Article

Polyadic Rings of *P*-adic Integers

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Abstract. In this note we, first, recall that the sets of all representatives of some special ordinary residue classes become (m,n)-rings. Second, we introduce a possible p-adic analog of the residue class modulo a p-adic integer. Then, we find the relations which determine, when the representatives form a (m,n)-ring.

Keywords: polyadic semigroup, polyadic ring, arity, querelement, residue class, representative, p-adic integer

1. Introduction

The fundamental conception of p-adic numbers is based on a special extension of rational numbers which is alternative to the real and complex numbers. The main idea is a completion of rational numbers with respect to the p-adic norm, which is non-Archimedian. Nowadays, p-adic methods are widely used in number theory [1,2] and arithmetic geometry [3,4], mathematical physics [5,6] and algorithmic computations [7]. For general reviews, see [8–10].

We have found that in the study of p-adic integers some polyadic structures, that is (m,n)-rings, can appear naturally, if we introduce informally a p-adic analog of the residue classes for ordinary integers and investigate the set of its representatives along the lines of [11–13].

2. (m,n)-rings of integer numbers from residue classes

Here we recall that representatives of special residue (congruence) classes can form polyadic rings, as was found in [11,12] (see also notation from [13]).

Let us denote the residue (congruence) class of an integer a modulo b by

$$[a]_b = \{\{r_k(a,b)\} \mid k \in \mathbb{Z}, a \in \mathbb{Z}_+, b \in \mathbb{N}, 0 \leq a \leq b-1\}, \tag{2.1}$$

where $r_k(a,b) = a + bk$ is a generic representative element of the class $[a]_b$. The canonical representative is the least nonnegative number among these. Informally, a is the remainder of $r_k(a,b)$ when divided by b. The corresponding equivalence relation (congruence modulo b) is denoted by

$$r = a(\operatorname{mod} b). \tag{2.2}$$

Introducing the binary operations between classes $(+_{cl}, \times_{cl})$, the addition $[a_1]_b +_{cl} [a_2]_b = [a_1 + a_2]_b$ and multiplication $[a_1]_b \times_{cl} [a_2]_b = [a_1 a_2]_b$, the residue class (binary) finite commutative ring $\mathbb{Z}/b\mathbb{Z}$ (with identity) is defined in the standard way (which was named "external" [11]). If $a \neq 0$ and b is prime, then $\mathbb{Z}/b\mathbb{Z}$ becomes a finite field.

The set of representatives $\{r_k(a,b)\}$ in a given class $[a]_b$ does not form a binary ring, because there are no binary operations (addition and multiplication) which are simultaneously closed for arbitrary a and b. Nevertheless, the following polyadic operations on representatives $r_k = r_k(a,b)$, m-ary addition ν_m

$$\nu_m\left[r_{k_1}, r_{k_2}, \ldots, r_{k_m}\right] = r_{k_1} + r_{k_2} + \ldots + r_{k_m}, \tag{2.3}$$

and n-ary multiplication μ_n

$$\mu_n\left[r_{k_1}, r_{k_2}, \ldots, r_{k_n}\right] = r_{k_1} r_{k_2} \ldots r_{k_n}, \quad r_{k_i} \in [a]_b,\ k_i \in \mathbb{Z}, \tag{2.4}$$

can be closed, but only for special values of $a = a_q$ and $b = b_q$, which defines the nonderived (m,n)-ary ring

$$\mathbb{Z}_{(m,n)}\left(a_q, b_q\right) = \left\langle \left[a_q\right]_{b_q} \mid \nu_m, \mu_n \right\rangle \tag{2.5}$$

of polyadic integers (that was called the “internal” way [11]). The conditions of closure for the operations between representatives can be formulated in terms of the (arity shape [12]) invariants (which may be seen as some kind of “quantization”)

$$(m-1)\frac{a_q}{b_q} = I_m\left(a_q, b_q\right) \in \mathbb{N}, \tag{2.6}$$

$$a_q^{n-1}\frac{a_q - 1}{b_q} = J_n\left(a_q, b_q\right) \in \mathbb{N}, \tag{2.7}$$

or, equivalently, using the congruence relations [11]

$$m a_q \equiv a_q \left(\operatorname{mod} b_q\right), \tag{2.8}$$

$$a_q^n \equiv a_q \left(\operatorname{mod} b_q\right), \tag{2.9}$$

where we have denoted by a_q and b_q the concrete solutions of the “quantization” equations (2.6)–(2.9). The arity shape of the ring of polyadic integers $\mathbb{Z}_{(m,n)}\left(a_q, b_q\right)$ (2.5) is the (surjective) mapping

$$\left(a_q, b_q\right) \Longrightarrow (m,n). \tag{2.10}$$

The mapping (2.10) for the lowest values of a_q, b_q is given in TABLE 1 ($I = I_m\left(a_q, b_q\right)$, $J = J_n\left(a_q, b_q\right)$).

Table 1. The arity shape mapping (2.10) for the polyadic ring $\mathbb{Z}_{(m,n)}\left(a_q, b_q\right)$ (2.5).

$a_q \setminus b_q$	2	3	4	5	6	7	8	9	10
1	$m = \mathbf{3}$ $n = \mathbf{2}$ $I = 1$ $J = 0$	$m = \mathbf{4}$ $n = \mathbf{2}$ $I = 1$ $J = 0$	$m = \mathbf{5}$ $n = \mathbf{2}$ $I = 1$ $J = 0$	$m = \mathbf{6}$ $n = \mathbf{2}$ $I = 1$ $J = 0$	$m = \mathbf{7}$ $n = \mathbf{2}$ $I = 1$ $J = 0$	$m = \mathbf{8}$ $n = \mathbf{2}$ $I = 1$ $J = 0$	$m = \mathbf{9}$ $n = \mathbf{2}$ $I = 1$ $J = 0$	$m = \mathbf{10}$ $n = \mathbf{2}$ $I = 1$ $J = 0$	$m = \mathbf{11}$ $n = \mathbf{2}$ $I = 1$ $J = 0$
2		$m = \mathbf{4}$ $n = \mathbf{3}$ $I = 2$ $J = 2$		$m = \mathbf{6}$ $n = \mathbf{5}$ $I = 2$ $J = 6$	$m = \mathbf{4}$ $n = \mathbf{3}$ $I = 1$ $J = 1$	$m = \mathbf{8}$ $n = \mathbf{4}$ $I = 2$ $J = 2$		$m = \mathbf{10}$ $n = \mathbf{7}$ $I = 2$ $J = 14$	$m = \mathbf{6}$ $n = \mathbf{5}$ $I = 1$ $J = 3$
3			$m = \mathbf{5}$ $n = \mathbf{3}$ $I = 3$ $J = 6$	$m = \mathbf{6}$ $n = \mathbf{5}$ $I = 3$ $J = 48$	$m = \mathbf{3}$ $n = \mathbf{2}$ $I = 1$ $J = 1$	$m = \mathbf{8}$ $n = \mathbf{7}$ $I = 3$ $J = 312$	$m = \mathbf{9}$ $n = \mathbf{3}$ $I = 3$ $J = 3$		$m = \mathbf{11}$ $n = \mathbf{5}$ $I = 3$ $J = 24$
4				$m = \mathbf{6}$ $n = \mathbf{3}$ $I = 4$ $J = 12$	$m = \mathbf{4}$ $n = \mathbf{2}$ $I = 2$ $J = 2$	$m = \mathbf{8}$ $n = \mathbf{4}$ $I = 4$ $J = 36$		$m = \mathbf{10}$ $n = \mathbf{4}$ $I = 4$ $J = 28$	$m = \mathbf{6}$ $n = \mathbf{3}$ $I = 2$ $J = 6$
5					$m = \mathbf{7}$ $n = \mathbf{3}$ $I = 5$ $J = 20$	$m = \mathbf{8}$ $n = \mathbf{7}$ $I = 11$ $J = 11160$	$m = \mathbf{9}$ $n = \mathbf{3}$ $I = 5$ $J = 15$	$m = \mathbf{10}$ $n = \mathbf{7}$ $I = 5$ $J = 8680$	$m = \mathbf{3}$ $n = \mathbf{2}$ $I = 1$ $J = 2$
6						$m = \mathbf{8}$ $n = \mathbf{3}$ $I = 6$ $J = 30$			$m = \mathbf{6}$ $n = \mathbf{2}$ $I = 3$ $J = 3$
7							$m = \mathbf{9}$ $n = \mathbf{3}$ $I = 7$ $J = 42$	$m = \mathbf{10}$ $n = \mathbf{4}$ $I = 7$ $J = 266$	$m = \mathbf{11}$ $n = \mathbf{5}$ $I = 7$ $J = 1680$
8								$m = \mathbf{10}$ $n = \mathbf{3}$ $I = 8$ $J = 56$	$m = \mathbf{6}$ $n = \mathbf{5}$ $I = 4$ $J = 3276$
9									$m = \mathbf{11}$ $n = \mathbf{3}$ $I = 9$ $J = 72$

The binary ring of ordinary integers $\mathbb{Z}$ corresponds to $\left(a_q = 0, b_q = 1\right) \Longrightarrow (2,2)$ or $\mathbb{Z} = \mathbb{Z}_{(2,2)}(0,1)$, $I = J = 0$.

3. Representations of p-adic integers

Let us explore briefly some well-known definitions regarding p-adic integers to establish notations (for reviews, see [8,9,14]).

A p-adic integer is an infinite formal sum of the form

$$\boldsymbol{x}=\boldsymbol{x}(p)=\alpha_0+\alpha_1 p+\alpha_2 p^2+\ldots+\alpha_{i-1}p^{i-1}+\alpha_i p^i+\alpha_{i+1}p^{i+1}+\ldots,\quad \alpha_i\in\mathbb{Z}, \tag{3.1}$$

where the digits (denoted by Greek letters from the beginning of alphabet) $0\leq\alpha_i\leq p-1$, and $p\geq 2$ is a fixed prime number. The expansion (3.1) is called standard (or canonical), and α_i are the p-adic digits which are usually written from the right to the left (positional notation) $\boldsymbol{x}=\boldsymbol{.}\ldots\alpha_{i+1}\alpha_i\alpha_{i-1}\ldots\alpha_2\alpha_1\alpha_0$ or sometimes $\boldsymbol{x}=\{\alpha_0,\alpha_1,\alpha_2,\ldots,\alpha_{i-1},\alpha_i,\alpha_{i+1}\ldots\}$. The set of p-adic integers is a commutative ring (of p-adic integers) denoted by $\mathbb{Z}_p=\{\boldsymbol{x}\}$, and the ring of ordinary integers (sometimes called "rational" integers) $\mathbb{Z}$ is its (binary) subring.

The so called coherent representation of $\mathbb{Z}_p$ is based on the (inverse) projective limit of finite fields $\mathbb{Z}\diagup p^l\mathbb{Z}$, because the surjective map $\mathbb{Z}_p\longrightarrow\mathbb{Z}\diagup p^l\mathbb{Z}$ defined by

$$\alpha_0+\alpha_1 p+\alpha_2 p^2+\ldots+\alpha_i p^i+\ldots\mapsto\left(\alpha_0+\alpha_1 p+\alpha_2 p^2+\ldots+\alpha_{l-1}p^{l-1}\right)\bmod p^l \tag{3.2}$$

is a ring homomorphism. In this case, a p-adic integer is the infinite Cauchy sequence that converges to

$$\boldsymbol{x}=\boldsymbol{x}(p)=\{x_i(p)\}_{i=1}^{\infty}=\{x_1(p),x_2(p),\ldots,x_i(p)\ldots\}, \tag{3.3}$$

where

$$x_i(p)=\alpha_0+\alpha_1 p+\alpha_2 p^2+\ldots+\alpha_{l-1}p^{l-1} \tag{3.4}$$

with the coherency condition

$$x_{i+1}(p)\equiv x_i(p)\bmod p^i,\quad\forall i\geq 1, \tag{3.5}$$

and the p-adic digits are $0\leq\alpha_i\leq p-1$.

If $0\leq x_i(p)\leq p^i-1$ for all $i\geq 1$, then the coherent representation (3.3) is called reduced. The ordinary integers $x\in\mathbb{Z}$ embed into p-adic integers as constant infinite sequences by $x\mapsto\{x,x,\ldots,x,\ldots\}$.

Using the fact that the process of reducing modulo p^i is equivalent to vanishing the last i digits, the coherency condition (3.5) leads to a sequence of partial sums [14]

$$\boldsymbol{x}=\boldsymbol{x}(p)=\{y_i(p)\}_{i=1}^{\infty}=\{y_1(p),y_2(p),\ldots,y_i(p)\ldots\}, \tag{3.6}$$

where

$$y_1(p)=\alpha_0,\ y_2(p)=\alpha_0+\alpha_1 p,\ y_3(p)=\alpha_0+\alpha_1 p+\alpha_2 p^2,\ y_4(p)=\alpha_0+\alpha_1 p+\alpha_2 p^2+\alpha_3 p^3,\ \ldots. \tag{3.7}$$

Sometimes the partial sum representation (3.6) is simpler for p-adic integer computations.

4. (m,n)-rings of p-adic integers

As may be seen from Section 2 and [11,12], the construction of the nonderived (m,n)-rings of ordinary ("rational") integers $\mathbb{Z}_{(m,n)}\left(a_q,b_q\right)$ (2.5) can be done in terms of residue class representatives (2.1). To introduce a p-adic analog of the residue class (2.1), one needs some ordering concept, which does not exist for p-adic integers [14]. Nevertheless, one could informally define the following analog of ordering.

Definition 1. *A "componentwise strict order" $<_{comp}$ is a multicomponent binary relation between p-adic numbers $\boldsymbol{a} = \{\alpha_i\}_{i=0}^{\infty}$, $0 \le \alpha_i \le p-1$ and $\boldsymbol{b} = \{\beta_i\}_{i=0}^{\infty}$, $0 \le \beta_i \le p-1$, such that*

$$\boldsymbol{a} <_{comp} \boldsymbol{b} \Longleftrightarrow \alpha_i < \beta_i, \quad \text{for all } i = 0, \dots, \infty, \quad \boldsymbol{a}, \boldsymbol{b} \in \mathbb{Z}_p, \quad \alpha_i, \beta_i \in \mathbb{Z}. \tag{4.1}$$

A "componentwise nonstrict order" $\le_{comp}$ is defined in the same way, but using the nonstrict order $\le$ for component integers from $\mathbb{Z}$ (digits).

Using this definition we can define a p-adic analog of the residue class informally by changing $\mathbb{Z}$ to $\mathbb{Z}_p$ in (2.1).

Definition 2. *A p-adic analog of the residue class of **a** modulo **b** is*

$$[\boldsymbol{a}]_{\boldsymbol{b}} = \left\{\{\boldsymbol{r}_{\boldsymbol{k}}(\boldsymbol{a}, \boldsymbol{b})\} \mid \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{k} \in \mathbb{Z}_p, 0 \le \boldsymbol{a} < \boldsymbol{b}\right\}, \tag{4.2}$$

and the generic representative of the class is

$$\boldsymbol{r}_{\boldsymbol{k}}(\boldsymbol{a}, \boldsymbol{b}) = \boldsymbol{a} +_p \boldsymbol{b} \bullet_p \boldsymbol{k}, \tag{4.3}$$

where $+_p$ and $\bullet_p$ are the binary sum and the binary product of p-adic integers (we treat them componentwise in the partial sum representation (3.7)), and the ith component of (4.3) r.h.s. is computed by mod p^i.

As with the ordinary ("rational") integers (2.1), the p-adic integer $\boldsymbol{a}$ can be treated as some kind of remainder for the representative $\boldsymbol{r}_{\boldsymbol{k}}(\boldsymbol{a}, \boldsymbol{b})$ when divided by the p-adic integer $\boldsymbol{b}$. We denote the corresponding p-adic analog of (2.2) (informally, a p-adic analog of the congruence modulo $\boldsymbol{b}$) as

$$\boldsymbol{r} = \boldsymbol{a}\left(\mathrm{Mod}_p\, \boldsymbol{b}\right). \tag{4.4}$$

Remark 1. *In general, to build a nonderived (m, n)-ring along the lines of* SECTION 2*, we do not need any analog of the residue class at all, but only the concrete form of the representative (4.3). Then demanding the closure of m-ary addition (2.3) and n-ary multiplication (2.4), we obtain conditions on the parameters (now digits of p-adic integers), similarly to (2.6)–(2.7).*

In the partial sum representation (3.6), the case of ordinary ("rational") integers corresponds to the first component (first digit α_0) of the p-adic integer (3.7), and higher components can be computed using the explicit formulas for sum and product of p-adic integers [15]. Because they are too cumbersome, we present here the "block-schemes" of the computations, while concrete examples can be obtained componentwise using (3.7).

Lemma 1. *The p-adic analog of the residue class (4.2) is a commutative m-ary group $\langle [\boldsymbol{a}]_{\boldsymbol{b}} \mid \boldsymbol{\nu}_m \rangle$, if*

$$(m-1)\boldsymbol{a} = \boldsymbol{b} \bullet_p \boldsymbol{I}, \tag{4.5}$$

where $\boldsymbol{I}$ is a p-adic integer (addition shape invariant), and the nonderived m-ary addition $\boldsymbol{\nu}_m$ is the repeated binary sum of m representatives $\boldsymbol{r}_{\boldsymbol{k}} = \boldsymbol{r}_{\boldsymbol{k}}(\boldsymbol{a}, \boldsymbol{b})$

$$\boldsymbol{\nu}_m\left[\boldsymbol{r}_{\boldsymbol{k}_1}, \boldsymbol{r}_{\boldsymbol{k}_2}, \dots, \boldsymbol{r}_{\boldsymbol{k}_m}\right] = \boldsymbol{r}_{\boldsymbol{k}_1} +_p \boldsymbol{r}_{\boldsymbol{k}_2} +_p \dots +_p \boldsymbol{r}_{\boldsymbol{k}_m}. \tag{4.6}$$

Proof. The condition of closure for the m-ary addition $\boldsymbol{\nu}_m$ is $\boldsymbol{r}_{\boldsymbol{k}_1} +_p \boldsymbol{r}_{\boldsymbol{k}_2} +_p \dots +_p \boldsymbol{r}_{\boldsymbol{k}_m} = \boldsymbol{r}_{\boldsymbol{k}_0}$ in the notation of (4.2). Using (4.3) it gives $m\boldsymbol{a} + \boldsymbol{b} \bullet_p \left(\boldsymbol{k}_1 +_p \boldsymbol{k}_2 +_p \dots +_p \boldsymbol{k}_m\right) = \boldsymbol{a} +_p \boldsymbol{b} \bullet_p \boldsymbol{k}_0$, which is equivalent to (4.5), where $\boldsymbol{I} = \boldsymbol{k}_0 -_p \left(\boldsymbol{k}_1 +_p \boldsymbol{k}_2 +_p \dots +_p \boldsymbol{k}_m\right)$. The querelement $\boldsymbol{r}_{\bar{\boldsymbol{k}}}$ [16] satisfies

$$\boldsymbol{\nu}_m[\boldsymbol{r}_{\boldsymbol{k}}, \boldsymbol{r}_{\boldsymbol{k}}, \dots, \boldsymbol{r}_{\boldsymbol{k}}, \boldsymbol{r}_{\bar{\boldsymbol{k}}}] = \boldsymbol{r}_{\boldsymbol{k}}, \tag{4.7}$$

which has a unique solution $\bar{\boldsymbol{k}} = (2-m)\boldsymbol{k} - \boldsymbol{I}$. Therefore, each element of $[\boldsymbol{a}]_{\boldsymbol{b}}$ is invertible with respect to $\boldsymbol{\nu}_m$, and $\langle [\boldsymbol{a}]_{\boldsymbol{b}} \mid \boldsymbol{\nu}_m \rangle$ is a commutative m-ary group. □

Lemma 2. *The p-adic analog of the residue class (4.2) is a commutative n-ary semigroup* $\langle [\boldsymbol{a}]_{\boldsymbol{b}} \mid \boldsymbol{\mu}_n \rangle$*, if*

$$\boldsymbol{a}^n - \boldsymbol{a} = \boldsymbol{b} \bullet_p \boldsymbol{J}, \tag{4.8}$$

where $\boldsymbol{J}$ *is a p-adic integer (multiplication shape invariant), and the nonderived m-ary multiplication* $\boldsymbol{\nu}_m$ *is the repeated binary product of n representatives*

$$\boldsymbol{\mu}_n\left[\boldsymbol{r}_{k_1}, \boldsymbol{r}_{k_2}, \ldots, \boldsymbol{r}_{k_n}\right] = \boldsymbol{r}_{k_1}\boldsymbol{r}_{k_2}\ldots\boldsymbol{r}_{k_n}. \tag{4.9}$$

Proof. The condition of closure for the n-ary multiplication $\boldsymbol{\mu}_n$ is $\boldsymbol{r}_{k_1} \bullet_p \boldsymbol{r}_{k_2} \bullet_p \ldots \bullet_p \boldsymbol{r}_{k_m} = \boldsymbol{r}_{k_0}$. Using (4.3) and opening brackets we obtain $n\boldsymbol{a} + \boldsymbol{b} \bullet_p \boldsymbol{J}_1 = \boldsymbol{a} +_p \boldsymbol{b} \bullet_p \boldsymbol{k}_0$, where $\boldsymbol{J}_1$ is some p-adic integer, which gives (4.8) with $\boldsymbol{J} = \boldsymbol{k}_0 -_p \boldsymbol{J}_1$. □

Combining the conditions (4.5) and (4.8), we arrive at

Theorem 1. *The p-adic analog of the residue class (4.2) becomes a* (m, n)*-ring with m-ary addition (4.6) and n-ary multiplication (4.9)*

$$\mathbb{Z}_{(m,n)}\left(\boldsymbol{a}_q, \boldsymbol{b}_q\right) = \left\langle \left[\boldsymbol{a}_q\right]_{\boldsymbol{b}_q} \mid \boldsymbol{\nu}_m, \boldsymbol{\mu}_n \right\rangle, \tag{4.10}$$

when the p-adic integers $\boldsymbol{a}_q, \boldsymbol{b}_q \in \mathbb{Z}_p$ *are solutions of the equations*

$$m\boldsymbol{a}_q = \boldsymbol{a}_q\left(\mathrm{Mod}_p\, \boldsymbol{b}_q\right), \tag{4.11}$$

$$\boldsymbol{a}_q^n = \boldsymbol{a}_q\left(\mathrm{Mod}_p\, \boldsymbol{b}_q\right). \tag{4.12}$$

Proof. The conditions (4.11)–(4.12) are equivalent to (4.5) and (4.8), respectively, which shows that $\left[\boldsymbol{a}_q\right]_{\boldsymbol{b}_q}$ (considered as a set of representatives (4.3)) is simultaneously an m-ary group with respect to $\boldsymbol{\nu}_m$, and an n-ary semigroup with respect to $\boldsymbol{\mu}_n$, and is therefore a (m, n)-ring. □

If we work in the partial sum representation (3.7), the procedure of finding the digits of p-adic integers $\boldsymbol{a}_q, \boldsymbol{b}_q \in \mathbb{Z}_p$ such that $\left[\boldsymbol{a}_q\right]_{\boldsymbol{b}_q}$ becomes a (m, n)-ring with initially fixed arities is recursive. To find the first digits α_0 and β_0 that are ordinary integers, we use the equations (2.6)–(2.9), and for their arity shape TABLE 1. Next we consider the second components of (3.7) to find the digits α_1 and β_1 of $\boldsymbol{a}_q$ and $\boldsymbol{b}_q$ by solving the equations (4.5) and (4.8) (these having initially given arities m and n from the first step) by application of the exact formulas from [15]. In this way, we can find as many digits $(\alpha_0, , \alpha_{i_{\max}})$ and $(\beta_0, , \beta_{i_{\max}})$ of $\boldsymbol{a}_q$ and $\boldsymbol{b}_q$, as needed for our accuracy preferences in building the polyadic ring of p-adic integers $\mathbb{Z}_{(m,n)}\left(\boldsymbol{a}_q, \boldsymbol{b}_q\right)$ (4.10).

Further development and examples will appear elsewhere.

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