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Mathematical singularities in the farthest confines of the Universe —and a brief report on its evolutionary history

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Abstract: It is advisable to avoid and, even better, demystify such grandiose terms as "infinity" or "singularity" in the description of the cosmos. Its proliferation does not positively contribute to the understanding of key concepts that are essential for an updated account of its origin and evolutionary history. It will be here argued that, as a matter of fact, there are no infinities in physics, in the real world: all which appear, in any given formulation of nature by means of mathematical equations, actually arise from extrapolations, which are made beyond the bounds of validity of the equations themselves. Such crucial point is rather well-known, but too often forgotten, as discussed in this paper with several examples; namely, the famous Big Bang singularity and others, which appeared before in classical mechanics and electrodynamics, and notably in the quantization of field theories. A brief description of the Universe's history and evolution follows. Special emphasis is put on what is presently known, from detailed observations of the cosmos and, complementarily, from advanced experiments of very-high-energy physics. To conclude, a future perspective on how this knowledge might soon improve is given.

Keywords: singularity; infinite; Big Bang; universe evolution; scientific theory.

1. Introduction: Fleeting biblical reference

The first book of the Bible, the Book of Genesis, contains an account of the creation of the world. We read there that God created the world in seven days. That narrative, although precise in some details, is quite schematic. It conveys the idea that, at the beginning of time and starting up from nothing, God made the universe we now have (Fig. 1). But, rather than from nothing, according to John 1:1 in the New Testament, it could have been from an "energy": the "Word" –which John says it was with God and was God himself.

Technically speaking, this is a fairly static and somewhat mechanistic vision, although on a very grand scale; and one should notice that, eventually, the whole work is completely finished. The evolution of the cosmos is not sufficiently considered. It should be observed, too, that the biblical creation of the world also contains another crucial stage, namely the creation of Man, from dust of the ground, God breathing into him the breath of life and consciousness, and then into Eve, to become partners. This paper will not deal at all with the last issue, which is of an enormous complexity, and is today being investigated by other means.

This very brief introduction should be taken as a mere historical and comparative reference, without further ado. It is not the purpose of the present work to delve into the biblical description, nor to criticize it. Some consider it as a poetic story, inherited from others coming from older cultures; but it is certainly fascinating and has some points of contact with the much more accurate view that science gives us today. It is the latter that

will be considered below, in an affordable way, but, at the same time, with all the rigor that, what has been so far discovered and experimentally verified, provides.

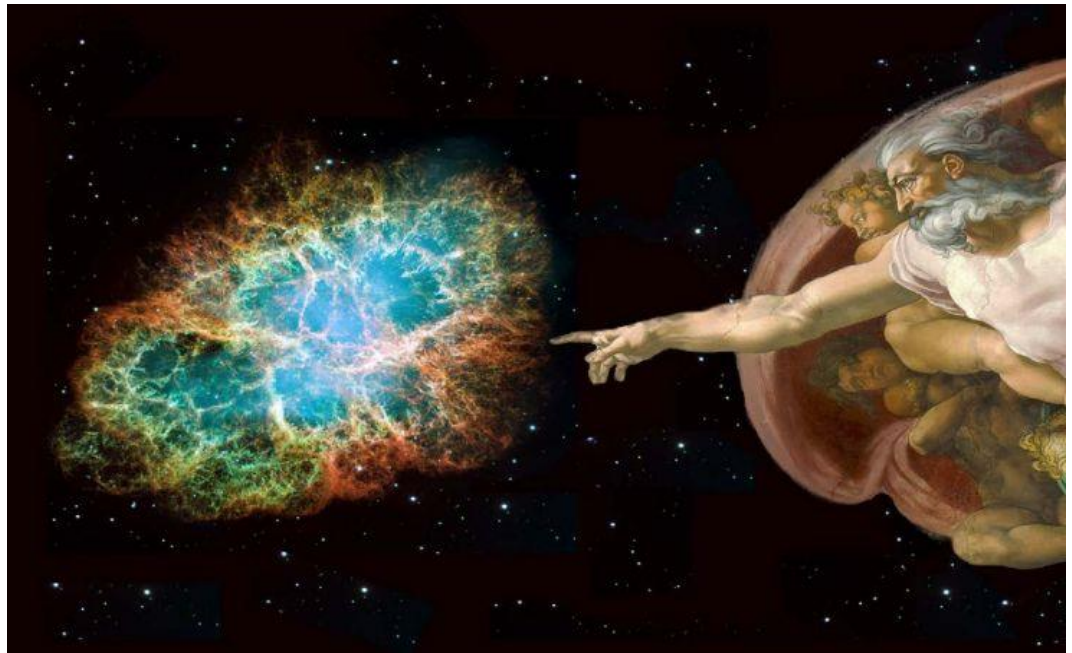



Figure 1. Artistic representation of God's creation of the universe.

At the risk of simplifying the comparison excessively, one could already advance, as an appetizer, that it might look as if, in the modern descriptions of the origin of everything, the latent (and, for many, explicit) mystery in the version of The Genesis had been replaced nowadays by another equally unfathomable one, which is however, for our rational intellect, much more precise, well posed, and apparently rigorous: The Big Bang singularity. The paper will deal with such singularity, as well as with others that so often arise in the mathematical equations corresponding to physical theories of different kinds.

What will follow is a transcription --though substantially expanded in several aspects-- of the content of two recent talks given by the author: one of them in a removable booth in the middle of the vivid Rambla del Raval, in Barcelona, within the framework of the Barcelona Science Festival, and the other at the stylish Auditorium Barradas of Hospitalet de Llobregat, also in Spain, on the occasion of the European Researcher's Night 2022, respectively. The thorough preparation of these talks led the author to reflect in depth on some of the points that will follow. We should praise, by the way, such great initiatives, which seek to bring top scientific subjects to the ordinary people in a solvent way, far from the cartoonish simplification, and to awaken the interest and passion for true science, which we miss so much in our society.

The structure of the paper is as follows. In Sect. 2 the Big Bang singularity and other relevant infinities are considered. Some key issues are discussed in several subsections devoted, respectively, to the limited range of validity of scientific theories, the definition of a mathematical function, and the ways infinites are dealt with, by means of regularization and renormalization processes. In fact, only a brief, historical account of some of them will be provided, with the purpose to stress the main point here, which is namely that there are no infinities in nature and that they only arise because the mathematical equations describing the corresponding physical theory are extended beyond their actual domain of validity. In Sect. 3 the first stages of the origin and evolution of our universe are discussed, up to the reheating epoch. Emphasis is put in the fact that they involve deep knowledge of particle physics at very high energies, indeed, the highest ones that are presently attainable in particle accelerators. In Sect. 4 a brief discussion of the recombination and reionization epochs of the universe evolution is given. Some current and future mission for the exploration of the cosmos will be presented in Sect. 5. In Sect. 6, a general

discussion of the main issues considered in the paper is collected. Finally, Sect. 7 contains some closing conclusions.

2. The Big Bang singularity and other relevant divergences

The traditional theory of the Big Bang (BB) uses the Theory of General Relativity (GR) to describe the Universe and, in particular, its origin; that is, the origin of everything, of space and time, of matter and energy, in a word, of all that we see. In the current, most commonly accepted description, everything begins, somehow mysteriously one must add, in the so-called Big Bang singularity. But the first (and serious) problem is that this extraordinary theory, for many, the most beautiful theory ever created (there are no adjectives to qualify it as it deserves), turns out to be not fitted for studying the Universe when it was very small, let alone its origin.

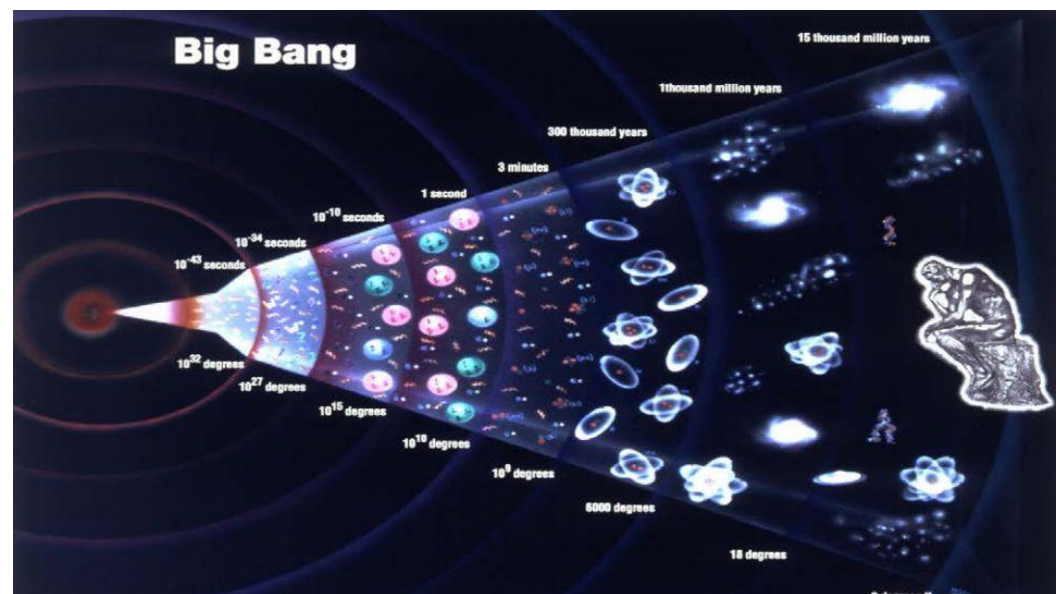



Figure 2. Schematic representation of the Big Bang singularity and subsequent universe evolution. Credit: grandunificationtheory.com.

Let us now reflect for a while --as the image on the right side of the picture (Fig. 2) invites us to do. At a much more comprehensible level --setting our feet on the ground and without having to travel that far back in time-- just consider the atomic level, which is much more familiar and closer to our understanding. Every physicist knows, since at least a whole century now, that GR is of no use to describe the behavior of atoms, let alone of their nuclei (several orders of magnitude smaller). But, yet, all of them are nowadays ordinary objects, on which much of our present scientific and technological developments are based. But there, at the atomic and nuclear levels, the Einstein equations fail with exactly the same crash as Newtonian mechanics does; since, on those scales, GR does not add anything new to the Newtonian results! At the very least we, physicists (and scientists of any sort I would dare to say) should be fully aware of this circumstance; although sometimes, perhaps enchanted by the extreme beauty of the Einsteinian theory, it does seem as if we forgot.

And now, just reflect a bit more. The consequence of the above is that, all those ramblings that, with the intention of impressing, are so often written about the "initial singularity", about a "point like" primitive universe of "infinite density and temperature", and many other nonsense statements like those, which even manage to sneak into some versions of the Wikipedia (fortunately, not in all of them) are mere mathematical extrapolations without the slightest *physical* sense. We could continue arguing on this in many ways, apart from the reasons already given.

2.1. The limited range of validity of scientific theories

The following message must stay clear: in the real world, in nature, there are *no* infinities! All those that have appeared, on so many occasions, in different theories and circumstances, all those that arise from the mathematical formulations of physical reality when attempting to describe it –such as the Newtonian laws, Einstein's field equations, or Maxwell's equations of electrodynamics– are, without exception, mathematical artifacts! They only pop up because it so happens that the equations of Physics are always, when we are outside certain domains or specific conditions, not precise enough or even completely inappropriate to describe the real world.

A key issue to be understood here is that the equations that express, in mathematical language, the principles of a physical theory, whatever this be, *always* have a *limited* range of validity. When one leaves such domain, then the equations start to become imprecise and then stop to be useful at all.

A very clear example of this issue are the Newtonian equations. For several centuries they were believed to be universal, literarily: the laws of universal gravitation, true under *any* circumstances or conditions. Until it became dramatically clear that this was not at all the case; rather, they had to be drastically corrected, both at small distances (quantum physics) and at high speeds (theory of relativity). But it is also a fact that these brand-new theories have, in turn, their own *limits*, their respective ranges of validity: they are not universal, either. In no way are they the final theories to describe our universe; at most, really good, complementary approximations to such.

2.2. Mathematics for comparison: The so precise definition of a mathematical function, and the unreasonable effectiveness of mathematics in the physical world

Mathematicians, in the realm of their axiomatic world – parallel, in some way, to the physical one-- have a perfect understanding of this point, which is already embedded in the very definition of a concept as basic as that of a "mathematical function". As is taught in high-school or freshman courses, a mathematical function is not merely a "correspondence", f (that is, a mathematical expression or formula), but it also includes its "domain", A, and "range", B, that is, the initial and final sets on which the correspondence is defined and it is strictly valid, that is, where it actually makes sense. A mathematical function is defined, therefore, as the triple (A,B,f) --and never just as f. Note the absolute parallelism with what precedes, on the domain of validity of a physical theory [1,2].

A recurrent issue that has puzzled many renowned scientists and engineers over time, is that of the unreasonable effectiveness of mathematics in the physical world. Of the several references that are immediately brought to mind (Einstein himself expressed this idea on several occasions), I have selected two, namely the all-famous paper by Eugene P. Wigner entitled "The Unreasonable Effectiveness of Mathematics in the Natural Sciences" [3], where this issue is magnificently explained, and another one by Richard W. Hamming, with the shorter title "The Unreasonable Effectiveness of Mathematics", where the subject is treated from a complementary viewpoint and further examples are given, coming from engineering and other scientific disciplines [4]. Little will be added here to what can be found there but, in relation with the basic discussion, the following remark is in order.

It need be recalled that, in addition to classical mechanics, we owe to Isaac Newton another invention of extreme importance, which he did in parallel with Gottfried Wilhelm Leibniz, namely, infinitesimal calculus [5,6]. Many of the equations in science are differential ones and integration is performed according to the rules of calculus, too. Infinitesimal calculus is the mathematical study of continuous change, and its starting scenario is a rigorous, perfectly well stablished definition of the mathematical continuum. This concept corresponds to the dense distribution of real numbers on the real line, then extended to two, three, and to any number of dimensions (even beyond those of the physical space), by Bernhard Riemann. In particular, the construction, step by step, of the real line, is one of the most exciting lessons of any university course on infinitesimal calculus. No student of physics or engineering should ever miss those lectures.

When attending, later, the lectures on atomic physics, it becomes suddenly clear how far apart are the two understandings: the one of the lower and lower levels of physical reality and that of the mathematical conception of the same dimensions. They have nothing to do with each other! And the mystery of the extraordinary effectiveness of mathematics in modeling physics grows and grows at the same pace.

Further to that, and looking again through a powerful microscope, one also realizes of a sudden that already the simple definition of a mathematical "point" or "dot" is an abstraction of reality; and the same is true for those of a line, a plane, etc. There are *no* physical objects which do correspond to such mathematical, purely axiomatic constructions, in physics. Think, e.g., that graphene sheets --the thinnest to have been constructed ever-- are at least one atom thick (which is not much, but still quite far from zero depth). And, what would be our physical candidate for a mathematical dot? of course not a quantum dot, a quark perhaps? Maybe a neutrino, which is much smaller, but still rather big, as compared to zero-diameter. In few words, a point-particle has no real sense. When confronted with this, a physicist should always ask something like: How small you mean is your dot? Does it have a diameter of 10^{-10} cm? of 10^{-35} cm? of 10^{-120} cm? Likewise, the term "infinity" is also devoid of real sense. When one hears "infinite temperature" at the moment of the Big Bang, the immediate question should follow, like: What do you mean? Higher than 10^{50} K, or something? There is still a long, very long way to infinity, or to zero, in both these examples. The actually strange thing –a great puzzle-- is that working consistently with those abstract mathematical entities one does obtain results that are extremely close to the real ones of the laboratory.

Whatever the case, and to summarize: what is undoubtedly wrong and inappropriate is to talk of point physical particles, infinite matter or charge densities, infinite temperatures, and the like; in short, divergences galore, when there is none. It is my point that even quick, vague or cartoonish descriptions of reality should avoid these expressions, which simply make no sense and do not help to understand the actual physical concepts.

2.3. Taming infinities: Cantor, Euler, Riemann, and a few drops of zeta regularization

It so happens that the present author, aside from being both a physicist and a mathematician, has dedicated a good part of his life to considering questions related to "infinity". Although, rather, only to some aspects of this concept, since the "infinity" has many faces. So many, that more than one mathematician, like the great Georg Cantor, who dedicated a good part of his life to studying it, and decisively advanced his knowledge of it, in the end he went mad, literally [7,8].

Another famous mathematician, Leonard Euler, was convinced that all infinite series made sense; specifically, that with some reasonable criterion they could always be assigned a finite value [9,10].

To give a couple of examples of this fact, the simplest infinite series:

$$1+1+1+1+1+1+ ... \tag{1}$$

is better represented as corresponding to the value -1/2,

$$1+1+1+1+1+1+ ... \approx -1/2; \tag{2}$$

while the series

$$1+2+3+4+5+6+7+ ... \tag{3}$$

yields with the same method (namely, analytic continuation by means of the zeta function) the value -1/12 [11-14]

$$1+2+3+4+5+6+7+ ... \approx -1/12. \tag{4}$$

And it happens that, when the most advanced physical theories make an appropriate use of such a discovery, this apparent "foolishness" or "arbitrariness" (once well worked out

in the complex field [11-14]) acquires an extraordinary value in describing with incredible precision the phenomena of the real world.

Leonhard Euler devoted great efforts to studying infinite series. The harmonic series

$$1+1/2+1/3+1/4+1/5+ \ldots \tag{5}$$

fascinated him. Its terms correspond to the wavelengths of the successive notes of a vibrating string of a musical instrument –and each one of them is the harmonic mean of its neighboring terms. Mathematically, it turns out to be a logarithmically divergent series, but Euler realized that, by putting an exponent, s, on the natural number for the frequency

$$1+1/2^s+1/3^s+1/4^s+1/5^s+ \ldots = \sum_{n=1}^{\infty}\frac{1}{n^s} \, , \tag{6}$$

he obtained for the sum of the infinite terms spectacular results, such as the following. When the exponent s was 2, the result was

$$\pi^2 / 6, \tag{7}$$

and for s = 4, 6, 8, he got, respectively,

$$\pi^4 / 90, \quad \pi^6 / 945, \quad \pi^8 / 9450. \tag{8}$$

Furthermore, Euler obtained all results up to the power 26. As a matter of fact, there is a closed expression for any even exponent, n, in terms of π^n and Bernoulli numbers.

But, when the exponent is 1, we are back in the initial case of the harmonic series, and there is no way to get a finite value for it. This one is indeed a genuine logarithmic divergence; what means that, if we sum a number N of terms in the series, the value of the sum behaves as $log\ N$.

Euler derived important properties of this modified series as a function of s, not only for even or natural values of s, but also for other real values of this variable. One of the most important he got is the relation of this function to prime numbers: the zeta function is equal to an infinite product extended to all prime numbers, p, namely

$$\sum_{n=1}^{\infty}\frac{1}{n^s} = \prod_{p}\frac{1}{1-p^s}. \tag{9}$$

Some years later, Bernhard Riemann went much further and discovered the extraordinary importance of such function of s, when s is allowed to be a complex number. He defined the zeta function, $\zeta(s)$ – today called the Riemann zeta function – as follows:

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s} \, , \quad \text{when} \quad \text{Re } s > 1. \tag{10}$$

that is, when the real part of the variable s is bigger than *1*. There, the series is absolutely convergent, can be added without any problem and yields a finite result. But, what happens for the other values of s on the complex plane that do not satisfy this condition? Riemann showed that this series can be *uniquely* extended to all the rest of the complex plane, C. At any point, this analytic extension gives a finite value, with a single exception, that is, when *s = 1*, in which case one recovers the harmonic series, inevitably divergent. This one, therefore, remains as the only singularity of the zeta function, in the entire complex plane, a simple pole with residue equal to *1* (Fig. 3).

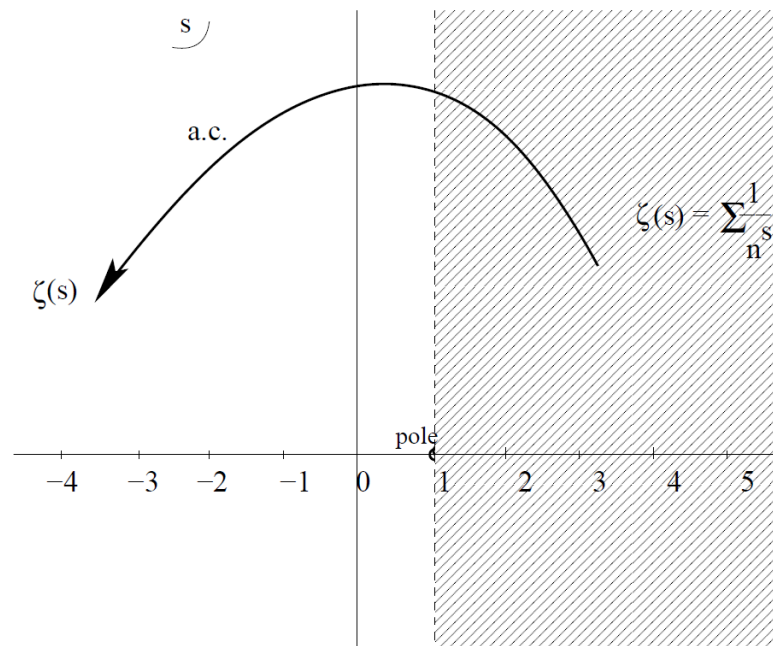



Figure 3. Definition of the Riemann zeta function on the complex plane.

To be noted is that the vertical line of the complex plane Re $s = 1/2$ (parallel to the ordinate axis) is of exceptional importance. Specifically, the Riemann conjecture (or Riemann hypothesis, considered by many to be the most important unsolved mathematical problem in history) states that the points in the complex plane where the zeta function is zero are all located on this line (one leaves apart, of course, the so-called trivial zeros, which are well known and obtained when $s = -2n$, for n any arbitrary positive integer).

The zeta function is a wonderful function and at the same time full of mystery. Many consider it the most important function ever constructed. We can look at her as an oracle, or a prophetess, we will see why. The author has devoted a significant part of his life to study a few physical applications of it and also of several of its non-trivial generalizations, as zeta functions build from the spectral values of pseudodifferential operators, which are associated with physical quantities [13,14] (one zeta function bears the author's name). Many references on these results can be found in [15].

In few words, by means of the zeta function, terribly divergent infinite series can be driven to give finite values. And, what is more important, these values, which often may seem absurd at first sight, do make full sense in the end, in the physical world, having been validated by the most precise and accurate laboratory experiments ever conducted to date.

2.4. A couple of simple examples, and a word of warning

If you look deeply into the matter, you will discover that all secrets of the zeta function are hidden in the extraordinary properties of the complex field. There, the most amazing of all mathematical formulas appears, namely

$$e^{i\pi} + 1 = 0. \tag{11}$$

It contains, in a single and very simple expression, the most important mathematical numbers. No matter how much one went on, it would be impossible to give even a faint idea of the exceptional importance that complex numbers have in physics and in all of the sciences. Every scientist has a particular glimpse of this crucial importance, in his/her own field. Simply said, without them no theory would be possible. Although, of course, they are not real numbers and, consequently, the result of the measurement of whatever quantity of nature will not be expressed under the form of a complex number.

We have already seen that in Riemann's definition of the zeta function these numbers play a key role, allowing the function to be defined in a unique (unambiguous) way, as a meromorphic function.

Let us now consider our first series (1) and assume that it is indeed a form (a particular case) of the zeta function. Actually, it is so, for $s = 0$, that is

$$1+1+1+1+1+1+ ... \approx \zeta(0) = -1/2. \tag{12}$$

For the other case (3), we have similarly,

$$1+2+3+4+5+6+7+ ... \approx \zeta(-1) = -1/12. \tag{13}$$

For those are the values one obtains when one looks at the values of the zeta function at the positions $s = 0$, and $s = 1$ of the complex field, respectively. To repeat, this may look at first sight as a mere mathematical trick, but it is indeed of an enormous value, because of its deep implications in the real world.

Nevertheless, dealing with these sums is not so simple as they might seem from the above. The present author has devoted a lot of time to correct the numerous mistakes in the mathematical-physics literature coming from erroneous application of the method of zeta regularization. Even if this one is, actually, a rigorously defined procedure, which we cannot go into any detail here.

Let us just discuss a typical mistake that has been committed very often. Consider the series

$$\sum_{n=0}^{\infty}\left(n + \frac{1}{2}\right). \tag{14}$$

A common mistake, which could be made by anyone who thinks that, with what has been explained above, has already become an expert in those calculations (this actually happens too often!), would be to split it into ½ (corresponding to $n = 0$), plus two series, both starting from $n=1$, as it corresponds to the zeta function definition. So that the result would then be

$$1/2 + \zeta(-1) + ½ + \zeta(0) = 1/2\text{-}1/12\text{-}1/4 = 1/6. \tag{15}$$

This value is in error. The actual one is obtained by considering the whole series, as it is (*without* any splitting), which constitutes a form of an extended Riemann zeta function, namely the Hurwitz zeta function. At the value $s= -1$ it yields 1/24. The deviation (a factor of 4) could be crucial, if this would correspond to an actual physical experiment [12].

2.5. Dealing with divergences in quantum field theory: Renormalization

This brings to the author's mind conferences that two very competent theoretical physicists, Francisco Ynduráin and Andrei Slavnov, gave some years ago at the University of Barcelona. In them, both speakers invoked (independently from each other, on two different occasions) the above procedure and correspondences (in particular, Eq. (1)) as the basis of the regularization methods they used. Both also coincided to kindly suggest to the audience that anyone who did not know about them would do well to leave the classroom, since they would not understand anything of the subsequent discussion. It was, frankly, amazing, as if they had copied their gestures from each other.

Such is the dependence of current quantum theories on the renormalization procedures of the infinities that so often appear in them. The story would be too long to tell here in any detail. Its origin can be found perhaps in investigations by Hendrik Lorentz and Max Abraham, who already at the beginning of the last century tried to develop a classical field theory for the electron, which eliminated the problems associated with infinities coming from considering them as point particles.

Later, the terrifying infinities that appeared in the perturbative series calculations of quantum field theories led, in the 1930s, the greatest physicists of that time (including Paul Dirac and Albert Einstein, among many others) upside down. Robert Oppenheimer showed in 1930 that higher-order perturbative calculations in quantum electrodynamics (QED) always resulted in infinite quantities, such as the electron self-energy and the vacuum zero-point energy of the electron and photon fields [16,17]. Max Born, Werner

Heisenberg, Pascual Jordan, and Paul Dirac, while developing quantum electrodynamics in the 1930s, arrived to similar conclusions, too. In papers written between 1934 and 1938, a relativistic invariant formulation of quantum field theory (QFT) was established by Ernst Stueckelberg, who by 1947 developed a complete renormalization procedure (achievement that was not properly recognized by the theoretical community of the time). It was not until some years later that a systematic and convincing approach to remove all the infinities was developed [18,19].

But one had to wait for QED, which was the first successful case of renormalizable quantum field theory. Renormalization was first developed for QED, in order to make sense of infinite integrals in perturbation theory. Initially viewed as a suspect provisional procedure, even by some of its originators, renormalization eventually was embraced as an important and self-consistent meaningful mechanism of scale physics in several other fields of physics and mathematics. This breakthrough came around 1950 when a robust method for eliminating infinities was developed by Julian Schwinger, Richard Feynman, Freeman Dyson, and Shinichiro Tomonaga, who in 1965 received the Nobel Prize in Physics for the quantization of electromagnetism, QED. All these works were later systematized by Freeman Dyson.

The main idea is to replace the calculated values of the mass and charge, infinite in principle, by their finite measured values, in a fully consistent way. This systematic computational procedure is known as renormalization and can be applied to arbitrary order in perturbation theory. By using the renormalization procedure, calculations could finally explain the electron's anomalous magnetic moment (deviation from 2 of the electron *g*-factor) and also vacuum polarization. The results perfectly agreed with the experimental measurements, putting an end to the very harsh discussions, in which some reputed physicists had even predicted that those methods could lead to the end of Physics.

Eventually, all the divergences could be renormalized by means of this precious theory: the theory of renormalization [16-19]. To have a physical feeling of what it is about, consider, as an example, a theory for an electron, which may start by postulating an electron with an initial mass and charge. But it so happens that, in quantum field theory, a cloud of virtual particles, such as photons, positrons, and other, surround and interact with the initial electron. Accounting for the interactions of the surrounding particles (e.g. collisions at different energies) shows that the electron-system behaves as if it had a different mass and charge, from the ones initially postulated. Renormalization mathematically replaces the initially postulated mass and charge of an electron with the experimentally observed mass and charge, where these physical interactions have all been taken into account. Positrons and more massive particles like protons exhibit exactly the same observed charge as the electron –even in the presence of much stronger interactions and more intense clouds of virtual particles. Renormalization specifies relationships between parameters in the theory when parameters describing large distance scales differ from parameters describing small distance scales. These relationships or transformations constitute a mathematical group, termed the renormalization group.

According to Wilson's picture, every QFT is fundamentally accompanied by its respective energy cut-off, Λ, with the meaning that the theory is no longer valid at energies higher than Λ, and that all degrees of freedom above the scale Λ are to be omitted. For an example, the cut-off could be the inverse of the atomic spacing in a condensed matter system, and in elementary particle physics it could be associated with the fundamental "graininess" of spacetime caused by quantum fluctuations in gravity. The cut-off scale of theories of particle interactions lies far beyond current experiments. Even if the theory were very complicated at that scale, as long as its couplings are sufficiently weak, it must be described at low energies by a renormalizable effective field theory.

The difference between renormalizable and non-renormalizable theories is that the first are insensitive to details at high energies, whereas the last do depend on them. According to this view, non-renormalizable theories are to be seen as low-energy effective theories of a more fundamental theory. The failure to remove the cut-off Λ from calculations in such a theory merely indicates that new physical phenomena appear at scales

above Λ, where a new theory, which we still do not have, is necessary. The understanding and the way to deal with effective field theories was decisively improved by the late Steven Weinberg, as mentioned somewhere else in the present article.

The idea of renormalization was later applied to other quantum field theories, described as gauge invariance theories. And the last great finding was to show that such gauge field theories, corresponding to the weak and strong interactions, were also renormalizable. What was also recognized with the Nobel Prize in Physics, awarded to the discoverers of this fact, Gerardus 't Hooft and Martinus Veltman, in 1999 [18,19]. An award with which several other physicists, who have applied the renormalization group to other very diverse disciplines, have also been recognized, as Kenneth G. Wilson. Further, the names of Nicolai Bogoliubov, Curtis C. Callan and Kurt Symanzik must also be mentioned in this context. For completeness, it must also be added that the regularization method for the infinities, used in this case and leading to the proof of renormalizability, was dimensional regularization, which was proven to work to all loop orders, in the perturbative calculation [18,19]. In that respect, the situation is not the same with zeta function regularization, which has a more limited scope. There is no place here to go into more details.

In short, just briefly summarizing the key issue posed at the beginning and sufficiently discussed in the preceding paragraphs, the main conclusion to be drawn from all those developments is that, actually, there are *no* infinities in physics, in the real world. This constitutes an extremely valuable lesson. All the infinities that appear in any description or model of nature, expressed by means of mathematical equations, are always due to extrapolations that are made beyond the range of validity of the equations used. They never correspond to physical reality and they directly point out to the failure, to the breakdown of the equations in the region or under the conditions that originate the singularity.

2.6. Further on the Big Bang singularity

Returning to the specific case of GR, the singularities that arise at the origin of the universe, as well as inside each of the numerous black holes that exist in the cosmos, are not real. And this is so, as much as we read in the best encyclopedias and textbooks that such singularities are mathematically rigorous results, the outcome of flawless theorems, proved by the most brilliant minds, like Stephen Hawking, Roger Penrose and others, and that have led to Nobel Prize awards. All of the latter is absolutely true: they are in fact rigorous results obtained by exceptional geniuses and for the most brilliant of theories: GR. But it turns out that, from the physical point of view, there is an Achilles' heel: as has been made clear before, this theory, GR, is not applicable, it is useless at very small scales; we enter there in a domain in which the GR ceases to be valid! It is not better than Newtonian physics that, as everybody knows, it had to be drastically modified at small scales and replaced by quantum physics.

But there is more to that. Namely, it has been actually possible to demonstrate that, sometimes, it is enough to correct GR just a bit, to patch it up by adding a pinch of quantum physics to the classical GR formulas (in technical language: first-order quantum corrections) and the infinities suddenly disappear, like magically. In a similar way to how the singularities of classical electrodynamics vanished when the theory was quantized, giving rise to quantum electrodynamics (Feynman-Schwinger-Tomonaga, see above). Although in gravitation it has not yet been possible to complete this process thoroughly, nor is there any promising glimpse of how this could be done. That was one of the objectives of supergravity, which still remains pending.

The question now is: were the great scientists who proposed these rigorous singularity theorems aware of the limitations just mentioned? The answer is yes, of course. One just had to go to the website of the late Stephen Hawking to see everything that has been here argued masterfully explained. Hawking was precisely one of the first to add the quantum corrections that have been mentioned, after a visit he made to the Soviet Union in 1973, where he discussed --as he himself confessed in his memoirs-- with Yakov B. Zeldovich (who did the first calculation of the value of the cosmological constant and interpreted it as coming from quantum physics) and his students in Moscow, pioneers in

this field. Corrections that, for the case of a black hole, led him to find the famous "Hawking effect", the most important discovery of his life [20].

But this "tagline" of the theorems, the one that has to do with the limitations of GR, is hardly explained in very few places; because --despite its enormous significance-- for the purpose of the apotheotic narrative usually pursued, it would be like the rain that puts out fireworks, suddenly vanishing all singularities and infinities. And, again, there is more. Taking such quantum corrections to Einstein's field equations very seriously, and extending them to higher orders, theorists have built families of what are known as modified gravity theories, in particular the *f(R)* theories, tensor-vector theories, and others, which are very promising with a view to being able to solve some of the fundamental problems to be solved, such as the nature of dark energy (and the cosmological constant issue) and dark matter. Several of these theories have been proven to be able to provide a unified cosmic history within the same given model and also workable answers to the dark energy and dark matter problems [21,22].

Before going into that, let us close the present section with the earnest request that, from now on, no one else be fooled again by all these "half-hearted" stories, unprofessionally told, with infinites and singularities galore, that only seek what the French call "épater le bourgeois". It seems that, today, if there is not a good dose of spectacularism in what is being said, there is no chronicle or article, because it does not sell.

3. The real Big Bang: Inflation

From now on, let us focus on what is actually known, either by very clear evidence from laboratory experiments or through increasingly precise observations of celestial objects. To begin with, it is known that our universe was born some 13.78 billion years ago. Since, on one side, no matter how careful observations of the sky have been made, no older objects have ever been found (aside from some exceptional one, from time to time, which have required some additional explanation). This clearly indicates that our universe was born then. But there are also several other evidences, independent of each other, of this fact. They are collected in the working Big Bang cosmological model.

And it is also rigorously known that, shortly after its origin, when the Universe was between a trillionth and a millionth of a second of age, it was all a very hot plasma, made up of leptons, quarks and gluons. This is known as the primordial plasma, or as the quark-gluon plasma, or lepton-quark-gluon plasma. Leptons are the lightest elementary particles, quarks are the simplest components of non-lepton elementary particles, and gluons the interaction forces that hold them together, to form mesons (particles of intermediate mass, made up of two quarks), and baryons (the heaviest particles, consisting of three quarks). The properties of the primordial plasma have been very well determined in laboratories such as CERN in Geneva, Switzerland. But, how the hell did that plasma come to be? And from what preexisting stuff?

There is no definitive answer to those questions, but evidence is accumulating, which indicates that it was as a consequence of a sudden and very powerful expansion of the fabric of space, which enormously swelled it, as if it actually were a balloon (whose radius would correspond to time). Rigorously, the simplest model implementing this view would be a three-dimensional surface of a four-dimensional sphere, with the fourth (radial) dimension being time (multiplied by the speed of light squared). This is roughly what the Big Bang cosmological theory says. By the way, one should never confuse it with the Big Bang singularity [1,2], which has become the main meaning of the tag "Big Bang", among common people. And neither of them corresponds to the genuine, original meaning of this term, namely, what is now known as "inflation" (see, e.g., Ref. [1]).

The evidence supporting the BB theory is very extensive and precise. The thermal fingerprint of the primordial plasma, so-called cosmic microwave background (CMB) radiation that permeates the Universe, has been discovered and carefully studied. In few words, this theory is a very successful one, as it also explains the current composition of the cosmos, as well as many other phenomena. But it turns out that a few years ago, just before the end of the last century, it was discovered that the Universe is in fact expanding

at an ever-increasing rate, in other words, the expansion is accelerating. This has generated a great mystery that the Big Bang theory fails to explain convincingly. Some theorists relate it to the accelerated expansion of the early cosmos, and modified gravity models have been built (in particular, those of the *f(R)* type already mentioned above), which promise to provide a *unified* description of its entire history, corroborated by the data obtained from the most precise astronomical observations. Be that as it may, we still keep asking ourselves: What exactly happened when our universe was born? And how did it take the form that it now has?

There is currently no valid physical theory to describe the origin of the universe but only a few plausible proposals. The most advanced scientific ideas that are in no contradiction with known physics --but not yet tested in the laboratory, either-- allow us to go back to when the cosmos was only 10^{-35} or 10^{-36} seconds of age. That is, up to a trillionth of a trillionth of a second, after everything started.

At that time, the universe underwent an extremely brief and dramatic stage of inflation, expanding much faster than the speed of light. It doubled in size probably as much as one hundred times, in a tiny fraction of a second, and its temperature dropped from some 10^{27} to some 10^{22} K.

Here a comment is in order. It might seem that inflation would violate the main principle of the special theory of relativity, but actually it does not. In fact, relativity holds that neither information nor matter can be transported between two positions in space at a speed faster than that of light. But inflation did not involve any space travel, as it was just an expansion of the fabric of space itself.

Before inflation, there was hardly anything; just a handful few ingredients. Only those strictly necessary to set everything in motion are needed: space-time itself, the Higgs field, the field that receives the name of *inflaton*, to spark inflation, and little else. As of today, there is no proven scientific evidence on how these ingredients appeared. But what does seem quite clear is that it took something like inflation to suddenly enlarge the Universe and thus endow it with the properties we now observe. It should be mentioned, however, that there are alternative approaches that can dispense with the whole inflationary paradigm, completely replacing the image of the universe, as a whole, by a new one. Among them, those that consider our universe to be inside a black hole solution of Einstein's equations [23-26].

Such rapidly expanding universe was practically devoid of matter, but, according to the most accepted theory, it harbored large amounts of dark energy. And this one is precisely the same force that is also believed to be driving the current accelerated expansion of the cosmos.

3.1. Reheating

Dark energy caused inflation and when, after a very brief moment of time, it stopped, the energy it harbored was transformed into ordinary matter and radiation, through a process called reheating. The Universe, which had been very cold during inflation, heated up to a very high temperature again (about 10^{25} K, according to some models) and the created matter was constituted in a primordial plasma, made up of leptons, quarks and gluons, which are the most elementary components of all the matter we see now. However, there is no clear scientific evidence, yet, on how and why inflation appeared. Alternative mechanisms are being considered, too.

3.2. Inflation or a cyclic universe?

Inflation is, today, the most appropriate theory to explain the current characteristics of the Universe; in particular, the fact that it is, on large scales, so flat, homogeneous and isotropic, with more or less the same amount of matter distributed equally at all its points and in all directions. Several pieces of evidence point to the fact that inflation did occur, but there is as yet no conclusive proof that this was the case.

Consequently, other alternative theoretical models cannot be ruled out, such as the so-called cyclical model, which recalls the Stoic philosophers' concept of the ekpyrotic

universe. It argues that the universe did not appear from a point, out of nowhere, but rather bounced from another pre-existing universe into expansion, at a much calmer rate than inflation theory predicts, from a previous universe that it would have been cooling and then contracting. If that theory were correct, our universe would have undergone an indeterminate succession of previous expansions and contractions.

The cyclic model relies on string and brane theories. In accordance with them, it posits that the universe actually has eleven dimensions, only four of which can be observed (the so-called space-time brane). Branes actually exist in more dimensions than the conventional 4, being the brane the whole structure, and there are theories with different space-time dimensions. There could be other branes lurking in eleven-dimensional space, and a collision between two branes could have rocked the cosmos, shifting it from contraction to expansion and stimulating the CMB radiation that we now detected, and which would closely match that of the Big Bang model.

4. The Universe we know takes shape

Just after inflation, the universe was a very dense, absolutely dark, very hot plasma. When it was about a trillionth of a second (10^{-12} s) young, it entered the electroweak phase, in which massless vector and scalar bosons were present. Shortly after, the photons were formed, from the first ones, and the Higgs boson acquired mass to, subsequently, provide it to the rest of the bosons; all this in perfect agreement with the electroweak quantum field theory [16,17]. The universe was then about 300 light-seconds across (comparable to the distance between the Sun and Venus).

As time passed, the cosmos continued to cool, until it entered the stage, already mentioned above, in which quarks and gluons were formed. When the cosmos was already a microsecond of age (10^{-6} s), from the quarks and gluons the first protons and neutrons could be constituted and, to an even greater extent, also the first mesons (muons and pions, fundamentally). In addition to their corresponding antiparticles, although their production was soon halted by an asymmetry, providential for our existence and that of the universe as we now know it, namely the baryon asymmetry, which has not yet been fully explained, but which is most crucial.

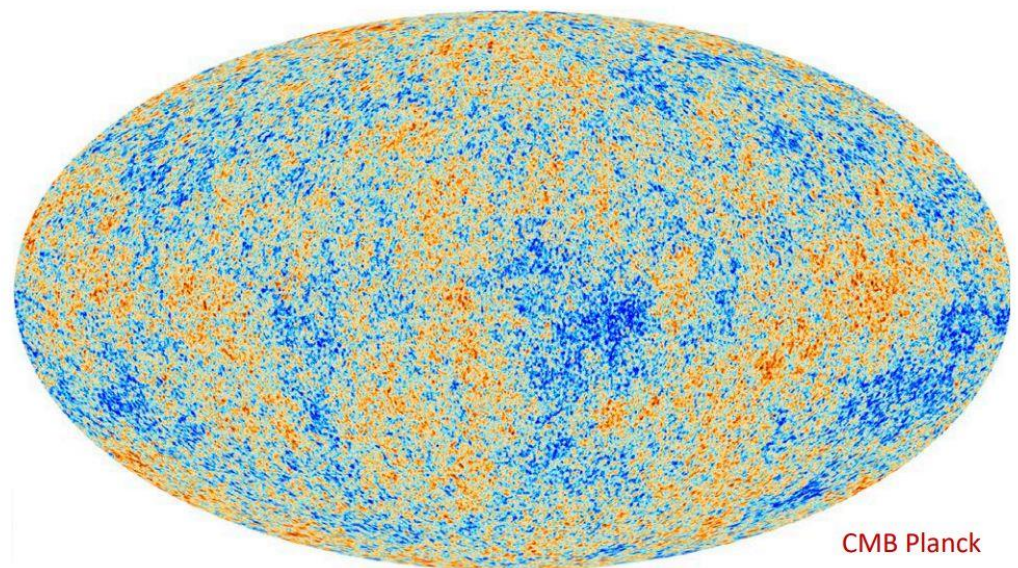
All of the above has been reproduced very faithfully, step by step, in high-energy laboratories, such as the LHC at CERN, Geneva (which, in fact, can reach up to 10^{-15} s of age of the cosmos), and also in large heavy ion colliders. When the cosmos had one second of existence, the neutrinos managed to free themselves from the original soup, and they could already travel everywhere through the universe. This was then about ten light-years across (a couple of times greater than the distance to the closest stars to the Sun).

After the first three minutes [27], protons and neutrons began to merge, forming deuterium nuclei; and these, on its turn, joined with each other, to form helium-4 nuclei.

4.1. Recombination: the Universe becomes transparent

Atomic nuclei could not yet constitute atoms, since the universe was too hot to allow the capture of electrons by the protons. The first atom that could be formed was hydrogen, the simplest atom of all (one proton - one electron). This happened when the universe was 378,000 years old, a period that is known as recombination. In a relatively short time span, the hydrogen ions, and also those of helium, were able to capture electrons, thus forming electrically neutral atoms, which, as we know, are extraordinarily transparent to light. And thus, of a sudden, the photons were finally free to cross the entire universe. And, for

the first time ever, there was light! which definitively separated from the darkness that had reigned until then (Fig. 4).

Figure 4. The cosmic microwave background radiation as reported by Planck.

Recombination radically changed the appearance of the universe, which until that period had been very dense and opaque, a dark mist, suddenly becoming transparent. The cosmic microwave background radiation (CMB) that we continue observing today, although very faint, is the light of that time.

4.2. Reionization

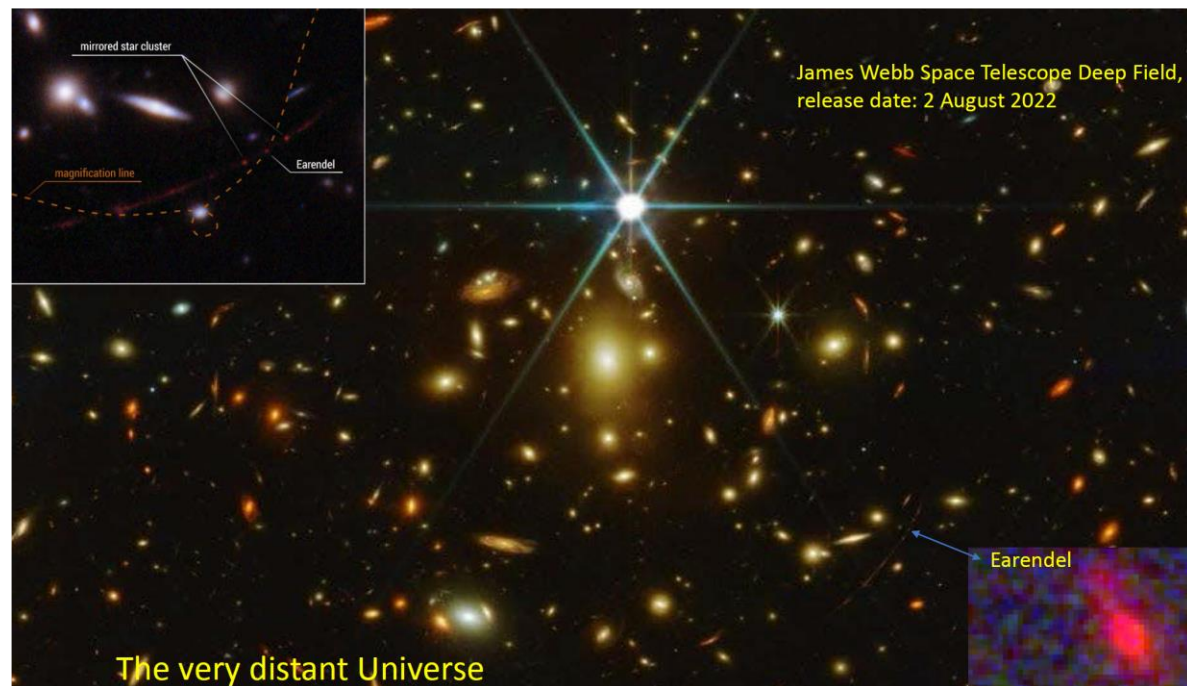



Figure 5. The very distant Universe. James Webb Space Telescope Deep Field, release date: 2 August 2022. Earendel is to be seen at the crossing of two gravitational lense images.

But the universe, as a whole, was still pretty dark for a long time after recombination. The expansion was constantly enlarging it, and there were still no objects of great luminosity around, apart from the faint background radiation. The cosmos only truly lit up

when the first young stars began to shine, some 300 million years after its birth. The early stars, and perhaps some other sources as yet little known, or still to be discovered (remote objects that no one had counted on previously, are continually turning up in cosmos research), blasted enough radiation into space to separate, again, most of the hydrogen atoms present in its constituent protons and electrons. Earendel is one of the most distant stars having been observed from that time (Fig. 5).

This process, known as reionization, appears to have run its course at least a billion years after the Big Bang. But keep in mind that now the universe was no longer opaque, as it had been before the recombination, since it had already greatly expanded and its density was much lower. Today, photon scattering interactions are in fact quite rare, on a large scale, although they remain of paramount importance on local scales, within large galaxies.

Over time, and by the action of gravity, the stars formed larger and larger galaxies and cosmic structures, interconnected on a large scale; it is what is called the cosmic web (cosmic web, Fig. 6). The planets were later formed around some newly formed stars, as was the case with our Sun. And 3.8 billion years ago, life took root on Earth. But this would already be a separate chapter, just as or even more exciting than the present one, as was said at the beginning.

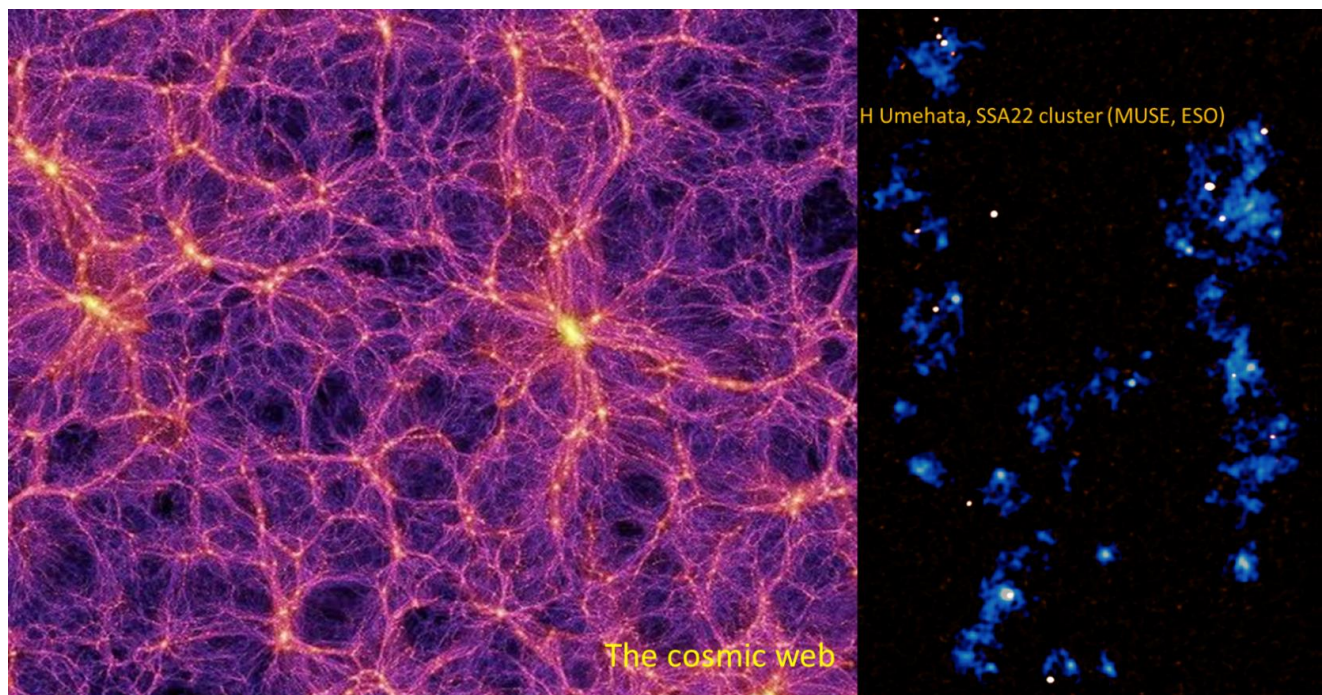



Figure 6. A comparison of the cosmic web as generated by computer simulations with a real observation by H. Humehata of the SSA22 cluster (MUSE, ESO).

5. Some current and future missions

The Planck mission of the European Space Agency (ESA), which orbited the Earth between 2009 and 2013, helped further refine theories about the nature of the cosmos and its origin [28,29]. The detailed map of the cosmic microwave background, generated by the mission, clearly revealed that our universe - although it cannot yet be completely ruled out that it might have arisen from a predecessor - will most likely never shrink again in the future. future. And that goes against the possibility of a cyclic universe (Fig. 7).

Using the Planck data, estimates of the age of the universe and of the amounts of visible matter, dark matter and dark energy it contains have been refined. The mission did

not in fact provide any great surprises, largely confirming existing theories and corroborating that the future evolution of the universe will be, in short, quite sad or boring.

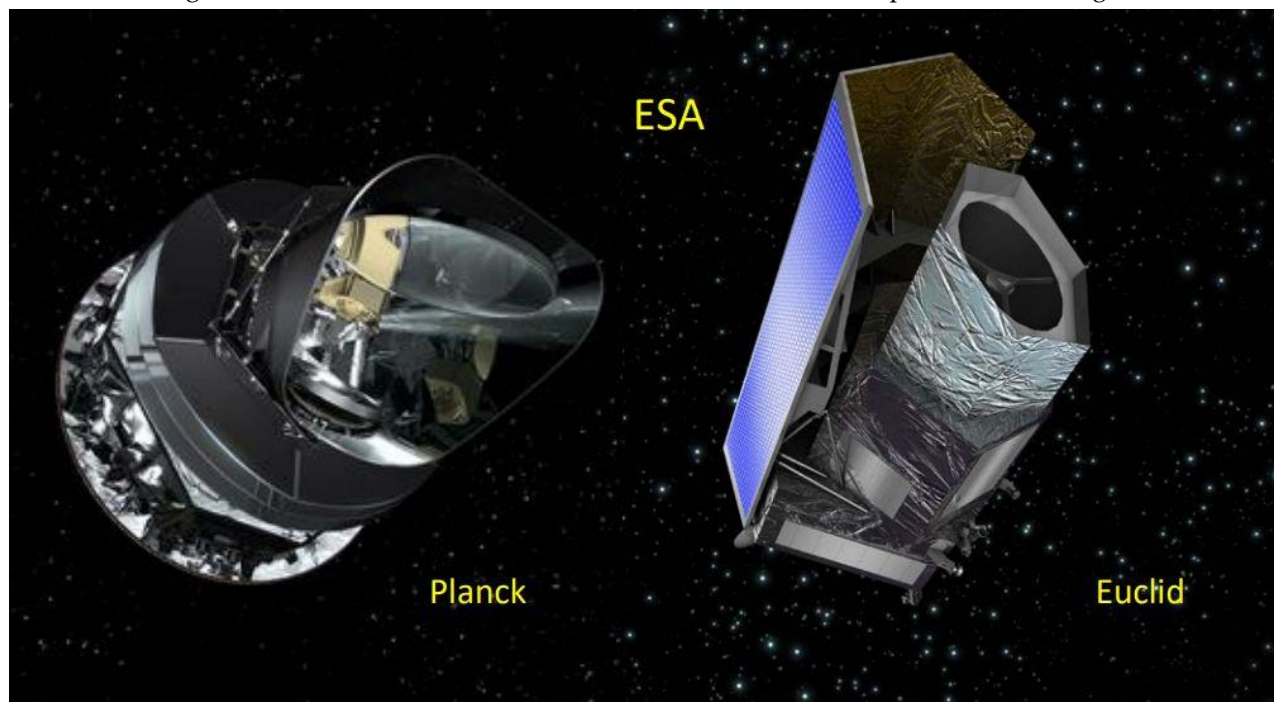



Figure 7. Artistic representation of the two ESA missions Planck and Euclid put together for visual comparison.

However, their results have also raised new questions. For example, the Hubble constant --which describes the rate of expansion of the cosmos-- measured by Planck in the distant universe, gives a different value than that obtained with the Hubble Space Telescope in the nearby universe. This important tension between the two outcomes is now being feverishly investigated.

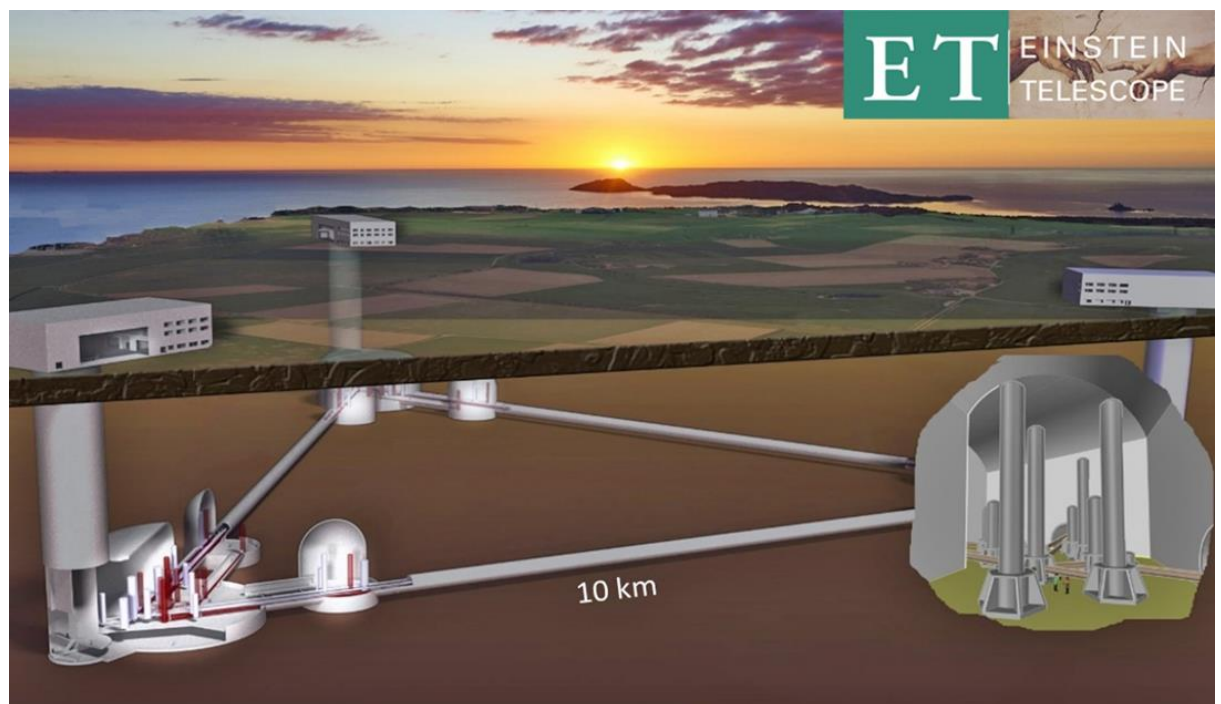



Figure 8. Schematic representation of the Einstein Telescope set up. The final site where it is going to be built has not been chosen yet.

From the field of theory, while returning now to the infinities, with which we began this journey --and also answering questions that are often formulated by the audience when presenting these results-- it must be added that modified gravity theories, such as those of the *f(R)* type, although they are capable (as pointed out already) of eliminating singularities that appear in GR, they often create new ones, both in the past and in the future of the universe. Thus, they do not constitute, in fact, any panacea in that sense. This warns us, being consistent, that they cannot be in any case final theories to describe the cosmos. But one should be aware, too, that not even a future theory of quantum gravity, nor probably the use of string theories, will literally be able to bridge the temporary distance between t=0 and t=10^{-44} s, the Planck time. Although it is most likely that, in fact, being able to go beyond those t=10^{-44} seconds might completely lack any physical sense, if it can be shown that quantum physics is, de facto, the last (and, as such, insurmountable) frontier of Physics. There are some theorists working in that direction.

As far as observations of the cosmos are concerned, it is expected that ESA's Euclid mission, which will be launched in 2023 and, a decade later, in the 2030's, the Einstein Telescope of gravitational waves (ET Project, Fig. 8), the LISA mission, as well as a handful other projects, will help us advance in understanding the nature of the dark side of the universe, namely dark energy, dark matter, black holes, as well as in resolving the other great unknowns already mentioned, and which still remain open for a comprehensive knowledge of the cosmos [30-37].

6. Discussion

The high status of Einstein's General Theory of Relativity as the fundamental theory for explaining, in precise mathematical language, the origin and the behavior of the Universe is still unchallenged. This theory, conceived by his author as a model for the rather small, static and eternal universe of his time --which just comprised the Milky Way-- has now proven able to deal with a large and very diverse number of unforeseen scenarios playing in a million times larger cosmos, which is expanding at an accelerated pace. This has brought, as a consequence, the idea that GR is a kind of all-powerful theory and to believe wholeheartedly all the results and outcomes from it, whatever the situation and circumstances. What is analogous, in a way, to what happened with the Newtonian universal gravity laws, at the time of their inception. This is true, in special, for the abundant popular accounts of the Universe, that have proliferated since the Hubble telescope started to send spectacular images of galaxies and other celestial objects [1], making of the observation of the cosmos a thrilling social endeavor.

But we should recall that, notably, it was Einstein himself the first to point out, when constructing his theory over one-hundred years ago, that his GR was in fact a truncated theory, at most a good approximation to a really fundamental one that would incorporate all physical ideas of his time, e.g. Mach's principle and the like, and which has been called by some total relativity. Following Einstein, many other relevant physicists have stressed the shortcomings of GR; for dealing, e.g. with very large curvatures, energies, densities or pressures [38,1]. Some of those situations have started to be crucial recently in different observations. Some of them have to, and can (at least to some extent) be dealt with QFTs of gauge interactions; in a word, the standard model of particle physics, which constitutes, so to say, the other (and already unified) side of the present-day fundamental description of nature.

It was Franck Wilczek who, in an obituary for the late Steven Weinberg, stressed the importance of his work by saying namely that "he played a central part in formulating and establishing theoretical physics' two standard models — the standard model of fundamental interactions and the standard model of cosmology" [39]. The most important, and still missing, task is to merge both together by building a model of quantum gravity, what seems at the moment impossible, by usual means. Now, my point here is that, outside the community of specialists, many scientists, and a vast majority of science popularizers seem to forget about this.

Some paragraphs have been devoted to discussing these issues, in special, concerning the Big Bang singularity, with hints towards its possible disappearing when quantum corrections are introduced in the game. We are far from having a reliable understanding of the origin of the Universe but, rather than spending much time on that (what is done too often, not being of any practical use), it has been argued above that it is much wiser to spend it in getting a good knowledge of what is really known, which starts just a tiny fraction of a second after the singularity at t = 0. This precious knowledge humankind should be extremely proud of; that is, to start with, of the fact that we, human beings, have been able to explain with really simple models the behavior of the universe we live in. It is this positive view, which should be conveyed to students and to popular audiences, instead of spending most of the time on the so many questions whose answers we do not yet now. And it is in this sense that a vision of the state of the art concerning the epoch, very close to the universe's origin, which can be carefully investigated at the laboratory, and its subsequent evolution (more and more data becoming available from space missions and terrestrial observatories), have been discussed.

But on top of that, back to the issue of the singularities, some specific points have been picked up and conveniently highlighted: i) The fact that all fundamental theories of physics (GR included) have a *limited* range of applicability. ii) That, when extrapolated *beyond* their domain or any other conditions of validity, the theories produce singularities. iii) That, vice-versa, when one encounters a singularity, in a physical situation, it is always because the mathematical equations employed to implement the physical law or principle are simply *not valid* in the region where the singularity occurs.

 Needless to say, singularities are most useful, and they have been so very often during the history of physics; e.g., as indications that the theory at hand must be improved or completely changed in the circumstances that lead to the singularity. Several, important examples of this issue have been provided above, and also some particular ways how the infinities can be, and have actually been dealt with in the past, already.

Finally, some terrestrial projects and space missions to be launched have been highlighted. There is a general conviction that a lot many surprises are waiting for us in our present explorations of the cosmos. Several of them are even expected to be already there, in the lots of observational data having been taken already but still waiting to be processed by powerful computers. And indeed, a good number of celestial objects have already been found that defy our present models of the universe and their standard descriptions, both at the GR level and also at that of the QFTs. Starting with the well-known problems of the missing explanations for the nature of its dark side, namely dark matter, dark energy and black holes, but notably also concerning the possibility of quark stars having been already detected, as further alternatives to the omnipresent neutron stars and leading to a new paradigm of matter compression (with a, still unknown, new equation of state) [40,41]. Neutron stars and black holes with strange masses and properties have been found that defy the corresponding standard wisdom, too.

We are constrained to explaining all these phenomena with the laws of the standard models at disposal, although some alternatives have been proposed already, as different families of the so-called modified gravity theories and other kind of alternatives to GR; and also, some extensions of the standard model of particle physics, leaving room for pentaquarks and other exotic phenomena [42,43]. Of supersymmetry and superstring theories, not many impacting news are being heard recently, in the waiting for specific results from CERN or elsewhere, but they are of course latent and very well-grounded possibilities whose existence --in spite of the lack yet of actual experimental evidence-- we should always keep in mind.

7. Conclusions

To put it simply, by means of the zeta function, awfully divergent infinite series can be brought to produce finite values. And, what is more important, these values, which often may seem absurd, at first, do make full sense, in the end, as results of the physical

world, being fully validated by the most accurate laboratory experiments ever conducted to date.

A lesson to be taken home is that all fundamental theories of physics (GR included) have a limited range of applicability. When extrapolated beyond their respective working domains or any other conditions of validity, the theories tend to produce singularities. Vice-versa, when one encounters a singularity in a physical situation, it is because the mathematical equations employed to implement the physical law or principle are plainly not valid in the region where the singularity pops up.

The moral. From now on, no one else should be fooled by all these abundant "half-hearted" stories, some of them unprofessionally told, with infinites and singularities galore, and which only seek what the French call *"épater le bourgeois"*. It seems that, today, if there is not a good dose of spectacularism in what is being said, there is no chronicle or article, because it does not sell. Needless to say, I am here only talking of some bad popular articles, not at all of those written in serious professional journals or by competent popularizers of science, of which, fortunately, there are many, too. Not in vain, this paper emerged from some serious popular talks, which I feel it is the duty of all professionals to deliver, at least from time to time.

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