10.20944/preprints202211.0450.v1

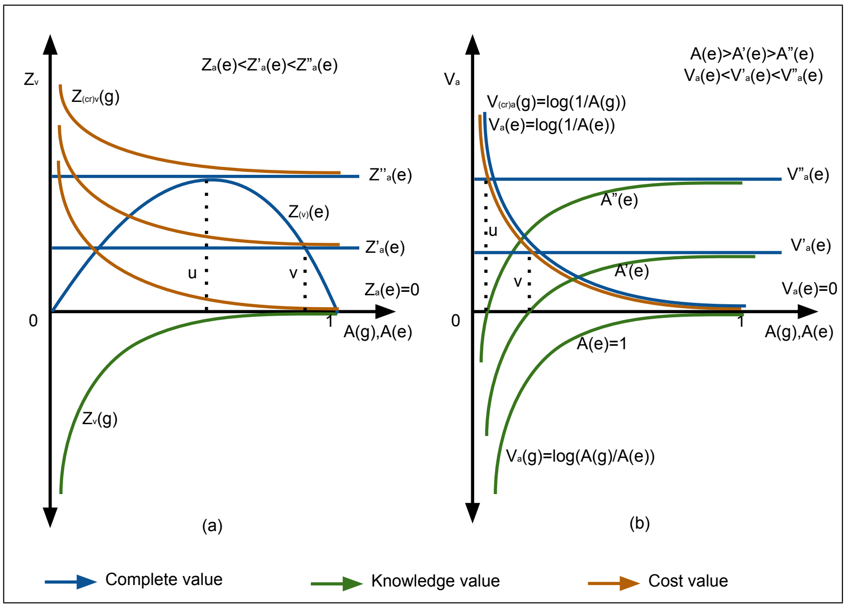

Figure 15. knowledge variation of an agent and environment in an *endopistemic process*. a) binary state target, b) single state target.

So, in the context of this research, we can generalize on the limits toward the asymptotic values of $f(a/b)$, as follows:

Proposition 21.

$$\text{If } a \to b, \lim_{(a,b))\to(0,0)} f(a/b) = \lim_{a\to 0}(\lim_{b\to 0} f(a/b)), \tag{4.131}$$

$$\text{If } a \leftarrow b, \lim_{(a,b))\to(0,0)} f(a/b) = \lim_{b\to 0}(\lim_{a\to 0} f(a/b)), \tag{4.132}$$

where a and b are single state values.

The environment influence on the expected knowledge value of an agent for multi-state target may vary differently as compared to a single state target. Figure 15(a) shows the variation for a complementary action with continuous state over a binary state target in an endopistemic process.

We can make the following conclusion about the process:

Proposition 22. *For any action $A(g)$, as $A(e) \to 1$*

$$A(e) \leq 0.5; \Delta Z_v(e) > 0, \Delta Z_{(cr)}(g) > 0, \Delta Z_v(g) > 0, \tag{4.133}$$

$$A(e) \geq 0.5; \Delta Z_v(e) < 0, \Delta Z_{(cr)}(g) < 0, \Delta Z_v(g) > 0, \tag{4.134}$$

where, $Z_v(e)_t = Z_v(g)_t + Z_{(cr)v}(g)_t$ (endopistemic process).

This implies that the influence of the environment value $Z_v(e)$ on the knowledge of an agent about a binary state target varies based on the action value $A(e)$ of the environment on the target. If $A(e) < 0.5$, an increase in $A(e)$ will increase $Z_v(e)$, which in turn will increase the cost and relative knowledge required by the agent about the target. If $A(e) > 0.5$, an increase in $A(e)$ will decrease $Z_v(e)$, which will decrease the cost, while the relative knowledge continue to increase.

Actually, Figure 15(a) represents an endopistemic learning process, in which the relative knowledge $Z_v(g)_t$ of an agent on a binary state target is required to increase. For this to be possible in an environment with a specific influence, we can either indirectly vary the cost value (i.e., cost dynamics) or directly vary the relative knowledge value (i.e., knowledge dynamics). Using the *cost dynamics*, while the environment value is constant, the cost of the agent is required to decrease in order to increase the knowledge over time.

$$Z_v(e)_t = \max[\Delta Z_v(g)_t]_\tau + \min[\Delta Z_{(cr)v}(g)_t]_\tau \tag{4.135}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 24 November 2022 doi:10.20944/preprints202211.0450.v1

The main issue in this cost oriented learning process is how much change of cost is required to reach complete value. This is given by the difference between the *actual cost* of the agent and the *complete cost* required by the agent for the process.

$$[\Delta Z_{(cr)v}(g)_t]_\tau = [Z_v(e)_t - Z_{(cr)v}(g)_t]_\tau \tag{4.136}$$

Applying Gibbs inequality to the cost change,

$$Gibbs\ \ inequality: \ Z_{(cr)v}(g)_t \geq Z_v(e)_t, \tag{4.137}$$

where $Z_{(cr)v}(g)_t \geq 0$, $Z_v(e)_t \geq 0$.

This implies that any change with the complete cost will always be positive, but how much positive it will be, depends on the value of the actual cost and complete cost value. Based on Proposition 22 and as shown in Figure 15(a), these values depend on the environment action $A(e)$, which is considered to be constant, will imply the complete cost $Z_v(e)_t$ is also constant. In this case, only the actual cost of the agent is required to fluctuate in order to vary this change and the amount of change required for an actual cost to reach complete cost defines the *learning cost* of the agent.

Since the actual cost is affected by the environment uncertainty, the learning cost is also affected by the environment uncertainty. The main difference is that, the actual cost is simply the *cost of value acquisition* while the learning cost is the *cost of value optimizing*. We may have a high actual cost but a small learning cost depending on the complete cost defined by the environment on the target. This is important in the calibration of the learning process of an agent.

These phenomena can be used to explain and address the difficulty of an agent learning endopistemically from an *imbalanced* dataset (environment observations) [99] with discrete multistates. In this case, the environment value will be low, making the complete cost value of any agent in that environment to be low. Such low complete cost will create a higher learning cost for the agent to achieve, and this will require more intelligence resource beyond what conventional agents can provide. A common solution without the use of additional intelligence is to balance the dataset, thereby increasing its complete value, which will imply increasing its complete cost and reducing its learning cost.

Also, we can use these concepts to compare the intelligence of agents. For example, two agents g_1 and g_2 with intelligence ϕ_1 and ϕ_2, are in environment e_1 and e_2 with different uncertainties about same binary state target t, can be logically related over the knowledge property as follows:

$$K(g_1)_t = K(g_2)_t, A(e_1)_t < A(e_2)_t \Rightarrow (\phi_1)_t > (\phi_2)_t, \tag{4.138}$$
$$K(g_1)_t = K(g_2)_t, A(e_1)_t > A(e_2)_t \Rightarrow (\phi_1)_t < (\phi_2)_t, \tag{4.139}$$
$$K(g_1)_t = K(g_2)_t, A(e_1)_t = A(e_2)_t \Rightarrow (\phi_1)_t = (\phi_2)_t, \tag{4.140}$$
$$K(g_1)_t > K(g_2)_t, A(e_1)_t > A(e_2)_t \Rightarrow (\phi_1)_t \leq (\phi_2)_t, \tag{4.141}$$
$$K(g_1)_t > K(g_2)_t, A(e_1)_t < A(e_2)_t \Rightarrow (\phi_1)_t > (\phi_2)_t, \tag{4.142}$$
$$K(g_1)_t > K(g_2)_t, A(e_1)_t = A(e_2)_t \Rightarrow (\phi_1)_t > (\phi_2)_t. \tag{4.143}$$

This implies, ceteris paribus, an agent g_1 in a highly uncertain environment has more intelligence on a single state target t than g_2 in a less uncertain environment if $K(g_1)_t \geq K(g_2)_t$. Similarly, for a multi-state target such as a binary target, a different comparison logic may occur between agents based on the environment uncertainty, which can also be considered as the *entropy* about the target according to Shannon theory.

An intuitive explanation we can give about the difference in the flow of knowledge during learning on a single state target and a binary state target as shown in Figure 15 is that, if we have only one choice to make about something, our satisfaction on it will vary proportionally with its definition, but if we have more options, our satisfaction get distributed over the options, and becomes unproportional with choice, making it difficult

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 24 November 2022 doi:10.20944/preprints202211.0450.v1

to make a choice. This is analogous to the application of the law of diminishing return [100] on a scale of preference.

Furthermore, the environment can be considered to be *isolated*, where only its influence on the target counts, or *non-isolated* where it interacts with different external entities which influences its influence on the target. This gives us two environmental settings which we present in the next sections.

4.5.1. Uncertainty in an Isolated Environment

In an isolated environment, no other influence is on the target apart of the influences from the set of environments that are in direct relationship (i.e., contact) with the target. Cognition in such environment may involve single or multiple action on a target. We described the complete cost, resultant value and the factors affecting cognition of multiple actions in this environment.

i. Complete cost from multiple actions

The complete cost of multiple actions on a target is defined by the environment value on the target and is related to the individual knowledge values of each action as follows:

Theorem 5. *The complete cost of multiple actions on a target in an environment is defined by the logical sum of their knowledge values on the target in the environment.*

Multiple actions in same environment (collinear),

$$\begin{aligned} &Z_{V_1}(g_1||e)_t + Z_{V_2}(g_2||e)_t + Z_{V_3}(g_3||e)_t - Z_{V_{12}}(g_{12}||e)_t, \\ &- Z_{V_{13}}(g_{13}||e)_t - Z_{V_{23}}(g_{23}||e)_t + Z_{V_{123}}(g_{123}||e)_t = Z_{V_R}(e)_t. \end{aligned} \quad (4.144)$$

Multiple actions in different environment (non-collinear),

$$\eta_{1\to R} Z_{V_1}(g)_t + \eta_{2\to R} Z_{V_2}(g)_t - \eta_{12\to R} Z_{V_{12}}(g_{12})_t = Z_{V_R}(e)_t, \quad (4.145)$$

where $Z_{V_{1,2..n}}(g_{1,2..n}||e)_t$ is a conjunction value in e,

$$Z_{V_{1,2..n}}(g_{1,2..n})_t = \sum_i A(e)_{t_i} log \frac{\prod_{j=1}^{n} A(g_j)_{t_i}}{A(e)_{t_i}}. \quad (4.146)$$

This theorem is considered as *the law of complete cost* for actions on a target in an environment. Semantic differences are considered using KTF for non-collinear action sources. The action value on the target can be discrete or continuous but we consider a discrete target and continuous action states.

If $\sum_{i=1}^{n} A(g_i) = 1$, then the action states are complementary on the target. Also, if the action states are discrete and mutually exclusive over the target, their conjunction is null, otherwise they are continuous and non-mutually exclusive.

ii. Total value from multiple actions on a target

Apart from the complete cost, there exist, a total value acquired by a set of agents at any given moment of value acquisition about a target in an environment. We consider this total value as a *resultant value* of the agents on the target, which respects a *law of total value* as follows:

Theorem 6. *The total value on a target defined by a set of directly related environments is the sum of the value from each environment and their dependencies.*

$$Z_{V_R}(g_R||e_R)_t = \sum_{i=1}^{n} \alpha_i Z_{V_i}(g_i||e_i)_t, \alpha = [0,1], \quad (4.147)$$

$$\sum_{i=1}^{n} \alpha_i Z_{V_i}(g_i||e_i)_t = \varpi_{dep} + \varpi_{ind}, \quad (4.148)$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are *actual* proportion of value contributed to the resultant value by an agent in an environment, ϖ_{dep} and ϖ_{ind} are values from the dependent and Independent

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 24 November 2022 doi:10.20944/preprints202211.0450.v1

actions of the contributed values. However, not all total value expressions can be separated into these two parts.

If the contributions to the resultant value is complementary, then $\sum_{i=1}^{n} \alpha_i = 1$. Also, the contributed environment influences $A(e)$ and agent actions $A(g)$ can be complementary on the target; $\sum_{i=1}^{n} A(\rho_i)_t = 1$, where $\rho = \{g, e\}$.

iii. Relation between resultant and contributed action

The resultant value can be expressed as a relationship between the actions of the resultant value and contributed value by applying *the principle of linearity of expectation*[79].

$$\lambda E[V] + \beta = E[\lambda V + \beta] \tag{4.149}$$

where $V = \{V_1, ..V_n\}$, and $\lambda = \{\lambda_1, ..\lambda_n\}$, $\beta = \{\beta_1, ..\beta_n\}$.

This can be written as summation over V.

$$\sum_{i=1}^{n} E[V_i] = E[V_R]. \tag{4.150}$$

For independent actions,

$$V_R = log \frac{\prod_{i=1}^{n} A(g)_i}{\prod_{i=1}^{n} A(e)_i}, \tag{4.151}$$

$$A(\rho_R)_t = \prod_{i=1}^{n} A(\rho_i)_t. \tag{4.152}$$

But for dependent actions, the mutual value is required.

$$V_{R'} = log \frac{\bigcap_{i=1}^{n} A(g_i)_t}{\bigcap_{i=1}^{n} A(e_i)_t} = V_R + \log \frac{r_g}{r_e}, \tag{4.153}$$

where $r_\rho = \frac{\bigcap_{i=1}^{n} A(\rho_i)_t}{\prod_{i=1}^{n} A(\rho_i)_t} \geq 1$.

This expectation can be generalized as follows:

Proposition 23.

$$\sum_{i=1}^{n} E[V_i] = E[V_{R'}] \text{ if } r_e = r_g, \tag{4.154}$$

$$\sum_{i=1}^{n} E[V_i] > E[V_{R'}] \text{ if } r_e > r_g, \tag{4.155}$$

$$\sum_{i=1}^{n} E[V_i] < E[V_{R'}] \text{ if } r_e < r_g. \tag{4.156}$$

Therefore, using the joint action as the resultant action, Theorem 6 can be generalized when all $\alpha_i = 1$ as follows:

Proposition 24.

$$\sum_{i=1}^{n} Z_{V_i}(g_i||e_i)_t = Z_{V_R}(g_R||e_R)_t \text{ if } r_e = r_g, \tag{4.157}$$

$$\sum_{i=1}^{n} Z_{V_i}(g_i||e_i)_t - Z_V(g_1; ..; g_n||e)_t = Z_{V_R}(g_R||e_R)_t \text{ if } r_e > r_g, \tag{4.158}$$

$$\sum_{i=1}^{n} Z_{V_i}(g_i||e_i)_t + Z_V(g_1; ..; g_n||e)_t = Z_{V_R}(g_R||e_R)_t \text{ if } r_e < r_g. \tag{4.159}$$

Alternatively, considering $0 < \alpha_i \leq 1$, we defined the resultant value as a *mixture model* [101] where the average of the individual actions is used as the contributed action.

Proposition 25.

$$A(\rho)_t = \frac{1}{n} \sum_{i=1}^{n} A(\rho_i)_t, \rho = \{g, e\}. \tag{4.160}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 24 November 2022 doi:10.20944/preprints202211.0450.v1

Applying Proposition 25 to Theorem 6. 1677
1) If $A(g)_t = \frac{1}{n}\sum_{i=1}^{n} A(g_i)_t$ and $A(e)_t = \frac{1}{n}\sum_{i=1}^{n} A(e_i)_t$,

$$Z_{V_R}(g_R||e_R)_t = \sum_{i=1}^{n} \alpha_i [\sum_{j}^{m} A(e)_{t_j} \log \frac{A(g)_{t_j}}{A(e)_{t_j}}]_i, \tag{4.161}$$

$$Z_{V_R}(g_R||e_R)_t = \frac{1}{n}[\sum_{j}^{m}(\sum_{i=1}^{n} A(e_i)_{t_j}) \log \frac{\sum_{i=1}^{n} A(g_i)_{t_j}}{\sum_{i=1}^{n} A(e_i)_{t_j}}] \sum_{i=1}^{n} \alpha_i, \tag{4.162}$$

$$\text{if } \sum_{i=1}^{n} \alpha_i = 1 \Rightarrow \frac{1}{n}[\sum_{j}^{m}(\sum_{i=1}^{n} A(e_i)_{t_j}) \log \frac{\sum_{i=1}^{n} A(g_i)_{t_j}}{\sum_{i=1}^{n} A(e_i)_{t_j}}], \tag{4.163}$$

$$\text{if } \alpha_i = \alpha_{i-1} = \cdots = \alpha_n,$$

$$\Rightarrow \alpha[\sum_{j}^{m}(\sum_{i=1}^{n} A(e_i)_{t_j}) \log \frac{\sum_{i=1}^{n} A(g_i)_{t_j}}{\sum_{i=1}^{n} A(e_i)_{t_j}}]. \tag{4.164}$$

2) If $A(g)_t = A(g_i)_t$ and $A(e)_t = \frac{1}{n}\sum_{i=1}^{n} A(e_i)_t$,

$$Z_{V_R}(g_R||e_R)_t = \sum_{i=1}^{n} \alpha_i [\sum_{j}^{m} A(e)_{t_j} \log \frac{1}{A(e)_{t_j}} - A(e)_{t_j} \log \frac{1}{A(g_i)_{t_j}}]_i, \tag{4.165}$$

$$Z_{V_R}(g_R||e_R)_t = \sum_{i=1}^{n} \alpha_i [Z_V(e)_t - Z_{(cr)V_i}(g_i||e)_t]_i, \tag{4.166}$$

$$\text{if } \sum_{i=1}^{n} \alpha_i = 1 \Rightarrow Z_V(e)_t - \sum_{i=1}^{n} \alpha_i [Z_{(cr)V_i}(g_i||e)_t]_i, \tag{4.167}$$

$$\text{if } \alpha_i = \alpha_{i-1} = \cdots = \alpha_n \Rightarrow Z_V(e)_t - \alpha \sum_{i=1}^{n} Z_{(cr)V_i}(g_i||e)_t]. \tag{4.168}$$

1678

3) If $A(g)_t = \frac{1}{n}\sum_{i=1}^{n} A(g_i)_t$ and $A(e)_t = A(e_i)_t$, 1679

$$Z_{V_R}(g_R||e_R)_t = \sum_{i=1}^{n} \alpha_i [\sum_{j}^{m} A(e_i)_{t_j} \log \frac{1}{A(e_i)_{t_j}} - A(e_i)_{t_j} \log \frac{1}{A(g)_{t_j}}]_i, \tag{4.169}$$

$$Z_{V_R}(g_R||e_R)_t = \sum_{i=1}^{n} \alpha_i [Z_V(e_i)_t - Z_{(cr)V_i}(g||e_i)_t]_i, \tag{4.170}$$

$$\text{if } \sum_{i=1}^{n} \alpha_i = 1 \Rightarrow \sum_{i=1}^{n} \alpha_i [Z_V(e_i)_t - Z_{(cr)V_i}(g||e_i)_t]_i, \tag{4.171}$$

$$\text{if } \alpha_i = \alpha_{i-1} = \cdots = \alpha_n \Rightarrow \alpha \sum_{i=1}^{n} [Z_V(e_i)_t - Z_{(cr)V_i}(g||e_i)_t]_i. \tag{4.172}$$

In a cognition where the endopistemic action is directly related to the exopistemic action, the environment and agent actions can be defined as a function of each other.

$$A(e) = f(A(g_1), A(g_1), \ldots A(g_n)), \tag{4.173}$$

$$A(g) = f(A(e_1), A(e_2), \ldots A(e_n)). \tag{4.174}$$

1680

The definition the function $f(\cdot)$ over the action values will depend on the relationship 1681
between the exopistemic and endopistemic action. Considering a resultant action value 1682
defined in Equation (3.57), where the resultant action is the average value of the contributed 1683
actions, then, 1684

Proposition 26.

$$A(e)_t = \frac{\sum_{i=1}^{n} A(g)_t}{n}, \tag{4.175}$$

$$A(g)_t = \frac{\sum_{i=1}^{m} A(g)_t}{m}, \tag{4.176}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 24 November 2022 doi:10.20944/preprints202211.0450.v1

where n and m are the number of contributed agent and environment actions, respectively.

Equation (4.176) will actually yield a total value identical to negative *Jensen-Shannon divergence* (JSD) [102] for two actions with $\alpha = 0.5$.

Applying Proposition 26 to Theorem 6.

1) If $A(g)_t = A(e)_t$,

$$Z_{V_R}(g_R||e_R)_t = 0. \tag{4.177}$$

2) If $A(g)_t = A(g_i)_t$ *and* $A(e)_t = \frac{1}{n}\sum_{i=1}^n A(g_i)_t$,

$$Z_{V_R}(g_R||e_R)_t = \sum_{i=1}^{n} \alpha_i [Z_V(e)_t - Z_{(cr)V_i}(g_i||e)_t]_i. \tag{4.178}$$

3) If $A(g)_t = \frac{1}{n}\sum_{i=1}^n A(e_i)_t$ *and* $A(e)_t = A(e_i)_t$,

$$Z_{V_R}(g_R||e_R)_t = \sum_{i=1}^{n} \alpha_i [Z_V(e_i)_t]_i - \sum_{i=1}^{n} \alpha_i [Z_{(cr)V_i}(g||e_i)_t]_i, \tag{4.179}$$

$$\text{if } n = 2, \alpha_i = 0.5 \Rightarrow 0.5 \sum_{i=1}^{2} [Z_V(e_i)_t]_i - Z_{(c)V}(g)_t = -JSD. \tag{4.180}$$

These propositions provide different ways of expressing a total (i.e., resultant) value on a target in a mixed environment based on the contributed values. For a more direct and naive approach, we can define the average or joint values of the contributed actions as the resultant action. That is,

Proposition 27.

$$A(\rho_R)_t = \frac{1}{n}\sum_{i=1}^{n} A(\rho_i)_t, \rho = \{g, e\}. \tag{4.181}$$

Applying Proposition 27 to Theorem 6.

1) If $A(g_R)_t = \frac{1}{n}\sum_{i=1}^n A(g_i)_t$ and $A(e_R)_t = \frac{1}{n}\sum_{i=1}^n A(e_i)_t$,

$$Z_{V_R}(g_R||e_R)_t = \frac{1}{n}[\sum_{j}^{m}(\sum_{i=1}^{n} A(e_i)_{t_j}) \log \frac{\sum_{i=1}^n A(g_i)_{t_j}}{\sum_{i=1}^n A(e_i)_{t_j}}]. \tag{4.182}$$

2) If $A(g_R)_t = \prod_{i=1}^n A(g_i)_t$ and $A(e_R)_t = \frac{1}{n}\sum_{i=1}^n A(e_i)_t$,

$$Z_{V_R}(g_R||e_R)_t = \sum_{j=1}^{m} A(e_R)_{t_j} \log \frac{\prod_{i=1}^n A(g_i)_{t_j}}{A(e_R)_{t_j}} = \sum_{i=1}^{n} Z_{V_i}(g_i||e_R)_t. \tag{4.183}$$

3) If $A(g_R)_t = \frac{1}{n}\sum_{i=1}^n A(g_i)_t$ and $A(e_R)_t = \prod_{i=1}^n A(e_i)_t$,

$$Z_{V_R}(g_R||e_R)_t = \sum_{j=1}^{m} A(e_R)_{t_j} \log \frac{\frac{1}{n}\sum_{i=1}^n A(g_i)_t}{A(e_R)_{t_j}} = Z_{V_i}(g_R||e_R)_t. \tag{4.184}$$

iv. Resultant value across target and agent

In general, the exopistemic (i.e., influence) and endopistemic (i.e., dependency) process on a target can be considered respectively as an *input and output* process of agents on the target. Logically, if the output equals the output, then the resultant (i.e., total) effect on the target will be zero. We used this notion to generalized on the relationship between the exopistemic (i.e., input) V_i and endopistemic (i.e., output) V_o values on a target as follows:

Proposition 28.

$$V_o - V_i = V_R \text{ (values on single state target)}, \tag{4.185}$$

$$Z_o - Z_i = Z_R \text{ (expected value for multistates)}. \tag{4.186}$$

Where for an endopistemic resultant across the target,

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 24 November 2022 doi:10.20944/preprints202211.0450.v1

$V_o = V_o(g) = \log A(g)$, $V_i = V_i(e) = \log A(e)$, $V_R = V_R(g||e) = \log \frac{A(g)}{A(e)}$, $Z_o = Z_o(g) = \sum A(e) \log A(g)$, $Z_i = Z_i(e) = \sum A(e) \log A(e)$, *and* $Z_R = Z_R(g||e) = \sum A(e) \log \frac{A(g)}{A(e)}$

Where for an exopistemic resultant across the target,

$V_o = V_o(g) = \log \frac{1}{A(g)}$, $V_i = V_i(e) = \log \frac{1}{A(e)}$, $V_R = V_R(e||g) = \log \frac{A(e)}{A(g)}$, $Z_o = Z_o(e) = \sum A(e) \log \frac{1}{A(g)}$, $Z_i = Z_i(e) = \sum A(e) \log \frac{1}{A(e)}$, *and* $Z_R = Z_R(e||g) = \sum A(e) \log \frac{A(e)}{A(g)}$

These have different implications on the target. We established these implications below and illustrate it in Figure 16.

Proposition 29.

For single state target, Proposition 28 implies that,

$$V_R = 0 \text{ iff } V_o = V_i \quad \text{(neutral resultant)}, \tag{4.187}$$
$$V_R < 0 \text{ iff } V_o < V_i \text{ (endopistemic resultant)}, \tag{4.188}$$
$$V_R > 0 \text{ iff } V_o > V_i \text{ (exopistemic resultant)}. \tag{4.189}$$

For multistate target, Proposition 28 implies that,

$$Z_R = 0 \text{ iff } Z_o = Z_i \quad \text{(neutral resultant)}, \tag{4.190}$$
$$Z_R < 0 \text{ iff } Z_o < Z_i \text{ (endopistemic resultant)}, \tag{4.191}$$
$$Z_R > 0 \text{ iff } Z_o > Z_i \text{ (exopistemic resultant)} . \tag{4.192}$$

It is important to realise that Equation (4.192) represent *Gibbs inequality*. This shows that the resultant value flow based on input and output values across a target obeys some natural laws defined in Proposition 29.

This implies that, for example if the resultant value on a target is negative, then there is more exopistemic (i.e., input or influence) values on the target than endopistemic (i.e., output or dependency) values, and vice versa.

For multiple types (or state) of actions (e.g., observation, actuation, etc.) on a target, the total resultant value becomes,

Proposition 30. *The total resultant value on a target is the sum of the resultant values of all action types on the target.*

$$\omega_R(g)_t = \omega_R(g_1)_t + ... + \omega_R(g_n)_t + ... + \omega_R(g_{m;n})_t, \tag{4.193}$$

where $\omega_R(g_i)_t$ is the resultant value of action type i, $\omega_R(g_{i;j})_t$ is the resultant mutual value between action types i and j, $\omega_R(g)_t$ is the total resultant value on the target.

So, for example if there is more exopistemic resultant values on a target than endopistemic resultant values, then the target will be more under influence than dependency.

In a similar way, the resultant value across an agent can be defined based on input and output values of its action on a target. Since an agent does work on a target to generate value, this work can be its absolute or relative value depending on the *cost dynamics* it is using, the input and output values of any type of its action will differ from that of a target. Considering an *absolute value cost dynamic, i.e., input*, and a *relative value as output*, then the resultant value of any action type of an agent based on its input and output values are:

Proposition 31.

$$V_o - V_i = V_R \text{ (values on single state target)}, \tag{4.194}$$
$$Z_o - Z_i = Z_R \text{ (expected value for multistates)}. \tag{4.195}$$

Where for an endopistemic input and output of an agent,

$V_o = V_o(g||e) = \log \frac{A(g)}{A(e)}$, $V_i = V_i(g) = \log A(g)$, $V_R = V_R(e) = \log \frac{1}{A(e)}$, $Z_o = Z_o(g||e) =$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 24 November 2022 doi:10.20944/preprints202211.0450.v1

$\sum A(e) \log \frac{A(g)}{A(e)}$, $Z_i = Z_i(g) = \sum A(e) \log A(g)$, *and* $Z_R = Z_R(e) = \sum A(e) \log \frac{1}{A(e)}$.
where for an exopistemic input and output of an agent,
$V_o = V_o(e||g) = \log \frac{A(e)}{A(g)}$, $V_i = V_i(g) = \log \frac{1}{A(g)}$, $V_R = V_R(e) = \log A(e)$, $Z_o = Z_o(e||g) = \sum A(e) \log \frac{A(e)}{A(g)}$, $Z_i = Z_i(g) = \sum A(e) \log \frac{1}{A(g)}$, *and* $Z_R = Z_R(e) = \sum A(e) \log A(e)$.

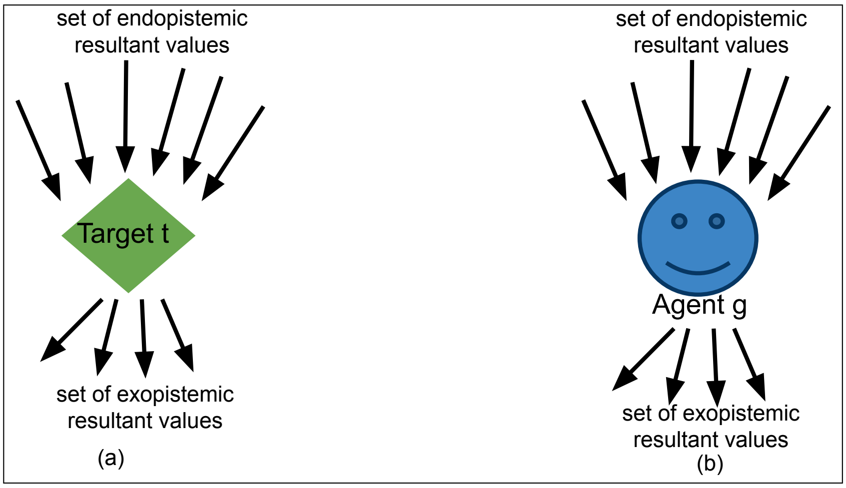

Figure 16. set of exopistemic and endopistemic resultant values across (a) target, (b) agent

This have different implications on the agent. We establish these implications below.

Proposition 32.
For single state target, Proposition 31 imply that,

$$V_R = 0 \text{ iff } V_o = V_i \quad \text{(neutral resultant)} \tag{4.196}$$
$$V_R < 0 \text{ iff } V_o < V_i \text{ (endopistemic resultant)} \tag{4.197}$$
$$V_R > 0 \text{ iff } V_o > V_i \text{ (exopistemic resultant).} \tag{4.198}$$

For multistate target, Proposition 31 imply that,

$$Z_R = 0 \text{ iff } Z_o = Z_i \quad \text{(neutral resultant),} \tag{4.199}$$
$$Z_R < 0 \text{ iff } Z_o < Z_i \text{ (endopistemic resultant),} \tag{4.200}$$
$$Z_R > 0 \text{ iff } Z_o > Z_i \text{ (exopistemic resultant).} \tag{4.201}$$

From Proposition 31, the resultant value is the value of the environment on the target, and it is endopistemic when the input and output of an agent are exopistemic and vice versa. Hence, the resultant value on an agent acts as a *resistance* to the acquisition and flow of value across the agent.

This implies that, for example if the resultant value across an agent is negative (i.e., endopistemic), then the agent have more exopistemic input than exopistemic output and vice versa.

For multiple types of actions (e.g., observation, actuation, etc) of an agent, the total resultant value is given as,

Proposition 33. *The total resultant value on an agent is the sum of the resultant values of all its action types.*

$$\omega_R(g)_t = \omega_R(g_1)_t + ... + \omega_R(g_n)_t + ... + \omega_R(g_{m;n})_t, \tag{4.202}$$

where $\omega_R(g_i)_t$ *is the resultant value on action type* i, $\omega_R(g_{i;j})_t$ *is the resultant mutual value between action types* i *and* j, *and* $\omega_R(g)_t$ *is the total resultant value on the agent.*

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 24 November 2022 doi:10.20944/preprints202211.0450.v1

So, for example, if an agent has more exopistemic resultant values than endopistemic resultant values, then the agent will be more under influence than dependency.

v. Number of actions and targets in an environment

Based on the resultant value expressions in the previous section, we distinguish two other factors influences value acquisition; *number of actions* and, their *degree of dependency*.

Using Theorem 6, Proposition 30, and Proposition 33 we deduce the relationship between the *number of actions* and total resultant value as follows:

Proposition 34.

$$\omega_R(g)_t = \sum_{j=1}^{j=m} \sum_{i=1}^{i=n} \omega_R(g_i)_{t_i} + \sum_{j=1}^{j=m} \sum_{i=1}^{i=n} \omega_R(g_{m;n})_{t_j}, \tag{4.203}$$

if $n \to \infty$ *or* $m \to \infty \Rightarrow \omega_R(g)_t \to \infty$,
if $\omega_R(g_i)_{t_j} \to \infty$ *or* $\omega_R(g_{m;n})_{t_j} \to \infty \Rightarrow \omega_R(g)_t \to \infty$.

So, resultant values will increase when the number of actions and target types increases and when their mutual dependency values increase. But, since such actions may seek to influence (i.e., exopistemic) while others may seek to depend (i.e., endopistemic), the addition of more exopistemic actions and targets for example, will lead to an increase in the exopistemic value of the total resultant value but a decrease in the endopistemic value. Thus, for an effective cognition, the agent will need to consider which type of action and target it needs to optimize the value it seeks.

According to Proposition 31, the resultant value on an agent is the environment value on the agent. Increasing the number of environments will thus increase the total resultant value on the agent. This is important as it also implies more output value will be expected from such high influence. This can be used to tackle complex target, by dividing it into different sub environments. This concept can be used to explain the high performance of the *Ensemble learning methods* [103] of Machine learning, where environment partitioning is used.

Also, increasing the mutual value of the total resultant value on an agent will increase the total resultant value on the agent. This will increase the output value of the agent. One way to achieve this during cognition is to extract as much information about the dependencies of the environment domains (input and output domains of a target) as possible. This concept is used in *Variational learning* [?],[105],[106] and *Information maximization* [107] methods of Machine learning.

4.5.2. Uncertainty in a Non-Isolated environment

In a non-isolated environment, external forces can influence the environment's action on a target. We describe the external influence in two types of entity relationship structures; *hierarchical and network structures*.

i. Hierarchical structure

In Proposition 3, we propose a dependency relationship between entities. Such dependency can be unidirectional or bidirectional and can flow from top-down or bottom-up. We can define such dependencies between entities as a hierarchical relationship using the following logic:

If $\rho_1 \to \rho_2 \cdots \to \rho_n$,
then by using Proposition 4.3,

$$Z(\rho_n) = Z(\rho_{n-1}) + C(\rho_{n-1}), \tag{4.204}$$

and Proposition 1,

$$Z(\rho_n) = Z(\rho_{n-1}) + \bar{Z}(\rho_{n-1}) - Z\bar{Z}(\rho_{n-1}). \tag{4.205}$$

Referencing from ρ_i to ρ_n and applying Proposition 16,

$$Z(\rho_i) = Z(\rho_n) - [C(\rho_{n-1}) + \cdots + C(\rho_i)] \tag{4.206}$$

$$Z(\rho_i) = Z(\rho_n) - [(\bar{Z}(\rho_{n-1}) - Z\bar{Z}(\rho_{n-1})) + \\ ...(\bar{Z}(\rho_i) - Z\bar{Z}(\rho_i))], \tag{4.207}$$

Since $C(\rho) > 0 \Rightarrow Z(\rho_i) < Z(\rho_n)$.
Referencing from ρ_n to ρ_i and applying Proposition 16,

$$Z(\rho_n) = Z(\rho_i) + [C(\rho_i) + \cdots + C(\rho_{n-1})], \tag{4.208}$$

$$Z(\rho_n) = Z(\rho_i) + [(\bar{Z}(\rho_i) - Z\bar{Z}(\rho_i)) + ..(\bar{Z}(\rho_{n-1}) - Z\bar{Z}(\rho_{n-1})], \tag{4.209}$$

Since $C(\rho) > 0 \Rightarrow Z(\rho_i) < Z(\rho_n)$.
Where using Proposition 4.3, and 1,

$$C(\rho_i) = \bar{Z}(\rho_i) - Z\bar{Z}(\rho_i). \tag{4.210}$$

From the above logic, we can deduce that,

Proposition 35. *Giving any hierarchical dependency of entities on a target, the complete cost increases from the dependent entity to the influencing entity and decreases from the influencing entity to the dependent entity.*

This implies that, the value acquisition process about targets existing in higher value levels is difficult to be achieved by lower level agents. This is because the lower level agents need to overcome additional *cost* imposed by the environment value at each higher level in order to achieve complete value about the targets at the higher levels.

So, the lower level (or dependent) entities have much uncertainty or noise when referenced to a higher level (or influential) entities, and this uncertainty acts as a weight on them that prevented them from achieving the complete values of a target in the higher cognitive level.

Furthermore, considering an entity in an infinite level in the hierarchical chain, its value relative to any entity is,

If $\rho_1 \rightarrow \rho_2 \cdots \rightarrow \rho_\infty$, where $n = \infty$, then,
referencing from ρ_i to ρ_n

$$Z(\rho_i) = Z(\rho_n) - [\sum_{u=i}^{u=n-1} C(\rho_u)] = -\infty, \tag{4.211}$$

referencing from ρ_n to ρ_i,

$$Z(\rho_n) = Z(\rho_i) + [\sum_{u=i}^{u=n-1} C(\rho_u)] = \infty. \tag{4.212}$$

Hence, the relationship between an agent at any dependency level with those at the higher and lower infinite level can be generalized as follows:

Proposition 36. *An entity at any dependency level is infinitely inferior on value about a target relative to agents in an infinitely higher level but is infinitely superior relative to agents in an infinitely lower level.*

It should be noted that the infinite level in this context is relative to each level and its agents because the infinite level of an agent in a lower level may not be the infinite level of another agent at a higher level. So, the closer an agent gets to its infinite level, through *level value acquisition and cost reduction*, the less infinite the level becomes.

Moreover, apart from the increase in cost of an agent as its *dependency chain* increases, the stability and relative change of value between cognitive levels, will also change as proposed below.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 24 November 2022 doi:10.20944/preprints202211.0450.v1

Proposition 37. *From a low to a high dependency level, the relative change of value is positive and the stability over the value decreases, and vice versa.*

Let $R^Z_{i\to n}$ be value change from ith to nth level, $n > i$, then,
from any dependent entity ρ_i to an nth level influencer ρ_n,

$$\text{relative } change;\ R^Z_{i\to n} = \frac{Z(\rho_n) - Z(\rho_i)}{Z(\rho_i)} \geq 0, \tag{4.213}$$

$$\text{stability};\ S_{R_{i\to n}} = \lim_{Z_i \to Z_n} \left(\frac{\partial R^Z_{i\to n}}{\partial Z_i}\right) \approx \frac{1}{Z_n}, \tag{4.214}$$

$$\tag{4.215}$$

from any nth level influencer ρ_n to a dependent entity ρ_i,

$$\text{relative } change;\ R^Z_{n\to i} = \frac{Z(\rho_i) - Z(\rho_n)}{Z(\rho_n)} \leq 0, \tag{4.216}$$

$$\text{stability};\ S_{R_{n\to i}} = \lim_{Z_n \to Z_i} \left(\frac{\partial R^Z_{n\to i}}{\partial Z_n}\right) \approx \frac{1}{Z_i}. \tag{4.217}$$

$$\text{Hence, since } Z_n > Z_i \Rightarrow \frac{1}{Z_i} > \frac{1}{Z_n}. \tag{4.218}$$

where $Z(\rho_n)$ can be the complete cost value (i.e., environment value) or the knowledge at level n.

Thus, as an agent approach its environment level, the lesser is the stability of the relative change with respect to its value and vice versa. This is illustrated in Figure 17.

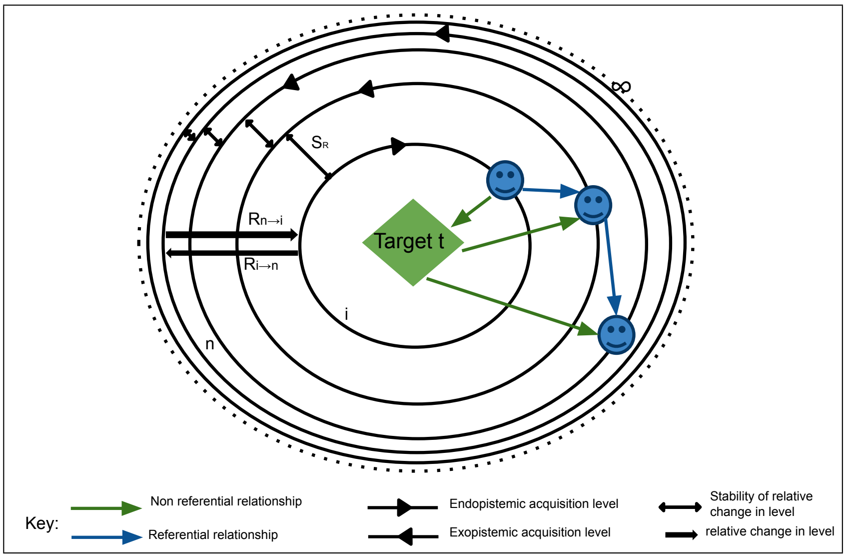

Figure 17. Valuation of agents from different hierarchical level on a target

This relative relationship between dependence levels represents another dimension of value quantification which is external to an agent. We consider this as an *external value quantification* of an agent in a non-isolated environment. All the value operations and their logic we have discussed are also applicable to this external value.

ii. Network structure

Set of entities may operate in same or different environment space over separate times. Thus, their values may differ in time and space. If such entities are related over the time and space domain, such that their values can be transferred or exchanged, then a *network of dependencies or influences* between the entities can be established. Such dependency or influence can be unidirectional or bidirectional and can flow from back-front or front-back.

In this situation, each set of entities during a time period form an *entity generation*, and each set of entities in a space form an *environment* such as a community, world or universe

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 24 November 2022 doi:10.20944/preprints202211.0450.v1

as describe in Section 2.2. In both cases, we make the following assumption on the variation of the value of each entity in a network.

Axiom 4.5. *The change in value of an entity in a network does not only on its personal intelligence and actions but also on the values acquired from related set of entities.*

This assumption forms a fundamental concept in *value transfer* as we shall see in Section 6. Also, the value estimations in a network of entities can be obtained using the propositions under hierarchical entities.

For example, let t be the time period for value acquisition in an environment e, by a networked of n entity generations.

$\rho_1, \rho_2, \rho_3, \ldots, \rho_n$, where each generation depends on the preceding one as follows: $\rho_1 \rightarrow \rho_2$, $(\rho_2, \rho_1) \rightarrow \rho_3$, $(\rho_3, \rho_1) \rightarrow (\rho_4, \rho_2), \ldots, \rho_{n-1} \rightarrow \rho_n$.

ρ_n is an independent and back-edge entity in the network, then to get the value for any *ith* entity generation, we simply use Proposition 35. This also applies for network over different spaces.

4.5.3. Uncertainty in a Close or Open Environment

A non-isolated environment can further be considered to be open or close as described in Section 2.2.2. The influence by such type of environments are same as described in Section 4.5.2, but their difference is on the exchange of entities and properties between the environment and its surrounding. Such exchange will definitely distort the dependency structure of entities and properties of the environment, thereby affecting value acquisition in such environment.

A common example is the addition of new observation sequences to the environment during cognition, which distorts the value of the environment observation, thereby affecting value acquisition of any agent referencing such environment. More on such dynamics will be discussed in a future publication.

Conventionally, complete value acquisition about a target in an environment is done using the information provided by the environment. Actually, such acquisition may have two main challenges; information provided by the environment may be insufficient for the agent to attain complete value, the agent does not have the full ability to attain complete value using the type of information provided by the environment.

To resolve this, the agent may instead benefit and learn from values generated by other agents in same or different environment, executing same or related actions about same or related targets. We discuss this in Section 6.

5. Value Optimization

Value optimization is an action operation in agents that enable them to maximize or minimize a cognitive value they have acquired about a target in an environment in order to attain the complete value of a target.

In this section, we present two types of optimization operations on cognitive value in an agent; the *learning and unlearning operations*. In Section 2.1.2, learning was defined as an optimization process which involves an increase in cognitive value. In same way, we define unlearning as follows:

Definition 87. ***Unlearning**, $\bar{\gamma}$, is an optimization process which involves the decrease in cognitive value through revision and update.*

Thus, learning a cognitive value implies unlearning the inverse of the cognitive value. For example, maximizing knowledge implies minimizing ignorance. Learning not to know a target is same as unlearning what is known about the target.

Since optimization is a dynamic process through which value changes, one key issue is to define the value that drives the dynamism of such a process. The absolute and relative

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 24 November 2022 doi:10.20944/preprints202211.0450.v1

values are two of such values that are dependent can use for such purpose, while the influencer agents can further access the complete cost in addition to those twos.

Any of these values can be used as a source value to drive the optimization process of both the dependent and influencer agent. In this regard, the exopistemic and endopistemic nature of the optimized value need to be taken into account. The optimized values are considered as the output values of an agent and a target for both agent and target optimization, respectively. Such output values where given in Section 4.5.1(iv).

Target optimization is to optimize the output values of a target, while agent optimization is to optimize the output value of an agent. Optimizing the output value entails optimizing the corresponding resultant value. Nevertheless, direct optimization of the resultant value can also be done if the value is reachable to the agent. This will indirectly also optimize the output value. We shall focus on output optimization of agents.

Using Theorem 2 *(law of conservation of value)*, we define the optimization process for relative exopistemic and endopisemic process using absolute and relative values as cost values. From Theorem 2,

$$\Delta_{g,e} Z_v(e) = \Delta_{g,e} Z_v(g) + \Delta_{g,e} Z_{(cr)v}(g). \tag{5.1}$$

1) Relative endopistemic process optimization of an agent

$$\underset{g}{\text{opt}}[Z_v(e)] = \underset{g}{\text{opt}}[Z_v(g)] + \underset{g}{\text{opt}}[Z_{(cr)v}(g)], \tag{5.2}$$

where $Z_v(g) = \sum_{i=1}^{n} A(e) \log \frac{A(g)_{t_i}}{A(e)_{t_i}}$, $Z_v(e) = \sum_{i=1}^{n} A(e) \log \frac{1}{A(e)_{t_i}}$, and $Z_{(cr)v}(g) = \sum_{i=1}^{n} A(e) \log \frac{1}{A(g)_{t_i}}$, $\text{opt} = \{\max, \min\}$.

If absolute value is used as input and relative value as output.

$$\underset{g}{\text{opt}}[Z_v(g)] = Z_v(e) - \underset{g}{\text{opt}}[Z_{(cr)v}(g)]. \tag{5.3}$$

If relative value is used as input and absolute value as output.

$$\underset{g}{\text{opt}}[Z_{(cr)v}(g)] = Z_v(e) - \underset{g}{\text{opt}}[Z_v(g)]. \tag{5.4}$$

2) Relative exopistemic process optimization of an agent.

$$\underset{e}{\text{opt}}[Z_v(e)] = \underset{e}{\text{opt}}[Z_v(g)] + \underset{e}{\text{opt}}[Z_{(cr)v}(g)], \tag{5.5}$$

where $Z_v(g) = \sum_{i=1}^{n} A(e) \log \frac{A(e)_{t_i}}{A(g)_{t_i}}$, $Z_v(e) = \sum_{i=1}^{n} A(e) \log A(e)_{t_i}$, and $Z_{(cr)}(g) = \sum_{i=1}^{n} A(e) \log A(g)_{t_i}$, $\text{opt} = \{\max, \min\}$.

If absolute value is used as input and relative value as output.

$$\underset{e}{\text{opt}}[Z_v(g)] = \underset{e}{\text{opt}}[Z_v(e)] - \underset{e}{\text{opt}}[Z_{(cr)v}(g)]. \tag{5.6}$$

If relative value is used as input and absolute value as output.

$$\underset{e}{\text{opt}}[Z_{(cr)v}(g)] = \underset{e}{\text{opt}}[Z_v(e)]_e - \underset{e}{\text{opt}}[Z_v(g)]. \tag{5.7}$$

Since exopistemic process and endopistemic process are inverse of each other, Equation (5.2) and (5.5) are obtain from each other by multiplying all terms with an inverse factor, -1.

Also, an endopistemic optimization can be carried on an exopistemic value and vice versa, by changing the optimizer. That is, a relative endopistemic value is optimized over e instead of g, and a relative exopistemic value is optimized over g instead of e.

Furthermore, the optimization of an agent is a process done over space and time, as described in the next two sections.

5.1. Optimization Over Space

The optimization operation is conventionally done with respect to the intelligence space of an agent. Nevertheless, other space domains of an agent such as the information,

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 24 November 2022 doi:10.20944/preprints202211.0450.v1

action and cognitive value domains can be used. In this research, we stick with the conventional approach.

Focusing on relative endopistemic knowledge, the *law of conservation of value* will separate the relative endopistemic value into the complete value and the cost as follows:

$$Z_v(g) = Z_{(c)v}(g) - Z_{(cr)v}(g), \tag{5.8}$$

where $Z_{(c)v}(g)$ is the complete Znetropy about the target, and $Z_{(cr)v}(g)$ is cross Znetropy about the target.

Optimizing $Z_v(g)$ with respect to ϕ of g can then be expressed for both learning and unlearning processes as follows:

1) Learning process

$$\max_{\phi \in g}[Z_v(g)] = \max_{\phi \in g}[Z_{(c)v}(g)] - Z_{(cr)v}(g)], \tag{5.9}$$

$$\max_{\phi \in g}[Z_v(g)] = \max_{\phi \in g}[Z_{(c)v}(g)] + \min_{\phi \in g}[Z_{(cr)v}(g)]. \tag{5.10}$$

Since $Z_{(c)v}(g)$ is the complete cost over which the intelligence ϕ of agent g can generate the complete (i.e., optimum) value about the target t, then given the intelligence of g, $Z_{(c)v}(g)$ cannot be maximized, hence, we can safely write,

$$\max_{\phi \in g}[Z_v(g)] = [Z_{(c)v}(g)] + \min_{\phi \in g}[Z_{(cr)v}(g)]. \tag{5.11}$$

The intelligence space that will enable the action that generates the maximum value is expressed as,

$$\phi = \operatorname*{argmax}_{\phi \in g}[Z_v(g)] = \operatorname*{argmax}_{\phi \in g}[Z_{(c)v}(g) - Z_{(cr)v}(g)]. \tag{5.12}$$

2) Unlearning process

$$\min_{\phi \in g}[Z_v(g)] = \min_{\phi \in g}[Z_{(c)v}(g)] - Z_{(cr)v}(g)], \tag{5.13}$$

$$\min_{\phi \in g}[Z_v(g)] = \min_{\phi \in g}[Z_{(c)v}(g)] + \max_{\phi \in g}[Z_{(cr)v}(g)]. \tag{5.14}$$

Similarly to learning, the agent g cannot minimize its complete value $Z_{(c)v}(g)$, hence, we can safely write,

$$\min_{\phi \in g}[Z_v(g)] = [Z_{(c)v}(g)] + \max_{\phi \in g}[Z_{(cr)v}(g)]. \tag{5.15}$$

The intelligence space that will enable the action that generates the minimum value is expressed as:

$$\phi = \operatorname*{argmin}_{\phi \in g}[Z_v(g)] = \operatorname*{argmin}_{\phi \in g}[Z_{(c)v}(g) - Z_{(cr)v}(g)]. \tag{5.16}$$

Furthermore, since the optimization process is itself a type of action of an agent, it can also be optimize, thereby implying a double optimization. Hence, we can say an agent can *learn to learn, unlearn to learn, learn to unlearn and unlearn to unlearn* as describe below.

Let,

$$\max_{\phi_\gamma \in g}[Z_v(g)] \equiv [Z_v(g)]_{\phi_\gamma} \tag{5.17}$$

$$\min_{\phi_{\bar{\gamma}} \in g}[Z_v(g)] \equiv [Z_v(g)]_{\phi_{\bar{\gamma}}}, \tag{5.18}$$

then, $[[Z_v(g)]_{\phi_\gamma}]_{\phi_\gamma}$ is learning to learn,
$[[Z_v(g)]_{\phi_\gamma}]_{\phi_{\bar{\gamma}}}$ is unlearning to learn,
$[[Z_v(g)]_{\phi_{\bar{\gamma}}}]_{\phi_\gamma}$ is learning to unlearn,
$[[Z_v(g)]_{\phi_{\bar{\gamma}}}]_{\phi_{\bar{\gamma}}}$ is unlearning to unlearn.

This can also be extended to multiple dimension of optimizations or actions. In this research, we stick with the one dimensional optimization of an agent.

5.2. Optimization Over Time

Time is an important property for any process because it can be used to capture the evolution of the process, and how time is defined depends on the process it is associated with. In this section, we define a time property based on the optimization process of an agent. We call this property the *optimization time* or simply the *learning time*.

Definition 88. ***Optimization time***, *τ, is one complete cycle of optimization based on a number of actions (observations, reasoning, actuations, etc.) of an agent on a target.*

Conventionally, this is also called learning epoch or cycle.

A time property can also be defined for each of the different actions in a learning cycle such as observation, actuation, etc., but we focus on the learning time in this research.

Based on Definition 88, the optimization over intelligence space of the relative endopistemic value will be further optimized over time as follows:

$$\underset{\phi\in g}{\text{opt}}[Z_v(g)]_\tau = \underset{\phi\in g}{\text{opt}}[Z_{(c)v}(g) - Z_{(cr)v}(g)]_\tau, \tag{5.19}$$

where opt $= \{\max, \min\}$, τ is the learning time.

The intelligence space that will enable the action to generate the optimized value over time is expressed as,

$$\phi_\tau = \underset{\phi\in g}{\text{argopt}}[Z_v(g)]_\tau = arg\,\underset{\phi\in g}{\text{opt}}[Z_{(c)v}(g) - Z_{(cr)v}(g)]_\tau. \tag{5.20}$$

Achieving this intelligence is done through an update process of its space using the generated cognitive value. This process is repeated over time until the cognitive value reaches a maximum during learning or minimum during unlearning.

For example, in a *gradient based optimization process*, the intelligence is updated using the gradient of the cognitive value over intelligence. This gradient is same as the stability of the cognitive value over the intelligence space. Taking *gradient descent and ascent* techniques as use cases,

$$[\phi]_{\tau_{i+1}} = [\phi]_{\tau_i} - [\delta\phi]_{\tau_i} \ \text{(gradient descend)}, \tag{5.21}$$

$$[\phi]_{\tau_{i+1}} = [\phi]_{\tau_i} + [\delta\phi]_{\tau_i} \ \text{(gradient ascent)}. \tag{5.22}$$

1) During Learning

$$[\delta\phi]_\tau = \underset{\delta\phi\in g}{\text{argmax}}[\delta Z_v(g)]_\tau = \gamma_L \max_{\phi\in g}[\frac{\partial Z_v(g)}{\partial\phi}]_\tau, \tag{5.23}$$

$$but, \max_{\phi\in g}[\frac{\partial Z_v(g)}{\partial\phi}]_\tau = \max_{\phi\in g}[\frac{\partial Z_{(c)v}(g)}{\partial\phi}]_\tau + \min_{\phi\in g}[\frac{\partial Z_{(cr)v}(g)}{\partial\phi}]_\tau. \tag{5.24}$$

Since agent g cannot optimize $Z_{(c)v}(g)$, and if $Z_{(c)v}(g)$ is constant, then,

$$\max_{\phi\in g}[\frac{\partial Z_v(g)}{\partial\phi}]_\tau = \min_{\phi\in g}[\frac{\partial Z_{(cr)v}(g)}{\partial\phi}]_\tau. \tag{5.25}$$

The intelligence update function during gradient descent learning becomes,

$$[\phi]_{\tau_{i+1}} = [\phi]_{\tau_i} - \gamma_L \min_{\phi\in g}[\frac{\partial Z_{(cr)v}(g)}{\partial\phi}]_{\tau_i}. \tag{5.26}$$

This can also be expressed using gradient ascent as follows:

$$[\phi]_{\tau_{i+1}} = [\phi]_{\tau_i} + \gamma_L \min_{\phi\in g}[\frac{\partial(-Z_{(cr)v}(g))}{\partial\phi}]_{\tau_i}, \tag{5.27}$$

$$[\phi]_{\tau_{i+1}} = [\phi]_{\tau_i} - \gamma_L \max_{\phi\in g}[\frac{\partial(-Z_{(cr)v}(g))}{\partial\phi}]_{\tau_i}, \tag{5.28}$$

where $[\frac{\partial Z_v(g)}{\partial\phi}]$,$[\frac{\partial Z_{(cr)v}(g)}{\partial\phi}]$, $[\frac{\partial Z_{(c)v}(g)}{\partial\phi}]$ are respective stability values $[S_{Z_v}(g)_\phi]$, $[S_{Z_{(cr)v}}(g)_\phi]$, $[S_{Z_{(c)v}}(g)_\phi]]$ over the intelligence space, and γ_L is learning rate.

2) During Unlearning

$$[\delta\phi]_\tau = \underset{\delta\phi\in g}{\text{argmin}}[\delta Z_v(g)]_\tau = \gamma_{UL} \min_{\phi\in g}[\frac{\partial Z_v(g)}{\partial\phi}]_\tau, \tag{5.29}$$

$$but, \min_{\phi\in g}[\frac{\partial Z_v(g)}{\partial\phi}]_\tau = \min_{\phi\in g}[\frac{\partial Z_{(c)v}(g)}{\partial\phi}]_\tau + \max_{\phi\in g}[\frac{\partial Z_{(cr)v}(g)}{\partial\phi}]_\tau. \tag{5.30}$$

Since agent g cannot optimize $Z_{(c)v}(g)$, and if $Z_{(c)v}(g)$ is constant, then,

$$\min_{\phi\in g}[\frac{\partial Z_v(g)}{\partial\phi}]_\tau = \max_{\phi\in g}[\frac{\partial Z_{(cr)v}(g)}{\partial\phi}]_\tau. \tag{5.31}$$

The intelligence update function during gradient descent unlearning becomes,

$$[\phi]_{\tau_{i+1}} = [\phi]_{\tau_i} - \gamma_{UL} \max_{\phi\in g}[\frac{\partial Z_{(cr)v}(g)}{\partial\phi}]_{\tau_i}. \tag{5.32}$$

This can also be expressed using gradient ascent as follows:

$$[\phi]_{\tau_{i+1}} = [\phi]_{\tau_i} + \gamma_{UL} \max_{\phi\in g}[\frac{\partial(-Z_{(cr)v}(g))}{\partial\phi}]_{\tau_i}, \tag{5.33}$$

$$[\phi]_{\tau_{i+1}} = [\phi]_{\tau_i} - \gamma_{UL} \min_{\phi\in g}[\frac{\partial(-Z_{(cr)v}(g))}{\partial\phi}]_{\tau_i}, \tag{5.34}$$

where γ_{UL} is unlearning rate.

So during optimization of an endopistemic relative value, using a gradient descent learning of the endopistemic cost is similar to gradient ascent of an exopistemic cost. Logically, for the optimization of an exopistemic relative value, using a gradient descent learning of the exopistemic cost will be similar to gradient ascent of an endopistemic cost.

Other optimization approaches such as meta-heuristic approaches e.g., Genetic algorithms, differential evolution, etc., can also be used. We shall use the gradient based approach for the experiment in this research. More on cognitropy learning and unlearning techniques will be presented in future publications.

This is a generalized framework of learning in agent development, where both time and space are considered. The time and space consideration is very important in modern intelligent agent learning logic such as *Continual learning* [108]. We shall provide more articles on this topic in future.

5.3. Relationship with Kinematics

The formalism of an optimization process is analogous to the formalism of physical object dynamics using *Lagrangian* and *Hamiltonian* mechanics. This similarities can be seen when using a gradient based optimization technique.

Based on physical object (i.e., entity) dynamics, the Lagrangian and Hamiltonian are given as,

$$\text{Lagrangian}: \quad L = T - V, \tag{5.35}$$

$$\text{Hamiltonian}: \quad H = T + V, \tag{5.36}$$

$$\Rightarrow H - L = 2V, \tag{5.37}$$

where L is the Lagrangian, H is the Hamiltonian, T is kinetic energy, and V is potential energy of the dynamic entity.

In a situation where all the variables change during the dynamic motion of the entity, the exopistemic optimization process presented in Equation (5.5) gives a suitable representation of the Lagrangian - Hamiltonian formalism of cognimatics.

$$L \text{ cognimatics}: \quad Z_v(e||g) = Z_{cr}(e) - Z_v(e), \tag{5.38}$$

$$H \text{ cognimatics}: \quad Z_{cr}(e) = Z_v(e||g) + Z_v(e), \tag{5.39}$$

$$\Rightarrow Z_{cr}(e) - Z_v(e||g) = Z_v(e), \tag{5.40}$$

 doi:10.20944/preprints202211.0450.v1

where, $Z_v(e||g) = \sum_{i=1}^{i-n} A(e)_{t_i} \log \frac{A(e)_{t_i}}{A(g)_{t_i}}$ is the relative exopistemic value of agent e and is analogous to the Lagrangian of agent e, $Z_{cr}(e) = \sum_{i=1}^{i-n} A(e)_{t_i} \log \frac{1}{A(g)_{t_i}}$ is the absolute exopistemic value agent e and is analogous to the Hamiltonian of agent e, $Z_v(e) = \sum_{i=1}^{i-n} A(e)_{t_i} \log \frac{1}{A(e)_{t_i}}$ is the exopistemic influence to dependent agents of agent e and is analagous to the potential energy of agent e.

Other cognitive values can also be used in such formalism, such as in the case of a fixed potential energy or complete value the endopistemic cognition defined in Equation (5.2) can be used. Nevertheless, an endopistemic optimization can still be carried on an exopistemic value and vice versa, by simply changing the optimizer.

Another important analogy is that related to action. Depending on nature of the curve (i.e., path) defined by the optimization process, i.e., concave or convex, we can find the action value that will produce the least divergence between the agent and its environment as described in Equation (4.103).

For a convex path, or when complete cost defined by $A(e)$ is convex, the action is,

$$A(g||e) = \underset{A(e)\in e}{\operatorname{argmin}}[Z(g||e)_t], \tag{5.41}$$

where, $Z(g||e)_t = \sum A(e) \log \frac{A(e)}{A(g)}$.

This implies the action in a convex path follows the path with least or minimum value. The dynamics of the motion is base on the *minimization* of the divergence between g and e.

For a concave path, or when complete cost defined by $A(e)$ is concave, the action is,

$$A(g||e) = \underset{A(e)\in e}{\operatorname{argmax}}[Z(g||e)_t], \tag{5.42}$$

where $Z(g||e)_t = \sum A(e) \log \frac{A(g)}{A(e)}$.

This implies the action in a concave path follows the path with maximum value. The dynamics of the motion is based on the maximization of the divergence between g and e.

The action in a convex path can be considered to be analogous to the *principle of least action* in Lagrangian and Hamiltonian mechanics when using gradient-based optimization.

$$\text{Action} = \int_{\tau_1}^{\tau_2} L d\tau, \tag{5.43}$$

where L is Lagrangian as mentioned in Equation (5.35).

Nevertheless, same action can be generated for a Hamiltonian value. Also, since the concave optimization entails maximization, the action follows a *principle of maximum action*.

These analogies are important as they give a clear picture of cognimatics in relation to kinematics, and can change the way conventional machine learning and data analytic are carried out. More of these will be presented in future publications.

5.4. Semantic Learning

Conventionally, learning is mostly classified based on the nature of the environment; supervised, unsupervised, and reinforcement learning. Well, we can also classify learning based on the dependencies between entities; *conditional learning, mutual learning, joint learning, and semantic learning*. Conditional learning, which entails conditional action optimization, is widely used in intelligent agent design. In this section, we introduce semantic learning.

Semantic learning is the process of optimizing the semantic value between environments influences on an agent about a target. We consider the semantic divergence in Section 4.4 as the relative dependency between non-collinear entities.

An agent learning a target based on an environment which has different influence from and relatively related to its environment, will need to not only learn the target based on the influence of its environment, but also based on the relative influence between its environment and the remote environment.

Leaning the value of the collinear environment can be done using conventional learning and intelligence design models, where the objective is simply to reduce the relative

 doi:10.20944/preprints202211.0450.v1

value between the agent and its environment, i.e., the cost of the agent in its environment. But, learning the value of the remote (non-collinear) environment will require a different learning and intelligence design approach. This is because,

1. The action defining the semantic environment is out of the agent's direct *sphere of dependency*, hence, it is non-collinear with the agent local environment.
2. The environments may not be identical and equal.

Based on these factors, we describe a learning technique of an agent with respect to a "fictitious" environment, defined by the relative knowledge value between the local and remote environment. As explained in Section 4.4, we provide an approximation of this environment to the agent, and the agent then takes optimization action on this approximated environment. With such approximation, the optimization process of the agent will be to find the best fit of its knowledge in this approximated environment as compared to the complete cost of the remote environment.

Actually, the agent does not need to learn the remote environment separately, as this will imply it has direct access to the input information space of the environment. In such case, conventional computational learning techniques will suffice. On the other hand, with semantic learning, an agent may have access to the output information of the remote agent, but not the input, and the objective is to learn or recreate this output, with values and information from a referentially related local environment.

Two types of semantic learning was discussed in Section 4.4; sequential and parallel semantic learning.

5.4.1. Parallel Semantic Learning

During parallel semantic learning, the agent takes optimization actions on both its local environment and the "fictitious" environment simultaneously. To achieve this, we estimate the action of the "fictitious" environment and use it in the optimization process.

As given in Section 4.4, the action of the "fictitious" environment on target can be based on any of these equations:

$$A(e_i') = \frac{A(e_i) + A(e_j)}{2}, \tag{5.44}$$

$$A(e_i') = \underset{A(e_j)\in e_j}{\text{argmax}}(Z(e_i||e_j), \tag{5.45}$$

where e_j is remote environment, e_i is local environment, e_i' is "fictitious" environment, $A(e_j)$ is remote environment action, and $A(e_j')$ is "fictitious" environment action.

Then, the semantic learning process of an agent will simply be to optimize the agent's action on this environment influence, rather than on the remote environment influence.

$$\underset{\phi_s\in g}{\text{opt}}[Z_v(g||e_i')]_{\tau_s} = \underset{\phi_s\in g}{\text{opt}}[Z_v(e_i') - Z_{(cr)v}(g)]_{\tau_s}, \tag{5.46}$$

where ϕ_s is the semantic intelligence, τ_s is semantic learning time.

However, the knowledge and cost generated during this learning process will further be compared with those of the actual output action or complete cost of the actual remote environment. This is described in Algorithm 1.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 24 November 2022 doi:10.20944/preprints202211.0450.v1

Algorithm 1 Parallel semantic learning

Require: $A(e_i') = \frac{A(e_i)+A(e_j)}{2}, g \rightarrow e_i \leftrightarrow e_j$ ▷ Initial constrain
Ensure: $[Z_{(cr)v}(g||e_i')] = [Z_c(e_j)]$ ▷ Final constrain
  But $Z_v(g||e_i') = Z_v(e_i') - Z_{(cr)v}(g||e_i')$
  while $[Z_{(cr)v}(g||e_i')] \neq [Z_c(e_j)]$ **do**
    $\max_{\phi_s}([Z_v(g||e_i')])_{\tau_s}$
    if $[Z_{(cr)v}(g||e_i')] > [Z_c(e_j)]$ **then**
      $\max_{\phi_s}([Z_v(g||e_i')])_{\tau_s}$
    else if $[Z_{(cr)v}(g||e_i')] < [Z_c(e_j)]$ **then**
      $\min_{\phi_s}([Z_v(g||e_i')])_{\tau_s}$
    end if
  end while

This implies that semantic learning does not only require a unidirectional learning action, e.g., only maximization or minimization, as conventional learning algorithms. It also requires a bi-directional learning, i.e., both maximization and minimization processes. This implies that a semantic learning agent is a bidirectional learning system.

In reality, this is analogous to how humans learn. For example, learning a new language if we already know another language, we use the knowledge of our original language to gain knowledge of the new language, and compare our acquired learning knowledge with that of an advanced speaker of the language. Thus, semantic learning can be important for language translation agents and a solution to other environment divergence problem.

5.4.2. Sequential Semantic Learning

Sequential semantic learning, on the other hand, involves completing the optimization of the local environment and then uses its complete cost to do optimization on the remote environment. This is presented in Algorithm 2.

Algorithm 2 Sequential semantic learning

Require: $g \rightarrow e_i \leftrightarrow e_j$ ▷ Initial constrain
Ensure: $[Z_{(cr)v}(g||e_i)] = [Z_c(e_j)]$ ▷ Final constrain
  But $Z_v(g||e_j) = Z_v(g||e_i) + Z_v(e_i||e_j)$
  while $[Z_{(cr)v}(g||e_i)] \neq [Z_c(e_j)]$ **do**
    $\max_{\phi_l}([Z_v(g||e_i)])_{\tau_s}$
    if $\max_{\phi_l}([Z_v(g||e_i)])_{\tau_s}$ **then**
      $A(g_i) = \text{argmax}_{A(g)}([Z_v(g||e_i)])_{\tau_s}$
    end if
    $\max_{\phi_s}([Z_v(g_1||e_j)])_{\tau_s}$
  end while

Where, ϕ_l is intelligence of g on the local environment, ϕ_s semantic intelligence of g on the remote environment e_j and its local environment e_i.

This represents a sequential learning where the agent first gains complete value about its local environment, then it uses its action and cost at that local completeness to acquire complete value in the remote environment.

Further discussions and applications on semantic learning and intelligence will be presented in a future publications.

5.5. Generalization

The downside of optimization over a target is the risk of a poor generalization over the target. This phenomenon leads to the issue of over-fitting and under-fitting of an agent on a target during non optimization period. For this reason, a regularization factor is introduced in conventional learning processes, especially gradient based learning processes of an agent. This is illustrated in the Figure 18.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 24 November 2022 doi:10.20944/preprints202211.0450.v1

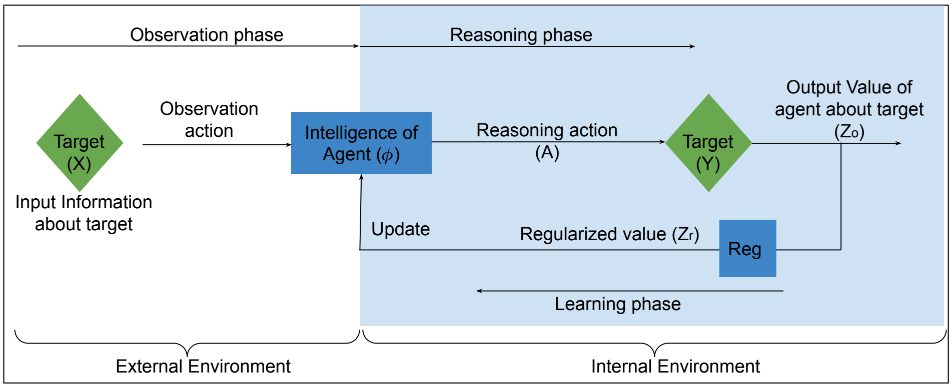

Figure 18. learning process with a regularization function (RF).

There are many regularization techniques in the literature [109]. These are important because learning a target does not guarantee knowledge on unseen information about the target. Also, the verification of the level of generalization is important, and many techniques such as Probably Approximately Correct (PAC) learning [110], Mistake Bound (MB) learning [111], and Cross validation estimates [112] are used for this purpose but are not solutions to the generalization problem.

In this research, we seek to provide theories that will not only lead to a flexible and scale-able agent optimization process, but also that will lead to a reduced generalization error. Values obtained after the application of a generalization solution can be considered as *refined values*.

To solve the generalization problem, we introduce the concept of a *revision process*, where the cognitive value of an agent is revised by values from remote agents or itself. This is presented in the next section.

6. Value Transfer

The transfer of value between entities is necessary if interactivity between the entities is permitted. In this section, we describe different characteristics of a cognitive transfer process. These include: transfer methods, revision, and optimization of cognitive properties.

6.1. Transfer Methods

Like any physical quantity, the value of a cognitive property can move from one entity to another, being transformed or remained unchanged at the destination. We thus distinguish two groups of transfer methods; *transformative value transfer* (TVT) and *non-transformative value transfer* (NTVT).

Transferring a value over space (a group of interactive entities) or time is considered as a *value propagation*. Conventionally, different models are used to describe both transformative and non-transformative value propagation. *transformative value propagation* (TVP) models include the *diffusion models, wave models*, etc., and the *non-transformative value propagation* (NTVP) models include the *SI, SIR*, etc., models.

In fact, NTVP process is commonly considered as a multiplication of same value over a group of entities, and its model focuses on counting the variation of this multiplicity over space and time, e.g., modeling of virus spread, information spread, etc. Whereas, TVP models focus on the degree of transformation that a source value undergoes over space and time, e.g., modeling of heat flow, sound flow, etc. Hence, NTVP and TVP are respectively quantitative and qualitative analytic processes of a value over space and time.

Using these concepts, the transfer process for the values of cognitive properties defined in this research can be modelled using TVP or NTVP models, based on the ability of the value to be transformed during the transfer process.

Considering each entity as a value processing unit, the values that passes through them can be transformed. Based on this, we adopt the transformative method of value transfer, with major focus on the diffusion model of value propagation, and reserving the wave model for a future article.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 24 November 2022 doi:10.20944/preprints202211.0450.v1

6.1.1. Diffusion Model of Value Transfer

Diffusion is a phenomena where a quantity Ω_t spreads from a region of high concentration to that of low concentration, and it is derived mathematically based on the following two laws of nature:

$$\frac{\partial \Omega_t(S,\tau)}{\partial \tau} + \frac{\partial J(S,\tau)}{\partial S} = 0 \text{ (Law of Continuity)}, \tag{6.1}$$

where $J(S,\tau) = -D\frac{\partial \Omega_t(S,\tau)}{\partial S}$ (Firk's First law),

$$\Rightarrow \frac{\partial \Omega_t(S,\tau)}{\partial \tau} = D\frac{\partial^2 \Omega_t(S,\tau)}{\partial^2 S} \text{ (Diffusion equation)}, \tag{6.2}$$

where, $\Omega_t(S,\tau)$ is the value of Ω_t at space S and time τ, D is the diffusion coefficient and J is the diffusion flux.

There are many techniques in the literature to solve the diffusion equation, such as the use of the Laplace-Fourier transform and the Laplace operator, applied during spectral analysis in image processing [113],[114] and in Graphical neural network [115],[116].

But modeling the diffusion process of a quantity Ω_t on a discrete space S can easily be done using a *random walk* of the quantity on the space [117]. This is the approach we shall use to model the diffusion of value in this research.

Consider a group of agents, g_1, g_2, g_3, g_4, interacting with each other based on the following dependency network:

$g_1 \to (g_2, g_3), g_2 \to (g_3), g_3 \to (g_2, g_4), g_4 \to (g_1, g_3, g_4)$.

Let g_i depends on g_j, and g_i generates action value $A(g_i)_t$ and cognitive value $Z_v(g_i)_t$ on a target t. Based on Proposition 1, the influence of g_i on g_j is a measure of its action on g_i, which represents the complete value $Z_{(c)v}(g_j||g_i)_t$ of g_j relative to g_i. The propagation of such influence over space and time may increase, decrease, or remain constant.

The cause of such transformation over time and space of the generated value of an influencer is mainly due to the medium through which the value is transferred. This medium defines an environment, which can be considered as a *mediating agent* that takes the action of transferring the value of g_i to g_j.

We define the following quantities to describe the propagation of the influence $[\Omega_t]_i$ generated by g_i to $[\Omega_t]_j$ of g_j.

$[[\Omega_t]_j]_\tau$ is $A(g_j)_t$ or $Z_{(c)v}(g_j)_t$ of g_j at time τ from g_i.

$[[\Omega_t]_j]_{\tau+1}$ is value of $[\Omega_t]_j$ at time $\tau + 1$.

$[[\Omega_t]_{i,j}]_\tau$ is $A(g_{ij})_t$ or $Z_{(c)v}(g_{ij})_t$ that defines the transformation of the value $[[\Omega_t]_i]_\tau$ of g_i to the value $[[\Omega_t]_j]_{\tau+1}$ of g_j over space S and time τ.

Since the propagation involves value transformation, we simply define a transformation process that represents the value transfer from g_i to g_j. We represent this transformation as an *additive or multiplicative (scalar)* operation on the value of g_i with a transformation factor for value increment over space and time, and a *subtraction or division* operation for a value decrements.

Proposition 38.

$$[[\Omega_t]_j]_{\tau+1} = \sum_i [[\Omega_t]_{i,j}]_\tau \otimes [[\Omega_t]_i]_\tau \forall i,j \in S, \tag{6.3}$$

$\otimes$ is addition or subtraction if $\Omega_t = \{Z(g)_t\}$, $\otimes$ is dot product or division if $\Omega_t = \{A(g)_t, Z(g)_t\}$, $Z(g)_t$ is complete value, and $A(g)_t$ is action value.

For a finite network space S, we can multiply $[\Omega_t]_{i,j}$ with an *adjacency matrix* $[M_t]_{i,j}$ of the space, over the target t.

$$[[\Omega_t]_j]_{\tau+1} = [M_t]_{i,j}[[\Omega_t]_{i,j}]_\tau \otimes [[\Omega_t]_i]_\tau. \tag{6.4}$$

The definition of the transformation factor $[[\Omega_t]_{i,j}]_\tau$ during the transfer process is important. As earlier mentioned, this can be considered as a value from a mediating agent g_{ij} defining the relationship between g_i and g_j. This is considered in this research as the *Attention property* of the agent g_j.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 24 November 2022 doi:10.20944/preprints202211.0450.v1

Definition 89. ***Attention*** *is the property of an agent g_j that regulates the flow of value from any source to the agent.*

$$[[\Omega_t]_{i,j}]_\tau = f(G_i, G_j, \tau), \tag{6.5}$$

where $G_i = \{g_i, ..g_{i+n}\}$, $G_j = \{g_j, ..g_{j+m}\}$, $m = \#j$ *of* i.

This implies that the *attention property* of an agent is a type of *cognitive control property* [118] of the agent on its input values. In general, the cognitive control properties of an agent control all its output and input values. The control of output values is called *executive attention* in the literature [118]. More so, attention can be *exogenous* or *endogenous* [119],[118], which corresponds respectively to the exopistemic and endopistemic processes we introduced in this research.

The valuation of the attention property depends on the transfer model of the agent. For example, in a *diffusion based transfer model*, where values from g_i to g_j decrease over space such that the complete value from g_i is transformed until all agents in the space have same value, can be modelled with an *attention (transformer) value* as proposed below.

Proposition 39.

$$\text{Given } [[\Omega_t]_j]_{\tau+1} = \sum_i [\Omega_t]_{i,j} \otimes [[\Omega_t]_i]_\tau, \tag{6.6}$$

if $[[\Omega_t]_i]_\tau = [[\Omega_t]_j]_\tau$, *no diffusion occurs between* g_i *and* g_j,
if $[[\Omega_t]_i]_\tau > [[\Omega_t]_j]_\tau$ *diffusion of* $[\Omega_t]_i]_\tau$ *occurs from* g_i *to* g_j,
if $[[\Omega_t]_i]_\tau < [[\Omega_t]_j]_\tau$ *diffusion of* $[\Omega_t]_j]_\tau$ *from* g_j *to* g_i.

Using a random walk model of diffusion, we define a simple *transformer* over the multiplicative operator as follows:

$$[\Omega_t]_{i,j} = \frac{1}{n(G_j)_i}, \tag{6.7}$$

$$\Rightarrow [[\Omega_t]_j]_{\tau+1} = \sum_i \frac{1}{n(G_j)_i}[[\Omega_t]_i]_\tau[[\Omega_t]_j]_{\tau+1}, \tag{6.8}$$

$$\Rightarrow \sum_i \frac{1}{n(G_j)_i}[[\Omega_t]_i]_\tau[[\Omega_t]_j]_{\tau+1} = [M_t]_{i,j}[D_t]^{-1}_{i,j}[[\Omega_t]_i]_\tau, \tag{6.9}$$

where $[D_t]_{i,j}$ is the degree matrix of G_i in S, $n(G_j)_i$ is number of g_j on g_i.

This represents a random walk of $[\Omega_t]_i$ on a space S defined by $[M_t]_{i,j}$. Also, according to Proposition 38, this implies that both the action value $A(g_i)$ and complete value $Z_{(c)v}(g_j||g_i)$ from g_i can be used in such a diffusion operation.

Before any walk, i.e., at time = 0, the value $[[\Omega_t]_i]_\tau$ from the *initial influencer* g_i reduce by a factor $[[\Omega_t]_{i,j}]_\tau$ after each layer of dependency, and considering each walk to represent one time step through a layer, then the value of $[[\Omega_t]_{i,j}]_\tau$ at n step will simply be given as,

$$[[\Omega_t]_j]_{\tau=n} = [[\Omega_t]_{i,j}]^n[[\Omega_t]_i]_{\tau=0}. \tag{6.10}$$

One significance of using a diffusion model for the modeling of influence flow across a network of agents is the effect of this influence on the cost, knowledge, and ignorance values of the individual dependent agents. Based on Proposition 22, for binary target, the required cost increases with reduced complete cost. This implies that, agents far from the value source will require more cost to acquire the complete value. This is intuitive in the sense that the complete truth about something is deformed as we move further away from it and much work will be required to uncover such truth.

In the next section, we describe the effect of a remote values such as influences on the self generated values of dependent agents, i.e., a microscopic view of the transfer process.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 24 November 2022 doi:10.20944/preprints202211.0450.v1

6.2. Transfer Value Revision

Apart of the influence received from its referenced dependencies, i.e., environments, an agent can experience other influences based on values from non-referenced dependencies. We consider the former as a *referential influence* and the later as a *non-referential influence*. Likewise, their transfer processes are considered as *referential and non referential value transfer*.

In Section 6.1.1, we discussed about the diffusion of environment influence over a network of interactive agents, and in Sections 4.4 and 4.5, we discussed about the influence of the divergence and variation of this environment on an agent. These are all related to the referential value transfer. In this section, we focus on non-referential value transfer.

In a *non-referential value transfer*, the influence on an agent is based on non-referential dependencies such as conditional, mutual and joint dependencies. This can be considered as *conscious conditioning* [98] of an agent. During this process, an agent receives values from other agents and carries a non referential dependency action on it. We consider this process as a *value revision process*, which we define below.

Axiom 6.1. *The revision process defines and is defined by a relationship between agents on a target over a value property.*

$$\text{Revised value} = R_v(g_j, G_i)_t, \tag{6.11}$$

where $R(\cdot)$ *is* revision function.

Related to the *resultant value* concept in Section 4, the *revised value* represents a type of resultant value generated by an agent based on already generated local and remote values.

If the revision process of an agent is carried out using only its values, then such process is considered here to be a *replay process*[120],[121], an *introspection, retrospection*, or a *prospection*[122], all of which require the use of a *memory*. We shall present more on these aspects in future publications.

In a non-referential transfer process, the relationship of a revision process can be defined using *logical dependency* such as conditional, joint, and mutual dependencies, or *functional dependency* such as min, max and avg dependencies.

In this regard, the revision of the values generated by an agent g_j based on non-referential values from a set of agents G_i, can be described using any of these logical operations.

$$R_v(g_j, G_i)_t = \Omega_{a_j|a_i}(g_j)_t \text{ (conditioned on } G_i), \tag{6.12}$$

$$R_v(g_j, G_i)_t = \Omega_{a_j;a_i}(g_j)_t \text{ (mutuality with } G_i), \tag{6.13}$$

$$R_v(g_j, G_i)_t = \Omega_{a_j,a_i}(g_j)_t \text{ (joint with } G_i), \tag{6.14}$$

where $\Omega = \{A(g), Z_a(g), Z_{(cr)a}(g)\}$, $G_i = \{g_1, g_2, ...g_n\}$. The inverse $\bar{\Omega} = \{\bar{A}(g), \bar{Z}_a(g), \bar{Z}_{(cr)a}(g)\}$ can also be used.

The expression for each logical operation was given in Section 3; logical operations on action and cognitropy values.

For functional dependency, we define the following:

$$R_v(g_j, G_i)_t = \max(\Omega(g_j), \Omega(G_i))_t \text{ (maximum } \Omega), \tag{6.15}$$

$$R_v(g_j, G_i)_t = \min in(\Omega(g_j), \Omega(G_i))_t \text{ (minimum } \Omega), \tag{6.16}$$

$$R_v(g_j, G_i)_t = \text{avg}(\Omega(g_j), \Omega(g_i))_t \text{ (average } \Omega). \tag{6.17}$$

An agent may use any of these revision logic but choosing a right logic will depend on the objective of the revision process. For example, if the aim is to revise only if another agent provides a value, then a conditional revision dependency is used, and if the aim is to have a general value based on others opinion, then a joint or average revision logic is preferable. In general, a revision value is a type of *resultant value* of an agent on a target, so the resultant values defined in Section 4.5.1 can also be used.

Furthermore, the revision process of a destination agent g_j can be expressed as a function of its self value Ω_j and the values Ω_i from the remote agents G_i.

$$\Omega_{r_j} = f(\Omega_j) + f(\Omega_i) \text{ (for } \Omega = \{A(g), Z(g)\}), \tag{6.18}$$

$$\Omega_{r_j} = f(\Omega_j) f(\Omega_i) \text{ (for } \Omega = \{A(g), Z_{(c)a}(g)\}). \tag{6.19}$$

Considering the knowledge value, we can express its revision operations in this format using expressions in Section 3, where the revision environment is defined by the joint value $\Omega_{ij}(e)$ of the environments of g_j and G_i.

$$Z_{a_j|a_i}(g_j)_t = Z_{a_j}(g_j)_t - Z_{a_j;a_i}(g_j)_t, \tag{6.20}$$

$$Z_{a_j;a_i}(g_j)_t = Z_{a_j}(g_j)_t - Z_{a_j|a_i}(g_j)_t, \tag{6.21}$$

$$Z_{a_j,a_i}(g_j)_t = Z_{a_j}(g_j)_t + Z_{a_i|a_j}(g_j)_t, \tag{6.22}$$

$$\text{avg}(Z(g_j), Z(g_i))_t = \frac{1}{n} Z_{a_j}(g_j)_t + \frac{1}{n}\sum_{i=1}^{n} Z_{a_i}(g_i)_t. \tag{6.23}$$

We consider the self value Ω_j of a destination agent g_j as *bias value* that influences its revision process. This can be a *confirmation bias* [123],[124] for conditional revision process. Also, bias value can represent either a *parallel or sequential* value acquisition process to the revision process, which will both require a parallel and sequential memory, respectively.

Ω_j is sequential to Ω_{r_j},

$$[\Omega_{r_j}]_{\tau+1} = [f(\Omega_j)]_\tau + [f(\Omega_i)]_\tau. \tag{6.24}$$

Ω_j is parallel to Ω_{r_j},

$$[\Omega_{r_j}]_{d_2} = [f(\Omega_j)]_{d_1} + [f(\Omega_i)]_{d_2}. \tag{6.25}$$

In the case of a parallel bias value, the input space and intelligence on which the bias value is generated differs from that of the revised value. We consider the former as the *background process* and the later as the *foreground process* of the agent g_j. In the sequential case, the bias value Ω_j can also be considered as the *prior* of the revised value Ω_{r_j}.

Based on these, a referential transfer process such as the diffusion process, can be considered as a revision process on the environment value based on a functional dependency defined by the diffusion equation given in Proposition 39.

Similar to referential influence, non referential influence can be experienced from *collinear and non-collinear* agents, hence, the divergence of the source and destination environment must be taken into account at the destination agent. This implies an *environment conversion* operation such as *semantic learning* is required by the destination agent.

In some cases, the transfer values may be reinserted as input to the background (bias) environment rather than the foreground (revision) environment. Such model will be identical to the *Ensemble learning* model [125],[103] if the transfer consists of action values. Other models such as *Transfer learning* [126] and *Federated learning* [127] focus on transfer of intelligence parameters rather than *values* as per this research.

6.3. Revision Process Optimization

Once an agent receives a set of values from different agents, it revises its bias or self generated value based on the received values. It then uses this revised value to update its *background* and *foreground intelligence* during learning.

Using the learning process discuss in Section 5, we can express the learning of the revised knowledge value as follows:

$$\underset{\phi \in g_i}{\text{argopt}}[R_v(g_j, G_i)_t]_\tau = \underset{\phi \in g_i}{\text{argopt}}[Z_r(g_j)_t]_\tau, \tag{6.26}$$

where argopt = {argmax, argmin}, and $r = f(a_j, a_i)$.

Based on the revision logic used by the agent, the revision learning can be divided into separate parts depending on the agent's ability to learn that value type and the type of already existing value that the agent already have.

For example, in a joint value revision logic, the optimization can be expressed as,

$$\underset{\phi\in g_i}{\text{argopt}}[R_v(g_j, G_i)_t]_\tau = \underset{\phi\in g_i}{\text{argopt}}[Z_{a_j}(g_j)_t + Z_{\cap a_i|a_j}(g_j)_t]_\tau. \tag{6.27}$$

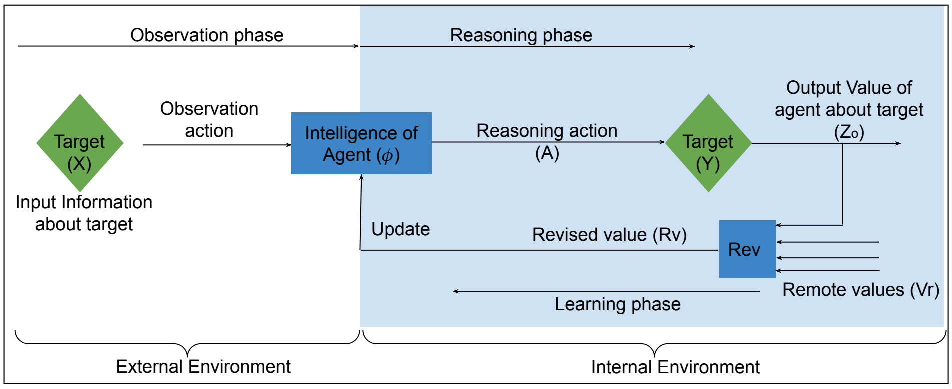

Figure 19. Optimization process of a revision function.

However, if the agent has more revision ability on mutual value than conditional value with the remote agents, it may learn the conditional value through the mutual value.

$$\underset{\phi\in g_i}{\text{argopt}}[Z_{a_j}(g_j)_t + Z_{\cap a_i}(g_j)_t + Z_{a_j;\cap a_i}(g_j)_t]_\tau. \tag{6.28}$$

If the agent considers the remote values are independent from each other and its self value, then the joint optimized revised value is considered the optimized sum of the independent values.

$$\underset{\phi\in g_i}{\text{argopt}}[Z_{a_j}(g_j)_t + Z_{a_i}(G_i)_t]_\tau. \tag{6.29}$$

In Figure 19, we present the revision process and its optimization. As already mentioned, such optimization process will not only achieve complete value but also increase generalization as we demonstrate in the experiment below.

7. Experiment and Simulations

7.1. Experimental Setup

Four agents (g_1, g_2, g_3, g_4) were used in the experiment. Three of the agents transfer value to the fourth agent, that acts as a value receptor. The fourth agent learns from these received values and its self value. The input (background) environment and target of all the four agents are based on the breast cancer diagnostics dataset textcolorblue[128].

Objective of experiment: To study the effect of transfer values on the generalization of the learning action of an agent about the target in cancer diagnostics dataset.

The design setup for the experiment is shown below.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 24 November 2022 doi:10.20944/preprints202211.0450.v1

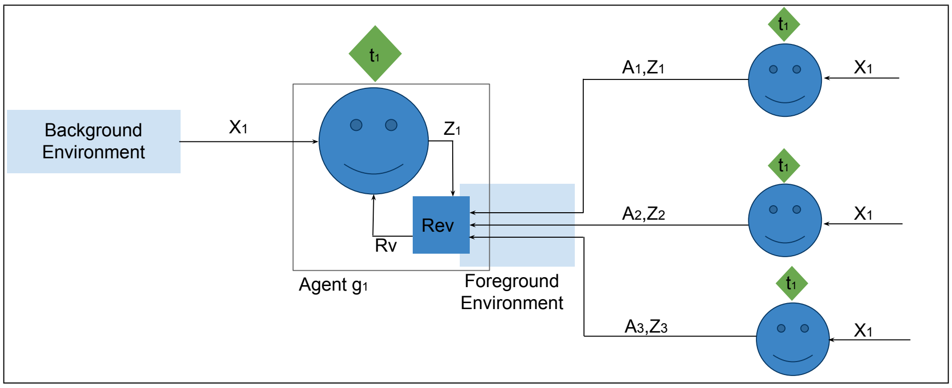

Figure 20. Experiment setup for a revision process of g_j.

7.2. Experimental Analysis 2280

All agents including the value receiving agent g_1 generate knowledge value on same target based on the definition of the target by their respective environment. But, the receiving agent g_1 is further enhanced with a revision logic which enables it to revise its generated value with the remote values. The revision logic used in this experiment is the *joint revision process* based on the non-referential actions of all related agents about the target. Nevertheless, the resultant knowledge value from the incoming knowledge value of all related agents can also be used. 2281–2287

First, we define all agents as value generators, and then we define the revision and learning logic of the receiving agent. 2288–2289

i) Value generators, g_i: 2290

Each agent generates two types of value: its absolute action value $A(g)_t$ and its knowledge value $Z(g)_t$ about the target.

$$A(g_i)_t = P(Y; X, \phi_i), \tag{7.1}$$

$$Z_a(g_i)_t = \sum_{s=1}^{s=n} A(e_i)V_a(g_i)_{t_s}, \tag{7.2}$$

where $i = 1, 2, 3, 4$ are generators, and s is state of t. 2291

ii) Value Receptor, g_j: 2292

The receptor receives the *absolute* actions $A(g_i)$ from all related agents and generate a value based on these actions.

$$Z_{a_j,a_i}(g_j)_t = Z_{a_j}(g_j)_t + Z_{\cap a_i|a_j}(g_j)_t \text{ (self incline)}, \tag{7.3}$$

$$= Z_{a_j|\cap a_i}(g_j)_t + \sum_{i=2}^{4} Z_{a_i}(g_j)_t \text{(remote incline)}, \tag{7.4}$$

where $a_2 \perp a_3 \perp a_4$, $j = \{1\}$ is a receptor, and $i = 1, 2, 3, 4$ are generators. 2293

iii) Optimization of receptor 2294

Since the objective of the experiment is to train the receptor g_j on a target given input information at its background environment, the optimization objective is to maximize the revised value, $Z_{a_j,a_i}(g_j)_t$, over the background intelligence of the agent. Also, the revision logic (i.e., foreground intelligence) can be optimized using the same revised value. 2295–2298

In this case, considering the receptor g_j is *self inclined*, the optimization of its revised value can be written as,

$$\max_{\phi \in g_j}[R_v(g_j, G_i)_t]_\tau = \max_{\phi \in g_j}[Z_{a_j}(g_j)_t + Z_{\cap a_i|a_j}(g_j)_t]_\tau. \tag{7.5}$$

2299

The first term on the right-hand side of Equation (7.5) is the self value of g_j in the revision environment, which is its ground truth as we are concerned in this experiment with only absolute action values. The second term is the conditional values of each g_i given g_j as perceived by g_i in the revision environment. 2300–2303

The optimization can be expressed separately as;

$$\max_{\phi \in g_j}[R_v(g_j, G_i)_t]_\tau = \max_{\phi \in g_j}[Z_{a_j}(g_j)_t]_\tau + \max_{\phi \in g_j}[Z_{\cap a_i|a_j}(g_j)_t]_\tau \tag{7.6}$$

where the first term in Equation 7.6 can be express as,

$$Z_{a_j}(g_j)_t = \sum_{i=1}^{i=n} A(e_r)_{t_i} J[\frac{A(g_j)_{t_i}}{A(e_r)_{t_i}}] = Z_{(c)a_j}(g_j)_t - Z_{(cr)a_j}(g_j)_t. \tag{7.7}$$

But, $Z_{(c)a_j}(g_j)_t = Z_{a_j}(e_r)_t$ and $\phi_g \perp \phi_r$, so g_j cannot optimize $Z_{(c)a_j}(g_j)_t$ because it is the value of its environment e_r and the optimum value it seeks. e_r is revision environment and represents the foreground and ground truth about the target t. In this experiment, e_r and $A(e_r)$ are same for all the agents, and represent the training data and true action value on the data, respectively. In some cases, their ground truth about the target can be different, leading to the transfer and analysis of *relative* actions.

The optimization of $Z_{a_j}(g_j)_t$ is limited to $Z_{(cr)a_j}(g_j)_t$.

$$\max_{\phi \in g_j}[Z_{a_j}(g_j)_t]_\tau = \max_{\phi \in g_j}[-Z_{(cr)a_j}(g_j)_t]_\tau, \tag{7.8}$$

$$\Rightarrow \max_{\phi \in g_j}[Z_{a_j}(g_j)_t]_\tau = \min_{\phi \in g_j}[Z_{(cr)a_j}(g_j)_t]_\tau. \tag{7.9}$$

The second term in Equation (7.6) can also be expressed as,

$$Z_{\cap a_i|a_j}(g_j)_t = \sum_{i=2}^{4} Z_{a_i|a_j}(g_j)_t, (A_2 \perp A_3 \perp A_4). \tag{7.10}$$

From Equation (3.112), using absolute actions,

$$Z_{a_i|a_j}(g_j)_t = \sum_{s=1}^{s=n} A(e_r)_{t_s} J[\frac{A(g_z)_{t'_s}}{A(e_r)_{t_s}}], \neg(A_i \perp A_j), \tag{7.11}$$

where $A(g_z)_{t'} = P(t'; A(g_j)_t, \phi_z)$, $t' = A(g_i)_t = P(t; X, \phi_i)$, $A(g_j)_t = P(t; X, \phi_j)$, and $A(e_r)_t = P(t; X, \phi_r)$

$A(g_z)_{t'}$ can be considered as the believe of g_j about the believe of g_i on the target, conditioned on its own believe $A(g_i)_t$ and referenced to its environment. That is, the action $A(g_z)_{t'}$ is conditioned on $A(g_j)_t$ and referenced on $A(e_r)_t$.

$$(A(g_j)_t \mid\leftarrow A(g_z)_{t'}) \rightarrow|| \; A(e_r)_t. \tag{7.12}$$

Similar to the first term optimization, the second term becomes,

$$\max_{\phi \in g_j}[Z_{\cap a_i|a_j}(g_j)_t]_\tau = \min_{\phi \in g_j}[\sum_{i=2}^{4} Z_{(cr)a_i|a_j}(g_j)_t]_\tau, \tag{7.13}$$

$$\max_{\phi \in g_j}[Z_{\cap a_i|a_j}(g_j)_t]_\tau = \sum_{i=2}^{4}(\min_{\phi \in g_j}[Z_{(cr)a_i|a_j}(g_j)_t]_\tau). \tag{7.14}$$

Finally, minimizing $[Z_{(cr)a}(g)_t]_\phi$ through updating of the intelligence ϕ can be done using any optimization technique such as gradient descent, differential evolution, etc. Using gradient descent for the learning process as explained in Section 5, the intelligence update process is expressed as,

$$[\phi]_{\tau_{i+1}} = [\phi]_{\tau_i} - [\delta\phi]_{\tau_i}, \tag{7.15}$$

$$[\phi]_{\tau_{i+1}} = [\phi]_{\tau_i} - \gamma_L \min_{\phi \in g}[\frac{\partial Z_{(cr)v}(g)}{\partial \phi}]_{\tau_i}, \tag{7.16}$$

where γ_L is learning rate, and τ is learning time (or epoch)

A simulation of the experiment based on this analysis was conducted, the simulation specifications and results are presented in the next section.

7.3. Simulation and Results

The simulation parameters and result of the experiment are given below.

7.3.1. Simulation Parameters of the Agents and Environments

_	Agent g_1	Agent g_2	Agent g_3	Agent g_4
Model	LR	KNN	RF	XGBoost
Parameters	learning rate = 1e-5, learning epoch = 100	neighbors = 5	estimators=100, max depth=3	estimators=100, maxdepth=3, learning rate=0.1

LR is logistic Regression, KNN is K-Nearest Neighbor, RF is Random Forest, and XGB is eXtreme Gradient Boosting.

The target is a binary output value about stages of breast cancer M (Malignant) and B (Bengin) in a health environment defined by the breast cancer diagnostic dataset [128], which has 569 observations and 30 input properties.

All agents uses the same dataset during training and testing.

7.3.2. Simulation Results

The simulation was done on the LR agent and the result was compare with that of a conventional LR as shown in Figure 21.

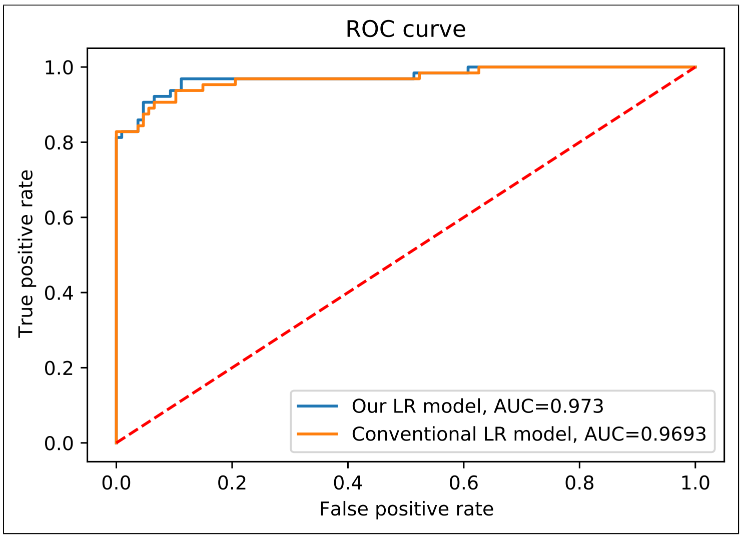

Figure 21. ROC curve of agent g_1 and an identical conventional model

7.3.3. Explanation of Simulation Results

The AUC of a LR based on our design approach is higher than conventional LR. This increase in performance is due to the use of the revision process on remote values. The background and foreground environments of g_j influence both the revised value and the generalization of g_j on the target.

We choose to use LR in this experiment because;

1. The gradient descent learning model defined in our experimental analysis can be deployed on the LR.
2. It is less complicated, and using it will ease understanding of the research.

In future publications, we shall apply the mathematical theory presented in this research to design new models and enhance other conventional learning models.

8. Conclusion

In this paper, we focused on describing a mathematical quantification and operability of cognitive properties, more precisely, the *knowledge* and *action* properties of an agent. Other properties such as *ignorance*, *understanding*, *trust*, *wisdom* and *attention* were also presented in this paper.

We went further to do mathematical analysis on these properties to describe their nature in an agent, using a quantity we called *cognitropy*. These includes, the concept of resultant value, environmental uncertainties and divergence, referential and non-referential influences, and value diffusion and revision. We also showed how the analysis of the optimization operation on this quantity is identical to that of conventional kinematics and dynamics of physical systems.

This research can be considered as a framework that can be used in the design and development of an intelligent and cognitive agent. More so, since the concepts presented are closely related to that of natural cognitive agents like humans, the research can be used not only to design an agent that does what we think but also what they think, thereby interacting, selecting and solving problems, as they want.

This is related to the statement of Alain Turing in his paper, *Computing machinery and intelligence* [1], which has shaped the AI field for decades, in which he asked the question, "Can machine think?", and responded to his own question by proposing that we can rather seek to build machines that do as we think rather those that think as we do.

In addition to this, the challenge in building such a machine is based on the logical definitions and operations of a thinking (reasoning) process. In this research, we provided a concise mathematical definitions and operations of a thinking process of any agent, which can be used in understanding, designing, and developing an agent. Hence, providing a pathway in the design and development of a rational *thinking machine*, that can also *communicate* and *interact* rationally in a conscious, subconscious, and unconscious cognition.

In the later part of the research, we use this knowledge framework to describe and provide solution to the generalization of a learning agent using remote values. The simulation results prove that *value transfer* is an important aspect to improve the generalization performance of a learn-able intelligent agent.

We look forward to provide more materials on the science, development and applications of intelligent and cognitive agents that do not only do as we think, but most especially, that do as they think.

Appendix A Proofs

Appendix A.1

Proof of Proposition 10,

1) $f_{k\to\phi} : (f_{a\to k} : A \to K) \to \Phi$.

2) $f_{\phi\to k} : (f_{\phi\to a} : \Phi \to A) \to K$.

Proof:

1) $\Phi = f_{k\to\phi}(K)$ but $K = (f_{a\to k}(A))$,

$\Rightarrow \Phi = f_{k\to\phi}(f_{a\to k}(A)) \vdash f_{k\to\phi} : (f_{a\to k} : A \to K) \to \Phi$.

2) $K = f_{\phi\to k}(\Phi)$ but $\Phi = f_{k\to\phi}(K)$,

$\Rightarrow K = [f_{\phi\to k}]^{-1}(\Phi), f_{\phi\to k} = [f_{k\to\phi}]^{-1} = [f_{k\to\phi}]^{-1}([f_{a\to k}(A)]^{-1})$

$f_{\phi\to k} : (f_{\phi\to a} : \Phi \to a) \to K$.

Appendix A.2

Proof of Proposition 19,

Given $Z_{cr}(g) = \eta_k Z_v(e)$, Let $\eta_k = \frac{1}{\sqrt{1-x}}$.

Proof:

$$\frac{1}{\sqrt{1-x}} = \frac{Z_{cr}(g)}{Z_v(e)} \Rightarrow x = 1 - (\frac{Z_v(e)}{Z_{cr}(g)})^2.$$

Taking the difference between two squares, we have that,

$$x = (1 - \frac{Z_v(e)}{Z_{cr}(g)})(1 + \frac{Z_v(e)}{Z_{cr}(g)}).$$

 doi:10.20944/preprints202211.0450.v1

Converting all values of e to relative endopistemic values of g using the law of conservation of values, then,

$$Z_v(g) = Z_v(e) - Z_(cr)v(g),$$
$$\Rightarrow x = (\frac{-Z_v(g)}{Z_{cr}(g)})(\frac{2Z_v(e) - Z_v(g)}{Z_{cr}(g)}).$$

But, since, η_K is an optimization factor over g, the $Z_v(e)$ is out of the optimization reach of g except at complete value and so we can safely ignore it.

$$\rightarrow x = (\frac{-Z_v(g)}{Z_{cr}(g)})(\frac{-Z_v(g)}{Z_{cr}(g)}) = (\frac{Z_v(g)}{Z_{cr}(g)})^2,$$
$$\text{Hence,} \quad \eta_K = \frac{1}{\sqrt{1 - (\frac{Z_v(g)}{Z_{cr}(g)})^2}}.$$

Appendix B List of Symbols

g	Agent
t	Target
e	Environment
τ	Action time
X	Input space
$(g \rightarrow t)$	Non referential dependency of g on t
$(g \rightarrow t) \mid\mid e, (g \mid\mid e)_t$	Referential dependency of g on e about t
$A((g \rightarrow t) \mid\mid e), A(g)_t$	Action of g about a single state t
$K((g \rightarrow t) \mid\mid e), K(g)_t$	Knowledge of g about a single state t
$I((g \rightarrow t) \mid\mid e), I(g)_t$	Ignorance of g about a single state t
$V((g \rightarrow t) \mid\mid e), V(g)_t$	Value (A, K, I) of a single state t
$A_Z((g \rightarrow t) \mid\mid e), A_Z(g)_t$	Action cognitropy of g about t
$Z_I((g \rightarrow t) \mid\mid e), Z_K(g)_t$	Knowledge cognitropy of g about t
$Z_I((g \rightarrow t) \mid\mid e), Z_I(g)_t$	Ignorance cognitropy of g about t
$Z_V((g \rightarrow t) \mid\mid e), Z_V(g)_t, Z_a(g)_t$	K, I cognitropy of g about t
$S_A((g \rightarrow t) \mid\mid e), S_A(g)_t$	Stability of action of g about t
$S_I((g \rightarrow t) \mid\mid e), S_K(g)_t$	Stability of Knowledge of g about t
$S_I((g \rightarrow t) \mid\mid e), S_I(g)_t$	Stability of Ignorance of g about t
$S_V((g \rightarrow t) \mid\mid e), S_V(g)_t$	Stability (A, K, I) of g about t

References

1. Turing, A.M. Computing machinery and intelligence. *Mind* **1950**, *LIX*, 433–460. https://doi.org/10.1093/mind/LIX.236.433.
2. Russell, S.; Norvig, P. *Artificial Intelligence: A Modern Approach*, 3 ed.; Prentice Hall, 2010.
3. Shrestha, A.; Mahmood, A. Review of Deep Learning Algorithms and Architectures. *IEEE Access* **2019**, *7*, 53040–53065. https://doi.org/10.1109/ACCESS.2019.2912200.
4. Sun, S.; Cao, Z.; Zhu, H.; Zhao, J. A Survey of Optimization Methods From a Machine Learning Perspective. *IEEE Transactions on Cybernetics* **2020**, *50*, 3668–3681. https://doi.org/10.1109/TCYB.2019.2950779.
5. de Grefte, J. Knowledge as Justified True Belief. *Erkenntnis* **2021**. https://doi.org/10.1007/s10670-020-00365-7.
6. Ichikawa, J.J.; Steup, M. The Analysis of Knowledge. In *The Stanford Encyclopedia of Philosophy*, Summer 2018 ed.; Zalta, E.N., Ed.; Metaphysics Research Lab, Stanford University, 2018.

7. Armstrong, D.M. *Belief, Truth And Knowledge*, 1 ed.; Cambridge University Press, 1973. https://doi.org/10.1017/CBO9780511570827.
8. Liew, A. DIKIW: Data, Information, Knowledge, Intelligence, Wisdom and their Interrelationships. *Business Management Dynamics* **2013**.
9. Rowley, J. The wisdom hierarchy: Representations of the DIKW hierarchy. *J Inf Sci* **2007**, *33*. https://doi.org/10.1177/0165551506070706.
10. Zhong, Y. Knowledge theory and information-knowledge-intelligence trinity. In Proceedings of the Proceedings of the 9th International Conference on Neural Information Processing, 2002. ICONIP '02., 2002, Vol. 1, pp. 130–133 vol.1. https://doi.org/10.1109/ICONIP.2002.1202145.
11. Dretske, F.I. *Knowledge and the Flow of Information*; MIT Press, 1981. https://doi.org/10.2307/2184492.
12. Topsoe, F. On truth, belief and knowledge. In Proceedings of the IEEE International Symposium on Information Theory - Proceedings, 2009, pp. 139 – 143. https://doi.org/10.1109/ISIT.2009.5205905.
13. Shannon, C.E. A mathematical theory of communication. *Bell Syst. Tech. J.* **1948**, *27*, 379–423.
14. Kolmogorov, A.N. Three approaches to the quantitative definition of information. *International Journal of Computer Mathematics* **1968**, *2*, 157–168.
15. Konorski, J.; Szpankowski, W. What is information? In Proceedings of the 2008 IEEE Information Theory Workshop, 2008, pp. 269–270. https://doi.org/10.1109/ITW.2008.4578666.
16. Valdma, M. A general classification of information and systems. *Oil Shale* **2007**, *24*.
17. Steup, M.; Neta, R. Epistemology. In *The Stanford Encyclopedia of Philosophy*, Fall 2020 ed.; Zalta, E.N., Ed.; Metaphysics Research Lab, Stanford University, 2020.
18. Jamie, Carlin, W. Epistemic Justification. In *The Internet Encyclopedia of Philosophy*.
19. Hasan, A.; Fumerton, R. Foundationalist Theories of Epistemic Justification. In *The Stanford Encyclopedia of Philosophy*, Fall 2018 ed.; Zalta, E.N., Ed.; Metaphysics Research Lab, Stanford University, 2018.
20. Lewin, K. Field Theory and Experiment in Social Psychology: Concepts and Methods. *American Journal of Sociology* **1939**, *44*, 868 – 896.
21. Martin, J. What is Field Theory? *American Journal of Sociology - AMER J SOCIOL* **2003**, *109*, 1–49. https://doi.org/10.1086/375201.
22. Audi, R. *The Architecture of Reason: The Structure and Substance of Rationality*; Oxford University Press, 2001.
23. Audi, R. Précis of the Architecture of Reason. *Philosophy and Phenomenological Research, Wiley-Blackwell* **2003**, *67*, 177–180. https://doi.org/10.1111/j.1933-1592.2003.tb00031.x.
24. Lenat, D. Ontological versus knowledge engineering. *IEEE Transactions on Knowledge and Data Engineering* **1989**, *1*, 84–88. https://doi.org/10.1109/69.43405.
25. Wang, W.; De, S.; Toenjes, R.; Reetz, E.; Moessner, K. A Comprehensive Ontology for Knowledge Representation in the Internet of Things. In Proceedings of the 2012 IEEE 11th International Conference on Trust, Security and Privacy in Computing and Communications, 2012, pp. 1793–1798. https://doi.org/10.1109/TrustCom.2012.20.
26. Lombardi, O. Dretske, Shannon's Theory and the Interpretation of Information. *Synthese* **2005**, *144*, 23–39. https://doi.org/10.1007/s11229-005-9127-0.
27. Cognition. *Oxford University Press and Dictionary.com*.
28. Kiely, K.M. *Cognitive Function*; Springer Netherlands: Dordrecht, 2014; pp. 974–978. https://doi.org/10.1007/978-94-007-0753-5_426.
29. Gudivada, V.N.; Pankanti, S.; Seetharaman, G.; Zhang, Y. Cognitive Computing Systems: Their Potential and the Future. *Computer* **2019**, *52*, 13–18. https://doi.org/10.1109/MC.2019.2904940.
30. Douglas, H. The Value of Cognitive Values. *Philosophy of Science* **2013**, *80*, 796–806. https://doi.org/10.1086/673716.
31. Hirsch Hadorn, G. On Rationales for Cognitive Values in the Assessment of Scientific Representations. *Journal for General Philosophy of Science* **2018**, *49*, 1–13. https://doi.org/10.1007/s10838-018-9403-6.
32. Todt, O.; Luján, J.L. Values and Decisions: Cognitive and Noncognitive Values in Knowledge Generation and Decision Making. *Science, Technology, & Human Values* **2014**, *39*, 720–743. https://doi.org/10.1177/0162243914521019.
33. Wirtz, P., Entrepreneurial finance and the creation of value: Agency costs vs. cognitive value; 2015; pp. 552–568. https://doi.org/10.4018/978-1-4666-8348-8.ch030.
34. Cleeremans, A. Conscious and unconscious cognition: A graded, dynamic perspective. *Int. J. Psychol* **2004**, *39*. https://doi.org/10.4324/9780203783122.
35. Epstein, S. Demystifying Intuition: What It Is, What It Does, and How It Does It. *Psychological Inquiry* **2010**, *21*, 295–312. https://doi.org/10.1080/1047840X.2010.523875.
36. Knowledge. *Oxford Dictionary*.
37. Biggam, J. Defining knowledge: an epistemological foundation for knowledge management. In Proceedings of the Proceedings of the 34th Annual Hawaii International Conference on System Sciences, 2001, pp. 7 pp.–. https://doi.org/10.1109/HICSS.2001.927102.
38. Beránková, M.; Kvasnička, R.; Houška, M. Towards the definition of knowledge interoperability. In Proceedings of the 2010 2nd International Conference on Software Technology and Engineering, 2010, Vol. 1, pp. V1–232–V1–236. https://doi.org/10.1109/ICSTE.2010.5608843.

39. Firestein, S. *Ignorance: How it drives science*; Oxford University Press, 2012. https://doi.org/10.5860/choice.49-6848.
40. ignorance. *Oxford Dictionary*.
41. Foo, N.; Vo, B. Reasoning about Action: An Argumentation - Theoretic Approach. *Journal of Artificial Intelligence Research* **2011**, *24*. https://doi.org/10.1613/jair.1602.
42. Giordano, L.; Martelli, A.; Schwind, C. Reasoning about actions in dynamic linear time temporal logic. *Logic Journal of the IGPL* **2001**, *9*, 273–288. https://doi.org/10.1093/jigpal/9.2.273.
43. Hartonas, C. Reasoning about types of action and agent capabilities. *Logic Journal of the IGPL* **2013**, *21*, 703–742. https://doi.org/10.1093/jigpal/jzs045.
44. Zhong, S.; Xia, K.; Yin, X.; Chang, J. The representation and simulation for reasoning about action based on Colored Petri Net. In Proceedings of the 2010 2nd IEEE International Conference on Information Management and Engineering, 2010, pp. 480–483. https://doi.org/10.1109/ICIME.2010.5477807.
45. Legg, S.; Hutter, M. A Collection of Definitions of Intelligence. *Advances in Artificial General Intelligence: Concepts, Architectures and Algorithms* **2007**, *157*.
46. Wilhelm, O.; Engle, R. *Handbook of understanding and measuring intelligence*; 2005; p. 542. https://doi.org/10.4135/9781452233529.n1.
47. Wang, Y.; Widrow, B.; Zadeh, L.; Howard, N.; Wood, S.; Bhavsar, V.; Budin, G.; Chan, C.; Gavrilova, M.; Shell, D. Cognitive Intelligence: Deep Learning, Thinking, and Reasoning by Brain-Inspired Systems. *International Journal of Cognitive Informatics and Natural Intelligence* **2016**, *10*, 1–20. https://doi.org/10.4018/IJCINI.2016100101.
48. Zhong, Y. Mechanism approach to a unified theory of artificial intelligence. In Proceedings of the 2005 IEEE International Conference on Granular Computing, 2005, Vol. 1, pp. 17–21 Vol. 1. https://doi.org/10.1109/GRC.2005.1547228.
49. Rens, G.; Varzinczak, I.; Meyer, T.; Ferrein, A. A Logic for Reasoning about Actions and Explicit Observations. 2010, Vol. 6464, pp. 395–404. https://doi.org/10.1007/978-3-642-17432-2_40.
50. Pereira, L.; Li, R. Reasoning about Concurrent Actions and Observations **1999**.
51. Kakas, A.C.; Miller, R.; Toni, F. E-RES: A System for Reasoning about Actions, Events and Observations. *ArXiv* **2000**, *cs.AI/0003034*.
52. Costa, A.; Salazar-Varas, R.; Iáñez, E.; Úbeda, A.; Hortal, E.; Azorín, J.M. Studying Cognitive Attention Mechanisms during Walking from EEG Signals. In Proceedings of the 2015 IEEE International Conference on Systems, Man, and Cybernetics, 2015, pp. 882–886. https://doi.org/10.1109/SMC.2015.162.
53. Wickens, C. Attention: Theory, Principles, Models and Applications. *International Journal of Human–Computer Interaction* **2021**, *37*, 403–417. https://doi.org/10.1080/10447318.2021.1874741.
54. Atkinson, R.C.; Shiffrin, R.M. Human Memory: A Proposed System and its Control Processes. In Proceedings of the Psychology of Learning and Motivation, 1968.
55. Vaswani, A.; Shazeer, N.; Parmar, N.; Uszkoreit, J.; Jones, L.; Gomez, A.N.; Kaiser, L.; Polosukhin, I. Attention is All You Need; Curran Associates Inc.: Red Hook, NY, USA, 2017; NIPS'17.
56. Miller, G.A. The magical number seven plus or minus two: some limits on our capacity for processing information. *Psychological review* **1956**, *63 2*, 81–97.
57. Fatemi, M.; Haykin, S. Cognitive Control: Theory and Application. *IEEE Access* **2014**, *2*, 698–710. https://doi.org/10.1109/ACCESS.2014.2332333.
58. Nunes, T. Logical Reasoning and Learning. In *Encyclopedia of the Sciences of Learning*; 2012; pp. 2066–2069. https://doi.org/https://doi.org/10.1007/978-1-4419-1428-6_790.
59. Stenning, K.; van Lambalgen, M. Reasoning, logic, and psychology. *WIREs Cognitive Science*, *2*, 555–567. https://doi.org/https://doi.org/10.1002/wcs.134.
60. Johnson-Laird, P.N. Mental models and human reasoning **2010**. *107*, 18243–18250. https://doi.org/10.1073/pnas.1012933107.
61. Atkinson, R.; Shiffrin, R. Human Memory: A proposed system and its control processes. In *Human Memory*; BOWER, G., Ed.; Academic Press, 1977; pp. 7–113. https://doi.org/https://doi.org/10.1016/B978-0-12-121050-2.50006-5.
62. Baddeley, A.D.; Hitch, G. Working Memory; Academic Press, 1974; Vol. 8, *Psychology of Learning and Motivation*, pp. 47–89. https://doi.org/https://doi.org/10.1016/S0079-7421(08)60452-1.
63. Baddeley, A.; Conway, M.; JP, A. Episodic Memory: New Directions in Research **2002**. https://doi.org/10.1093/acprof:oso/9780198508809.001.0001.
64. Angluin, D. Computational Learning Theory: Survey and Selected Bibliography. In Proceedings of the Proceedings of the Twenty-Fourth Annual ACM Symposium on Theory of Computing. Association for Computing Machinery, 1992, p. 351–369. https://doi.org/10.1145/129712.129746.
65. Vapnik, V. An overview of statistical learning theory. *IEEE Transactions on Neural Networks* **1999**, *10*, 988–999. https://doi.org/10.1109/72.788640.
66. Beránková, M.; Kvasnička, R.; Houška, M. Towards the definition of knowledge interoperability. In Proceedings of the 2010 2nd International Conference on Software Technology and Engineering, 2010, Vol. 1, pp. V1–232–V1–236. https://doi.org/10.1109/ICSTE.2010.5608843.
67. Chen, Z.; Duan, L.Y.; Wang, S.; Lou, Y.; Huang, T.; Wu, D.O.; Gao, W. Toward Knowledge as a Service Over Networks: A Deep Learning Model Communication Paradigm. *IEEE Journal on Selected Areas in Communications* **2019**, *37*, 1349–1363. https://doi.org/10.1109/JSAC.2019.2904360.

68. Corbett, D. Semantic interoperability of knowledge bases: how can agents share knowledge if they don't speak the same language? In Proceedings of the Sixth International Conference of Information Fusion, 2003. Proceedings of the, 2003, Vol. 1, pp. 94–98. https://doi.org/10.1109/ICIF.2003.177432.
69. Hunter, A.; Konieczny, S. Approaches to Measuring Inconsistent Information. 2005, Vol. 3300, pp. 191–236. https://doi.org/10.1007/978-3-540-30597-2_7.
70. Sayood, K. Information Theory and Cognition: A Review. *Entropy* **2018**, *20*, 706. https://doi.org/10.3390/e20090706.
71. Démuth, A. *Perception Theories*; Kraków: Trnavská univerzita, 2013.
72. Briscoe, R.; Grush, R. Action-based Theories of Perception. In *The Stanford Encyclopedia of Philosophy*; 2020.
73. Guizzardi, G. On Ontology, ontologies, Conceptualizations, Modeling Languages, and (Meta)Models. In Proceedings of the Seventh International Baltic Conference, DB&IS, 2006, pp. 18–39.
74. Chitsaz, M.; Hodjati, S.M.A. Conceptualization in ideational theory of meaning: Cognitive theories and semantic modeling. *Procedia - Social and Behavioral Sciences* **2012**, *32*, 450–455. https://doi.org/10.1016/J.SBSPRO.2012.01.068.
75. El Morr, C.; Ali-Hassan, H., Descriptive, Predictive, and Prescriptive Analytics: A Practical Introduction; 2019; pp. 31–55. https://doi.org/10.1007/978-3-030-04506-7_3.
76. Megha, C.; Madhura, A.; Sneha, Y. Cognitive computing and its applications. In Proceedings of the 2017 International Conference on Energy, Communication, Data Analytics and Soft Computing (ICECDS), 2017, pp. 1168–1172. https://doi.org/10.1109/ICECDS.2017.8389625.
77. Zhang, X.N. Formal analysis of diagnostic notions. In Proceedings of the 2012 International Conference on Machine Learning and Cybernetics, 2012, Vol. 4, pp. 1303–1307. https://doi.org/10.1109/ICMLC.2012.6359553.
78. Francesco, B.; Ahti-Veikko, P. Charles Sanders Peirce: Logic. In *The Internet Encyclopedia of Philosophy*; 2022.
79. Pishro-Nik, H. *Introduction to Probability, Statistics, and Random Processes*; Kappa Research LLC, 2014.
80. Lefcourt, H.M. *Locus of control: Current trends in theory and research*; Psychology Press, 1982. https://doi.org/https://doi.org/10.4324/9781315798813.
81. Shramko, Y.; Wansing, H. Truth Values. In *The Stanford Encyclopedia of Philosophy*; 2021.
82. Parent, T. Externalism and Self-Knowledge. In *The Stanford Encyclopedia of Philosophy*; 2017.
83. Ted, P. Epistemic Justification. In *The Internet Encyclopedia of Philosophy*; 2022.
84. Artemov, S.; Fitting, M. Justification Logic. In *The Stanford Encyclopedia of Philosophy*; 2021.
85. Smith, G. Newton's Philosophiae Naturalis Principia Mathematica. In *The Stanford Encyclopedia of Philosophy*; 2008.
86. Eddington, A.S.S. *The mathematical theory of relativity*; Cambridge University Press, 1924. https://doi.org/10.2307/2298148.
87. Kullback, S.; Leibler, R.A. On Information and Sufficiency. *The Annals of Mathematical Statistics* **1951**, *22*, 79 –86. https://doi.org/10.1214/aoms/1177729694.
88. Berge, T.; van Hezewijk, R. Procedural and Declarative Knowledge. *Theory & Psychology* **1999**, *9*, 605 – 624. https://doi.org/10.1177/0959354399095002.
89. Gärdenfors, P., Semantic Knowledge, Domains of Meaning and Conceptual Spaces; 2017; pp. 203–219. https://doi.org/10.1007/978-3-319-44588-5_12.
90. Amari, S. Information geometry and its applications. *Journal of Mathematical Psychology* **2005**, *49*, 101–102.
91. Nielsen, F. The Many Faces of Information Geometry. *Notices of the American Mathematical Society* **2022**, *69*, 36–45. https://doi.org/10.1090/noti2403.
92. Amari, S.; Cichocki, A. Information geometry of divergence functions. *Bulletin of the Polish Academy of Sciences Technical Sciences* **2010**, *58*. https://doi.org/10.2478/v10175-010-0019-1.
93. Guiasu, S.; Shenitzer, A. The principle of maximum entropy. *The Mathematical Intelligencer* **1985**, *7*, 42–48. https://doi.org/10.1007/BF03023004.
94. Duncan, P., Knowledge and Understanding. In *Virtue Epistemology Naturalized*; Springer International Publishing, 2014.
95. Duncan, P. Knowing the answer, understanding and epistemic value. *Grazer Philosophische Studien* **2008**, *77*, 325–339. https://doi.org/10.1163/18756735-90000852.
96. Einstein, A.B. Relativity: The Special and the General Theory. 1916.
97. Walter, S. Minkowski, Mathematicians, and the Mathematical Theory of Relativity. *The expanding worlds of general relativity, 45-86 (1999)* **1999**. https://doi.org/10.1007/978-1-4612-0639-2_2.
98. Greenwald, A.; De Houwer, J. Unconscious conditioning: Demonstration of existence and difference from conscious conditioning. *Journal of Experimental Psychology: General* **2017**, *146*. https://doi.org/10.1037/xge0000371.
99. Fernández, A.; García, S.; Galar, M.; Prati, R.; Krawczyk, B.; Herrera, F. *Learning from Imbalanced Data Sets*; Springer International Publishing, 2018. https://doi.org/10.1007/978-3-319-98074-4.
100. Shephard, R.W.; Färe, R. The law of diminishing returns. *Zeitschrift für Nationalökonomie* **1974**, *34*, 69–90. https://doi.org/doi:10.1007/BF01289147.
101. McLachlan, G.J.; Peel, D. *Finite Mixture Models*; Wiley Series in Probability and Statistics, 2000.
102. Lin, J. Divergence measures based on the Shannon entropy. *IEEE Transactions on Information Theory* **1991**, *37*, 145–151. https://doi.org/10.1109/18.61115.
103. Dietterich, T.G. Ensemble methods in machine learning. In Proceedings of the Multiple Classifier Systems: First International Workshop, MCS 2000. Springer, 2000, pp. 1–15. https://doi.org/10.1007/3-540-45014-9_1.

104.]varianinferprob Lawrence, N. Variational inference in probabilistic models. Phd thesis, Cambridge University, 2000 [Online].
105. Blei, D.; Kucukelbir, A.; McAuliffe, J. Variational Inference: A Review for Statisticians. *Journal of the American Statistical Association* **2016**, *112*. https://doi.org/10.1080/01621459.2017.1285773.
106. Mnih, A.; Gregor, K. Neural Variational Inference and Learning in Belief Networks. *31st International Conference on Machine Learning, ICML 2014* **2014**, *5*.
107. Tschannen, M.; Djolonga, J.; Rubenstein, P.K.; Gelly, S.; Lucic, M. On Mutual Information Maximization for Representation Learning. *ArXiv* **2020**, *abs/1907.13625*.
108. De Lange, M.; Aljundi, R.; Masana, M.; Parisot, S.; Jia, X.; Leonardis, A.; Slabaugh, G.; Tuytelaars, T. A Continual Learning Survey: Defying Forgetting in Classification Tasks. *IEEE Transactions on Pattern Analysis and Machine Intelligence* **2022**, *44*, 3366–3385. https://doi.org/10.1109/TPAMI.2021.3057446.
109. Kukačka, J.; Golkov, V.; Cremers, D. Regularization for Deep Learning: A Taxonomy. *ArXiv* **2017**, *abs/1710.10686*.
110. Haussler, D. Probably approximately correct learning. In Proceedings of the AAAI-90 Proceedings. Encyclopedia of Machine Learning and Data Mining, 1990. https://doi.org/10.1007/978-1-4899-7687-1_100377.
111. Mohri, M.; Rostamizadeh, A. Perceptron Mistake Bounds. *ArXiv* **2013**, *abs/1305.0208*.
112. Arlot, S.; Celisse, A. A Survey of Cross Validation Procedures for Model Selection. *Statistics Surveys* **2009**, *4*. https://doi.org/10.1214/09-SS054.
113. Cheung, G.; Magli, E.; Tanaka, Y.; Ng, M. Graph Spectral Image Processing. *Proceedings of the IEEE* **2018**, *PP*. https://doi.org/10.1109/JPROC.2018.2799702.
114. Bampis, C.G.; Maragos, P.; Bovik, A.C. Graph-Driven Diffusion and Random Walk Schemes for Image Segmentation. *IEEE Transactions on Image Processing* **2017**, *26*, 35–50. https://doi.org/10.1109/TIP.2016.2621663.
115. Chamberlain, B.P.; Rowbottom, J.R.; Gorinova, M.I.; Webb, S.; Rossi, E.; Bronstein, M.M. GRAND: Graph Neural Diffusion. In Proceedings of the ICML, 2021.
116. Atwood, J.; Towsley, D. Diffusion-convolutional neural networks. *Advances in neural information processing systems* **2016**, *29*.
117. Masuda, N.; Porter, M.; Lambiotte, R. Random walks and diffusion on networks. *Physics Reports* **2016**, *716*. https://doi.org/10.1016/j.physrep.2017.07.007.
118. Mackie, M.A.; Van Dam, N.; Fan, J. Cognitive Control and Attentional Functions. *Brain and cognition* **2013**, *82*, 301–312. https://doi.org/10.1016/j.bandc.2013.05.004.
119. Maclean, K.; Aichele, S.; Bridwell, D.; Mangun, G.; Wojciulik, E.; Saron, C. Interactions between Endogenous and Exogenous Attention during Vigilance. *Attention, perception & psychophysics* **2009**, *71*, 1042–58. https://doi.org/10.3758/APP.71.5.1042.
120. Wittkuhn, L.; Chien, S.; Hall-McMaster, S.; Schuck, N.W. Replay in minds and machines. *Neuroscience & Biobehavioral Reviews* **2021**, *129*, 367–388. https://doi.org/10.31234/osf.io/df7ga.
121. Zha, D.; Lai, K.H.; Zhou, K.; Hu, X. Experience Replay Optimization. In Proceedings of the Proceedings of the Twenty-Eighth International Joint Conference on Artificial Intelligence. IJCAI-19, 2019, pp. 4243–4249. https://doi.org/10.24963/ijcai.2019/589.
122. Schlanger, J. Introspection, rétrospection, prospection. *Revue De Metaphysique Et De Morale* **2001**, *32*. https://doi.org/10.3917/rmm.014.0527.
123. Peters, U. What Is the Function of Confirmation Bias? *Erkenntnis* **2022**, *87*, 1–26. https://doi.org/10.1007/s10670-020-00252-1.
124. Allahverdyan, A.; Galstyan, A. Opinion Dynamics with Confirmation Bias. *PloS one* **2014**, *9*, e99557. https://doi.org/10.1371/journal.pone.0099557.
125. Sagi, O.; Rokach, L. Ensemble learning: A survey. *Wiley Interdisciplinary Reviews: Data Mining and Knowledge Discovery* **2018**, *8*, e1249. https://doi.org/10.1002/widm.1249.
126. Weiss, K.; Khoshgoftaar, T.; Wang, D. A survey of transfer learning. *Journal of Big Data* **2016**, *3*. https://doi.org/10.1186/s40537-016-0043-6.
127. Abdulrahman, S.; Tout, H.; Ould-Slimane, H.; Mourad, A.; Talhi, C.; Guizani, M. A Survey on Federated Learning: The Journey From Centralized to Distributed On-Site Learning and Beyond. *IEEE Internet of Things Journal* **2021**, *8*, 5476–5497. https://doi.org/10.1109/JIOT.2020.3030072.
128. Dua, D.; Graff, C. UCI Machine Learning Repository, 2017.